

LOCAL CONSTANTS AND BOHR'S PHENOMENON FOR BANACH SPACES OF ANALYTIC POLYNOMIALS

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ABSTRACT. The primary aim of this work is to develop methods that provide new insights into the relationships between fundamental constants in Banach space theory—specifically, the projection constant, the unconditional basis constant and the Gordon-Lewis constant—for the Banach space $\mathcal{P}_J(X_n)$ of multivariate analytic polynomials. This class consists of all polynomials whose monomial coefficients vanish outside the set of multi-indices J , and it is equipped with the supremum norm on the unit sphere of the finite-dimensional Banach space $X_n = (\mathbb{C}^n, \|\cdot\|)$. We establish a general framework for proving quantitative results on the asymptotic optimal behavior of these constants, which depend on both the dimension of the space and the degree of the polynomials. Using the tools developed, we derive asymptotic estimates of the Bohr radius for general Banach sequence lattices. Additionally, we apply our results to the asymptotic study of local constants and the Bohr radius within finite-dimensional Lorentz sequence spaces, which requires a refined analysis of the combinatorial structure of the associated index sets. As a consequence, we obtain optimal results across a broad range of parameters.

INTRODUCTION

The study of local properties of infinite-dimensional Banach spaces, which are intricately linked to the structure of finite-dimensional subspaces, is fundamental to both the local and general theory of Banach spaces. This area of research is driven by various aspects of modern analysis. One of the core challenges in local Banach space theory is understanding the asymptotic behavior of certain fundamental constants that have significant applications.

The goal of this work is to establish estimates for key constants in Banach space theory—specifically the projection constant, the unconditional basis constant, and the Gordon-Lewis constant—within the context of the finite-dimensional Banach spaces $\mathcal{P}_J(X_n)$ of polynomials. In this setting, $\mathcal{P}_J(X_n)$ denotes the Banach space of multivariate analytic polynomials P defined on finite-dimensional Banach spaces $X_n = (\mathbb{C}^n, \|\cdot\|)$, where the monomial coefficients of P vanish outside the set of multi-indices J . This space is equipped with the supremum norm over the unit sphere of X_n . Additionally, we explore the relationships between the involved constants. Furthermore, we aim to unify these concepts in the study of the Bohr radius for unit spheres of finite-dimensional Banach spaces and demonstrate the application of our general results to a significant class of finite-dimensional

spaces generated by the scale of Lorentz spaces. To accomplish these objectives, we will employ techniques from various fields of analysis, including local Banach space theory, probability theory, complex analysis, Banach lattice theory, and combinatorics.

To clarify these aims, we will present key definitions and essential technical tools that naturally arise in our investigation.

The projection constant. The projection constant, in particular, is one of the most significant concepts in modern Banach space theory, closely related to the classical problem of describing complemented subspaces of Banach spaces—a problem that has been extensively studied since the inception of abstract operator theory.

Recall that if X is a closed subspace of a Banach space Y , then the relative projection constant of X in Y is defined by

$$\lambda(X, Y) = \inf \{ \|P\| : P \in \mathcal{L}(Y, X), P|_X = \text{id}_X \},$$

where id_X denotes the identity operator on X , and $\mathcal{L}(U, V)$ represents the Banach space of all bounded linear operators between Banach spaces U and V with the supremum norm. Here, $\mathcal{L}(U) := \mathcal{L}(U, U)$, and we adopt the convention that $\inf \emptyset = \infty$.

The (absolute) projection constant of X is expressed as:

$$\lambda(X) := \sup \lambda(I(X), Y),$$

where the supremum is taken over all Banach spaces Y and isometric embeddings $I: X \rightarrow Y$. Note that if X is a finite-dimensional Banach space and X_1 is a subspace of some $C(K)$ -space isometric to X , then (see, e.g., [60, III.B.5 Theorem]):

$$\lambda(X) = \lambda(X_1, C(K)).$$

Thus, determining $\lambda(X)$ is equivalent to finding the norm of a minimal projection from $C(K)$ onto X_1 .

