The matched projection and geodesics of the Grassmann manifold

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Abstract

Given an idempotent operator E in a complex Hilbert space $\mathcal{H}$, one can associate to it two orthogonal projections:

- The polar decomposition $2E-1=(2P-1)|2E-1|$ provides an orthogonal projection P. That the unitary part in the decomposition of $2E-1$ is of this form, i.e., a selfadjoint unitary operator, is a remarkable observation done by G. Corach, H. Porta and L. Recht (see references below).
- The question of which, among all orthogonal projections, is the one closest in norm to E, provides another projection, the so called *matched projection* $m(E)$, which answers this question. It was found by X. Tian, Q. Xu and C. Fu (see references below).

In this paper we show that these projections coincide. Moreover, we show that there exists a unique minimal geodesic of the Grassmann manifold of $\mathcal{H}$ (the manifold of closed subspaces of $\mathcal{H}$) that joins $R(E)$ and $R(E^*)$. The orthogonal projection onto the midpoint of this geodesic, also coincides with $m(E)$.

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1 Introduction

Let E be an idempotent in $\mathcal{B}(\mathcal{H})$, which written as a 2×2 matrix in terms of the decomposition $\mathcal{H}=R(E)\oplus R(E)^\perp$ is of the form

$$E=\begin{pmatrix} 1 & B \\ 0 & 0 \end{pmatrix}, \tag{1}$$

where $B:R(E)^\perp\to R(E)$. If $\mathcal{S}\subset\mathcal{H}$ is a closed subspace, denote by $P_{\mathcal{S}}$ the orthogonal projection onto $\mathcal{S}$. In this note we study the position of the projections $P_{R(E)}$ and $P_{R(E^*)}=P^\perp_{N(E)}$ as points in the Grassmann manifold $\mathcal{P}(\mathcal{H})$, of all orthogonal projections in $\mathcal{H}$. The points $P_{R(E)}$ and $P_{R(E^*)}$ lie at distance strictly less than 1, namely (Theorem 3.4)

$$\|P_{R(E)}-P_{R(E^*)}\|=\frac{\|B\|}{(1+\|B\|^2)^{1/2}}.$$

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In [11] H. Porta and L. Recht, introduced a natural linear connection in $\mathcal{P}(\mathcal{H})$, and proved that two projections $P, Q \in \mathcal{P}(\mathcal{H})$ such that $\|P - Q\| < 1$ are joined by a unique geodesic of $\mathcal{P}(\mathcal{H})$ (up to reparametrization), which is minimal for the Finsler metric given by the usual norm of $\mathcal{B}(\mathcal{H})$ at very tangent space. It follows that $P_{R(E)}$ and $P_{R(E^*)}$ are joined at times $t = 0$ and $t = 1$ by a (unique, minimal) geodesic δ_E, which is of the form $\delta_E(t) = e^{itX_E} P_{R(E)} e^{-itX_E}$, for $X_E^* = X_E$, $\|X_E\| < \pi/2$, co-diagonal in the above 2×2 matrix representation.

In [12] X. Tian, Q. Xu and C. Fu introduced the *matched projection* $m(E)$ of an idempotent operator E. Among the many remarkable properties, they prove that $m(E)$ is the orthogonal projection which is closest in norm to E:

$$\|m(E) - E\| \leq \|P - E\|, \quad \text{for all } P \in \mathcal{P}(\mathcal{H}).$$

In [7], G. Corach, H. Porta and L. Recht studied the inmersion $\mathcal{P}(\mathcal{H}) \hookrightarrow \mathcal{Q}(\mathcal{H})$, of $\mathcal{P}(\mathcal{H})$ into the larger space $\mathcal{Q}(\mathcal{H}) = \{E \in \mathcal{B}(\mathcal{H}) : E^2 = E\}$ of idempotent operators, or oblique projections. For instance, they prove that the C^∞-map

$$\pi : \mathcal{Q}(\mathcal{H}) \to \mathcal{P}(\mathcal{H}),$$

is a retraction, where $2E - 1 = (2\pi(E) - 1)|2E - 1|$ is the polar decomposition of $2E - 1$.

In this paper we show that

$$m(E) = \delta_E(\frac{1}{2}) = \pi(E).$$

The contents of the paper are the following. In Section 2 we give basic properties of $R(E)$ and $R(E^*)$. In Section 3 we compute the spectrum (and norm) of $P_{R(E)} - P_{R(E^*)}$. In Section 4 we characterize the geodesic δ_E, by giving an expression of the exponent X_E in terms of the operator B of (1). In Section 5 we consider the matched projection $m(E)$ of E [12], and show that $m(E) = \delta_E(\frac{1}{2})$. In Section 6 we use Dixmier's theory for two subspaces [9] (see also [10]) and Davis' results on the difference of orthogonal projections [8], to give further descriptions of $m(E)$. In Section 7 we study an example: $E_a = \frac{1}{2}(\Gamma_a - 1)$, where Γ_a is the composition operator $\Gamma_a f = f \circ \varphi_a$ acting in $\mathcal{H} = L^2(\mathbb{T}, dm)$ ($\mathbb{T}$ the unit circle, dm the normalized Lebesgue measure), and $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$, for $|a| < 1$. In Section 8 we consider the retraction $\pi : \mathcal{Q}(\mathcal{H}) \to \mathcal{P}(\mathcal{H})$ and prove that $\pi(E) = \delta_E(\frac{1}{2})$. We also examine certain metric properties of $m(E)$.

2 Preliminaries

We say that an operator C is a *reflection* if $C^2 = 1$; a *symmetry* S is a selfadjoint reflection. We identify a closed subspace $\mathcal{S} \subset \mathcal{H}$ with the orthogonal projection $P_\mathcal{S}$ onto $\mathcal{S}$, and with the symmetry $2P_\mathcal{S} - 1$ which equals the identity on $\mathcal{S}$ and minus the identity on $\mathcal{S}^\perp$. The set of subspaces, projections or symmetries will be denoted $\mathcal{P}(\mathcal{H})$. Similarly, we denote by $\mathcal{Q}(\mathcal{H})$ the space of idempotents Q or reflections $2Q - 1$.

When considering a pair of subspaces $\mathcal{S}, \mathcal{T} \subset \mathcal{H}$ (or projections $P_\mathcal{S}.P_\mathcal{T}$) as points in $\mathcal{P}(\mathcal{H})$, a key geometric fact are the intersections $\mathcal{S} \cap \mathcal{T}^\perp$ and $\mathcal{S}^\perp \cap \mathcal{T}$: the necessary and sufficient condition for the existence geodesics of $\mathcal{P}(\mathcal{H})$ joining $P_\mathcal{S}$ and $P_\mathcal{T}$ is that these intersections have the same dimension; the geodesic is unique (up to reparametrization) iff these intersections are trivial. The geodesic δ which joins $\delta(0) = \mathcal{S}$ and $\delta(1) = \mathcal{T}$ ($\mathcal{S}$ and $\mathcal{T}$ satisfying the above condiion) takes the form of a one-parameter group of unitaries acting on $\mathcal{S}$. The infinitesimal generator of this group is a bounded selfadjoint operator X, which can be chosen with norm $\|X\| \leq \pi/2$, and its 2×2 matrix with respect to $\mathcal{S} \oplus \mathcal{S}^\perp = \mathcal{H}$ (and also with respecto to $\mathcal{T} \oplus \mathcal{T}^\perp = \mathcal{H}$) is codiagonal. The geodesic is then, as a curve of subspaces

$$\delta(t) = e^{itX}\mathcal{S},$$

or

$$\delta(t) = e^{itX}P_{\mathcal{S}}e^{-itX}, \ \delta(t) = e^{itX}(2P_{\mathcal{S}} - 1)e^{-itX}$$

as projections or symmetries. Regarded as symmetries, it has the useful computational feature, that since X is codiagonal with respect to $\mathcal{S} \oplus \mathcal{S}^\perp = \mathcal{H}$, X anti-commutes with $2P_{\mathcal{S}} - 1$. Therefore δ takes the alternative forms

$$\delta(t) = e^{2itX}(2P_{\mathcal{S}} - 1) = (2P_{\mathcal{S}} - 1)e^{-2itX}.$$

These facts were shown in [11].

In our case,

$$R(E) \cap R(E^*)^\perp = R(E) \cap N(E) = \{0\} \text{ and } R(E)^\perp \cap R(E^*) = N(E^*) \cap R(E^*) = \{0\},$$

so that there exists a unique geodesic joining $P_{R(E)}$ and $P_{R(E^*)}$.

Also note that

$$R(E) \cap R(E^*) = \{f \in R(E) : B^*f = 0\} = N(B^*), \tag{2}$$

and

$$R(E)^\perp \cap R(E^*)^\perp = \{g \in R(E)^\perp : Bg = 0\} = N(B). \tag{3}$$

Two subspaces $\mathcal{S}, \mathcal{T}$ are in *generic position* [9], [10] if $\mathcal{S} \cap \mathcal{T} = \mathcal{S} \cap \mathcal{T}^\perp = \mathcal{S}^\perp \cap \mathcal{T} = \mathcal{S}^\perp \cap \mathcal{T}^\perp = \{0\}$. In general, we call *generic part* of $\mathcal{S}$ and $\mathcal{T}$ the subspace

$$\mathcal{H}_0 = \left(\mathcal{S} \cap \mathcal{T} \oplus \mathcal{S} \cap \mathcal{T}^\perp \oplus \mathcal{S}^\perp \cap \mathcal{T} \oplus \mathcal{S}^\perp \cap \mathcal{T}^\perp\right)^\perp.$$

In our case $R(E)$ and $R(E^*)$ are in generic position iff B has trivial nullspace and dense range. The generic part of $R(E)$ and $R(E^*)$

$$\mathcal{H}_0 = \left(R(E) \cap R(E^*) \oplus R(E)^\perp \cap R(E^*)^\perp\right)^\perp = (N(B) \oplus N(B^*))^\perp$$

is trivial iff $B = 0$, i.e. $E = E^*$ is an orthogonal projection. Since E and E^* coincide in $\mathcal{H}_0^\perp$, in order to characterize the unique geodesic between $P_{R(E)}$ and $P_{R(E^*)}$, we must focus on the subspace $\mathcal{H}_0$.

