Normal projections in Krein spaces II

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Dedicated to the memory of Prof. Heinz Langer

Abstract. We extend the study of normal projections in Krein spaces to the unbounded case. We characterize both weakly normal and normal projections, and prove that every normal projection admits a decomposition as the sum of a selfadjoint projection and a closed projection with neutral range. We show that every closed subspace $\mathcal{S}$ is the range of a (possibly unbounded) normal projection and parametrize the set of normal projections onto $\mathcal{S}$ using the notion of normal companions.

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1. Introduction

This work continues the study of normal projections in a Krein space started in [15], where bounded normal projections were characterized. A bounded operator T acting on a Krein space $(\mathcal{H}, [\ ,\ ])$ is normal if $T^{\#}T = TT^{\#}$, where $T^{\#}$ stands for the adjoint of T. Unlike the situation in Hilbert spaces, a bounded projection in a Krein space can be normal but not selfadjoint. The range of a bounded normal projection is a pseudo-regular subspace, i.e. a closed subspace $\mathcal{S}$ such that $\mathcal{S}+\mathcal{S}^{[\perp]}$ is also closed, for $\mathcal{S}^{[\perp]}$ the orthogonal companion of $\mathcal{S}$ in $\mathcal{H}$ [13]. Conversely, every pseudo-regular subspace is the range of a bounded normal projection. This projection is unique if and only if the subspace is regular (and in this case the projection is selfadjoint).

Normal projections appear naturally considering the spectral theory for (possibly unbounded) definitizable operators. If A is a definitizable operator acting on a Krein space $\mathcal{H}$, it can be shown that the Riesz projection onto a bounded spectral set of A is always a normal projection, and it is selfadjoint if and only if the spectral set is symmetric with respect to the real line, see [15, Example 1]. Spectral theory for definitizable operators was precisely one of the initial topics in Heinz Langer brilliant career.

Throughout this paper we study normal projections without assuming boundedness. A densely defined operator T is normal if T is closed, $\operatorname{dom} T = \operatorname{dom} T^{\#}$ and

$$[Tx, Tx] = \left[T^{\#}x, T^{\#}x\right]$$

for every $x \in \operatorname{dom} T$, see [4]. In particular, if T is normal then T is weakly normal, in the sense that $T^{\#}T = TT^{\#}$. The converse does not hold, see Example 3, for a weakly normal projection which is not normal. Given a closed densely defined projection Q, we show that Q is weakly normal if and only if $\ker Q^{\#}Q = \ker QQ^{\#}$, and it is normal if and only if both Q and $I - Q$ are weakly normal projections.

If Q is a bounded normal projection then it can be decomposed as $Q = E + P$, where $E = QQ^{\#}$ is a selfadjoint projection and $P = Q(I - Q^{\#})$ is a neutral normal projection i.e. it satisfies $P^{\#}P = PP^{\#} = 0$, see [15]. More generally, in Theorem 3.10 we show that a closed projection Q is normal if and only if $\operatorname{dom} Q = \operatorname{dom} Q^{\#}$ and there exist a selfadjoint projection E and a closed projection P with $PP^{\#}|_{\operatorname{dom} Q} = P^{\#}P|_{\operatorname{dom} Q} = 0|_{\operatorname{dom} Q}$ such that $Q = E + P$. In this case, $E = \overline{QQ^{\#}}$ and $P = \overline{Q(I - Q^{\#})}$.

Only closed non-degenerated subspaces of a Krein space are ranges of (densely defined) selfadjoint projections [12], but it is easy to check that every closed subspace is the range of a unique selfadjoint multivalued projection. Multivalued projections were introduced in [6] and later studied in [14]. They are a class of linear relations which preserve many properties of single-valued projections, for instance, they are fully described by their ranges and kernels. In particular, a multivalued projection E is selfadjoint if and only if $\operatorname{ran} E$ is closed and $\ker E = \operatorname{ran} E^{[\perp]}$.

If $E_{\mathcal{S}}$ is the selfadjoint multivalued projection onto a closed subspace $\mathcal{S}$, we show that the completely singular part is determined by the isotropic part $\mathcal{S}^{\circ}$ of $\mathcal{S}$, and the restrictions of normal projections onto $\mathcal{S}$ to $\mathcal{S} + \mathcal{S}^{[\perp]}$ provide a family of operator parts of $E_{\mathcal{S}}$ which are projections.

The paper is organized as follows. In Section 2, we collect key concepts on Krein spaces, as the notions of weak and strong dual pairs. Section 3 is devoted to describe normal projections. In particular, it is shown that every normal projection admits a unique decomposition into the sum of a selfadjoint projection and a closed projection with neutral range. In Section 4, we introduce the notion of a *normal companion* for a closed subspace and give a description and parametrization of the set of normal projections with a fixed range. To this end, we show that every closed subspace is the range of a normal projection and that all such projections can be constructed using normal companions. Section 5 deals with multivalued projections: we recall the framework of linear relations and prove that every closed subspace $\mathcal{S}$ of a Krein space is the range of a unique selfadjoint multivalued projection $E_{\mathcal{S}}$. Finally, we show that $E_{\mathcal{S}}$ admits a decomposition into an operator part and a singular part determined by $\mathcal{S}^{\circ}$. The operator part is not unique, but the set of normal projections onto $\mathcal{S}$ provides a family of operator parts for $E_{\mathcal{S}}$ that are projections.

2. Preliminaries

We assume that all Hilbert spaces are complex and separable. If $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces, $L(\mathcal{H},\mathcal{K})$ stands for the space of bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$. When $\mathcal{H}=\mathcal{K}$ we write, for short, $L(\mathcal{H})$. The direct sum of two subspaces $\mathcal{M}$ and $\mathcal{N}$ of $\mathcal{H}$ is indicated by $\mathcal{M}\dot{+}\mathcal{N}$, and $\mathcal{M}\oplus\mathcal{N}$ if $\mathcal{M}\subseteq\mathcal{N}^{\perp}$. In addition, if $\mathcal{N}\subseteq\mathcal{M}$, $\mathcal{M}\ominus\mathcal{N}$ means $\mathcal{M}\cap\mathcal{N}^{\perp}$. Given T an operator from $\mathcal{H}$ into $\mathcal{K}$, $\operatorname{dom}T$, $\operatorname{ran}T$ and $\ker T$ denote the domain, range and kernel of T, respectively.

Lemma 2.1 **([17, Eq. (1.1)]).** *Let T and S be operators from $\mathcal{H}$ to $\mathcal{K}$, R be an operator from $\mathcal{K}$ to $\mathcal{E}$ and U be an operator from $\mathcal{E}$ to $\mathcal{H}$. Then*

$$R(S+T)\supseteq RS+RT \text{ and } (S+T)U=SU+TU. \tag{2.1}$$

A densely defined operator P in $\mathcal{H}$ is said to be a *projection* if $P^2=P$. That is, $\operatorname{ran}P\subseteq\operatorname{dom}P$ and $P^2x=Px$ for every $x\in\operatorname{dom}P$. We denote by $P_{\mathcal{M}//\mathcal{N}}$ the projection with range $\mathcal{M}$ and nullspace $\mathcal{N}$. By [16, Lemma 3.5], $\operatorname{dom}P_{\mathcal{M}//\mathcal{N}}=\mathcal{M}\dot{+}\mathcal{N}$ and $P_{\mathcal{M}//\mathcal{N}}$ is a closed operator if and only if $\mathcal{M}$ and $\mathcal{N}$ are closed subspaces. Moreover, if $P_{\mathcal{M}//\mathcal{N}}$ is densely defined then $(P_{\mathcal{M}//\mathcal{N}})^*=P_{\mathcal{N}^{\perp}//\mathcal{M}^{\perp}}$. So that, if also $(P_{\mathcal{M}//\mathcal{N}})^*$ is densely defined then $\overline{P_{\mathcal{M}//\mathcal{N}}}=P_{\overline{\mathcal{M}}//\overline{\mathcal{N}}}$.

Lemma 2.2. *Let P and Q be (densely defined) projections such that $P+Q$ is also a projection. Then $PQ|_{\operatorname{dom}P\cap\operatorname{dom}Q}=QP|_{\operatorname{dom}P\cap\operatorname{dom}Q}=0|_{\operatorname{dom}P\cap\operatorname{dom}Q}$, $\operatorname{ran}(P+Q)=\operatorname{ran}P+\operatorname{ran}Q$ and $\ker(P+Q)=\ker P\cap\ker Q$.*

Krein Spaces

Although familiarity with operator theory on Krein spaces is presumed, we include some basic notions. Standard references on Krein spaces and operators on them are [1], [3], [5] and [11]. We also refer to [7] and [8] as authoritative accounts of the subject.

Consider a linear space $\mathcal{H}$ with an indefinite metric; i.e., a sesquilinear Hermitian form $[\ ,\ ]$. A vector $x\in\mathcal{H}$ is said to be *positive* if $[x,x]>0$. A subspace $\mathcal{S}$ of $\mathcal{H}$ is *positive* if every $x\in\mathcal{S}$, $x\neq 0$, is a positive vector. *Negative, nonnegative, nonpositive* and *neutral* vectors and subspaces are defined likewise.

Two closed subspaces $\mathcal{M}$ and $\mathcal{N}$ are *orthogonal* if $[m,n]=0$ for every $m\in\mathcal{M}$ and $n\in\mathcal{N}$. Denote the orthogonal sum of two closed subspaces $\mathcal{M}$ and $\mathcal{N}$ by $\mathcal{M}\,[+]\,\mathcal{N}$ and $\mathcal{M}\,[\dot{+}]\,\mathcal{N}$ if $\mathcal{M}\cap\mathcal{N}=\{0\}$.

An indefinite metric space $(\mathcal{H},[\ ,\ ])$ is a *Krein space* if it admits a decomposition as an orthogonal direct sum

$$\mathcal{H}=\mathcal{H}_+\,[\dot{+}]\,\mathcal{H}_-, \tag{2.2}$$

where $(\mathcal{H}_+,[\ ,\ ])$ and $(\mathcal{H}_-,-[\ ,\ ])$ are Hilbert spaces. Any decomposition with these properties is called a *fundamental decomposition* of $\mathcal{H}$.

Given a Krein space $(\mathcal{H},[\ ,\ ])$ with a fundamental decomposition like in (2.2), the (orthogonal) direct sum of the Hilbert spaces $(\mathcal{H}_+,[\ ,\ ])$ and $(\mathcal{H}_-,-[\ ,\ ])$ is a Hilbert space, $(\mathcal{H},\langle\ ,\ \rangle)$. Notice that the inner product $\langle\ ,\ \rangle$ and the corresponding norm $\|\ \|$ depend on the fundamental decomposition.

Given any subspace $\mathcal{S}$ of $\mathcal{H}$, the *orthogonal companion* of $\mathcal{S}$ in $\mathcal{H}$ is defined as

$$\mathcal{S}^{[\perp]} := \{x \in \mathcal{H} : [\,x, s\,] = 0 \text{ for every } s \in \mathcal{S}\}.$$

The *isotropic part* of $\mathcal{S}$, $\mathcal{S}^\circ := \mathcal{S} \cap \mathcal{S}^{[\perp]}$, can be a non-trivial subspace. A subspace $\mathcal{S} \subseteq \mathcal{H}$ is *non-degenerate* if $\mathcal{S} \cap \mathcal{S}^{[\perp]} = \{0\}$. Given a closed non-degenerate subspace $\mathcal{M}$ of $\mathcal{H}$, $E_{\mathcal{M}}$ denotes the densely defined selfadjoint projection onto $\mathcal{M}$ with nullspace $\mathcal{M}^{[\perp]}$.