General bounds for projection constants of various finite-dimensional Banach spaces have been studied by many authors. The most fundamental general upper bound is due to Kadets and Snobar [38]: For every finite-dimensional Banach space X , we have

$$(1) \quad \lambda(X) \leq \sqrt{\dim X}.$$

Regarding the Kadets-Snobar estimate (1), we also mention a remarkable result of Pisier [53, Corollary 10.8], who constructed an infinite dimensional Banach space X such that $\|P\| \geq \delta \sqrt{\text{rank } P}$ for some $\delta \in (0, 1)$ and all finite rank projections $P: X \rightarrow X$. König [41] proved that the Kadets-Snobar estimate (1) is asymptotically best possible. More precisely, he showed that there exists a sequence

$(X_{n_k})_{k=1}^\infty$ of finite-dimensional real Banach spaces such that $\dim(X_{n_k}) = n_k$, where $n_k \rightarrow \infty$ as $k \rightarrow \infty$, and

$$\lim_{k \rightarrow \infty} \frac{\lambda(X_{n_k})}{\sqrt{n_k}} = 1.$$

Another surprising result in this direction is due to Figiel, Lindenstrauss and Milman [31]. It states that there is a universal constant $c \in (0, 1)$ such that every n -dimensional Banach space E contains a subspace F such that $\lambda(F) \geq c\sqrt{n}$.

The unconditional basis constant. The unconditional basis constant of a basis $(e_i)_{i \in I}$ of a Banach space E is given by the infimum over all $K > 0$ such that for any finitely supported family $(\alpha_i)_{i \in I}$ of scalars and for any finitely supported family $(\varepsilon_i)_{i \in I}$ with $|\varepsilon_i| = 1$, $i \in I$ we have

$$\left\| \sum_{i \in I} \varepsilon_i \alpha_i e_i \right\| \leq K \left\| \sum_{i \in I} \alpha_i e_i \right\|.$$

We denote the unconditional basis constant of $(e_i)_{i \in I}$ by $\chi((e_i)_{i \in I}) = \chi((e_i)_{i \in I}; X)$. We also write $\chi((e_i)_{i \in I}) = +\infty$, whenever $(e_i)_{i \in I}$ is not unconditional, and say that $(e_i)_{i \in I}$ is a 1-unconditional basis, whenever $\chi((e_i)_{i \in I}) = 1$. The unconditional basis constant $\chi(X)$ of X is defined to be the infimum of $\chi((e_i)_{i \in I})$ taken over all possible unconditional basis $(e_i)_{i \in I}$ of X .

The Gordon-Lewis constant. A significant milestone in the study of unconditional basis constants was achieved by Gordon and Lewis in [35], who introduced the constant now known as the G-L constant. In light of our study of this constant for the space of polynomials, we need to introduce some additional concepts to properly present the original definition of this important concept in the context of local Banach space theory.

Given Banach spaces X , Y and $1 \leq p \leq \infty$, an operator $u \in \mathcal{L}(X, Y)$ is said to be p -factorable whenever there exist a measure space (Ω, Σ, μ) and operators $v \in \mathcal{L}(X, L_p(\mu))$, $w \in \mathcal{L}(L_p(\mu), Y^{**})$, satisfying the following factorization $\kappa_Y u: X \xrightarrow{v} L_p(\mu) \xrightarrow{w} Y^{**}$; here, as usual, $\kappa_Y: Y \rightarrow Y^{**}$ is the canonical injection. In this case the γ_p -norm of the p -factorable operator u is given by

$$(2) \quad \gamma_p(u) = \inf \|v\| \|w\|,$$

where the infimum is taken over all possible factorizations. We are mainly interested in the norms γ_p for operators acting between finite dimensional Banach spaces X and Y . In this case, the infimum in (2) is realized considering all possible factorizations of the more simple form

$$\begin{array}{ccc} X & \xrightarrow{u} & Y, \\ & \searrow v & \nearrow w \\ & & \ell_p^n \end{array}$$

where n is arbitrary. An operator $u \in \mathcal{L}(X, Y)$ is said to be 1-summing if there is a constant $C > 0$ such that for each choice of finitely many $x_1, \dots, x_N \in X$ one has

$$\sum_{j=1}^N \|ux_j\|_Y \leq C \sup \left\{ \sum_{j=1}^N |x^*(x_j)| : \|x^*\|_{X^*} \leq 1 \right\}.$$

By $\pi_1(u: X \rightarrow Y)$ we denote the least such C satisfying this inequality. It is known that for any n -dimensional Banach space X , we have

$$\lambda(X) \pi_1(\text{id}_X) \geq n,$$

and if X has enough symmetries (i.e., any operator $u: X \rightarrow X$ which commutes with all isometries on X , is a multiple of the identity map), then

$$\lambda(X) \pi_1(\text{id}_X) = n,$$

see, e.g., [34, 42]. A Banach space X has the Gordon-Lewis property if every 1-summing operator $u: X \rightarrow \ell_2$ is 1-factorable. In this case, there is a positive number C such that for all 1-summing operators $u: X \rightarrow \ell_2$

$$\gamma_1(u) \leq C \pi_1(u),$$

and the best such $C > 0$ is called the Gordon–Lewis constant of X and denoted by $\mathbf{gl}(X)$.