3 The distance between $R(E)$ and $R(E^*)$

Let us first recall a result by D. Buckholtz [6]:

Remark 3.1. [6] Let $\mathcal{S}, \mathcal{T} \subset \mathcal{H}$ be closed subspaces. Then the following are equivalent:

1. $\mathcal{S} \dot{+} \mathcal{T} = \mathcal{H}$.
2. $\|P_{\mathcal{S}} + P_{\mathcal{T}} - 1\| < 1$.

3. $P_{\mathcal{S}} - P_{\mathcal{T}}$ is invertible.

As an immediate consequence we get

Corollary 3.2. $\|P_{R(E)} - P_{R(E)}\| < 1$.

Proof. Put $\mathcal{S} = R(E)$ and $\mathcal{T} = N(E) = R(E^*)^\perp$. Then the first condition is verified, and therefore the second says that

$$\|P_{R(E)} + P_{N(E)} - 1\| = \|P_{R(E)} - P_{N(E)^\perp}\| = \|P_{R(E)} - P_{R(E^*)}\| < 1.$$

□

Let us now estimate this distance in terms of the operator $B = B_E : R(E)^\perp \to R(E)$ of (1). To this effect, it will be useful to have formulas of $P_{R(E)}$ and $P_{R(E^*)}$. In [1], it was shown that (note that $E + E^* - 1$ is invertible)

$$P_{R(E)} = E(E + E^* - 1)^{-1}, \tag{4}$$

and

$$P_{R(E^*)} = E^*(E^* + E - 1)^{-1}. \tag{5}$$

Therefore

$$P_{R(E)} - P_{R(E^*)} = (E - E^*)(E + E^* - 1)^{-1}.$$

Let us write this expression in terms of B (see (1)). To compute $(E + E^* - 1)^{-1}$, note that

$$E + E^* - 1 = \begin{pmatrix} 1 & B \\ B^* & -1 \end{pmatrix}$$

and

$$(E + E^* - 1)^2 = \begin{pmatrix} 1 + BB^* & 0 \\ 0 & 1 + B^*B \end{pmatrix}.$$

Then

$$\begin{aligned}(E + E^* - 1)^{-1} &= \begin{pmatrix} (1 + BB^*)^{-1} & 0 \\ 0 & (1 + B^*B)^{-1} \end{pmatrix} \begin{pmatrix} 1 & B \\ B^* & -1 \end{pmatrix} \\ &= \begin{pmatrix} (1 + BB^*)^{-1} & (1 + BB^*)^{-1}B \\ (1 + B^*B)^{-1}B^* & -(1 + B^*B)^{-1} \end{pmatrix}.\end{aligned}$$

Therefore

$$P_{R(E)} - P_{R(E^*)} = \begin{pmatrix} B(1 + B^*B)^{-1}B^* & -B(1 + B^*B)^{-1} \\ -B^*(1 + BB^*)^{-1} & -B^*(1 + BB^*)^{-1}B \end{pmatrix}. \tag{6}$$

Since $B^*(BB^*)^k = (B^*B)^k B^*$ for any $k \geq 0$, we get that

$$B^*(1 + BB^*)^{-1}B = (1 + B^*B)^{-1}B^*B = B^*B(1 + B^*B)^{-1}.$$

Similarly,

$$B(1 + B^*B)^{-1}B^* = (1 + BB^*)^{-1}BB^* = BB^*(1 + BB^*)^{-1}.$$

Then we have also

$$P_{R(E)} - P_{R(E^*)} = \begin{pmatrix} BB^*(1 + BB^*)^{-1} & -B(1 + B^*B)^{-1} \\ -B^*(1 + BB^*)^{-1} & -B^*B(1 + BB^*)^{-1} \end{pmatrix}. \tag{7}$$

We shall need the following result by G. Corach, H. Porta and L. Recht [7] (Corollary 1.7):

Remark 3.3. [7] Put $\mathcal{H}_1 = R(E) \cap N(E)^\perp$, $\mathcal{H}_2 = R(E)^\perp \cap N(E)$, $\mathcal{H}_3 = R(E) \ominus \mathcal{H}_1$ and $\mathcal{H}_4 = R(E)^\perp \ominus \mathcal{H}_2$. Then, in the decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \oplus \mathcal{H}_4$, E is written

$$E = 1 \oplus 0 \oplus \begin{pmatrix} 1 & -Q^{1/2}(1-Q)^{-1/2}D \\ 0 & 0 \end{pmatrix}.$$

Here Q is a positive contraction in $\mathcal{H}_3$ with $N(Q) = 0$ and $1 - Q$ invertible, and $D : \mathcal{H}_4 \to \mathcal{H}_3$ is an isometric isomorphism.

Denote by R the selfadjoint operator

$$R := -Q^{1/2}(1-Q)^{-1/2} \tag{8}$$

. Then, with these notations, we have

Theorem 3.4. *Suppose that $R(E)$ and $N(E)$ are infinite dimensional. Then*

$$\sigma(P_{R(E)} - P_{R(E^*)}) \setminus \{0\} = \{\pm \frac{t}{\sqrt{1+t^2}} : t \in \sigma(R), t \neq 0\}.$$

In particular,

$$\|P_{R(E)} - P_{R(E^*)}\| = \frac{\|B\|}{(1 + \|B\|^2)^{1/2}}.$$

Proof. **Case 1.** Suppose first that $\mathcal{H} = \mathcal{L} \times \mathcal{L}$ and $B : \mathcal{L} \to \mathcal{L}$ satisfies $B^* = B$. Then, using (7) we have

$$P_{R(E)} - P_{R(E^*)} = \begin{pmatrix} B^2(1+B^2)^{-1} & -B(1+B^2)^{-1} \\ -B(1+B^2)^{-1} & -B^2(1+B^2)^{-1} \end{pmatrix}.$$

We can factorize $P_{R(E)} - P_{R(E^*)} = ST$, where

$$S = \begin{pmatrix} B(1+B^2)^{-1/2} & -(1+B^2)^{-1/2} \\ -(1+B^2)^{-1/2} & -B(1+B^2)^{-1/2} \end{pmatrix} \text{ and } T = \begin{pmatrix} B(1+B^2)^{-1/2} & 0 \\ 0 & B(1+B^2)^{-1/2} \end{pmatrix}.$$

The operator S is a symmetry: $S^* = S$ and $S^2 = 1$. Also it is clear that S and T commute. Consider $\mathcal{A}$ the (commutative) C*-algebra generated by S and T. The spectrum $\sigma(P_{R(E)} - P_{R(E^*)})$ can be computed

$$\sigma(P_{R(E)} - P_{R(E^*)}) = \{\varphi(S)\varphi(T) : \varphi \text{ is a character in } \mathcal{A}\}.$$

Since S is a symmetry, $\varphi(S) = \pm 1$. On the other hand $\varphi(T)$ takes (all possible values) in the spectrum of T, i.e., $\sigma(T) = \sigma(B(1+B^2)^{-1/2}) = \{\frac{t}{\sqrt{1+t^2}} : t \in \sigma(B)\}$. Since $P_{R(E)} - P_{R(E^*)}$ is a difference of orthogonal projections, its spectrum is symmetric with respect to the origin (see for instance [8]): save for ± 1, which may or may not belong to this spectrum - and in this case they do not, because the norm of the difference is strictly less than 1, we have that $\lambda \in \sigma(P_{R(E)} - P_{R(E^*)})$ iff $-\lambda \in \sigma(P_{R(E)} - P_{R(E^*)})$. Therefore we have that in this case

$$\sigma(P_{R(E)} - P_{R(E^*)}) = \{\pm \frac{t}{\sqrt{1+t^2}} : t \in \sigma(B)\}.$$

In particular, since the continuous function $f(t)=t(1+t^2)^{-1/2}$ is increasing for all $t\in\mathbb{R}$, and B is selfadjoint (which implies that either $+\|B\|$ or $-\|B\|$ belong to $\sigma(B)$) we have that

$$\|B(1+B^2)^{-1/2}\| = \sup\{|f(t)| : t\in\sigma(B)\} = \max\{-f(-\|B\|), f(\|B\|)\} = f(\|B\|)$$
$$= \frac{\|B\|}{(1+\|B\|^2)^{1/2}}.$$

It follows that $\|P_{R(E)}-P_{R(E^*)}\| = \frac{\|B\|}{(1+\|B\|^2)^{1/2}}$ in this case.