Lemma 2.3 ([5, Ch.I Lemma 5.1])**.** *Given a closed subpace $\mathcal{S}$ of a Krein space $\mathcal{H}$, if $\mathcal{S} = \mathcal{M} \dot{+} \mathcal{S}^\circ$ then $\mathcal{S} = \mathcal{M}[\dot{+}]\mathcal{S}^\circ$ and $\mathcal{M}$ is non-degenerate.*

Lemma 2.4. *Let $\mathcal{S}$, $\mathcal{N}$ and $\mathcal{M}$ be closed subspaces of a Krein space $\mathcal{H}$, such that $\mathcal{M}$ is non-degenerate and $\mathcal{N}$ is neutral. If $\mathcal{S} = \mathcal{M}[\dot{+}]\mathcal{N}$ then $\mathcal{N} = \mathcal{S}^\circ$.*

If $\mathcal{H}$ is a Krein space, $L(\mathcal{H})$ stands for the Banach algebra of all the linear operators on $\mathcal{H}$ which are bounded in an associated Hilbert space $(\mathcal{H}, \langle\ ,\ \rangle)$. Since the norms generated by different fundamental decompositions of a Krein space $\mathcal{H}$ are equivalent (see, for instance, [3, Theorem 7.19]), $L(\mathcal{H})$ does not depend on the chosen underlying Hilbert space.

A subspace $\mathcal{S}$ of a Krein space $\mathcal{H}$ is *regular* if it is itself a Krein space in the indefinite metric of $\mathcal{H}$. A subspace $\mathcal{S}$ is regular if and only if $\mathcal{H} = \mathcal{S}\ [\dot{+}]\ \mathcal{S}^{[\perp]}$.

A closed subspace $\mathcal{S}$ of a Krein space $\mathcal{H}$ is *pseudo-regular* if the algebraic sum $\mathcal{S}[+]\mathcal{S}^{[\perp]}$ is closed. Clearly, $\mathcal{S}$ is (pseudo) regular if and only if $\mathcal{S}^{[\perp]}$ is (pseudo) regular. The following proposition compiles several conditions which are equivalent to pseudo-regularity, see for instance [13].

Proposition 2.5. *Let $\mathcal{S}$ be a closed subspace of a Krein space $\mathcal{H}$. Then the following are equivalent:*

- $i)$ *$\mathcal{S}$ is pseudo-regular;*
- $ii)$ $\mathcal{S}^{\circ[\perp]} = \mathcal{S} + \mathcal{S}^{[\perp]}$;
- $iii)$ *There exists a regular subspace $\mathcal{M}$ such that $\mathcal{S} = \mathcal{M}[\dot{+}]\mathcal{S}^\circ$;*
- $iv)$ *If $\mathcal{S} = \mathcal{T} \dot{+} \mathcal{S}^\circ$ with $\mathcal{T}$ closed then $\mathcal{T}$ is regular.*

From now on, $\mathcal{H}$ is assumed to be a Krein space. In the following we recall two useful notions, those of weakly and strongly dual pairs [3, 10].

Definition. Let $\mathcal{S}$ and $\mathcal{T}$ be two closed subspaces of $\mathcal{H}$. We say that $\mathcal{S}$ and $\mathcal{T}$ are *skewly linked* (or in *weak duality*) if $\mathcal{S} \cap \mathcal{T}^{[\perp]} = \mathcal{S}^{[\perp]} \cap \mathcal{T} = \{0\}$, and we write $\mathcal{S}\#\mathcal{T}$. We say that $\mathcal{S}$ and $\mathcal{T}$ are in *strong duality* or form a *strongly dual pair* if $\mathcal{S} \dot{+} \mathcal{T}^{[\perp]} = \mathcal{H}$.

Note that $\mathcal{S}$ and $\mathcal{T}$ are skewly linked if and only if $\mathcal{H} = \overline{\mathcal{S} \dot{+} \mathcal{T}^{[\perp]}}$, and that $\mathcal{S}$ and $\mathcal{T}$ form a strongly dual pair if and only if $\mathcal{S}^{[\perp]}$ and $\mathcal{T}^{[\perp]}$ form a strongly dual pair. It is straightforward that a closed subspace $\mathcal{S}$ is regular if and only if it is in strong duality with itself, and $\mathcal{S}$ is non-degenerate if and only if it is in weak duality with itself.

Every fundamental decomposition of $\mathcal{H}$ as in (2.2) has an associated *signature operator*: $J := P_+ - P_-$ with $P_\pm := P_{\mathcal{H}_\pm//\mathcal{H}_\mp}$. We denote by $\mathcal{J}$ the set of signature operators for $(\mathcal{H}, [\ ,\ ])$. The indefinite metric and the inner product corresponding to a fundamental decomposition of $\mathcal{H}$ with signature operator J are related to each other by

$$\langle x, y \rangle = [\, Jx, y\,] \quad x, y \in \mathcal{H}.$$

Since $J = J^{-1} = J^*$ (where T^* stands for the adjoint of T with respect to $\langle \cdot, \cdot \rangle$), it can be checked that, given a subspace $\mathcal{S} \subseteq \mathcal{H}$,

$$\mathcal{S}^{[\perp]} = J\mathcal{S}^{\perp} = (J\mathcal{S})^{\perp},$$

where $\mathcal{S}^{\perp}$ denotes the orthogonal complement of $\mathcal{S}$ with respect to $\langle \cdot, \cdot \rangle$. The symbols $\oplus$ and $\ominus$ are defined accordingly.

Also, if J is any signature operator of $\mathcal{H}$, since $(J\mathcal{S})^{[\perp]} = \mathcal{S}^{\perp}$ then $\mathcal{S}$ and $J\mathcal{S}$ form a strongly dual pair. But there are subspaces $\mathcal{S}$ and $\mathcal{T}$ that form a strongly dual pair but $\mathcal{S} \neq J\mathcal{T}$ for any signature operator J of $\mathcal{H}$, see [10, Example 2.3]. However, if $\mathcal{S}$ and $\mathcal{T}$ are neutral subspaces, it is always possible to find such a signature operator, see [10, Theorem 4.2].

Theorem 2.6. *Let $\mathcal{S}$ and $\mathcal{T}$ be two closed neutral subspaces of $\mathcal{H}$. Then the following are equivalent:*

$i)$ *$\mathcal{S}$ and $\mathcal{T}$ are a strongly dual pair;*
$ii)$ *$\mathcal{S} \dotplus \mathcal{T}$ is a regular subspace;*
$iii)$ *There exists a signature operator J of $\mathcal{H}$ such that $\mathcal{S} = J\mathcal{T}$.*

By [3, Remark 1.30], if $\mathcal{S}$ and $\mathcal{T}$ are neutral then $\mathcal{S}$ and $\mathcal{T}$ are skewly linked if and only if $\mathcal{S} \dotplus \mathcal{T}$ is non-degenerate. See [3, Example 1.33] for an example of a non trivial weakly dual pair that is not a strongly dual pair.

3. Normal projections in Krein spaces

In this section, we study normal projections. Bounded normal projections were studied in [15]. For a densely defined operator T in $\mathcal{H}$, $T^{\#}$ stands for the adjoint of T with respect to $[\cdot, \cdot]$.

The notion of normal operators in Krein spaces was given in [4]. We also consider a weaker notion of normality inspired by [17, Proposition 3.25]. Both definitions coincide if $T \in L(\mathcal{H})$.

Definition. We say that a densely defined operator T is *weakly normal* if $T^{\#}T = TT^{\#}$ and that T is *normal* if T is closed, $\operatorname{dom} T = \operatorname{dom} T^{\#}$ and $[\,Tx, Tx\,] = \left[\,T^{\#}x, T^{\#}x\,\right]$ for every $x \in \operatorname{dom} T$.

If T is normal then $T^{\#}$ and $I - T$ are normal. Following the lines of the proof of Proposition 3.25 in [17], it can be proved that if T is normal then T is weakly normal. Also, if T is weakly normal and closed then $T^{\#}$ is weakly normal.

Example 1. Let $\mathcal{S}$ and $\mathcal{T}$ be two neutral subspaces of $\mathcal{H}$, such that $\mathcal{S}\#\mathcal{T}$. Then $Q := P_{\mathcal{T}^{[\perp]}//\mathcal{S}}$ is weakly normal.

In fact, Q is densely defined with $Q^{\#} = P_{\mathcal{S}^{[\perp]}//\overline{\mathcal{T}}}$ and then, since $\mathcal{S}$ and $\mathcal{T}$ are neutral, $\overline{\mathcal{T}} \subseteq \mathcal{T}^{[\perp]} = \overline{\mathcal{T}}^{[\perp]}$ and

$$\operatorname{dom} Q^{\#}Q = \mathcal{S} + \mathcal{T}^{[\perp]} \cap (\mathcal{S}^{[\perp]} + \overline{\mathcal{T}}) = \mathcal{S} + \overline{\mathcal{T}} + \mathcal{T}^{[\perp]} \cap \mathcal{S}^{[\perp]}$$
$$\operatorname{dom} QQ^{\#} = \overline{\mathcal{T}} + \mathcal{S}^{[\perp]} \cap (\mathcal{S} + \mathcal{T}^{[\perp]}) = \overline{\mathcal{T}} + \mathcal{S} + \mathcal{S}^{[\perp]} \cap \mathcal{T}^{[\perp]}.$$

So that $\operatorname{dom} Q^{\#}Q = \operatorname{dom} QQ^{\#}$. If $x \in \operatorname{dom} Q^{\#}Q$ then $x = t + s + y$ with $t \in \overline{\mathcal{T}}$, $s \in \mathcal{S}$ and $y \in \mathcal{S}^{[\perp]} \cap \mathcal{T}^{[\perp]}$ so

$$Q^{\#}Qx = Q^{\#}(t + y) = y = Q(s + y) = QQ^{\#}x$$

and Q is weakly normal. But if $\mathcal{S}$ is not closed, then Q is not closed, and therefore not normal. See [3, Example 1.33] for an explicit pair of subspaces $\mathcal{S}$ and $\mathcal{T}$ satisfying these conditions.

Example 2. Consider $\mathcal{H}$ such that $\dim \mathcal{H}_+ = \dim \mathcal{H}_-$, where $\mathcal{H} = \mathcal{H}_+[\dot{+}]\mathcal{H}_-$ is a fundamental decomposition. If $(\mathcal{S}, \mathcal{T})$ is any pair of hypermaximal neutral subspaces in $\mathcal{H}$ (i.e. $\mathcal{S}^{[\perp]} = \mathcal{S}$ and $\mathcal{T}^{[\perp]} = \mathcal{T}$, see [3, Definition 4.15]) with $\mathcal{S}\#\mathcal{T}$ then $P := P_{\mathcal{S}//\mathcal{T}}$ is normal. In fact, since $\mathcal{S}^{[\perp]} = \mathcal{S}$ and $\mathcal{T}^{[\perp]} = \mathcal{T}$, $P^{\#} = I - P$, P is closed, $\operatorname{dom} P = \operatorname{dom} P^{\#}$ and if $x = s + t$ with $s \in \mathcal{S}$ and $t \in \mathcal{T}$ then $[Px, Px] = [s, s] = 0 = [t, t] = [P^{\#}x, P^{\#}x]$.

From now on we assume that $\mathcal{S}$ and $\mathcal{T}$ are closed, skewly linked subspaces of $\mathcal{H}$. In this case, $P_{\mathcal{S}//\mathcal{T}^{[\perp]}}$ and $P_{\mathcal{T}//\mathcal{S}^{[\perp]}}$ are closed densely defined projections, and $(P_{\mathcal{S}//\mathcal{T}^{[\perp]}})^{\#} = P_{\mathcal{T}//\mathcal{S}^{[\perp]}}$.

Proposition 3.1. *Let $Q = P_{\mathcal{S}//\mathcal{T}^{[\perp]}}$. The following are equivalent:*

$i)$ *Q is weakly normal;*
$ii)$ $\ker QQ^{\#} = \ker Q^{\#}Q$;
$iii)$ $\operatorname{dom} Q^{\#}Q = \operatorname{dom} QQ^{\#} = \mathcal{S} \cap \mathcal{T}[\dot{+}]\mathcal{S}^{[\perp]} \cap \mathcal{T}^{[\perp]}[\dot{+}](\mathcal{S}^{\circ} \dot{+} \mathcal{T}^{\circ})$.

Proof. The implication $i) \Rightarrow ii)$ is trivial.