We can further clarify the aforementioned comment about the crucial role of the Gordon-Lewis constant in the study of unconditionality in Banach spaces. Specifically, Gordon and Lewis [35] showed that for every Banach space X with an unconditional basis $(e_i)_{i \in I}$ the following estimate holds

$$(3) \quad \mathbf{gl}(X) \leq \chi(X) \leq \chi((e_i)_{i \in I}).$$

It is worth noting that in contrast the unconditional basis constant, the Gordon-Lewis constant has the useful (ideal) property that $\mathbf{gl}(X) \leq \|u\| \|v\| \mathbf{gl}(Y)$ whenever $u: X \rightarrow Y$ and $v: Y \rightarrow X$ are operators such that $\text{id}_X = uv$. In particular, if X_0 is an isometric subspace of X , we get the following estimate

$$\mathbf{gl}(X_0) \leq \lambda(X_0, X) \mathbf{gl}(X).$$

The projection constant of a Banach space X can be formulated in terms of the ∞ -factorization norm of the identity operator id_X . More precisely, if X is a Banach space and X_0 is any subspace of some $L_\infty(\mu)$ isometric to X , then

$$(4) \quad \lambda(X) = \gamma_\infty(\text{id}_X) = \lambda(X_0, L_\infty(\mu)).$$

In the general case of finite-dimensional spaces, determining these constants, or even finding their optimal asymptotic behavior as a function of the dimension of the space, is indeed a challenging problem.

Summary of key results. Our focus is on spaces of polynomials defined on n -dimensional Banach spaces.

We mention three fundamental examples for projection constants of multivariate polynomials. The first one is a theorem of Lozinski and Kharshiladze (see [60, IIIB. Theorem 22] and [49]), which shows a precise estimate of the projection constant of $\text{Trig}_{\leq m}(\mathbb{T})$, the space of trigonometric polynomials on the torus $\mathbb{T} \subset \mathbb{C}$ of degree at most m equipped with the sup-norm:

$$(5) \quad \lambda(\text{Trig}_{\leq m}(\mathbb{T})) = \frac{4}{\pi^2} \log(m+1) + o(1).$$

The second result is a beautiful formula from the 1980s, due to Ryll and Wojtaszczyk [56] (see also [60, IIIB. Theorem 15]), which provides the exact value of the projection constant for the space $\mathcal{P}_m(\ell_2^n)$ of all m -homogeneous polynomials on the n -dimensional complex Hilbert space ℓ_2^n :

$$(6) \quad \lambda(\mathcal{P}_m(\ell_2^n)) = \frac{\Gamma(n+m)\Gamma(1+\frac{m}{2})}{\Gamma(1+m)\Gamma(n+\frac{m}{2})}.$$

This result was a key step in addressing several open questions of the time in the context of Hardy and Bloch spaces on the complex ball, including one posed by Rudin, who asked whether there exists an inner function on the open unit ball of Hilbert spaces (see, e.g., Rudin's monograph [55] and Aleksandrov's article [1]).

Motivated by the study of multivariate variants of Bohr's famous power series theorem, Defant and Frerick [16] (see also [25, Section 22.2]) provided (implicitly) the following asymptotically sharp estimate

$$(7) \quad \lambda(\mathcal{P}_m(\ell_r^n)) \sim_{C^m} \left(1 + \frac{n}{m}\right)^{m\left(1 - \frac{1}{\min\{r, 2\}}\right)},$$

where $1 \leq r \leq \infty$ and \sim_{C^m} stands for equivalence up to C^m with a constant C only depending on r .

We note that the cases involving polynomial spaces on real and complex Hilbert spaces, Boolean cubes, as well as Dirichlet polynomials, have been explored in [19],[21], [18], and [20], respectively. These works, authored by Defant, Galicer, Mansilla, Mastysłó, and Muro, are part of a recent systematic study that also includes an analysis of the projection constant for the trace class [22].

We now provide a brief overview of some of the main results of the present article. In Section 1, we review the key definitions and notation used throughout the paper.