Case 2. (General case): by the result of Corach, Porta and Recht (Corollary 1.7 in [7]) recalled in Remark 3.3, we have that in certain orthogonal decomposition $\mathcal{H}=\mathcal{H}_1\oplus\mathcal{H}_2\oplus\mathcal{H}_3\oplus\mathcal{H}_4$, E can be written

$$E = 1\oplus 0\oplus \begin{pmatrix} 1 & RD \\ 0 & 0 \end{pmatrix},$$

where $R=-Q^{1/2}(1-Q)^{-1/2}$ is selfadjoint in $\mathcal{H}_3$ (as in Remark 3.3), and $D:\mathcal{H}_4\to\mathcal{H}_3$ is a unitary transformation. Denote by E' the idempotent $\begin{pmatrix} 1 & RD \\ 0 & 0 \end{pmatrix}$ in $\mathcal{H}_3\oplus\mathcal{H}_4$. Clearly, it suffices to establich our claim for E' (on $\mathcal{H}_1\oplus\mathcal{H}_2$, E is selfadjoint, and therefore coincides with E^*, $P_{R(E)}$ and $P_{R(E^*)}$). Consider the unitary operator $\mathbf{D}:\mathcal{H}_3\oplus\mathcal{H}_4\to\mathcal{H}_3\times\mathcal{H}_3$ given by

$$\mathbf{D}(f+g) = (f, D^*g).$$

Then $\mathbf{D}E'\mathbf{D}^*$ is a idempotent in $\mathcal{H}_3\times\mathcal{H}_3$, given by

$$\mathbf{D}E'\mathbf{D}^*(f,g) = \mathbf{D}E'(f+D^*g) = \mathbf{D}\begin{pmatrix} 1 & RD \\ 0 & 0 \end{pmatrix}\begin{pmatrix} f \\ D^*g \end{pmatrix}\begin{matrix}\mathcal{H}_3\\ \mathcal{H}_4\end{matrix} = \mathbf{D}\begin{pmatrix} f+RDD^*g \\ 0 \end{pmatrix}\begin{matrix}\mathcal{H}_3\\ \mathcal{H}_4\end{matrix}$$
$$= (f+Rg, 0),$$

i.e., $\mathbf{D}E'\mathbf{D}^* = \begin{pmatrix} 1 & R \\ 0 & 0 \end{pmatrix}$ in $\mathcal{H}_3\times\mathcal{H}_3$. Since $(\mathbf{D}E'\mathbf{D}^*)^* = \mathbf{D}(E')^*\mathbf{D}^*$ and

$$P_{R(\mathbf{D}E'\mathbf{D}^*)} - P_{R(\mathbf{D}(E')^*\mathbf{D}^*)} = \mathbf{D}\left(P_{R(E')} - P_{R((E')^*)}\right)\mathbf{D}^*,$$

we have, by the previous case,

$$\sigma(P_{R(E')} - P_{R((E')^*)}) = \{\pm\frac{t}{\sqrt{1+t^2}} : t\in\sigma(R)\},$$

and thus

$$\sigma(P_{R(E)} - P_{R((E)^*)})\setminus\{0\} = \{\pm\frac{t}{\sqrt{1+t^2}} : t\in\sigma(R), t\neq 0\}.$$

And also

$$\|P_{R(E)} - P_{R((E)^*)}\| = \|P_{R(E')} - P_{R((E')^*)}\| = \frac{\|R\|}{(1+\|R\|^2)^{1/2}} = \frac{\|B\|}{(1+\|B\|^2)^{1/2}},$$

because $\|B\| = \|RD\| = \|R\|$. □

Remark 3.5. We can write the spectrum $P_{R(E)} - P_{R((E)^*}$ in terms of Q (instead of R). Note that

$$R(1+R^2)^{-1/2} = -Q^{1/2}.$$

Thus we get

$$\sigma(P_{R(E)} - P_{R(E)^*}) \setminus \{0\} = \{s : s^2 \in \sigma(Q), s \neq 0\}.$$

Remark 3.6. The value 0 may or may not lie in the spectrum of $P_{R(E)} - P_{R(E^*)}$. Indeed, since

$$P_{R(E)} - P_{R(E^*)} = (E - E^*)(E + E^* - 1)^{-1},$$

it follows that $P_{R(E)} - P_{R(E^*)}$ is invertible if and only if $E - E^*$ is invertible. Note that

$$E - E^* = \begin{pmatrix} 0 & B \\ -B^* & 0 \end{pmatrix}$$

is invertible if and only if $B : R(E)^\perp \to R(E)$ is an isomorphism.

The geodesic distance $d_g(P, Q)$ between two projections P and Q is defined as the infinum of the lengths of all piecewise smooth curves joining P and Q inside $\mathcal{P}(\mathcal{H})$, where the length $\ell(\gamma)$ of a piecewise smooth curve γ parametrized in the interval I is measured by

$$\ell(\gamma) = \int_I \|\dot{\gamma}(t)\| dt,$$

using the usual norm of $\mathcal{B}(\mathcal{H})$. It has been shown that (see for instance [2]), that $d_g(P, Q) = \arcsin \|P - Q\|$.

Corollary 3.7. *Let* $E = \begin{pmatrix} 1 & B \\ 0 & 0 \end{pmatrix}$, *with* $R(E)$ *and* $N(E)$ *infinite dimensional. Then*

$$d_g(P_{R(E)}, P_{R(E^*)}) = \arctan \|B\|.$$

Proof. Denote $d = d_g(P_{R(E)}, P_{R(E^*)})$. Note that $0 \leq d \leq \pi/2$. We know that $\|P_{R(E)} - P_{R(E^*)}\| = \frac{\|B\|}{(1+\|B\|^2)^{1/2}} = \sin(d)$. Then $\cos(d) = (1 - \sin(d)^2)^{1/2} = \frac{1}{(1+\|B\|)^{1/2}}$. Thus $\tan(d) = \|B\|$. ☐

4 The geodesic between $R(E)$ and $R(E^*)$

Since $\|P_{R(E)} - P_{R(E^*)}\| < 1$, there exists a unique operator X_E, the velocity vector of the geodesic δ_E at $t = 0$, such that $\delta_E(0) = P_{R(E)}$ and $\delta_E(1) = P_{R(E^*)}$. The operator X_E thus satisfies

$$X_E^* = X_E, \ \|X_E\| < \pi/2, \ X_E \text{ is } R(E) \oplus R(E)^\perp \text{ co-diagonal},$$

(meaning that its matrix with respect to this decomposition is co-diagonal, or equivalently, $X_E(R(E)) \subset R(E)^\perp$ and $X_E(R(E)^\perp) \subset R(E)$) and

$$e^{iX_E} R(E) = R(E^*).$$

The geodesic si of the form $\delta_E(t) = e^{itX_E} P_{R(E)} e^{-itX_E}$ (and this last condition is $e^{iX_E} P_{R(E)} e^{-iX_E} = P_{R(E^*)}$ (see [7])).

For certain computations, it is useful to consider symmetries instead of projections: to the closed subspace $\mathcal{S} \subset \mathcal{H}$ the symmetry $2P_{\mathcal{S}} - 1$, which equals the identity in $\mathcal{S}$ and minus the indentity in $\mathcal{S}^{\perp}$. This is the standpoint in [11]. The fact that X_E is co-diagonal tranlates to the fact that X_E anti-commutes with $2P_{R(E)} - 1$. It follows that $e^{itX_e}(2P_{R(E)} - 1) = (2P_{R(E)} - 1)e^{-itX_E}$. Thus the geodesic $\epsilon(t) = 2\delta(t) - 1$ is given by (see[11])

$$\epsilon(t) = e^{2itX_e}(2P_{R(E)} - 1) = (2P_{R(E)} - 1)e^{-2itX_E}.$$

Let us compute the exponent X_E associated to E (with $R(E)$ and $N(E)$ infinite dimensional). First we analize the case when $\mathcal{H} = \mathcal{L} \times \mathcal{L}$ and $B : \mathcal{L} \to \mathcal{L}$ is selfadjoint.

Lemma 4.1. *If $\mathcal{H} = \mathcal{L} \times \mathcal{L}$ and $E = \begin{pmatrix} 1 & B \\ 0 & 0 \end{pmatrix}$, with $B : \mathcal{L} \to \mathcal{L}$ selfadjoint, we have that*

$$X_E = \begin{pmatrix} 0 & i\arctan(B) \\ -i\arctan(B) & 0 \end{pmatrix}.$$

Proof. In view of the above comment, we have that $e^{2iX_e}(2P_{R(E)} - 1) = 2P_{R(E^*)} - 1$, i.e.,

$$e^{2iX_E} = (2P_{R(E^*)} - 1)(2P_{R(E)} - 1) = \left(2E^*(E + E^* - 1)^{-1} - 1\right)(2P_{R(E)} - 1)$$

$$= (E^* - E + 1)(E + E^* - 1)^{-1}(2P_{R(E)} - 1) = \begin{pmatrix} 1 & -B \\ B & 1 \end{pmatrix}\begin{pmatrix} 1 & B \\ B & -1 \end{pmatrix}^{-1}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -B \\ B & 1 \end{pmatrix}\begin{pmatrix} 1 & B \\ B & -1 \end{pmatrix}\begin{pmatrix} (1 + B^2)^{-1} & 0 \\ 0 & (1 + B^2)^{-1} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

using that $\begin{pmatrix} 1 & B \\ B & -1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & B \\ B & -1 \end{pmatrix}\begin{pmatrix} (1 + B^2)^{-1} & 0 \\ 0 & (1 + B^2)^{-1} \end{pmatrix}$. Since the right hand two matrices in the product above commute, we have

$$e^{2iX_E} = \begin{pmatrix} 1 & -B \\ B & 1 \end{pmatrix}\begin{pmatrix} 1 & B \\ B & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} (1 + B^2)^{-1} & 0 \\ 0 & (1 + B^2)^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} (1 - B^2)(1 + B^2)^{-1} & -2B(1 + B^2)^{-1} \\ 2B(1 + B^2)^{-1} & (1 - B^2)(1 + B^2)^{-1} \end{pmatrix}.$$

Note that $C =: (1 - B^2)(1 + B^2)^{-1}$ and $S := 2B(1 + B^2)^{-1}$ are selfadjoint operators, with $-1 + d \leq C \leq 1 - d$, $-1 + d \leq S \leq 1 - d$ for some $0 < d < 1$, and $C^2 + S^2 = 1$. Then there exists $Z^* = Z$ in $\mathcal{B}(\mathcal{L})$, $\|Z\| < \pi/2$ such that $C = \cos(Z)$ and $S = \sin(Z)$. Thus

$$e^{2iX_E} = \begin{pmatrix} \cos(Z) & -\sin(Z) \\ \sin(Z) & \cos(Z) \end{pmatrix}, \text{ i.e. }, X_E = \begin{pmatrix} 0 & \frac{i}{2}Z \\ -\frac{i}{2}Z & 0 \end{pmatrix}.$$

Finally, note that $\frac{1}{2}Z = \frac{1}{2}\arcsin(2B(1 + B^2)) = \arctan(B)$, by the functional equality

$$\frac{1}{2}\arcsin(\frac{2t}{1 + t^2}) = \arctan(t)$$

valid for all $t \in \mathbb{R}$.