If $ii)$ holds then

$$\mathcal{S}^{[\perp]} \dot{+} \mathcal{T}^{\circ} = \ker QQ^{\#} = \ker Q^{\#}Q = \mathcal{S}^{\circ} \dot{+} \mathcal{T}^{[\perp]}.$$

Therefore, $\mathcal{S}^{[\perp]} = \mathcal{S}^{\circ}[\dot{+}]\mathcal{S}^{[\perp]} \cap \mathcal{T}^{[\perp]}$ and $\mathcal{T}^{[\perp]} = \mathcal{T}^{\circ}[\dot{+}]\mathcal{S}^{[\perp]} \cap \mathcal{T}^{[\perp]}$. Taking orthogonal companions, it follows that $\mathcal{S} \cap \mathcal{T}^{\circ[\perp]} = \mathcal{T} \cap \mathcal{S}^{\circ[\perp]}$ and then $\mathcal{S} \cap \mathcal{T}^{\circ[\perp]} = \mathcal{T} \cap \mathcal{S}^{\circ[\perp]} = \mathcal{S} \cap \mathcal{T}$. In particular, $\mathcal{S} \cap \mathcal{T} = \mathcal{S} \cap (\mathcal{T} + \mathcal{T}^{[\perp]})$. Then, since $\operatorname{dom} Q^{\#} = \mathcal{T} \dot{+} \mathcal{S}^{[\perp]} = \mathcal{S}^{\circ} \dot{+} (\mathcal{T} + \mathcal{T}^{[\perp]})$, it follows that

$$\begin{aligned}\operatorname{dom} Q^{\#}Q &= \mathcal{T}^{[\perp]} + \mathcal{S} \cap \operatorname{dom} Q^{\#}\\ &= \mathcal{T}^{[\perp]} + \mathcal{S} \cap (\mathcal{S}^{\circ} + \mathcal{T} + \mathcal{T}^{[\perp]})\\ &= \mathcal{T}^{[\perp]} + \mathcal{S}^{\circ} + \mathcal{S} \cap \mathcal{T} = \mathcal{S} \cap \mathcal{T}[\dot{+}]\mathcal{S}^{[\perp]} \cap \mathcal{T}^{[\perp]}[\dot{+}](\mathcal{S}^{\circ} \dot{+} \mathcal{T}^{\circ}).\end{aligned}$$

In the same fashion, $\mathcal{S}^{\circ[\perp]} \cap \mathcal{T} = \mathcal{S} \cap \mathcal{T} = (\mathcal{S} + \mathcal{S}^{[\perp]}) \cap \mathcal{T}$, and $\operatorname{dom} Q^{\#}Q = \operatorname{dom} QQ^{\#}$ so $iii)$ follows.

Finally, if $iii)$ holds, let $x \in \operatorname{dom} QQ^{\#} = \operatorname{dom} Q^{\#}Q$ then $x = x_1 + s_0 + x_2 + t_0$ with $x_1 \in \mathcal{S} \cap \mathcal{T}$, $s_0 \in \mathcal{S}^{\circ}$, $x_2 \in \mathcal{S}^{[\perp]} \cap \mathcal{T}^{[\perp]}$ and $t_0 \in \mathcal{T}^{\circ}$. So that $Qx = x_1 + s_0$ and $Q^{\#}x = x_1 + t_0$. Hence $Q^{\#}Qx = x_1 = QQ^{\#}x$. □

Corollary 3.2. *$P_{\mathcal{S}//\mathcal{T}^{[\perp]}}$ is weakly normal if and only if*

$$\mathcal{S}^{[\perp]} = \mathcal{S}^{\circ}[\dot{+}]\mathcal{S}^{[\perp]} \cap \mathcal{T}^{[\perp]} \text{ and } \mathcal{T}^{[\perp]} = \mathcal{T}^{\circ}[\dot{+}]\mathcal{S}^{[\perp]} \cap \mathcal{T}^{[\perp]}.$$

It follows from Lemma 2.4 and Corollary 3.2 that if $P_{\mathcal{S}//\mathcal{T}^{[\perp]}}$ is weakly normal then $\mathcal{M} := \mathcal{S}^{[\perp]} \cap \mathcal{T}^{[\perp]}$ is non-degenerate and

$$\mathcal{S}^{[\perp]} = \mathcal{M}[\dot{+}]\mathcal{S}^{\circ} \text{ and } \mathcal{T}^{[\perp]} = \mathcal{M}[\dot{+}]\mathcal{T}^{\circ}. \tag{3.1}$$

Also, $\mathcal{S}^{\circ}\#\mathcal{T}^{\circ}$. In fact,

$$\{0\} = \mathcal{S} \cap \mathcal{T}^{[\perp]} = (\mathcal{M}^{[\perp]} \cap \mathcal{S}^{\circ[\perp]}) \cap (\mathcal{M} \dot{+} \mathcal{T}^{\circ}) = \mathcal{S}^{\circ[\perp]} \cap \mathcal{T}^{\circ},$$

where we used that $\mathcal{S}^{\circ} + \mathcal{T}^{\circ} \subseteq \mathcal{M}^{[\perp]}$. Likewise, $\mathcal{T}^{\circ[\perp]} \cap \mathcal{S}^{\circ} = \{0\}$. Conversely, the following holds.

Proposition 3.3. *Suppose that $\mathcal{N}_1$ and $\mathcal{N}_2$ are closed neutral subspaces, $\mathcal{N}_1\#\mathcal{N}_2$, and there exists a non-degenerate subspace $\mathcal{M}$ such that $\mathcal{M}[\dot{+}]\mathcal{N}_1$ and $\mathcal{M}[\dot{+}]\mathcal{N}_2$ are closed. If $\mathcal{S} := \mathcal{M}^{[\perp]} \cap \mathcal{N}_1{}^{[\perp]}$ and $\mathcal{T} := \mathcal{M}^{[\perp]} \cap \mathcal{N}_2{}^{[\perp]}$ then the projection $P_{\mathcal{S}//\mathcal{T}^{[\perp]}}$ is weakly normal.*

Proof. First, notice that $\mathcal{S}^{[\perp]} = \mathcal{M}[\dot{+}]\mathcal{N}_1$ and $\mathcal{T}^{[\perp]} = \mathcal{M}[\dot{+}]\mathcal{N}_2$. So that, by Lemma 2.4, $\mathcal{N}_1 = (\mathcal{S}^{[\perp]})^{\circ} = \mathcal{S}^{\circ}$, $\mathcal{N}_2 = \mathcal{T}^{\circ}$. Also,

$$\mathcal{S} \cap \mathcal{T}^{[\perp]} = \mathcal{M}^{[\perp]} \cap \mathcal{N}_1{}^{[\perp]} \cap (\mathcal{M}[\dot{+}]\mathcal{N}_2) = \mathcal{M}^{[\perp]} \cap \mathcal{M} = \{0\},$$

because $\mathcal{N}_1 \subseteq \mathcal{M}^{[\perp]}$, $\mathcal{N}_1\#\mathcal{N}_2$ and $\mathcal{M}$ is non-degenerate. Likewise, $\mathcal{T} \cap \mathcal{S}^{[\perp]} = \{0\}$. Set $Q := P_{\mathcal{S}//\mathcal{T}^{[\perp]}}$ then Q is a closed densely defined projection. It holds that

$$\mathcal{S}^{[\perp]} \cap \mathcal{T}^{[\perp]} = (\mathcal{M}[\dot{+}]\mathcal{N}_1) \cap (\mathcal{M}[\dot{+}]\mathcal{N}_2) = \mathcal{M},$$

because $(\mathcal{M}[\dot{+}]\mathcal{N}_1) \cap \mathcal{N}_2 \subseteq \mathcal{N}_1{}^{[\perp]} \cap \mathcal{N}_2 = \{0\}$. Then, by Corollary 3.2, Q is weakly normal. □

Note that the weak normality of Q in Example 1 follows immediately from Proposition 3.3, taking $\mathcal{M} = \{0\}$.

Theorem 3.4. *The projection $P_{\mathcal{S}//\mathcal{T}^{[\perp]}}$ is weakly normal if and only if*

$$\mathcal{S}^{[\perp]} \subseteq \mathcal{S}^{\circ} \dot{+} \mathcal{T}^{\circ[\perp]}, \quad \mathcal{T}^{[\perp]} \subseteq \mathcal{T}^{\circ} \dot{+} \mathcal{S}^{\circ[\perp]}, \tag{3.2}$$

and

$$\mathcal{S}^{[\perp]} \cap \mathcal{T}^{\circ[\perp]} = \mathcal{T}^{[\perp]} \cap \mathcal{S}^{\circ[\perp]}. \tag{3.3}$$

Proof. If $P_{\mathcal{S}//\mathcal{T}^{[\perp]}}$ is weakly normal, by Corollary 3.2, $\mathcal{S}^{[\perp]} = \mathcal{T}^{[\perp]} \cap \mathcal{S}^{[\perp]}[\dot{+}]\mathcal{S}^{\circ} \subseteq \mathcal{S}^{\circ} \dot{+} \mathcal{T}^{\circ[\perp]}$. The inclusion for $\mathcal{T}^{[\perp]}$ is similar. Since, by the remark before Proposition 3.3, $\mathcal{S}^{\circ}\#\mathcal{T}^{\circ}$,

$$\mathcal{S}^{[\perp]} \cap \mathcal{T}^{\circ[\perp]} = (\mathcal{T}^{[\perp]} \cap \mathcal{S}^{[\perp]}[\dot{+}]\mathcal{S}^{\circ}) \cap \mathcal{T}^{\circ[\perp]} = \mathcal{T}^{[\perp]} \cap \mathcal{S}^{[\perp]}.$$

The other equality follows in the same fashion.

Conversely, suppose that (3.2) and (3.3) hold. Then $\mathcal{S}^{[\perp]} = \mathcal{S}^{\circ}[\dot{+}]\mathcal{S}^{[\perp]} \cap \mathcal{T}^{\circ[\perp]} = \mathcal{S}^{\circ} \dot{+} \mathcal{S}^{[\perp]} \cap \mathcal{T}^{[\perp]}$. Likewise, $\mathcal{T}^{[\perp]} = \mathcal{T}^{\circ} \dot{+} \mathcal{S}^{[\perp]} \cap \mathcal{T}^{[\perp]}$. Then, by Corollary 3.2, $P_{\mathcal{S}//\mathcal{T}^{[\perp]}}$ is weakly normal. □

The following result characterizes normal projections.

Theorem 3.5. *Let $Q = P_{\mathcal{S}//\mathcal{T}^{[\perp]}}$. The following are equivalent:*

$i)$ *Q is normal;*
$ii)$ *Q and $I - Q$ are weakly normal;*
$iii)$ $\operatorname{dom} Q = \operatorname{dom} Q^{\#} = \mathcal{S} \cap \mathcal{T}[\dotplus]\mathcal{S}^{[\perp]} \cap \mathcal{T}^{[\perp]}[\dotplus](\mathcal{S}^{\circ} \dotplus \mathcal{T}^{\circ})$.

Proof. If Q is normal then Q and $I - Q$ are weakly normal.

If $ii)$ holds, by Corollary 3.2,

$$\mathcal{S} = \mathcal{S} \cap \mathcal{T}[\dotplus]\mathcal{S}^{\circ}, \qquad \mathcal{T}^{[\perp]} = \mathcal{S}^{[\perp]} \cap \mathcal{T}^{[\perp]}[\dotplus]\mathcal{T}^{\circ},$$
$$\mathcal{T} = \mathcal{S} \cap \mathcal{T}[\dotplus]\mathcal{T}^{\circ}, \ \text{ and } \ \mathcal{S}^{[\perp]} = \mathcal{S}^{[\perp]} \cap \mathcal{T}^{[\perp]}[\dotplus]\mathcal{S}^{\circ};$$

then $iii)$ follows.