In Section 2, we establish the relationships between the Gordon-Lewis constant $\mathbf{gl}(\mathcal{P}_J(X_n))$, the unconditional basis constant $\chi_{\mathbf{mon}}(\mathcal{P}_J(X_n))$ (that is, the unconditional basis constant of the monomials $(z^\alpha)_{\alpha \in J}$ in $\mathcal{P}_J(X_n)$) and the projection constant $\lambda(\mathcal{P}_J(X_n))$ in the case that $X_n = (\mathbb{C}^n, \|\cdot\|)$ is a Banach lattice and J an index set. The main results here are Theorem 2.6 and Theorem 2.7.

Applying a famous theorem of Lozanovskii, we derive new upper estimates for $\lambda(\mathcal{P}_J(X_n))$ in the setting of Banach lattices $X_n = (\mathbb{C}^n, \|\cdot\|)$. We draw attention to Theorem 2.19 and Corollary 2.20. These results allow us to systematically extend estimates like (7) to a wider range of spaces X_n and index sets J . To obtain lower estimates for $\chi_{\mathbf{mon}}(\mathcal{P}_J(X_n))$, we use probabilistic methods. An application of the obtained estimates is Theorem 2.16, which states that in a class of 2-convex Banach lattices X_n for a large class of index sets J , the lower and upper bounds of the aforementioned constants $\chi_{\mathbf{mon}}(\mathcal{P}_J(X_n))$ and $\lambda(\mathcal{P}_J(X_n))$ are asymptotically correct.

We also revisit and expand on various results from the literature concerning Bohr radii in high dimensions in this section. Recall that given an n -dimensional Banach lattice $X_n = (\mathbb{C}^n, \|\cdot\|)$, the Bohr radius $K(B_{X_n})$ of its open unit ball B_{X_n} is the largest $0 < r < 1$ such that for every holomorphic function $f: B_{X_n} \rightarrow \mathbb{C}$, the following holds:

$$\sup_{z \in r B_{X_n}} \sum_{\alpha \in \mathbb{N}_0^{(n)}} \left| \frac{\partial^\alpha f(0)}{\alpha!} z^\alpha \right| \leq \sup_{z \in B_{X_n}} |f(z)|.$$

Bohr's famous power series theorem in [11] shows, for the unit ball \mathbb{D} of the Banach space \mathbb{C} , that $K(\mathbb{D}) = \frac{1}{3}$. As usual, the Banach space of all bounded holomorphic functions $f: B_{X_n} \rightarrow \mathbb{C}$ is denoted by $H_\infty(B_{X_n})$. We will focus on a more general setting. Fixing an n -dimensional Banach lattice $X_n = (\mathbb{C}^n, \|\cdot\|)$ together with an index set $J \subset \mathbb{N}_0^{(n)}$, we consider the closed subspace

$$H_\infty^J(B_{X_n}) := \{f \in H_\infty(B_{X_n}) : c_\alpha(f) = 0 \text{ for } \alpha \notin J\}$$

of the Banach space of all bounded holomorphic functions $H_\infty(B_{X_n})$, and define

$$(8) \quad K(B_{X_n}, J) := \sup \left\{ r \in (0, 1) : \sup_{z \in r B_{X_n}} \sum_{\alpha \in J} \left| \frac{\partial^\alpha f(0)}{\alpha!} z^\alpha \right| \leq \|f\|_\infty \text{ for all } f \in H_\infty^J(B_{X_n}) \right\},$$

the Bohr radius of B_{X_n} with respect to J , and $K(B_{X_n}) := K(B_{X_n}, \mathbb{N}_0^{(n)})$.

Based on our findings, we describe the behavior of Bohr radii with respect to the unconditional basis and projection constants of characteristics 'homogeneous' index sets, as well as the characteristic 'Homogeneous building blocks' $H_\infty^J(B_{X_n})$. Additionally, we explore their relationship to the convexity of the underlying Banach sequence lattices. The key result in this section is Theorem 2.34 which shows for a large class of Banach sequence lattices X and sets of indices J that $K(B_{X_n}, J) / \sqrt{\log n / n} \rightarrow 1$ as $n \rightarrow \infty$. This result is inspired by the remarkable work of Bayart, Pellegrino, and Seoane, who in

[5] established that the preceding limit holds for $X = \ell_\infty$ and the full set of multi-indices, with the asymptotic order initially determined by Defant, Frerick, Ortega-Cerdà, Ounaïes, and Seip in [17].