□

We analize now the general case. Again we invoke the result by Corach, Porta and Recht [7] (Remark 3.3 above): in the orthogonal decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \oplus \mathcal{H}_4$, $E = 1 \oplus 0 \oplus \begin{pmatrix} 1 & RD \\ 0 & 0 \end{pmatrix}$, where R is selfadjoint in $\mathcal{H}_3$ and $D : \mathcal{H}_4 \to \mathcal{H}_3$ is a unitary transformation. Then, using this decomposition,

$$E^* = 1 \oplus 0 \oplus \begin{pmatrix} 1 & 0 \\ D^*R & 0 \end{pmatrix}$$

and thus

$$\begin{aligned} e^{2iX_E} &= (E^* - E + 1)(E + E^* - 1)^{-1}(2P_{R(E)} - 1) \\ &= 1 \oplus 1 \oplus ((E')^* - E' + 1)(E' + (E')^* - 1)^{-1}(2P_{R(E')} - 1), \end{aligned}$$

where again

$$E' = \begin{pmatrix} 1 & RD \\ 0 & 0 \end{pmatrix} = \mathbf{D}^* \begin{pmatrix} 1 & R \\ 0 & 0 \end{pmatrix} \mathbf{D}$$

for the unitary $\mathbf{D} : \mathcal{H}_3 \oplus \mathcal{H}_4 \to \mathcal{H}_3 \times \mathcal{H}_3$, $\mathbf{D}(f + g) = (f, D^*g)$. Then $((E')^* - E' + 1)(E' + (E')^* - 1)^{-1}(2P_{R(E')} - 1)$ equals

$$\mathbf{D}^* \begin{pmatrix} (1 - R^2)(1 + R^2)^{-1} & -2R(1 + R^2)^{-1} \\ 2R(1 + R^2)^{-1} & (1 - R^2)(1 + R^2)^{-1} \end{pmatrix} \mathbf{D}.$$

Therefore we have

Corollary 4.2. *Let E be an idempotent with $R(E)$ and $N(E)$ infinite dimensional. Then, with the current notations, the exponent X_E of the unique geodesic in $\mathcal{P}(\mathcal{H})$ joining $P_{R(E)}$ a $t = 0$ and $P_{R(E^*)}$ at $t = 1$ is*

$$X_E = 0 \oplus 0 \oplus \begin{pmatrix} 0 & i \arctan(R)D \\ -iD^* \arctan(R) & 0 \end{pmatrix}.$$

Proof. In $\mathcal{H}_1 \oplus \mathcal{H}_2$, e^{2iX_E} equals the identity, so $X_E = 0$ there. The rest of X_E is deduced from Lemma 4.1:

$$\mathbf{D}^* \begin{pmatrix} 0 & i \arctan(R) \\ -i \arctan(R) & 0 \end{pmatrix} \mathbf{D} = \begin{pmatrix} 0 & i \arctan(R)D \\ -iD^* \arctan(R) & 0 \end{pmatrix}.$$

□

Remark 4.3. We can therefore compute de midpoint $\gamma(\frac{1}{2})$ of the geodesic joining $P_{R(E)}$ and $P_{R(E^*)}$. We choose to write the corresponding symmetry $2\gamma(\frac{1}{2}) - 1 = e^{iX_E}(2P_{R(E)} - 1)$. Note that in the decomposition $\mathcal{H} = \bigoplus_{i=1}^4 \mathcal{H}_i$ given in Remark 3.3, we have that

$$e^{iX_E} = 1 \oplus 1 \oplus e^{\begin{pmatrix} 0 & -\arctan(R)D \\ D^* \arctan(R) & 0 \end{pmatrix}},$$

which after elementary computations, equals

$$1 \oplus 1 \oplus \begin{pmatrix} \cos(\arctan(R)) & -\sin(\arctan(R))\,D \\ D^* \sin(\arctan(R)) & D^* \cos(\arctan(R))\,D \end{pmatrix}.$$

Since $\cos(\arctan(t)) = \frac{1}{\sqrt{1+t^2}}$ and $\sin(\arctan(t)) = \frac{t}{\sqrt{1+t^2}}$ we get that this expression above equals

$$1 \oplus 1 \oplus \begin{pmatrix} (1+R^2)^{-1/2} & -R(1+R^2)^{-1/2}D \\ D^*R(1+R^2)^{-1/2} & D^*(1+R^2)^{-1/2}D \end{pmatrix}.$$

Therefore we get

$$2\gamma(\frac{1}{2}) - 1 = 1 \oplus 1 \oplus \begin{pmatrix} (1+R^2)^{-1/2} & R(1+R^2)^{-1/2}D \\ D^*R(1+R^2)^{-1/2} & -D^*(1+R^2)^{-1/2}D \end{pmatrix}. \tag{9}$$

In the special case when $\mathcal{H} = \mathcal{L} \times \mathcal{L}$ and $B : \mathcal{L} \to \mathcal{L}$ is selfadjoint, we have

$$2\gamma(\frac{1}{2}) - 1 = \begin{pmatrix} (1+B^2)^{-1/2} & B(1+B^2)^{-1/2} \\ B(1+B^2)^{-1/2} & -(1+B^2)^{-1/2} \end{pmatrix}. \tag{10}$$

5 The matched projection

In [12], X. Tian, Q. Xu and C. Fu defined the *matched projection* $m(E)$ associated to an idempotent E. It has remarkable properties with respect to the operator order (see also [13]). For $E = \begin{pmatrix} 1 & B \\ 0 & 0 \end{pmatrix}$ in the decompostion $\mathcal{H} = R(E) \oplus R(E)^\perp$, $m(E)$ is defined as

$$m(E) = \frac{1}{2} \begin{pmatrix} (T+1)T^{-1} & T^{-1}B \\ B^*T^{-1} & B^*\left(T(T+1)\right)^{-1}B \end{pmatrix}, \quad \text{for } T := (1+BB^*)^{1/2}. \tag{11}$$

Remark 5.1. Let us state some of the properties of $m(E)$ (see [12]).

1. For any orthogonal projection P,

$$\|m(E) - E\| \le \|P - E\| \le \|1 - m(E) - E\|.$$

2. $\|m(E) - E\| = \frac{1}{2}\{\|E\| - 1 + \sqrt{\|E\|^2 - 1}\}$.
3. $m(E) = m(E^*)$, $m(1-E) = 1 - m(E)$.
4. If $\mathcal{K}$ is a Hilbert space and $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ is a $*$-homomorphism, then $m(\Phi(E)) = \Phi(m(E))$.

The matched projection has the following geometric interpretation:

Theorem 5.2. *Let E be an idempotent, and γ_E the unique geodesic of $\mathcal{P}(\mathcal{H})$ such that $\gamma_E(0) = P_{R(E)}$ and $\gamma_E(1) = P_{R(E^*)}$. Then*

$$m(E) = \gamma_E(\frac{1}{2}).$$

Proof. As before, we consider first the case $\mathcal{H} = \mathcal{L} \times \mathcal{L}$ and $B : \mathcal{L} \to \mathcal{L}$ selfadjoint. Then the symmetry $2m(E) - 1$ associated to $m(E)$ is

$$\begin{pmatrix} \left((1+B^2)^{1/2}+1\right)(1+B^2)^{-1} - 1 & (1+B^2)^{-1}B \\ B(1+B^2)^{-1} & B\left((1+B^2)^{1/2}\left((1+B^2)^{1/2}+1\right)\right)^{-1}B - 1 \end{pmatrix}$$

which after elementary computations equals

$$\begin{pmatrix} (1+B^2)^{-1/2} & B(1+B^2)^{-1/2} \\ B(1+B^2)^{-1/2} & -(1+B^2)^{-1/2} \end{pmatrix}$$

which is precisely the expression of $2\delta_E(\frac{1}{2})-1$ given in (10) in Remark 4.3.

Consider now the general case, and again the decomposition $\mathcal{H}=\bigoplus_{i=1}^4 \mathcal{H}_i$ in Remark 3.3. In $\mathcal{H}_1\oplus\mathcal{H}_2$, E is an orthogonal projection, and thus $m(E)=E=P_{R(E)}=P_{R(E^*)}$. Let us analize $\mathcal{H}_3\oplus\mathcal{H}_4$, and denote by E' the reduction of E there. Here $E'=\begin{pmatrix} 1 & RD \\ 0 & 0 \end{pmatrix}$, and according to (11) (see [12]): for $T:=(1+BB^*)^{1/2}$, after elementary computations

$$2m(E')-1=\begin{pmatrix} (T+1)T^{-1}-1 & T^{-1}B \\ B^*T^{-1} & B^*\left(T(T+1)\right)^{-1}B-1 \end{pmatrix}$$

$$=\begin{pmatrix} (1+BB^*)^{-1/2} & (1+BB^*)^{-1/2} \\ B^*(1+BB^*)^{-1/2} & B^*\left((1+BB^*)^{1/2}\left((1+BB^*)^{1/2}+1\right)\right)^{-1}-1 \end{pmatrix}.$$

Note that for any $k\geq 0$, $(BB^*)^kB=B(B^*B)^k$, and thus for any continuous function g in the spectrum of BB^* and B^*B, one has $g(BB^*)B=Bg(B^*B)$. Therefore the $2,2$ entry of the matrix above equals

$$B^*\left((1+BB^*)^{1/2}\left((1+BB^*)^{1/2}+1\right)\right)^{-1}-1$$

$$=B^*B\left((1+B^*B)^{1/2}\left((1+B^*B)^{1/2}+1\right)\right)^{-1}-1=-(1+B^*B)^{-1/2},$$

after elementary computations. That is

$$2m(E')-1=\begin{pmatrix} (1+BB^*)^{-1/2} & (1+BB^*)^{-1/2}B \\ B^*(1+BB^*)^{-1/2} & -(1+B^*B)^{-1/2} \end{pmatrix}. \tag{12}$$