Finally, assume that $iii)$ holds. If $x \in \operatorname{dom} Q$ then $x = x_1 + s_0 + x_2 + t_0$ with $x_1 \in \mathcal{S} \cap \mathcal{T}$, $s_0 \in \mathcal{S}^{\circ}$, $x_2 \in \mathcal{S}^{[\perp]} \cap \mathcal{T}^{[\perp]}$ and $t_0 \in \mathcal{T}^{\circ}$. So that

$$\begin{aligned}[\,Qx, Qx\,] &= [\,x_1 + s_0, x_1 + s_0\,] = [\,x_1, x_1\,] \\ &= [\,x_1 + t_0, x_1 + t_0\,] = \left[\,Q^{\#}x, Q^{\#}x\,\right].\end{aligned}$$

Also, Q is closed and $\operatorname{dom} Q = \operatorname{dom} Q^{\#}$. Therefore, Q is normal. ☐

Corollary 3.6. *The following are equivalent:*

$i)$ *$P_{\mathcal{S}//\mathcal{T}^{[\perp]}}$ is normal;*
$ii)$ *$\mathcal{S} + \mathcal{T}^{\circ} = \mathcal{S}^{\circ} + \mathcal{T}$ and $\mathcal{S}^{[\perp]} + \mathcal{T}^{\circ} = \mathcal{S}^{\circ} + \mathcal{T}^{[\perp]}$;*
$iii)$

$$\begin{aligned}&\mathcal{S} = \mathcal{S} \cap \mathcal{T}[\dotplus]\mathcal{S}^{\circ}, \qquad \mathcal{T}^{[\perp]} = \mathcal{S}^{[\perp]} \cap \mathcal{T}^{[\perp]}[\dotplus]\mathcal{T}^{\circ}, \\ &\mathcal{T} = \mathcal{S} \cap \mathcal{T}[\dotplus]\mathcal{T}^{\circ}, \ \textit{and} \ \mathcal{S}^{[\perp]} = \mathcal{S}^{[\perp]} \cap \mathcal{T}^{[\perp]}[\dotplus]\mathcal{S}^{\circ}.\end{aligned} \tag{3.4}$$

If $P_{\mathcal{S}//\mathcal{T}^{[\perp]}}$ is normal, it follows from Corollary 3.6 that

$$\mathcal{S} \dotplus \mathcal{T}^{[\perp]} = \mathcal{S}^{\circ} \dotplus (\mathcal{T} + \mathcal{T}^{[\perp]}) = \mathcal{T}^{\circ} \dotplus (\mathcal{S} + \mathcal{S}^{[\perp]}). \tag{3.5}$$

Definition. Given $\mathcal{S}$ and $\mathcal{T}$ skewly linked closed subspaces of $\mathcal{H}$, we say that $\mathcal{S}$ and $\mathcal{T}$ form a *normal pair*, and denote it by $\mathcal{S}\#_n\mathcal{T}$, if

$$\mathcal{S} + \mathcal{S}^{[\perp]} \subseteq \mathcal{S}^{\circ} \dotplus \mathcal{T}^{\circ[\perp]} \text{ and } \mathcal{T} + \mathcal{T}^{[\perp]} \subseteq \mathcal{T}^{\circ} \dotplus \mathcal{S}^{\circ[\perp]}. \tag{3.6}$$

Remark 1. If $\mathcal{S}\#_n\mathcal{T}$ then $\mathcal{S}^{\circ}\#\mathcal{T}^{\circ}$. In fact, from (3.6) we get that $\mathcal{S} \dotplus \mathcal{T}^{[\perp]} \subseteq \overline{\mathcal{S}^{\circ} \dotplus \mathcal{T}^{\circ[\perp]}}$ and $\mathcal{T} + \mathcal{S}^{[\perp]} \subseteq \mathcal{T}^{\circ} \dotplus \mathcal{S}^{\circ[\perp]}$. Since $\mathcal{S}\#\mathcal{T}$, it holds that $\mathcal{H} = \overline{\mathcal{S}^{\circ} \dotplus \mathcal{T}^{\circ[\perp]}} = \overline{\mathcal{T}^{\circ} \dotplus \mathcal{S}^{\circ[\perp]}}$. Therefore $\mathcal{S}^{\circ}\#\mathcal{T}^{\circ}$.

For neutral subspaces it is easy to check that $\mathcal{S}\#_n\mathcal{T}$ if and only if

$$\mathcal{S} \dotplus \mathcal{T}^{[\perp]} = \mathcal{T} \dotplus \mathcal{S}^{[\perp]}.$$

Corollary 3.7. *Let $\mathcal{S}$ and $\mathcal{T}$ be neutral (closed) subspaces of $\mathcal{H}$ and $P := P_{\mathcal{S}//\mathcal{T}^{[\perp]}}$. Then, the following conditions are equivalent:*

$i)$ *P is normal;*
$ii)$ *P is weakly normal;*
$iii)$ *$\mathcal{S}\#_n\mathcal{T}$;*

iv) $\mathcal{K} := \overline{\mathcal{S} \dotplus \mathcal{T}}$ *is a Krein space and* $\mathcal{S}, \mathcal{T}$ *are hypermaximal neutral in* $\mathcal{K}$;
v) $\overline{P + P^{\#}} = E_{\overline{\mathcal{S}+\mathcal{T}}} \in L(\mathcal{H})$, *the selfadjoint projection onto* $\overline{\mathcal{S} \dotplus \mathcal{T}}$,

$$\overline{\mathcal{S} \dotplus \mathcal{T}} \cap \mathcal{S}^{[\perp]} = \mathcal{S} \quad \textit{and} \quad \overline{\mathcal{S} \dotplus \mathcal{T}} \cap \mathcal{T}^{[\perp]} = \mathcal{T}.$$

Proof. The implication $i) \Rightarrow ii)$ is straightforward; $ii) \Rightarrow iii)$ follows from Corollary 3.2, considering that $\mathcal{S} \dotplus \mathcal{T}^{[\perp]} = \mathcal{T} \dotplus \mathcal{S}^{[\perp]}$, so that $\mathcal{S}\#_n\mathcal{T}$. If $iii)$ holds, then $\mathcal{S}^{[\perp]} = \mathcal{S}[\dotplus]\mathcal{S}^{[\perp]} \cap \mathcal{T}^{[\perp]}$ and $\mathcal{T}^{[\perp]} = \mathcal{T}[\dotplus]\mathcal{S}^{[\perp]} \cap \mathcal{T}^{[\perp]}$. Since $\mathcal{S}^{[\perp]}$ is pseudo-regular, $\mathcal{S}^{[\perp]} \cap \mathcal{T}^{[\perp]}$ is regular, or equivalently $\overline{\mathcal{S} \dotplus \mathcal{T}}$ is regular. Also, taking orthogonal complement to both sides of the equality $\mathcal{S}^{[\perp]} = \mathcal{S}[\dotplus]\mathcal{S}^{[\perp]} \cap \mathcal{T}^{[\perp]}$, we get that $\mathcal{S} = \overline{\mathcal{S} \dotplus \mathcal{T}} \cap \mathcal{S}^{[\perp]} = \mathcal{K} \cap \mathcal{S}^{[\perp]}$, and $\mathcal{S}$ is hypermaximal in $\mathcal{K}$. Likewise, $\mathcal{T}$ is hypermaximal in $\mathcal{K}$. Then $iii)$ implies $iv)$. If $iv)$ holds, $\mathcal{S}^{[\perp]_{\mathcal{K}}} = \mathcal{K} \cap \mathcal{S}^{[\perp]} = \mathcal{S}$. Taking orthogonal complement, $\mathcal{S}^{[\perp]} = \overline{\mathcal{S}^{[\perp]} \cap \mathcal{T}^{[\perp]} + \mathcal{S}} = \mathcal{S}^{[\perp]} \cap \mathcal{T}^{[\perp]}[\dotplus]\mathcal{S}$, because $\overline{\mathcal{S} \dotplus \mathcal{T}}$ is regular. In the same way, $\mathcal{T}^{[\perp]} = \mathcal{T}[\dotplus]\mathcal{S}^{[\perp]} \cap \mathcal{T}^{[\perp]}$. The equalities for $\mathcal{S}$ and $\mathcal{T}$ in (3.4) are automatic because $\mathcal{S}$ and $\mathcal{T}$ are neutral. Then, by Corollary 3.6, P is normal and $iv)$ implies $i)$. The equivalence $iv) \Leftrightarrow v)$ is straightforward. ☐

The following is an example of a closed weakly normal projection which is not normal.

Example 3. Let $\{e_j\}_{j\geq 1}$ be the standard basis of $\ell^2(\mathbb{N})$ and consider the shift operator $S \in L(\ell^2(\mathbb{N}))$ given by $Se_j = e_{j+1}$ for $j \geq 1$. Also, consider the Krein space $\mathcal{H} = \ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$ endowed with the form

$$[\,(u,v),(x,y)\,] = \langle\, u,x\,\rangle - \langle\, v,y\,\rangle$$

for $(x,y),(u,v) \in \mathcal{H}$. Let $\mathcal{S}$ and $\mathcal{T}$ be the neutral (closed) subspaces given by $\mathcal{S} := \{(x,Sx) : x \in \ell^2(\mathbb{N})\}$ and $\mathcal{T} := \{(S^2x,x) : x \in \ell^2(\mathbb{N})\}$.

Notice that $\mathcal{S}^{[\perp]} = \{(S^*y,y) : y \in \ell^2(\mathbb{N})\} = \mathcal{S}[\dotplus]\,\mathrm{span}\{(0,e_1)\}$ and $\mathcal{T}^{[\perp]} = \{(z,{S^*}^2z) : z \in \ell^2(\mathbb{N})\} = \mathcal{T}[\dotplus]\,\mathrm{span}\{(e_1,0),(e_2,0)\}$.

It holds that $\mathcal{T} \cap \mathcal{S}^{[\perp]} = \mathcal{S} \cap \mathcal{T}^{[\perp]} = \{0\}$. In fact, let $(u,v) \in \mathcal{T} \cap \mathcal{S}^{[\perp]}$ then

$$(u,v) = (S^2v,v) = (u,Su) + \alpha(0,e_1)$$

for some $\alpha \in \mathbb{C}$. Then

$$u = S^2v = S^2(Su + \alpha e_1) = S^3u + \alpha e_3.$$

If $u = \sum_{k\geq 1} \beta_k e_k \in \ell^2(\mathbb{N})$ then

$$\sum_{k\geq 1} \beta_k e_{k+3} + \alpha e_3 = S^3u + \alpha e_3 = u = \sum_{k\geq 1} \beta_k e_k.$$

Thus $\beta_1 = \beta_2 = 0, \beta_3 = \alpha$, and $\beta_k = \beta_{k-3},$ for $k \geq 4$.

So that $u = \alpha(\sum_{j\geq 1} e_{3j}) \in \ell^2(\mathbb{N})$ and then $\alpha = 0$. Therefore $\mathcal{T} \cap \mathcal{S}^{[\perp]} = \{0\}$. The equality $\mathcal{S} \cap \mathcal{T}^{[\perp]} = \{0\}$ follows in the same manner.

If $P := P_{\mathcal{S}//\mathcal{T}^{[\perp]}}$ then P is a densely defined closed projection. Note that

$$\mathcal{S}^{[\perp]} \dotplus \mathcal{T} = (\mathcal{S} \dotplus \mathcal{T}) \dotplus \mathrm{span}\{(0,e_1)\}$$

and

$$\mathcal{S} \dotplus \mathcal{T}^{[\perp]} = (\mathcal{S} \dotplus \mathcal{T}) \dotplus \mathrm{span}\{(e_1,0),(e_2,0)\}.$$

It is easy to see that $\mathcal{S}^{[\perp]} \dotplus \mathcal{T} \neq \mathcal{S} \dotplus \mathcal{T}^{[\perp]}$ and P is not weakly normal, see Proposition 3.1. But, since $\ker(I-P)$ is neutral, by Example 1, $I-P$ is weakly normal. Hence, by Theorem 3.5, $I-P$ is weakly normal but not normal.

Also it holds that $\mathcal{S}^{[\perp]} \cap \mathcal{T}^{[\perp]} = \{0\}$. Then, $\mathcal{S}$ and $\mathcal{T}$ are neutral subspaces such that $\overline{\mathcal{S} \dotplus \mathcal{T}}$ is a Krein space but $\mathcal{S}$ and $\mathcal{T}$ are not hypermaximal subspaces in $\mathcal{K}$.

In the general case, an extra condition should be added in order to get normality.

Proposition 3.8. *The projection $P_{\mathcal{S}//\mathcal{T}^{[\perp]}}$ is normal if and only if $\mathcal{S}\#_n\mathcal{T}$ and*

$$\mathcal{S} \cap \mathcal{T}^{\circ[\perp]} = \mathcal{T} \cap \mathcal{S}^{\circ[\perp]}. \tag{3.7}$$

Proof. If $Q := P_{\mathcal{S}//\mathcal{T}^{[\perp]}}$ is normal, by $ii)$ of Theorem 3.5, $\mathcal{S}\#_n\mathcal{T}$ and (3.7) follows.

Conversely, suppose that $\mathcal{S}\#_n\mathcal{T}$ and (3.7) holds. Then $\mathcal{S} = \mathcal{S}^\circ[\dotplus]\mathcal{S}\cap\mathcal{T}$ and $\mathcal{T} = \mathcal{T}^\circ[\dotplus]\mathcal{S}\cap\mathcal{T}$ and, by Corollary 3.2, $I-Q$ is weakly normal. Then $\mathcal{S}\dotplus\mathcal{T}^\circ = \mathcal{T}\dotplus\mathcal{S}^\circ$. Therefore, $\mathcal{S}^{[\perp]}\cap\mathcal{T}^{\circ[\perp]} = \mathcal{T}^{[\perp]}\cap\mathcal{S}^{\circ[\perp]}$ and, by Theorem 3.4, Q is weakly normal. Then, by Theorem 3.5, Q is normal. □

Next, we give a decomposition that characterizes normal projections, similar to the one given in [15, Theorem 3.1] for the bounded case. We start with the following proposition.

Proposition 3.9. *Let $Q := P_{\mathcal{S}//\mathcal{T}^{[\perp]}}$ be a weakly normal projection. Then*

$$QQ^\# = P_{\mathcal{S}\cap\mathcal{T}//\mathcal{T}^{[\perp]}+\mathcal{S}^{[\perp]}}.$$

If Q is normal then $QQ^\#$ is essentially selfadjoint with $\overline{QQ^\#} = E_{\mathcal{S}\cap\mathcal{T}}$ and $\overline{Q(I-Q^\#)} = P_{\mathcal{S}^\circ//\mathcal{T}^{\circ[\perp]}}$.

Proof. Since Q is a weakly normal projection then $(QQ^\#)^2 = QQ^\#$ and is easy to check that $\operatorname{ran} QQ^\# = \mathcal{S}\cap\mathcal{T}$. From the proof of Proposition 3.1,

$$\mathcal{S}^{[\perp]} \dotplus \mathcal{T}^\circ = \mathcal{S}^\circ \dotplus \mathcal{T}^{[\perp]} = \ker Q^\# Q, \tag{3.8}$$

therefore, $\ker QQ^\# = \mathcal{T}^{[\perp]} + \mathcal{S}^{[\perp]}$.

If Q is normal then $QQ^\#$ is symmetric. Therefore $QQ^\#$ is closable with $\overline{QQ^\#} = E_{\mathcal{S}\cap\mathcal{T}}$.

Finally, since Q is normal, $\operatorname{dom} Q(I-Q^\#) = \operatorname{dom} Q$, so that $Q(I-Q^\#) = Q - QQ^\# = Q - Q^\#Q = (I-Q^\#)Q$. Hence $Q(I-Q^\#) = P_{\mathcal{S}^\circ//\mathcal{T}+\mathcal{T}^{[\perp]}}$. Since $\mathcal{S}^\circ\#\mathcal{T}^\circ$, $Q(I-Q^\#)$ is closable with $\overline{Q(I-Q^\#)} = P_{\mathcal{S}^\circ//\mathcal{T}^{\circ[\perp]}}$. □

Theorem 3.10. *Let $Q = P_{\mathcal{S}//\mathcal{T}^{[\perp]}}$. Then Q is normal if and only if $\operatorname{dom} Q = \operatorname{dom} Q^\#$ and there exist a selfadjoint projection E and a closed projection P with $PP^\#|_{\operatorname{dom} Q} = P^\#P|_{\operatorname{dom} Q} = 0|_{\operatorname{dom} Q}$ such that*

$$Q = E + P. \tag{3.9}$$

In this case, $E = \overline{QQ^\#} = E_{\mathcal{S}\cap\mathcal{T}}$ and $P = \overline{Q(I-Q^\#)} = P_{\mathcal{S}^\circ//\mathcal{T}^{\circ[\perp]}}$.

Proof. Suppose that Q is normal. Then $\operatorname{dom} Q = \operatorname{dom} Q^{\#}$, and it is straightforward to see that

$$Q = QQ^{\#} + Q(I - Q^{\#}) = P_{\mathcal{S}\cap\mathcal{T}//\mathcal{S}^{[\perp]}+\mathcal{T}^{[\perp]}} + P_{\mathcal{S}^{\circ}//\mathcal{T}+\mathcal{T}^{[\perp]}}.$$

From Proposition 3.9, $\overline{QQ^{\#}} = E_{\mathcal{S}\cap\mathcal{T}} := E$ and $\overline{Q(I - Q^{\#})} = P_{\mathcal{S}^{\circ}//\mathcal{T}^{\circ[\perp]}} := P$. Then

$$Q^{\#} = (QQ^{\#} + Q(I - Q^{\#}))^{\#} \supseteq QQ^{\#} + P^{\#}.$$

Since $\operatorname{dom} QQ^{\#} = \operatorname{dom} Q$ and $\operatorname{dom} Q \subseteq \operatorname{dom} P^{\#}$ then $\operatorname{dom}(QQ^{\#} + P^{\#}) = \operatorname{dom} Q = \operatorname{dom} Q^{\#}$ and equality holds, i.e., $Q^{\#} = QQ^{\#} + P^{\#}$. Then $Q = (Q^{\#})^{\#} \supseteq E + P \supseteq QQ^{\#} + P_{\mathcal{S}^{\circ}//\mathcal{T}+\mathcal{T}^{[\perp]}} = Q$ and $Q = E + P$ where E is a selfadjoint projection and P is a closed projection. It is straightforward to see that $PP^{\#}|_{\operatorname{dom} Q} = P^{\#}P|_{\operatorname{dom} Q} = 0|_{\operatorname{dom} Q}$.

Conversely, on the one hand, $Q = E + P$ implies that $\operatorname{dom} Q = \operatorname{dom} E \cap \operatorname{dom} P$. Also $Q^{\#} \supseteq E^{\#} + P^{\#} = E + P^{\#}$. On the other hand, the condition $PP^{\#}|_{\operatorname{dom} Q} = 0|_{\operatorname{dom} Q}$ implies that $\operatorname{dom} Q \subseteq \operatorname{dom} P^{\#}$, and then $\operatorname{dom} Q^{\#} = \operatorname{dom} Q \subseteq \operatorname{dom}(E + P^{\#})$. Therefore

$$Q^{\#} = E + P^{\#}. \tag{3.10}$$

Now, let us show that $E \supseteq \overline{QQ^{\#}}$. In fact, since $Q = E + P$ is a projection, by Lemma 2.2, $PE|_{\operatorname{dom} Q} = EP|_{\operatorname{dom} Q} = 0|_{\operatorname{dom} Q}$. Likewise, since $Q^{\#} = E + P^{\#}$ is a projection, $EP^{\#}|_{\operatorname{dom} Q} = P^{\#}E|_{\operatorname{dom} Q} = 0|_{\operatorname{dom} Q}$. Then $QQ^{\#} = (E + P)(E + P^{\#}) \supseteq E|_{\operatorname{dom} Q} + PE|_{\operatorname{dom} Q} + EP^{\#}|_{\operatorname{dom} Q} + PP^{\#}|_{\operatorname{dom} Q} = E|_{\operatorname{dom} Q}$. Then $E \supseteq \overline{QQ^{\#}}$ because $\operatorname{dom} QQ^{\#} = \operatorname{dom} Q$ and $QQ^{\#} = E|_{\operatorname{dom} Q}$. In a similar way, $E|_{\operatorname{dom} Q} = Q^{\#}Q$ and Q is weakly normal. Since $\operatorname{ran} Q \subseteq \operatorname{dom} Q = \operatorname{dom} Q^{\#}$, then $\operatorname{dom} Q^{\#}Q = \operatorname{dom} Q$. So that, since $[Qx, Qx] = [Q^{\#}x, Q^{\#}x]$ for every $x \in \operatorname{dom} Q^{\#}Q = \operatorname{dom} Q$, Q is normal. Then, by Proposition 3.9, $QQ^{\#}$ is essentially selfajdoint. Hence, taking adjoint to $E \supseteq \overline{QQ^{\#}}$, it follows that $E = \overline{QQ^{\#}}$.

Note that $P|_{\operatorname{dom} Q} = Q - E|_{\operatorname{dom} Q} = Q - QQ^{\#} = Q(I - Q^{\#})$ because $\operatorname{dom} Q(I - Q^{\#}) = \operatorname{dom} Q$. Therefore $P \supseteq \overline{Q(I - Q^{\#})}$. By (3.10), $P^{\#}|_{\operatorname{dom} Q} = Q^{\#} - E|_{\operatorname{dom} Q} = Q^{\#} - QQ^{\#} = Q^{\#} - Q^{\#}Q = Q^{\#}(I - Q)$ because $\operatorname{dom} Q^{\#}(I - Q) = \operatorname{dom} Q^{\#}$. So that $P^{\#} \supseteq \overline{Q^{\#}(I - Q)}$. Taking adjoint and using Lemma 3.9, it follows that $P \subseteq (Q^{\#}(I - Q))^{\#} = \overline{Q(I - Q^{\#})}$. Then $P = \overline{Q(I - Q^{\#})}$. □

Corollary 3.11. *Let $P_{\mathcal{S}//\mathcal{T}^{[\perp]}}$ be normal. Then*

$$\overline{\mathcal{S}^{\circ} + \mathcal{T}^{\circ}} \cap \mathcal{T}^{\circ[\perp]} = \mathcal{T}^{\circ} \text{ and } \overline{\mathcal{S}^{\circ} + \mathcal{T}^{\circ}} \cap \mathcal{S}^{\circ[\perp]} = \mathcal{S}^{\circ},$$

$\overline{\mathcal{S}^{\circ} \dotplus \mathcal{T}^{\circ}}$ is non-degenerate and if $P := P_{\mathcal{S}^{\circ}//\mathcal{T}^{\circ[\perp]}}$ then $\overline{P + P^{\#}} = E_{\overline{\mathcal{S}^{\circ} \dotplus \mathcal{T}^{\circ}}}$.

Proof. By Proposition 3.8 and Corollary 3.6,

$$\mathcal{S} + \mathcal{S}^{[\perp]} = \mathcal{S}^{\circ} + \mathcal{S} \cap \mathcal{T} + \mathcal{S}^{[\perp]} \cap \mathcal{T}^{[\perp]} \subseteq \mathcal{S}^{\circ} + \mathcal{S}^{\circ[\perp]} \cap \mathcal{T}^{\circ[\perp]} \subseteq \mathcal{S}^{\circ[\perp]}.$$

Taking orthogonal companion, $\mathcal{S}^{\circ} \supseteq \mathcal{S}^{\circ[\perp]} \cap \overline{\mathcal{S}^{\circ} + \mathcal{T}^{\circ}} \supseteq \mathcal{S}^{\circ}$ and equality holds. In a similar fashion, we get that $\overline{\mathcal{S}^{\circ} \dotplus \mathcal{T}^{\circ}} \cap \mathcal{T}^{\circ[\perp]} = \mathcal{T}^{\circ}$. Since $\mathcal{S}^{\circ} \# \mathcal{T}^{\circ}$,

$$\overline{\mathcal{S}^{\circ} \dotplus \mathcal{T}^{\circ}} \cap (\mathcal{T}^{\circ[\perp]} \cap \mathcal{S}^{\circ[\perp]}) = \mathcal{T}^{\circ} \cap \mathcal{S}^{\circ[\perp]} = \{0\}.$$

Then $\overline{\mathcal{S}^{\circ} \dotplus \mathcal{T}^{\circ}}$ is non-degenerate.