On the other hand, in $\mathcal{H}_3\oplus\mathcal{H}_4$ (using the unitary $\mathbf{D}$ above)

$$2\delta_{E'}(\frac{1}{2})-1=e^{iX_{E'}}(2P_{R(E')}-1)=e^{\begin{pmatrix} 0 & i\arctan(R)D \\ -iD^*\arctan(R) & 0 \end{pmatrix}}(2P_{R(E')}-1)$$

$$=\mathbf{D}^*e^{\begin{pmatrix} 0 & i\arctan(R)D \\ -iD^*\arctan(R) & 0 \end{pmatrix}}\mathbf{D}(2P_{R(E')}-1).$$

Note that $\mathbf{D}(2P_{R(E')}-1)=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\mathbf{D}$. Then, using the first case

$$2\delta_{E'}(\frac{1}{2})-1=\mathbf{D}^*\begin{pmatrix} (1+R^2)^{-1/2} & (1+R^2)R \\ -R(1+R^2)^{-1/2} & (1+R^2)^{-1/2} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\mathbf{D}$$

$$=\begin{pmatrix} (1+R^2)^{-1/2} & (1+B^2)^{-1/2}RD \\ D^*R(1+R^2)^{-1/2} & -D^*(1+R^2)^{-1/2}D \end{pmatrix}. \tag{13}$$

Since $B=RD$, we have

- $(1+BB^*)^{-1/2} = (1+RDD^*R)^{-1/2} = (1+R^2)^{-1/2}$;
- $(1+BB^*)^{-1/2}B = (1+R^2)^{-1/2}RD$;
- $B^*(1+BB^*)^{-1/2} = D^*R(1+R^2)^{-1/2}$; and
- $-(1+B^*B)^{-1/2} = -(1+D^*R^2D)^{-1/2} = -D^*(1+R^2)^{-1/2}D$.

Therefore (12) equals (13). $\square$

Remark 5.3. Recall the properties of the matched projection cited in Remark 5.1. For instance, if δ_E is the unique minimal geodesic such that $\delta_E(0) = P_{R(E)}$ and $\delta_E(1) = P_{R(E^*)}$, then $\delta_E^\perp(t) = 1-\delta_E(t)$ is the unique minimal geodesic joining $\delta_E^\perp(0) = 1 - P_{R(E)} = P_{N(E^*)}$ and $\delta_E^\perp(1) = 1 - P_{R(E^*)} = P_{N(E)}$. It follows that the midpoints $\delta_E(\frac{1}{2})$ of $P_{R(E)}$, $P_{R(E^*)}$ and $\delta_E^\perp(\frac{1}{2})$ of $P_{N(E)}$, $P_{N(E^*)}$, satisfy

$$\|\gamma(\frac{1}{2}) - E\| \le \|P-E\| \le \|\gamma^\perp(\frac{1}{2}) - E\|,$$

for all $P \in \mathcal{P}(\mathcal{H})$.

Note also that the properties $m(E) = m(E^*)$ and $m(1-E) = 1-m(E)$ are (geometrically) evident. Indeed, in the first case, $m(E)$ is the midpoint of the unique geodesic between $P_{R(E)}$ and $P_{R(E^*)}$, which is the same as the midpoint beween $P_{R(E^*)}$ and $P_{R(E)}$, i.e., $m(E^*)$. In the latter, $m(1-E)$ is the midpoint beween $P_{R(1-E)} = P_{N(E)} = P_{R(E^*)}^\perp$ and $P_{R(1-E^*)} = P_{R(E)}^\perp$. Now the geodesic between $P^\perp$ and $Q^\perp$ is $1-\delta$, where δ is the geodesic between P and Q. The asssertion follows.

Finally, if $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ is a unital $*$-homomorphism, and $\delta(t) = e^{itX}P_{R(E)}e^{-itX}$ is the geodesic joining $\delta(0) = P_{R(E)}$ and $\delta(1) = P_{R(E^*)}$, then

$$\Phi(\delta(t)) = \Phi(e^{itX}P_{R(E)}e^{-itX}) = e^{it\Phi(X)}\Phi(P_{R(E)})e^{-it\Phi(X)}.$$

Note $\Phi(\delta)$ is also a geodesic: $\Phi(X)$ is selfadjoint and $\Phi(P_{R(E)})$-codiagonal. And also, due to the formulas (4) and (5),

$$\Phi(P_{R(E)}) = \Phi(E(E+E^*-1)^{-1}) = \Phi(E)(\Phi(E)+\Phi(E)^*-1)^{-1} = P_{R(\Phi(E))},$$

and similarly $\Phi(P_{R(E^*)}) = P_{R(\Phi(E)^*)}$. So that $\Phi(m(E)) = \Phi(\delta(\frac{1}{2})) = m(\Phi(E))$.

6 Computation of the midpoint in terms of Dixmier's theory and Davis' symmetry

J. Dixmier [9] (see also [10]) proved that given two projections P, Q in generic position in a Hilbert space $\mathcal{H}$, there exists a unitary isomorphism between $\mathcal{H}$ and a product space $\mathcal{L}\times\mathcal{L}$, and a positive operator X in $\mathcal{L}$ with $N(X) = \{0\}$ and $\|X\| \le \pi/2$ such that, the projections are carried (via this isomorphism) to

$$P \simeq \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } Q \simeq \begin{pmatrix} C^2 & CS \\ CS & S^2 \end{pmatrix},$$

where $C = \cos(X)$ and $S = \sin(X)$. In terms of this description (and modulo this spatial isomorphism) the unique geodesic between P and Q is easy to describe. Indeed, note that

$$\begin{pmatrix} C^2 & CS \\ CS & S^2 \end{pmatrix} = \begin{pmatrix} C & -S \\ S & C \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} C & S \\ -S & C \end{pmatrix},$$

and

$$\begin{pmatrix} C & -S \\ S & C \end{pmatrix} = e^{\begin{pmatrix} 0 & -X \\ X & 0 \end{pmatrix}} = e^{i\begin{pmatrix} 0 & iX \\ -iX & 0 \end{pmatrix}}.$$

Therefore

$$\begin{pmatrix} C & S \\ -S & C \end{pmatrix} = \begin{pmatrix} C & -S \\ S & C \end{pmatrix}^* = e^{-i\begin{pmatrix} 0 & iX \\ -iX & 0 \end{pmatrix}}.$$

That is, $\begin{pmatrix} 0 & iX \\ -iX & 0 \end{pmatrix}$ is the exponent of the unique geodesic

$$\delta(t) = e^{it\begin{pmatrix} 0 & iX \\ -iX & 0 \end{pmatrix}} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} e^{-it\begin{pmatrix} 0 & iX \\ -iX & 0 \end{pmatrix}}$$

joining $\delta(0) \simeq P$ and $\delta(1) \simeq Q$. Therefore, if one writes the curve $2\delta - 1$ of symmetries , associated to the geodesic δ, it takes the form

$$2\delta(t) - 1 = 2e^{it\begin{pmatrix} 0 & iX \\ -iX & 0 \end{pmatrix}} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} e^{-it\begin{pmatrix} 0 & iX \\ -iX & 0 \end{pmatrix}} - 1$$

$$= e^{it\begin{pmatrix} 0 & iX \\ -iX & 0 \end{pmatrix}} \left(2\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - 1\right) e^{-it\begin{pmatrix} 0 & iX \\ -iX & 0 \end{pmatrix}} = e^{2it\begin{pmatrix} 0 & iX \\ -iX & 0 \end{pmatrix}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then

$$2\delta(\frac{1}{2}) - 1 = e^{2i\frac{1}{2}\begin{pmatrix} 0 & iX \\ -iX & 0 \end{pmatrix}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} C & -S \\ S & C \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} C & S \\ S & -C \end{pmatrix}.$$

Therefore:

Theorem 6.1. *Let P, Q be in generic position. Then, modulo the Dixmier isomorphism, the midpoint of the geodesic bewteen P and Q is*

$$\delta(\frac{1}{2}) \simeq \frac{1}{2}\begin{pmatrix} C+1 & S \\ S & 1-C \end{pmatrix}.$$

In our case, $P = P_{R(E)}$ and $Q = P_{R(E^*)}$ may not be in generic position. The relevant decomposition to describe the position of $R(E)$ and $R(E^*)$, in terms of Dixmier's approach (recalling that $R(E) \cap R(E^*)^\perp = \{0\} = R(E)^\perp \cap R(E^*)$) is

$$\mathcal{H} = R(E) \cap R(E^*) \oplus R(E)^\perp \cap R(E^*)^\perp \oplus \mathcal{H}_0, \tag{14}$$

where $\mathcal{H}_0$ is the generic part. Then, the exponent X_E, of the geodesic between $P_{R(E)}$ and $P_{R(E^*)}$, in terms of the decomposition (14) is, modulo the Dixmier isomorphism

$$X_E \simeq 0 \oplus 0 \oplus \begin{pmatrix} 0 & iX \\ -iX & 0 \end{pmatrix}.$$

This is because in the first two subspaces, $P_{R(E)}$ and $P_{R(E^*)}$ act, respectively, as the identity and zero.

In [8], Chandler Davis established the relation between decompositions of a selfadjoint contraction A as a difference of two orthogonal projections P, Q (i.e., $A = P - Q$), and symmetries V such that $VAV = -A$. Specifically, in the case when P and Q satisfy that $R(P) \cap N(Q) = N(P) \cap R(Q) = \{0\}$, there is a special symmetry V_d, given by

$$V_d := D^{-1/2}(P + Q - 1), \tag{15}$$

where $D := 1 - (P - Q)^2$. Note that $N(1 - (P - Q)^2) = R(P) \cap N(Q) \oplus N(P) \cap R(Q) = \{0\}$, but $1 - (P_Q)^2$ need not be invertible, and thus $D^{-1/2}$ might be unbounded. Nevertheless, V_d turns out to be a symmetry, satisfying $V_d A V_d = -A$. In [3], we proved the relation between V_d and the geodesic joining P and Q (which in this case is unique), namely

$$V_d = e^{iZ}(2P - 1), \tag{16}$$

where Z is the (selfadjoint) exponent of the (unique) geodesic δ joining P and Q, or through Dixmier's isomorphism

$$V_d \simeq \begin{pmatrix} C & -S \\ S & C \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In other words:

Corollary 6.2. *The midpoint $\delta(\frac{1}{2})$ of the geodesic δ joining P and Q is*

$$\delta(\frac{1}{2}) = \frac{1}{2}(V_d + 1),$$

i.e., $V_d = 2\delta(\frac{1}{2}) - 1$.