Finally, $\mathrm{dom}(P + P^{\#}) = \mathcal{S}^\circ \dotplus \mathcal{T}^\circ \dotplus \mathcal{S}^{\circ[\perp]} \cap \mathcal{T}^{\circ[\perp]}$ and

$$P + P^{\#} = P_{\mathcal{S}^\circ \dotplus \mathcal{T}^\circ // \mathcal{S}^{\circ[\perp]} \cap \mathcal{T}^{\circ[\perp]}}$$

is densely defined. Therefore $(P + P^{\#})^{\#} \supseteq P + P^{\#}$. Hence $P + P^{\#}$ is closable with $\overline{P + P^{\#}} = E_{\overline{\mathcal{S}^\circ \dotplus \mathcal{T}^\circ}}$. □

Remark 2. Following the lines of the proof of Corollary 3.11, we get that, if $P_{\mathcal{S}//\mathcal{T}^{[\perp]}}$ is normal then the subspace $\overline{\mathcal{S} \cap \mathcal{T} [\dotplus] \mathcal{S}^{[\perp]} \cap \mathcal{T}^{[\perp]}}$ is non-degenerate and

$$\overline{E_{\mathcal{S}\cap\mathcal{T}} + E_{\mathcal{S}^{[\perp]}\cap\mathcal{T}^{[\perp]}}} = E_{\overline{\mathcal{S}\cap\mathcal{T}[\dotplus]\mathcal{S}^{[\perp]}\cap\mathcal{T}^{[\perp]}}}.$$

Proposition 3.12. *If $P_{\mathcal{S}//\mathcal{T}^{[\perp]}}$ is normal, the following are equivalent:*

i) $P_{\mathcal{S}^\circ//\mathcal{T}^{\circ[\perp]}}$ is normal;

ii) $\overline{\mathcal{S} \cap \mathcal{T}[\dotplus]\mathcal{S}^{[\perp]} \cap \mathcal{T}^{[\perp]}}$ is regular;

iii) $\mathcal{S}^{\circ[\perp]} = \mathcal{S}^\circ[\dotplus]\overline{\mathcal{S} \cap \mathcal{T} + \mathcal{S}^{[\perp]} \cap \mathcal{T}^{[\perp]}}$ or $\mathcal{T}^{\circ[\perp]} = \mathcal{T}^\circ[\dotplus]\overline{\mathcal{S} \cap \mathcal{T} + \mathcal{S}^{[\perp]} \cap \mathcal{T}^{[\perp]}}$.

In this case, $\mathcal{S}^{\circ[\perp]} \cap \mathcal{T}^{\circ[\perp]} = \overline{\mathcal{S} \cap \mathcal{T} + \mathcal{S}^{[\perp]} \cap \mathcal{T}^{[\perp]}}$.

Proof. If $P_{\mathcal{S}//\mathcal{T}^{[\perp]}}$ is normal then by Corollary 3.6,

$$\mathcal{S} + \mathcal{S}^{[\perp]} = \mathcal{S}^\circ[\dotplus]\mathcal{S} \cap \mathcal{T}[\dotplus]\mathcal{S}^{[\perp]} \cap \mathcal{T}^{[\perp]},$$

and by Corollary 3.11,

$$\mathcal{S}^{\circ[\perp]} = \overline{\mathcal{S}^\circ[\dotplus]\mathcal{S}^{\circ[\perp]} \cap \mathcal{T}^{\circ[\perp]}}.$$

Set $P := P_{\mathcal{S}^\circ//\mathcal{T}^{\circ[\perp]}}$. Suppose that P is normal, then by $iv)$ of Corollary 3.7, $\mathcal{S}^{\circ[\perp]} \cap \mathcal{T}^{\circ[\perp]}$ is regular. Since $\mathcal{S} \cap \mathcal{T} + \mathcal{S}^{[\perp]} \cap \mathcal{T}^{[\perp]} \subseteq \mathcal{S}^{\circ[\perp]} \cap \mathcal{T}^{\circ[\perp]}$ it follows that

$$\mathcal{S}^{\circ[\perp]} = \mathcal{S}^\circ[\dotplus]\overline{\mathcal{S} \cap \mathcal{T} + \mathcal{S}^{[\perp]} \cap \mathcal{T}^{[\perp]}} \subseteq \mathcal{S}^\circ[\dotplus]\mathcal{S}^{\circ[\perp]} \cap \mathcal{T}^{\circ[\perp]} = \mathcal{S}^{\circ[\perp]}.$$

Then

$$\overline{\mathcal{S} \cap \mathcal{T} + \mathcal{S}^{[\perp]} \cap \mathcal{T}^{[\perp]}} = \mathcal{S}^{\circ[\perp]} \cap \mathcal{T}^{\circ[\perp]}$$

is regular and $ii)$ follows.

Suppose that $ii)$ holds, then since $\overline{\mathcal{S} \cap \mathcal{T}[\dotplus]\mathcal{S}^{[\perp]} \cap \mathcal{T}^{[\perp]}}$ is regular

$$\mathcal{S}^{\circ[\perp]} = \overline{\mathcal{S} + \mathcal{S}^{[\perp]}} = \mathcal{S}^\circ[\dotplus]\overline{\mathcal{S} \cap \mathcal{T} + \mathcal{S}^{[\perp]} \cap \mathcal{T}^{[\perp]}}.$$

In the same way, $\mathcal{T}^{\circ[\perp]} = \mathcal{T}^\circ[\dotplus]\overline{\mathcal{S} \cap \mathcal{T} + \mathcal{S}^{[\perp]} \cap \mathcal{T}^{[\perp]}}$.

If $iii)$ holds, suppose for example that $\mathcal{S}^{\circ[\perp]} = \mathcal{S}^\circ[\dotplus]\overline{\mathcal{S} \cap \mathcal{T} + \mathcal{S}^{[\perp]} \cap \mathcal{T}^{[\perp]}}$ then

$$\mathcal{S}^{\circ[\perp]} \subseteq \mathcal{S}^\circ[\dotplus]\mathcal{S}^{\circ[\perp]} \cap \mathcal{T}^{\circ[\perp]} \subseteq \mathcal{S}^{\circ[\perp]}.$$

Then

$$\mathcal{S}^{\circ[\perp]} = \mathcal{S}^\circ[\dotplus]\mathcal{S}^{\circ[\perp]} \cap \mathcal{T}^{\circ[\perp]}.$$

Therefore $\mathcal{S}^{\circ[\perp]} \cap \mathcal{T}^{\circ[\perp]}$ is regular, or equivalently, $\overline{\mathcal{S}^\circ + \mathcal{T}^\circ}$ is a Krein space. Since by Corollary 3.11, $\mathcal{S}^\circ$ and $\mathcal{T}^\circ$ are hypermaximal in $\overline{\mathcal{S}^\circ + \mathcal{T}^\circ}$ by $iv)$ of Corollary 3.7, P is normal. □

Proposition 3.13. *Let $P_{\mathcal{S}//\mathcal{T}^{[\perp]}}$ be normal. The following are equivalent:*

i) $\mathcal{S}$ is pseudo-regular;

ii) $\mathcal{T}$ is pseudo-regular;

iii) $\mathcal{S}\cap\mathcal{T}[\dot{+}]\mathcal{S}^{[\perp]}\cap\mathcal{T}^{[\perp]}$ *is regular.*

In this case, $P_{\mathcal{S}^\circ//\mathcal{T}^{\circ[\perp]}}$ *is normal.*

Proof. $i) \Leftrightarrow ii)$: If $\mathcal{S}$ is pseudo-regular then, since $\mathcal{S}=\mathcal{S}^\circ[\dot{+}]\mathcal{S}\cap\mathcal{T}$ then, applying Proposition 2.5, $\mathcal{S}\cap\mathcal{T}$ is regular. Therefore, applying again Proposition 2.5, $\mathcal{T}=\mathcal{T}^\circ[\dot{+}]\mathcal{S}\cap\mathcal{T}$ is pseudo-regular. By the same arguments, the converse holds.

$i) \Leftrightarrow iii)$: If $\mathcal{S}$ is pseudo-regular then $\mathcal{S}^{[\perp]}$ is also pseudo-regular. Then, from (3.4), we get that $\mathcal{S}\cap\mathcal{T}$ and $\mathcal{S}^{[\perp]}\cap\mathcal{T}^{[\perp]}$ are regular orthogonal subspaces. Therefore $\mathcal{S}\cap\mathcal{T}[\dot{+}]\mathcal{S}^{[\perp]}\cap\mathcal{T}^{[\perp]}$ is regular. The converse follows from the general fact that given $\mathcal{M},\mathcal{M}_1$ and $\mathcal{M}_2$ closed subspaces such that $\mathcal{M}=\mathcal{M}_1[\dot{+}]\mathcal{M}_2$, then $\mathcal{M}$ is regular if and only if $\mathcal{M}_1$ and $\mathcal{M}_2$ are regular.

The final assertion follows applying Proposition 3.12.

□

4. The set of normal projections onto a fixed range

Let $P_{\mathcal{S}//\mathcal{T}^{[\perp]}}$ be a normal projection. By (3.5),

$$\mathcal{S}+\mathcal{S}^{[\perp]}\subseteq\mathcal{S}^\circ\dot{+}\mathcal{T}^{\circ[\perp]}.$$

Also, since $\mathcal{S}^{[\perp]}\cap\mathcal{T}^{\circ[\perp]}+\mathcal{T}^\circ=\mathcal{T}^{[\perp]}$,

$$\overline{\mathcal{S}+\mathcal{T}^\circ}\cap\mathcal{T}^{\circ[\perp]}=\mathcal{T}=\mathcal{S}\cap\mathcal{T}^{\circ[\perp]}+\mathcal{T}^\circ.$$

Likewise,

$$\overline{\mathcal{S}^{[\perp]}+\mathcal{T}^\circ}\cap\mathcal{T}^{\circ[\perp]}=\mathcal{T}^{[\perp]}=\mathcal{S}^{[\perp]}\cap\mathcal{T}^{\circ[\perp]}+\mathcal{T}^\circ. \tag{4.1}$$

These properties motivate the following definition.

Definition. Let $\mathcal{S}$ be a closed subspace of $\mathcal{H}$. We say that a neutral closed subspace $\mathcal{N}$ of $\mathcal{H}$ is a *normal companion of* $\mathcal{S}$ if $\mathcal{S}^\circ\#\mathcal{N}$,

$$\mathcal{S}+\mathcal{S}^{[\perp]}\subseteq\mathcal{S}^\circ\dot{+}\mathcal{N}^{[\perp]}, \tag{4.2}$$

and

$$\begin{aligned}\overline{\mathcal{S}+\mathcal{N}}\cap\mathcal{N}^{[\perp]}&=\mathcal{S}\cap\mathcal{N}^{[\perp]}+\mathcal{N},\\ \overline{\mathcal{S}^{[\perp]}+\mathcal{N}}\cap\mathcal{N}^{[\perp]}&=\mathcal{S}^{[\perp]}\cap\mathcal{N}^{[\perp]}+\mathcal{N}.\end{aligned} \tag{4.3}$$

Remark 3. $i)$ By definition, if $P_{\mathcal{S}//\mathcal{T}^{[\perp]}}$ is normal then $\mathcal{T}^\circ$ is a normal companion of $\mathcal{S}$ and $\mathcal{S}^\circ$ is a normal companion of $\mathcal{T}$. $ii)$ Let $\mathcal{S}$ be a closed subspace of $\mathcal{H}$ then, for any $J\in\mathcal{J}$, $\mathcal{N}:=J\mathcal{S}^\circ$ is a normal companion of $\mathcal{S}$. In fact, since $\mathcal{H}=\overline{\mathcal{S}+\mathcal{S}^{[\perp]}}\oplus\mathcal{N}=\mathcal{S}^{\circ[\perp]}\oplus\mathcal{N}$, $\mathcal{S}^\circ$ and $\mathcal{N}$ form a strongly dual pair and (4.2) holds trivially. Finally, the subspaces $\mathcal{S}^{[\perp]}+\mathcal{N}$ and $\mathcal{S}+\mathcal{N}$ are closed, (4.3) holds and $\mathcal{N}$ is a normal companion of $\mathcal{S}$.

Proposition 4.1. *Let* $\mathcal{S}$ *be a closed subspace of* $\mathcal{H}$. *If* $\mathcal{N}$ *is a normal companion of* $\mathcal{S}$ *then* $P_{\mathcal{S}//\mathcal{S}^{[\perp]}\cap\mathcal{N}^{[\perp]}+\mathcal{N}}$ *is normal.*

Proof. Let $\mathcal{N}$ be any normal companion of $\mathcal{S}$. From (4.2), $\mathcal{S} = \mathcal{S}^\circ + \mathcal{S} \cap \mathcal{N}^{[\perp]}$ and $\mathcal{S}^{[\perp]} = \mathcal{S}^\circ + \mathcal{S}^{[\perp]} \cap \mathcal{N}^{[\perp]}$. From (4.3), the subspace $\mathcal{T} := \mathcal{N} + \mathcal{S} \cap \mathcal{N}^{[\perp]}$ is closed and, $\mathcal{T}^{[\perp]} = \mathcal{N} + \mathcal{S}^{[\perp]} \cap \mathcal{N}^{[\perp]}$.

It holds that $\mathcal{S} \cap \mathcal{T} = \mathcal{S} \cap (\mathcal{N} + \mathcal{S} \cap \mathcal{N}^{[\perp]}) = \mathcal{S} \cap \mathcal{N}^{[\perp]}$. In the same way, $\mathcal{S}^{[\perp]} \cap \mathcal{T}^{[\perp]} = \mathcal{S}^{[\perp]} \cap (\mathcal{N} + \mathcal{S}^{[\perp]} \cap \mathcal{N}^{[\perp]}) = \mathcal{S}^{[\perp]} \cap \mathcal{N}^{[\perp]}$.

Also, $\mathcal{S} \cap \mathcal{T}^{[\perp]} = \{0\}$. In fact, if $x \in \mathcal{S} \cap \mathcal{T}^{[\perp]}$ then $x = s_0 + s = n_0 + n$ with $s_0 \in \mathcal{S}^\circ$, $s \in \mathcal{S} \cap \mathcal{N}^{[\perp]}$, $n_0 \in \mathcal{N}$ and $n \in \mathcal{S}^{[\perp]} \cap \mathcal{N}^{[\perp]}$. So that $s_0 = n_0 + n - s \in \mathcal{S}^\circ \cap \mathcal{N}^{[\perp]} = \{0\}$. Hence $s_0 = 0$ and $s = n_0 + n$, so $n_0 = s - n \in \mathcal{N} \cap (\mathcal{S} + \mathcal{S}^{[\perp]}) = \{0\}$. Therefore $n_0 = 0$ and $s = n \in \mathcal{S} \cap \mathcal{N}^{[\perp]} \cap \mathcal{S}^{[\perp]} = \mathcal{S}^\circ \ \cap \mathcal{N}^{[\perp]} = \{0\}$. Then $x = 0$. Finally, $\mathcal{T} \cap \mathcal{T}^{[\perp]} = (\mathcal{N} + \mathcal{S} \cap \mathcal{N}^{[\perp]}) \cap \mathcal{T}^{[\perp]} = \mathcal{N} + \mathcal{S} \cap \mathcal{N}^{[\perp]} \cap \mathcal{T}^{[\perp]} = \mathcal{N}$. Thus,

$$\mathcal{T}^\circ = \mathcal{N}. \tag{4.4}$$

Since $\mathcal{S} = \mathcal{S}^\circ + \mathcal{S} \cap \mathcal{T}$, $\mathcal{S}^{[\perp]} = \mathcal{S}^\circ + \mathcal{S}^{[\perp]} \cap \mathcal{T}^{[\perp]}$, $\mathcal{T} = \mathcal{T}^\circ + \mathcal{S} \cap \mathcal{T}$ and $\mathcal{T}^{[\perp]} = \mathcal{T}^\circ + \mathcal{S}^{[\perp]} \cap \mathcal{T}^{[\perp]}$, by Corollary 3.6, $P_{\mathcal{S}//\mathcal{T}^{[\perp]}}$ is a normal projection. □

Corollary 4.2. *Every closed subspace $\mathcal{S}$ of $\mathcal{H}$ is the range of a normal projection.*

Proof. Take $\mathcal{N} := J\mathcal{S}^\circ$ for any $J \in \mathcal{J}$ and apply Proposition 4.1. □

Corollary 4.3. *The projection $P_{\mathcal{S}//\mathcal{T}^{[\perp]}}$ is normal if and only if $\mathcal{T}^\circ$ is a normal companion of $\mathcal{S}$, $\mathcal{S}^\circ$ is a normal companion of $\mathcal{T}$ and $\mathcal{S}^{\circ[\perp]} \cap \mathcal{T}^{[\perp]} = \mathcal{S}^{[\perp]} \cap \mathcal{T}^{\circ[\perp]}$.*

Proof. If $P_{\mathcal{S}//\mathcal{T}^{[\perp]}}$ is normal then the conditions follow from Remark 3 $i)$, and taking orthogonal complement to the equalities in $ii)$ of Theorem 3.5.

Conversely, since $\mathcal{S}^\circ$ is a normal companion of $\mathcal{T}$, it holds that $\mathcal{T}^{[\perp]} = \mathcal{T}^\circ + \mathcal{S}^{\circ[\perp]} \cap \mathcal{T}^{[\perp]} = \mathcal{T}^\circ + \mathcal{S}^{[\perp]} \cap \mathcal{T}^{\circ[\perp]}$. So that, since $\mathcal{T}^\circ$ is a normal companion of $\mathcal{S}$, by Proposition 4.1, $P_{\mathcal{S}//\mathcal{T}^{[\perp]}}$ is normal. □

Corollary 4.4. *Let $\mathcal{S}$ and $\mathcal{N}$ be neutral subspaces of $\mathcal{H}$. Then the following are equivalent:*

$i)$ *$P_{\mathcal{S}//\mathcal{N}^{[\perp]}}$ is normal;*
$ii)$ *$\mathcal{N}$ is a normal companion of $\mathcal{S}$;*
$iii)$ *$\mathcal{S}$ is a normal companion of $\mathcal{N}$.*

Proof. Let $\mathcal{N}$ be a normal companion of $\mathcal{S}$. Then, by Proposition 4.1, $\mathcal{T} := \mathcal{N} + \mathcal{S} \cap \mathcal{N}^{[\perp]} \subseteq \mathcal{N} + \mathcal{S}^{[\perp]} \cap \mathcal{N}^{[\perp]} = \mathcal{T}^{[\perp]}$. Therefore, by (4.4), $\mathcal{T} = \mathcal{T}^\circ = \mathcal{N}$ and $P_{\mathcal{S}//\mathcal{N}^{[\perp]}}$ is normal. The converse follows from Remaark 3 $i)$. Hence $i) \Leftrightarrow ii)$. Finally, since $P_{\mathcal{S}//\mathcal{N}^{[\perp]}}$ is normal if and only if $P_{\mathcal{N}//\mathcal{S}^{[\perp]}}$ is normal, the equivalence $i) \Leftrightarrow iii)$ follows from $i) \Leftrightarrow ii)$. □

By Corollary 4.2, the set $\tilde{\mathcal{Q}}_{\mathcal{S}}$ of normal projections onto $\mathcal{S}$ is not empty for any closed subspace $\mathcal{S}$. Moreover, by Propositions 4.1 and Remark 3 $i)$,

$$\tilde{\mathcal{Q}}_{\mathcal{S}} = \{P_{\mathcal{S}//\mathcal{S}^{[\perp]}\cap\mathcal{N}^{[\perp]}[\dot{+}]\mathcal{N}} : \mathcal{N} \text{ is a normal companion of } \mathcal{S}\}. \tag{4.5}$$

Proposition 4.5. *Consider $\mathcal{S}$ and $\mathcal{T}$ closed subespaces such that $Q = P_{\mathcal{S}//\mathcal{T}^{[\perp]}} \in L(\mathcal{H})$. Then Q is normal if and only if there exists $J \in \mathcal{J}$ such that $\mathcal{T}^\circ = J\mathcal{S}^\circ$ and*

$$\mathcal{S} \cap \mathcal{T}^{\circ[\perp]} = \mathcal{T} \cap \mathcal{S}^{\circ[\perp]} \tag{4.6}$$

Proof. Since $Q \in L(\mathcal{H})$, if Q is normal then, by Proposition 3.8, condition (4.6) holds. Also, if $P := Q(I - Q^{\#}) = P_{\mathcal{S}^\circ // \mathcal{T}^{\circ[\perp]}}$, then

$$P + P^{\#} = P_{\mathcal{S}^\circ \dotplus \mathcal{T}^\circ // \mathcal{S}^{\circ[\perp]} \cap \mathcal{T}^{\circ[\perp]}} \in L(\mathcal{H}).$$

So that $\mathcal{S}^\circ \dotplus \mathcal{T}^\circ$ is closed. Therefore $\mathcal{S}^\circ \dotplus \mathcal{T}^\circ$ is regular and, by Theorem 2.6, there is a $J \in \mathcal{J}$ such that $\mathcal{T}^\circ = J\mathcal{S}^\circ$.

Conversely, if $\mathcal{T}^\circ = J\mathcal{S}^\circ$, by Theorem 2.6 again, $\mathcal{S}^\circ \dotplus \mathcal{T}^\circ$ is regular and $\mathcal{H} = (\mathcal{S}^\circ \dotplus \mathcal{T}^\circ)[\dotplus](\mathcal{S}^{\circ[\perp]} \cap \mathcal{T}^{\circ[\perp]})$. Then

$$\mathcal{S}^{[\perp]} = \mathcal{S}^{[\perp]} \cap \mathcal{H} = \mathcal{S}^\circ \dotplus \mathcal{S}^{[\perp]} \cap (\mathcal{T}^\circ \dotplus \mathcal{S}^{\circ[\perp]} \cap \mathcal{T}^{\circ[\perp]}) \subseteq \mathcal{S}^\circ \dotplus \mathcal{T}^{\circ[\perp]}.$$

In the same fashion, $\mathcal{T}^{[\perp]} \subseteq \mathcal{S}^\circ \dotplus \mathcal{T}^{\circ[\perp]}$ and (3.2) holds. Since (4.6) holds, by Theorem 3.4, Q is weakly normal and then, since Q is bounded, Q is normal. □

Following the notation introduced in [15], consider the set $\mathcal{Q}_{\mathcal{S}}$ of bounded normal projections onto $\mathcal{S}$, i.e. $\mathcal{Q}_{\mathcal{S}} = \tilde{\mathcal{Q}}_{\mathcal{S}} \cap L(\mathcal{H})$.

Proposition 4.6. *The set $\mathcal{Q}_{\mathcal{S}}$ is not empty if and only if $\mathcal{S}$ is pseudo-regular. In this case,*

$$\mathcal{Q}_{\mathcal{S}} = \{P_{\mathcal{S}//\mathcal{S}^{[\perp]} \cap J(\mathcal{S}^\circ)^{[\perp]} [\dotplus] J\mathcal{S}^\circ}\}_{J \in \mathcal{J}}.$$

In particular, if $\mathcal{S}$ is neutral

$$\mathcal{Q}_{\mathcal{S}} = \{P_{\mathcal{S}//(J\mathcal{S})^{[\perp]}}\}_{J \in \mathcal{J}}.$$

Proof. Use Proposition 4.5 and (4.5). □

5. Multivalued projections

A classical result for Hilbert spaces is that every closed subspace of a Hilbert space is the range of a (unique, linear operator) selfadjoint projection. The next paragraphs are devoted to show that every closed subspace of a Krein space is the range of a multivalued selfadjoint projection.

A linear relation in a vector space $\mathcal{H}$ is a subspace of $\mathcal{H} \times \mathcal{H}$. The set of linear relations in $\mathcal{H}$ is denoted by $\mathrm{lr}(\mathcal{H})$. Given $T \in \mathrm{lr}(\mathcal{H})$, $\operatorname{dom} T$, $\operatorname{ran} T$ and $\ker T$ denote the domain, range and kernel of T, respectively,

$$\operatorname{dom} T = \{x \in \mathcal{H} : (x, y) \in T \text{ for some } y \in \mathcal{H}\},$$
$$\operatorname{ran} T = \{y \in \mathcal{H} : (x, y) \in T \text{ for some } x \in \mathcal{H}\},$$
$$\ker T = \{x \in \mathcal{H} : (x, 0) \in T\}.$$

The multivalued part of T is defined by $\operatorname{mul} T := \{y \in \mathcal{H} : (0, y) \in T\}$. If $\operatorname{mul} T = \{0\}$ then T is (the graph of) a linear operator.

For $T, S \in \mathrm{lr}(\mathcal{H})$, $T \hat{+} S$ stands for the sum of T and S as subspaces. The product ST is the linear relation in $\mathcal{H}$ defined by

$$ST := \{(x, y) : (x, z) \in T \text{ and } (z, y) \in S \text{ for some } z \in \mathcal{H}\}.$$

If $\mathcal{H}$ is a normed space, the *closure* $\overline{T}$ of T is the closure of T in $\mathcal{H} \times \mathcal{H}$ endowed with the product topology. The linear relation T is *closed* when $T = \overline{T}$.