Proof. We emphasize that in our case, $P = P_{R(E)}$ and $Q = P_{R(E^*)}$, the condition $R(P) \cap N(Q) = N(P) \cap R(Q) = \{0\}$ holds. ☐

Let us write down the symmetry V_d in our special case. I turns out that it has a nicer expression in terms of reflections: write $C = C_E = 2E - 1$ (so that $C^* = 2E^* - 1$). After routine computations, we obtain that

$$P_{R(E)} = (1 + C)(C + C^*)^{-1} \quad \text{and} \quad P_{R(E^*)} = (1 + C^*)(C + C^*)^{-1}. \tag{17}$$

Notice that in our case, $D = 1 - A^2$ is invertible (i.e., D^{-1} is bounded), because $\|A\| = \|P_{R(E)} - P_{R(E^*)}\| < 1$.

Theorem 6.3. *With the current notations, we have that for $P_{R(E)}$ and $P_{R(E^*)}$, the Davis symmetry V_d is the unitary part in the polar decomposition of $C + C^*$.*

Proof. We compute first

$$D = 1 - A^2 = (1-A)(1+A)$$

$$= \{1-(1+C)(C+C^*)^{-1}+(1+C^*)(C+C^*)^{-1}\}\{1+(1+C)(C+C^*)^{-1}-(1+C^*)(C+C^*)^{-1}\}$$

$$= 4C^*(C+C^*)^{-1}C(C+C^*)^{-1}.$$

Then

$$D^{-1} = \frac{1}{4}(C+C^*)C(C+C^*)C^* = \frac{1}{4}(2+C^*C+CC^*) = \frac{1}{4}(C+C^*)^2,$$

and therefore

$$D^{-1/2} = \frac{1}{2}\{(C+C^*)^2\}^{1/2} = |C+C^*|.$$

On the other hand,

$$P_{R(E)} + P_{R(E^*)} - 1 = (1+C)(C+C^*)^{-1} + (1+C^*)(C+C^*)^{-1} - 1 = 2(C+C^*)^{-1}.$$

Thus,

$$V_d = |C+C^*|(C+C^*)^{-1}, \text{ i.e., } C+C^* = V_d|C+C^*|$$

is the (unique) polar decomposition of the invertible element $C+C^*$. □

7 An example

Let us consider the following example:

Example 7.1. Let $\mathcal{H} = L^2 = L^2(\mathbb{T})$ (with normalized Lebesgue measure), and for $a \in \mathbb{D}$, consider the map $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$. This map satisfies $\varphi_a(\mathbb{D}) = \mathbb{D}$ and $|\varphi_a(z)| = 1$, so that it also is a map from $\mathbb{T}$ onto $\mathbb{T}$, and is its own inverse: $\varphi_a(\varphi_a(z)) = z$. Therefore it induces a reflection in L^2,

$$\Gamma_a f = f \circ \varphi_a, \; f \in L^2.$$

Clearly $\Gamma_a^2 = 1$. The idempotent $E_a = \frac{1}{2}(\Gamma_a + 1)$ associated to Γ_a has range $R(E_a) = N(\Gamma_a - 1)$, with $R(E_a^*) = N(\Gamma_a + 1)^\perp$. These subspaces were studied in [4]. There it was shown that $N(\Gamma_a - 1)$ and $N(\Gamma_a + 1)$ are in generic position (Theorem 6.3). Also it was shown (Theorem 3.2) that $N(\Gamma_a - 1) = \Gamma_{\omega_a}(\mathcal{E})$, where $\omega_a = \frac{1}{\bar{a}}(1-\sqrt{1-|a|^2})$ is the (unique) fixed point of φ_a inside $\mathbb{D}$, and $\mathcal{E}$ is the subspace of elements in $\mathcal{H}$ which have only even powers in their Fourier series:

$$\mathcal{E} = \{f \in L^2 : f = \sum_{k=-\infty}^{\infty} \hat{f}(2k) z^{2k}\},$$

so that

$$R(E_a) = \{g \in L^2 : g = \sum_{k=-\infty}^{\infty} a_k \left(\frac{\omega_a - z}{1-\bar{\omega}_a z}\right)^{2k}\}.$$

Accordingly, $N(\Gamma_a + 1) = \Gamma_{\omega_a}\mathcal{O}$, for $\mathcal{O}$ the subspace of elements which have only non nil Fourier coefficients for odd indices, and then

$$R(E_a^*) = \{h \in L^2 : h = \sum_{k=-\infty}^{\infty} a_k \left(\frac{\omega_a - z}{1-\bar{\omega}_a z}\right)^{2k+1}\}^\perp.$$

Let us compute the polar decomposition of $\Gamma_a + \Gamma_a^*$. It is easy to see that $\Gamma_a^* = M_{|\psi_a|^2}\Gamma_a = \Gamma_a M_{1/|\psi_a|^2}$, where $\psi_a(z) = \frac{(1-|a|^2)^{1/2}}{1-\bar{a}z}$ is the normalized Szego kernel. Then

$$|\Gamma_a + \Gamma_a^*|^2 = (\Gamma_a + \Gamma_a^*)^2 = (\Gamma_a + M_{|\psi_a|^2}\Gamma_a)(\Gamma_a + \Gamma_a M_{1/|\psi_a|^2}) = 2 + M_{|\psi_a|^2} + M_{1/|\psi_a|^2},$$

and then $|\Gamma_a + \Gamma_a^*| = M_{\gamma_a}$, where

$$\gamma_a = (2 + |\psi_a|^2 + 1/|\psi_a|^2)^{1/2} = \frac{1 + |\psi_a|^2}{|\psi_a|}.$$

Then

$$V_d = |\Gamma_a + \Gamma_a^*|(\Gamma_a + \Gamma_a^*)^{-1} = M_{\gamma_a}(\Gamma_a + \Gamma_a M_{1/|\psi_a|^2})^{-1} = M_{\gamma_a}\{\Gamma_a(1 + M_{1/|\psi_a|^2}\Gamma_a)\}^{-1}$$
$$= M_{\gamma_a(1+1/|\psi_a|^2)^{-1}}\Gamma_a = M_{|\psi_a|}\Gamma_a.$$

Then. the midpoint $\delta(\frac{1}{2})$ of the geodesic between $P_{R(E_a)}$ and $P_{R(E_a^*)}$ is

$$\delta(\frac{1}{2}) = \frac{1}{2}(M_{|\psi_a|}\Gamma_a + 1).$$

Using the formula in [1] for $P_{R(E)}$ ((4) above), for the idempotent $\frac{1}{2}(\Gamma_a + 1)$ that projects onto $N(\Gamma_a - 1)$ and for $1 - \frac{1}{2}(\Gamma_a + 1)$ that projects onto $N(\Gamma_a + 1)$, it is easy to see that

$$P_{N(\Gamma_a - 1)} = (1 + \Gamma_a)M_{(1+|\psi_a|^2)^{-1}} \quad \text{and} \quad P_{N(\Gamma_a + 1)} = (1 - \Gamma_a)M_{(1+|\psi_a|^2)^{-1}} \tag{18}$$

Then

$$P_{N(\Gamma_a - 1)} - P^{\perp}_{N(\Gamma_a + 1)} = P_{N(\Gamma_a - 1)} + P_{N(\Gamma_a + 1)} - 1 = M_{\kappa_a},$$

where $\kappa_a := 2(1 + |\psi_a|^2)^{-1} - 1 = \frac{1-|\psi_a|^2}{1+|\psi_a|^2}$. An elementary calculation shows that the image $\{\kappa_a(z) : z \in \mathbb{T}\}$ of this map is $[-|a|, |a|]$. Therefore, in particular,

$$\|P_{N(\Gamma_a - 1)} - P^{\perp}_{N(\Gamma_a + 1)}\| = |a|. \tag{19}$$

The operator $B_a : N(\Gamma_a - 1)^{\perp} \to N(\Gamma_a - 1)$ in the matrix representation $E_a = \frac{1}{2}(\Gamma_a + 1) = \begin{pmatrix} 1 & B_a \\ 0 & 0 \end{pmatrix}$ in terms of $\mathcal{H} = N(\Gamma_a - 1) \oplus N(\Gamma_a - 1)^{\perp}$ has trivial nullspace and dense range (this is a consequence of the fact the eigenspaces of Γ_a are in generic position, see Section 2). Also B_a can be explicitely computed:

$$B_a = \frac{1}{2}P_{N(\Gamma_a - 1)}(1 + \Gamma_a)(1 - P_{N(\Gamma_a)}) = \frac{1}{2}M_{\kappa_a}(\Gamma_a - 1).$$

From Theorem 3.4 and (19) above, it follows that

$$\|B_a\| = \frac{|a|}{\sqrt{1 - |a|^2}}.$$

Consider the polar decomposition $B_a = U_a|B_a|$, with $U_a : N(\Gamma_a - 1)^{\perp} \to N(\Gamma_a - 1)$ isometric. Then, with the similar computations as in Lemma 4.1 and Corollary 4.2, we have that the exponent of the geodesic γ joining $\gamma(0) = P_{R(E_a)} = P_{N(\Gamma_a - 1)}$ and $\gamma(1) = P_{R(E_a^*)} = P^\perp_{N(\Gamma_1+1)}$ is (in terms of the decomposition $\mathcal{H} = N(\Gamma_a - 1) \oplus N(\Gamma_a - 1)^\perp$)

$$X_a = \begin{pmatrix} 0 & iU_a \arctan(|B_a|) \\ -i\arctan(|B_a|)U_a^* & 0 \end{pmatrix}.$$