Multivalued projections in Hilbert spaces were introduced by Cross and Wilcox in [6] and later studied by Labrousse in [14].

Definition. Let $E \in \operatorname{lr}(\mathcal{H})$ such that $\operatorname{ran} E \subseteq \operatorname{dom} E$. We say that E is a *multivalued projection* if E is *idempotent*, that is $E^2 = E$.

Given a subspace $\mathcal{M}$ of $\mathcal{H}$, we denote by $I_{\mathcal{M}} := \{(u,u) : u \in \mathcal{M}\}$. The following characterization can be found in [6, 14].

Proposition 5.1. *Let $E \in \operatorname{lr}(\mathcal{H})$. Then, E is a multivalued projection if and only if $E = I_{\operatorname{ran} E} \hat{+} (\ker E \times \{0\})$.*

Thus, given subspaces $\mathcal{M}$ and $\mathcal{N}$ of $\mathcal{H}$, we write

$$P_{\mathcal{M},\mathcal{N}} := I_{\mathcal{M}} \hat{+} (\mathcal{N} \times \{0\}),$$

for the multivalued projection with range $\mathcal{M}$ and kernel $\mathcal{N}$. It is easy to check that $\operatorname{dom} P_{\mathcal{M},\mathcal{N}} = \mathcal{M} + \mathcal{N}$ and $\operatorname{mul} P_{\mathcal{M},\mathcal{N}} = \mathcal{M} \cap \mathcal{N}$.

Given a Krein space $(\mathcal{H}, [\,\cdot,\cdot\,])$, endow $\mathcal{H} \times \mathcal{H}$ with the usual indefinite inner product:

$$[\,(x_1,y_1),(x_2,y_2)\,] = [\,x_1,x_2\,] + [\,y_1,y_2\,], \quad (x_1,y_1),(x_2,y_2) \in \mathcal{H} \times \mathcal{H}.$$

Given $T \in \operatorname{lr}(\mathcal{H})$, the adjoint of T (in the Krein space sense) is the linear relation in $\mathcal{H}$ defined by

$$T^{\#} := \{(x,y) \in \mathcal{H} \times \mathcal{H} : [\,g,x\,] = [\,f,y\,] \text{ for all } (f,g) \in T\}.$$

We say that $T \in \operatorname{lr}(\mathcal{H})$ is *selfadjoint* if $T = T^{\#}$.

If $J \in L(\mathcal{H})$ is any signature operator for $\mathcal{H}$, it is straightfordward to see that $T^{\#} = \tilde{J}T^*$, where $\tilde{J}(x,y) := (Jx, Jy)$, for every $(x,y) \in \mathcal{H} \times \mathcal{H}$. Using this property and the well known properties of T^* [2], the operator $T^{\#}$ is a closed linear relation, $\overline{T}^{\#} = T^{\#}$ and $T^{\#\#} := (T^{\#})^{\#} = \overline{T}$. Also, $\operatorname{mul} T^{\#} = (\operatorname{dom} T)^{[\perp]}$ and $\ker T^{\#} = (\operatorname{ran} T)^{[\perp]}$.

Proposition 5.2. *Given subspaces $\mathcal{M}$ and $\mathcal{N}$ of a Krein space $\mathcal{H}$, it holds that*

$$P^{\#}_{\mathcal{M},\mathcal{N}} = P_{\mathcal{N}^{[\perp]},\mathcal{M}^{[\perp]}}.$$

Proof. For multivalued projections in Hilbert spaces it holds that $P^*_{\mathcal{M},\mathcal{N}} = P_{\mathcal{N}^\perp,\mathcal{M}^\perp}$, see e.g. [6, 14]. Then $P^{\#}_{\mathcal{M},\mathcal{N}} = \tilde{J}P^*_{\mathcal{M},\mathcal{N}} = \tilde{J}P_{\mathcal{N}^\perp,\mathcal{M}^\perp}$. But

$$\begin{aligned} \tilde{J}P_{\mathcal{N}^\perp,\mathcal{M}^\perp} &= \{(J(u+v), Ju) : u \in \mathcal{N}^\perp, v \in \mathcal{M}^\perp\} \\ &= \{(u+v, u) : u \in \mathcal{N}^{[\perp]}, v \in \mathcal{M}^{[\perp]}\} = P_{\mathcal{N}^{[\perp]},\mathcal{M}^{[\perp]}}. \end{aligned}$$ □

Proposition 5.3. *Let E be a multivalued projection in $\mathcal{H}$. Then, $E^{\#} = E$ if and only if $\operatorname{ran} E$ is closed and $\ker E = \operatorname{ran} E^{[\perp]}$.*

Proof. By Proposition 5.2, if $E = P_{\operatorname{ran} E,\ker E}$ then $E^{\#} = P_{\ker E^{[\perp]},\operatorname{ran} E^{[\perp]}}$. Since a multivalued projection is uniquely determined by its range and its kernel, $E^{\#} = E$ if and only if $\ker E = \operatorname{ran} E^{[\perp]}$ and $\operatorname{ran} E = \ker E^{[\perp]}$. □

An immediate consequence of the above result is that every closed subspace of a Krein space is the range of a selfadjoint multivalued projection. Moreover, different properties of the subspace translate into conditions on the selfadjoint multivalued projection.

Corollary 5.4. *Given a closed subspace $\mathcal{S}$ of a Krein space $\mathcal{H}$ there exists a unique selfadjoint multivalued projection $E_{\mathcal{S}}$ onto $\mathcal{S}$. Moreover,*

$$\operatorname{dom} E_{\mathcal{S}} = \mathcal{S} + \mathcal{S}^{[\perp]}, \quad \operatorname{mul} E_{\mathcal{S}} = \mathcal{S}^{\circ}, \tag{5.1}$$

and the following conditions hold:

1. *$\mathcal{S}$ is pseudo-regular if and only if $E_{\mathcal{S}}$ is a continuous multivalued projection.*
2. *$\mathcal{S}$ is non-degenerated if and only if $E_{\mathcal{S}}$ is a (densely defined) linear projection.*
3. *$\mathcal{S}$ is regular if and only if $E_{\mathcal{S}} \in L(\mathcal{H})$.*

Proof. If $\mathcal{S}$ is a closed subspace of $\mathcal{H}$, consider $E_{\mathcal{S}} := P_{\mathcal{S},\mathcal{S}^{[\perp]}}$. Then, by Proposition 5.3, $E_{\mathcal{S}}$ is the unique selfadjoint multivalued projection onto $\mathcal{S}$. By the remarks above, $\operatorname{dom} E_{\mathcal{S}} = \mathcal{S} + \mathcal{S}^{[\perp]}$ and $\operatorname{mul} E_{\mathcal{S}} = \mathcal{S}^{\circ}$.

Since $E_{\mathcal{S}}$ is closed, $\mathcal{S}$ is pseudo-regular if and only if $\operatorname{dom} E_{\mathcal{S}} = \mathcal{S}[+]\mathcal{S}^{[\perp]}$ is closed. But the closedness of $\operatorname{dom} E_{\mathcal{S}}$ is equivalent to the continuity of $E_{\mathcal{S}}$, see e.g. [6, Theorem 3.2].

Note that $\mathcal{S}$ is non-degenerated if and only if $\operatorname{mul} E_{\mathcal{S}} = \mathcal{S}^{\circ} = \{0\}$. This implies that $E_{\mathcal{S}}$ is a linear operator with $\overline{\operatorname{dom} E_{\mathcal{S}}} = (\mathcal{S}^{\circ})^{[\perp]} = \mathcal{H}$. Conversely, if $E_{\mathcal{S}}$ is a linear operator then, by (5.1), $\mathcal{S}^{\circ} = \operatorname{mul} E_{\mathcal{S}} = \{0\}$.

Finally, by (5.1), if $\mathcal{S}$ is regular then $\operatorname{dom} E_{\mathcal{S}} = \mathcal{S}[\dot{+}]\mathcal{S}^{[\perp]} = \mathcal{H}$ and $\operatorname{mul} E_{\mathcal{S}} = \{0\}$, i.e. $E_{\mathcal{S}}$ is a closed operator with domain $\mathcal{H}$. Then $E_{\mathcal{S}} \in L(\mathcal{H})$. Conversely, if $E_{\mathcal{S}} \in L(\mathcal{H})$ then $\operatorname{mul} E_{\mathcal{S}} = \mathcal{S}^{\circ} = \{0\}$ and $\mathcal{S}[\dot{+}]\mathcal{S}^{[\perp]} = \mathcal{H} = \operatorname{dom} E_{\mathcal{S}}$. Therefore, $\mathcal{S}$ is regular. □

To conclude this section, we show that for any closed subspace $\mathcal{S}$ of $\mathcal{H}$, $E_{\mathcal{S}}$ can be decomposed in terms of the normal companions of $\mathcal{S}^{\circ}$.

Proposition 5.5. *If $\mathcal{S}$ is a closed subspace of $\mathcal{H}$, then*

$$E_{\mathcal{S}} = P_{\mathcal{S}//\mathcal{S}^{[\perp]}\cap\mathcal{N}^{[\perp]}} \; [\hat{+}] \; (\{0\} \times \mathcal{S}^{\circ}),$$

where $\mathcal{N}$ is any normal companion of $\mathcal{S}^{\circ}$.

Proof. If $\mathcal{N}$ is any normal companion of $\mathcal{S}^{\circ}$ then, from Proposition 4.1, $\mathcal{S}^{\circ}\#\mathcal{N}$, $\mathcal{S} = \mathcal{S}^{\circ}[\dot{+}]\mathcal{S} \cap \mathcal{N}^{[\perp]}$ and $\mathcal{S}^{[\perp]} = \mathcal{S}^{\circ}[\dot{+}]\mathcal{S}^{[\perp]} \cap \mathcal{N}^{[\perp]}$. So that $\mathcal{S} \cap \mathcal{S}^{[\perp]} \cap \mathcal{N}^{[\perp]} = \mathcal{S}^{\circ} \cap \mathcal{N}^{[\perp]} = \{0\}$ and $\mathcal{S} \dot{+} \mathcal{S}^{[\perp]} \cap \mathcal{N}^{[\perp]} = \mathcal{S}^{\circ} + \mathcal{S} \cap \mathcal{N}^{[\perp]} + \mathcal{S}^{[\perp]} \cap \mathcal{N}^{[\perp]} = \mathcal{S} + \mathcal{S}^{[\perp]}$.

It holds that $E_{\mathcal{S}} \subseteq P_{\mathcal{S}//\mathcal{S}^{[\perp]}\cap\mathcal{N}^{[\perp]}} \; [\hat{+}] \; (\{0\} \times \mathcal{S}^{\circ})$. In fact, if $(s+t, s) \in E_{\mathcal{S}}$ with $s \in \mathcal{S}$ and $t \in \mathcal{S}^{[\perp]}$ we have that $t = s_0 + n$ with $s_0 \in \mathcal{S}^{\circ}$ and $n \in \mathcal{S}^{[\perp]} \cap \mathcal{N}^{[\perp]}$. So that $(s+t, s) = (s + s_0 + n, s + s_0) + (0, -s_0) \in P_{\mathcal{S}//\mathcal{S}^{[\perp]}\cap\mathcal{N}^{[\perp]}} \hat{+} (\{0\} \times \mathcal{S}^{\circ})$. But since both relations have the same domain and multivalued part, equality holds, [2, 2.02].

It is easy to see that $P_{\mathcal{S}//\mathcal{S}^{[\perp]}\cap\mathcal{N}^{[\perp]}} [\perp] (\{0\} \times \mathcal{S}^{\circ})$. If $(x, y) \in P_{\mathcal{S}//\mathcal{S}^{[\perp]}\cap\mathcal{N}^{[\perp]}} \cap (\{0\} \times \mathcal{S}^{\circ})$ then $y = s_0$ and $x = s_0 + t = 0$ with $t \in \mathcal{S}^{[\perp]} \cap \mathcal{N}^{[\perp]}$ and $s_0 \in \mathcal{S}^{\circ}$. So that $s_0 = -t \in \mathcal{S}^{\circ} \cap \mathcal{N}^{[\perp]} = \{0\}$. Hence, $t = s_0 = 0$ and the sum is direct. □

Declarations

Competing interests

The authors have no competing interests to declare that are relevant to the content of this article.

Data availability

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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