Let us compute $|B_a|$ and U_a explicitely. Recall that

$$(E_a + E_a^* - 1)^2 - 1 = \begin{pmatrix} B_a B_a^* & 0 \\ 0 & B_a^* B_a \end{pmatrix}$$

On the other hand

$$(E_a + E_a^* - 1)^2 - 1 = E_a E_a^* + E_a^* E_a - E_a - E_a^*$$

$$= \frac{1}{4}(1+\Gamma_a)(1+\Gamma_a^*) + \frac{1}{4}(1+\Gamma_a^*)(1+\Gamma_a) - \frac{1}{2}(1+\Gamma_a) - \frac{1}{2}(1+\Gamma_a^*) = \frac{1}{4}\Gamma_a\Gamma_a^* + \frac{1}{4}\Gamma_a^*\Gamma_a - \frac{1}{2}$$

$$= \frac{1}{4}M_{|\psi_a|^2} + \frac{1}{4}M_{\frac{1}{|\psi_a|^2}} - \frac{1}{2} = M_{\frac{(|\psi_a|^2-1)^2}{4|\psi_a|^2}}.$$

Then

$$\begin{pmatrix} |B_a^*| & 0 \\ 0 & |B_a| \end{pmatrix} = \frac{1}{2} M_{\frac{||\psi_a|^2-1|}{|\psi_a|}}.$$

It follows that $|B_a|$ is the compression of $\frac{1}{2}M_{\frac{||\psi_a|^2-1|}{|\psi_a|}}$ with $P^\perp_{N(\Gamma_a-1)}$. Note that

$$P^\perp_{N(\Gamma_a-1)} = 1 - P_{N(\Gamma_a-1)} = 1 - (1+\Gamma_a)M_{(1+|\psi_a|^2)^{-1}} = M_{1-(1+|\psi_a|^2)^{-1}} - \Gamma_a M_{(1+|\psi_a|^2)^{-1}}.$$

Clearly $\Gamma_a M_{(|\psi_a|^2+1)^{-1}} = M_{\frac{|\psi_a|^2}{1+|\psi_a|^2}}\Gamma_a$. Then

$$P^\perp_{N(\Gamma_a-1)} = M_{\frac{|\psi_a|^2}{1+|\psi_a|^2}}(1-\Gamma_a).$$

Thus

$$|B_a| = \frac{1}{2} M_{\frac{|\psi_a|^2}{1+|\psi_a|^2}}(1-\Gamma_a) M_{\frac{||\psi_a|^2-1|}{|\psi_a|^2}} M_{\frac{|\psi_a|^2}{1+|\psi_a|^2}}(1-\Gamma_a),$$

which after straightforwad computations (involving $\Gamma_a M_{f(|\psi_a|^2)} = M_{f(1/|\psi_a|^2)}\Gamma_a$, for f continuous in $(0, +\infty)$), give

$$|B_a| = M_{|\psi_a|^2 \frac{||\psi_a|^2-1|}{(1+|\psi_a|^2)^2}}(1-\Gamma_a). \tag{20}$$

To complete the computation of the exponent X_a, we must describe the isometry $U_a : N(\Gamma_a - 1)^\perp \to N(\Gamma_a - 1)$. That is

$$\frac{1}{2}M_{\kappa_a}(\Gamma_a - 1) = U_a M_{|\psi_a|^2 \frac{||\psi_a|^2-1|}{(1+|\psi_a|^2)^2}}(1-\Gamma_a) \text{ restricted to } N(\Gamma_a - 1)^\perp.$$

Since $\Gamma_a - 1$ is an isomorphism when restricted to $N(\Gamma_a - 1)^\perp$, this amounts to

$$\frac{1}{2}M_{\kappa_a} = U_a M_{|\psi_a|^2 \frac{||\psi_a|^2-1|}{(1+|\psi_a|^2)^2}} \text{ on the range of } \Gamma_a - 1|_{N(\Gamma_a-1)^\perp}, \text{ i.e., on } N(\Gamma_a+1).$$

This implies that the (non invertible) operator $M_{|\psi_a|^2 \frac{||\psi_a|^2-1|}{(1+|\psi_a|^2)^2}}$ maps $N(\Gamma_a+1)$ onto a dense subspace of $\mathcal{S} \subset N(\Gamma_a-1)=\overline{R(B_a^*)}$. It follows that on this subspace $\mathcal{S}$, we have

$$U_a\Big|_{\mathcal{S}} = M_{\kappa_a} M_{\frac{(1+|\psi_a|^2)^2}{|1-|\psi_a|^2|}}\Big|_{\mathcal{S}} = M_{sgn(1-|\psi_a|^2)} M_{\frac{1}{2}(1+|\psi_a|^2)}\Big|_{\mathcal{S}},$$

where $sgn(t)$ is the sign function: $sign(t)=\begin{cases} t/|t| \text{ if } t\neq 0 \\ 0 \text{ if } t=0\end{cases}$. Since the right hand operator is bounded, it follows that

$$U_a = M_{sgn(1-|\psi_a|^2)} M_{\frac{1}{2}(1+|\psi_a|^2)}\Big|_{N(\Gamma_a-1)}.$$

Note that $M_{sgn(1-|\psi_a|^2)}$ is a global symmetry: the set $\{t\in[-\pi,\pi] : |\psi_a|=1\}$ has measure zero: it consists of two points (for instance, for $a=r\in(0,1)$, it is $\{\arccos(r), -\arccos(r)\}$). This implies that $M_{\frac{1}{2}(1+|\psi_a|^2)}\Big|_{N(\Gamma_a-1)}$ is isometric.

8 A retraction from idempotents onto projections

In [11] it was noted that the unitary part R in the polar decomposition $C=R|C|$ of a reflection C, is a symmetry. In [7], G. Corach, H. Porta and L. Recht studied the geometry of the map from the set $\mathcal{Q}(\mathcal{H})=\{C\in\mathcal{B}(\mathcal{H}) : C^2=1\}$ of reflections onto the set $\mathcal{P}(\mathcal{H})=\{S\in\mathcal{Q}(\mathcal{H}) : S^*=S\}$ of symmetries:

$$\pi:\mathcal{Q}(\mathcal{H})\to\mathcal{P}(\mathcal{H}),\ \pi(C)=R=C|C|^{-1}. \tag{21}$$

Notice that in Example 7.1, the symmetry $V_d=M_{|\psi_a|}\Gamma_a$ also coincides with the unitary part in the polar decomposition of Γ_a:

$$\Gamma_a^*\Gamma_a = M_{\frac{(1-|a|^2}{|1-\bar{a}z|^2}}\Gamma_a\Gamma_a = M_{|\psi_a|^2},$$

and thus $|\Gamma_a| = \left(M_{|\psi_a|^2}\right)^{1/2} = M_{|\psi_a|}$. Therefore the unitary part R_a of the polar decomposition $\Gamma_a = R_a|\Gamma_a|$ is $R_a=|\Gamma_a|\Gamma_a = M_{|\psi_a|}\Gamma_a = V_d$. It is fair to ask if this is always the case.

Proposition 8.1. *Let C be a reflection, then the unitary part in the polar decomposition of $C+C^*$ coincides with the unitary part of the polar decomposition of C.*

Proof. Let $C=R_C|C|$ and $C+C^*=R_{C+C^*}|C+C^*|$ be the polar decompositions of C and $C+C^*$. We must show that

$$C|C|^{-1}=(C+C^*)|C+C^*|^{-1}.$$

This is equivalent to the equality of the inverses $|C|C=|C+C^*|(C+C^*)^{-1}$, i.e.,

$$|C|C(C+C^*)=|C+C^*|. \tag{22}$$

Both terms in (22) are positive:

$$|C|C(C+C^*)=|C|(1+CC^*)=(CC^*)^{-1/2}(1+CC^*)$$

is positive. Then it suffices to prove the equality of the squares in (22):

$$(|C|(1+CC^*))^2 = |C+C^*|^2.$$

The left hand term equals (again, since $|C|$ and $1+CC^*$ commute)

$$C^*C(1+CC^*)^2 = C^*C(1+2CC^*+(CC^*)^2) = C^*C+2+CC^*,$$

which coincides with

$$|C+C^*|^2 = (C+C^*)^2 = 2+CC^*+C^*C.$$

□

We have then yet another characterization of the midpoint between $R(E)$ and $R(E^*)$:

Corollary 8.2. *The midpoint $\delta_E(\frac{1}{2})$ between $P_{R(E)}$ and $P_{R(E^*)}$ is the unitary part R_E in the polar decomposition $2P_{R(E)}-1 = R_E|2P_{R(E)}-1|$.*

In other words,

$$\frac{1}{2}(\pi(C)+1) = m(\frac{1}{2}(C+1)).$$

In [7], a Finsler metric was introduced in $\mathcal{Q}(\mathcal{H})$: for $C \in \mathcal{Q}(\mathcal{H})$ and $X \in (T\mathcal{Q}(\mathcal{H}))_C$,

$$|X|_C := \||C|^{1/2}X|C|^{-1/2}\|. \tag{23}$$

Note that in $\mathcal{P}(\mathcal{H}) \subset \mathcal{Q}(\mathcal{H})$, this is the usual norm of $\mathcal{B}(\mathcal{H})$. Among the facts proved in [7], it was shown that $(T\pi)_Q$ is contractive at every point $Q \in \mathcal{Q}(\mathcal{H})$. Also, that the metric behaves as a non positively curved metric when restricted to the fibers $\pi^{-1}(S)$, for S in $\mathcal{P}$. In particular, any two points in the fiber $\pi^{-1}(S)$ can be joined by a unique geodesic, which is minimal along its path. The geodesic $C(t)$ joining $C(1) = C$ with $C(0) = S = \pi(C)$ inside $\pi^{-1}(S)$ is

$$C(t) = S|C|^t = |C|^{-t/2}S|C|^{t/2} = |C|^{-t}S, \ t \in \mathbb{R}. \tag{24}$$

Remark 8.3. Note that the following maps in $\mathcal{Q}(\mathcal{H})$ are isometric for the Finsler structure of [7]:

1. The map $E \mapsto 1-E$ between idempotents, at the reflection level is $C = 2E-1 \mapsto 2(1-E)-1 = -C$, and is clearly isometric.
2. The adjoint map $C \mapsto C^*$ is also isometric. Indeed, for $X \in (T\mathcal{Q}(\mathcal{H}))_C$ (using that $|C^*| = |C|^{-1}$)

$$|X^*|_{C^*} = \||C^*|^{1/2}X^*|C^*|^{-1/2}\| = \||C|^{-1/2}X^*|C|^{1/2}\| = \|(|C|^{1/2}X|C|^{-1/2})^*\|$$
$$= \||C|^{1/2}X|C|^{-1/2}\| = |X|_C.$$

We may combine these facts, with the norm inequalities proved by X. Tian, Q. Xu and C. Fu in [12], to obtain:

Corollary 8.4. *Let E be an idempotent (with infinite rank and nullity). Then, for any orthogonal projection P and any $t \in \mathbb{R}$ we have that*

$$\|m(E)-P\| \leq \||2E-1|^{-t/2}m(E)|2E-1|^{t/2}-P\|.$$

Proof. Let $C = 2E - 1$ and $C(t) = |C|^{-t/2}\pi(C)|C|^{t/2}$ the geodesic joining C with $\pi(C) = 2m(E) - 1$. Then using Remark 5.1.1, for the idempotent $\frac{1}{2}C(t)$

$$\|m(\frac{1}{2}(C(t)+1)) - P\| \leq \|\frac{1}{2}(C(t)+1) - P\|,$$

for all $t \in \mathbb{R}$. Note that

$$\frac{1}{2}(C(t)+1) = \frac{1}{2}(|C|^{-t/2}\pi(C)|C|^{t/2}+1) = |C|^{-t/2}\frac{1}{2}(\pi(C)+1)|C|^{t/2} = |C|^{-t/2}E|C|^{t/2}.$$

Also note that all elements in the fiber $\pi^{-1}(S)$ project onto S: $\pi(C(t)) = S$, i.e., $m(\frac{1}{2}(C(t)+1) = \frac{1}{2}(S+1)$. Then the norm inequality above reads

$$\|\frac{1}{2}(S+1) - P\| \leq \|\frac{1}{2}(C(t)+1) - P\|,$$

which proves our claim. $\square$

Remark 8.5. With the same argument, it can be shown that for any idempotent F such that $2F - 1$ belongs to the fiber $\pi^{-1}(S)$ of $S = 2E - 1$, one has that for any projection $P \in \mathcal{P}(\mathcal{H})$,

$$\|m(E) - P\| \leq \|F - P\|.$$

Remark 8.6. Also the property that the tangent map $(T\pi)$ of π is contractive can be used. Denote by d the Finsler metric in $\mathcal{Q}(\mathcal{H})$:

$$d(E,F) = \inf\{\ell(\gamma) : \gamma \text{ is smooth and joins } E \text{ and } F \text{ in } \mathcal{Q}\},$$

where, for γ joining $\gamma(a) = E$ and $\gamma(b) = F$

$$\ell(\gamma) = \int_a^b |\dot{\gamma}(t)|_{\gamma(t)} dt.$$

Therefore, for $E, F \in \mathcal{Q}$ we have that

$$d(\pi(E), \pi(F)) \leq d(E,F).$$

For any curve γ in $\mathcal{Q}$ joining $\gamma(a) = E$ and $\gamma(b) = F$, we have that $\pi(\gamma)$ is smooth and joins $\pi(E)$ and $\pi(F)$, and since $(T\pi)$ is contractive,

$$|\dot{\pi(\gamma)}(t)|_{\pi(\gamma)(t)} = |(T\pi)_{\gamma(t)}\dot{\gamma}(t)|_{\pi(\gamma)(t)} \leq |\dot{\gamma}(t)|_{\gamma(t)}.$$

It follows that $\ell(\pi(\gamma)) \leq \ell(\gamma)$, and the assertion follows.

In particular, for any orhogonal projection P (in the same connected component as $\pi(E)$, i.e., with the same rank and nullity as E), one has that

$$d(P, m(E)) \leq d(P, E). \tag{25}$$

We may describe the situation with the following figure:

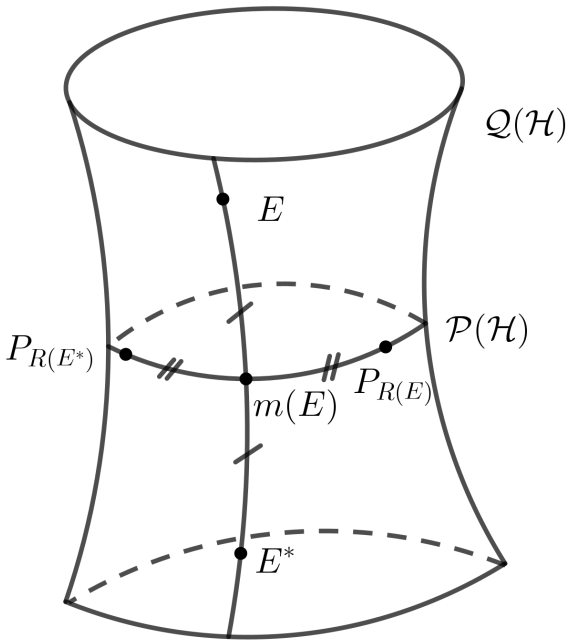

Figure 1

Regarding this figure we have that

Proposition 8.7. *Identifying an idepotent E with its associated reflection $2E-1$, and a projection P with the symmetry $2P-1$.*

1.

$$d(E, m(E)) = d(E^*, m(E)) = \| \log |2E-1| \|$$

and

$$\|E - m(E)\| = \|E^* - m(E)\| = \frac{1}{2}\left(\sqrt{\|B\|^2 - 1} + \|B\|\right).$$

2.

$$d(P_{R(E)}, m(E)) = d(P_{R(E^*)}, m(E)) = \frac{1}{2}\arctan(\|B\|)$$

and

$$\|P_{R(E)} - m(E)\| = \|P_{R(E^*)} - m(E)\| = \sin(\frac{1}{2}\arctan(\|B\|)).$$

Proof. The first part of assertion 1 is from [7]: the unique minimal geodesic γ joining $\gamma(1) = C = 2E-1$ and $\gamma(0) = \pi(C) = 2m(E)-1$ is $\gamma(t) = \pi(C)|C|^t$, and thus $|\dot{\gamma}(t)|_{\gamma(t)} = \| \log |C| \|$. The second assertion is Theorem 3.17 of [12]: there it was shown that $\|m(E) - E\| = \frac{1}{2}\left(\|E\| - 1 + \sqrt{\|E\|^2 - 1}\right)$, and clearly $\|E\| = \sqrt{\|B\|^2 + 1}$.

For the second assertion, note that $d(P_{R(E)}, m(E)) = \frac{1}{2}\|X_E\|$, and $\|X_E\| = \arctan\|B\|$. $\square$

Remark 8.8. We may compute explicitely $d(E_a, m(E_a)) = \|\log(\Gamma_a)\|$ for Γ_a of Example 7.1 in Section 7. Since $\log(|C|) = \frac{1}{2}\log(|C|^2) = -\frac{1}{2}\log(|C^*|^{-2})$ and $|\Gamma_a^*|^2 = \Gamma_a\Gamma_a^* = M_{1/|\psi_a|^2} = M_{\frac{|1-\bar{a}z|^2}{1-|a|^2}}$, we have that

$$\log(|\Gamma_a^*|^2) = M_{\log(\frac{|1-\bar{a}z|^2}{1-|a|^2})}.$$

the function $\log(\frac{|1-\bar{a}z|^2}{1-|a|^2})$ takes values between $\log(1-|a|^2)-\log(1+|a|^2)$ and $\log(1+|a|^2)-\log(1-|a|^2)$. It follows that

$$d(E_a, m(E_a)) = \frac{1}{2}\log\left(\frac{1+|a|}{1-|a|}\right).$$

Notice that the (unique, minimal) geodesic $\Gamma(t)$ of $\mathcal{Q}(\mathcal{H})$ which satisfies $\Gamma(0)=\pi(\Gamma_a)$ and $\Gamma(1)=\Gamma_a$ is (see [7]) $\Gamma(t)=\pi(\Gamma_a)|\Gamma_a|^t$. Then (since $\pi(\Gamma_a)$ is selfadjoint)

$$\Gamma(-1)=\pi(\Gamma_a)|\Gamma_a|^{-1}=|\Gamma_a|\pi(\Gamma_a)=(\pi(\Gamma_a)|\Gamma_a|)^*=\Gamma_a^*.$$

That is, $\Gamma_a^*, \pi(\Gamma_a)$ and Γ_a (in that order), belong to the same geodesic in the fiber of π. In particular, this implies that

$$d(\Gamma_a,\Gamma_a^*)=\log(\frac{1+|a|}{1-|a|}).$$

The fact that $C^*, \pi(C)$ and C (or $E^*, m(E), E$) belong to the same geodesic, at times $t=-1, t=0$ and $t=1$, holds in general, with the same proof:

Corollary 8.9. *Let $C\in\mathcal{Q}(\mathcal{H})$. The geodesic $C(t)$, $t\in\mathbb{R}$ with $C(0)=\pi(C)$ and $C(1)=C$, satisfies $C(-1)=C^*$. In other words, C and C^* belong to the same fiber of π, and $\pi(C)=2m(E)-1$ is the midpoint of the unique geodesic of this fiber joining C and C^*.*

This valids the scheme at Figure 1: we already have proved that $P_{R(E)}, m(E)$ and $P_{R(E^*)}$ belong o the same (unique. minimal) geodesic of $\mathcal{P}(\mathcal{H})\subset\mathcal{Q}$, at times $t=0, t=\frac{1}{2}$ and $t=1$.

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