In Section 3, we focus on polynomials defined on finite-dimensional Lorentz spaces $\ell_{r,s}^n$ and supported on a given finite index set J of multi-indices. We provide asymptotically accurate estimates for the projection constant and the unconditional basis constant of $\mathcal{P}_J(\ell_{r,s}^n)$, which depend on both the dimension n and the structure of J . In particular, we show in Theorem 3.3 that for the set of tetrahedral indices, the estimates for the space of homogeneous polynomials on ℓ_r^n (as given by (7)) remain valid for $\mathcal{P}_J(\ell_{r,s}^n)$ and for more general sets of indices J , independently of the second parameter s . To establish similar results for more general index sets (see the Theorems 3.5, 3.8, 3.9, and 3.19), we overcome subtle technical challenges that are more complex than those encountered in the case of ℓ_r^n , requiring the use of most of the techniques previously developed. Finally, we apply this to prove Theorem 3.22 on asymptotic estimates of Bohr radii in finite-dimensional Lorentz sequence spaces. Specifically, assuming that the index set J has a certain mild structure (including the tetrahedral indices), we provide the optimal asymptotic order of $K(B_{\ell_{r,s}^n}, J)$ for (almost) all r and s .

For the entire index set $J = \mathbb{N}_0^{(\mathbb{N})}$, this problem was addressed in [27, Corollary 10]. The results presented here significantly advance those findings by proving the limit in certain cases and providing a new asymptotically optimal result over a range of parameters, thus making a relevant progress on an open problem in the field. Our research contributes to the extensive recent literature on both one-dimensional and multidimensional Bohr radii. For the one-dimensional case, see, for example, [2, 6, 8, 12, 36, 39, 40, 51, 52], and for the multivariate case [3, 5, 9, 16, 17, 23, 27, 10].

1. BACKGROUND AND NOTATION

We will require significant notation from (local) Banach space theory, as used in the monographs [29, 44, 53, 59]. Unless otherwise specified, we consider complex Banach spaces. The dual Banach space of X is denoted by X^* . The symbol B_X (resp., \overline{B}_X) denotes the open (resp., closed) unit ball of X .

If X and Y are isomorphic spaces, that is, there is an invertible operator from X onto Y , we write $X \simeq Y$. We use the notion $X \equiv Y$ whenever X and Y may be identified isometrically. To indicate injective bounded mappings $T: X \rightarrow Y$, we sometimes write $T: X \hookrightarrow Y$.

For isomorphic Banach spaces X and Y , the Banach-Mazur distance between X and Y is defined to be

$$d(X, Y) := \inf\{\|T\| \|T^{-1}\| : T \text{ an isomorphism of } X \text{ onto } Y\}.$$

If X and Y are not isomorphic, we let $d(X, Y) = +\infty$.

Given two sequences (a_n) and (b_n) of non-negative real numbers we write $a_n <_C b_n$, if there is a constant $C > 0$ such that $a_n \leq C b_n$ for all $n \in \mathbb{N}$, while $a_n \sim_C b_n$ means that $a_n <_C b_n$ and $b_n <_C a_n$ holds. In the case that an extra parameter m is also involved, for two sequences of non-negative real numbers $(a_{n,m})$ and $(b_{n,m})$, we write $a_{n,m} <_{C^m} b_{n,m}$ when there is a hypercontractive comparison, that is, there is a constant $C > 0$ (independent of n and m) such that $a_{n,m} \leq C^m b_{n,m}$ for all $n, m \in \mathbb{N}$. Of course, if $a_{n,m} <_{C_1^m} b_{n,m}$ and $b_{n,m} <_{C_2^m} a_{n,m}$ we simply write $a_{n,m} \sim_{C^m} b_{n,m}$.

We denote the set of all sequences $\alpha \in \mathbb{N}_0^{(\mathbb{N})}$ with finite support as $\mathbb{N}_0^{(\mathbb{N})}$. As is customary, we refer to these sequences α as multi-indices. It is clear that

$$\mathbb{N}_0^{(\mathbb{N})} = \bigcup_{n \in \mathbb{N}} \mathbb{N}_0^n,$$

where \mathbb{N}_0^n is interpreted as a subset of $\mathbb{N}_0^{(\mathbb{N})}$. We say that \mathbb{N}_0^n forms the set of all multi indices of length n . For each multi-index $\alpha = (\alpha_i) \in \mathbb{N}_0^{(\mathbb{N})}$, we define the order of α as

$$|\alpha| = \sum \alpha_i.$$

Using this notion, for $m, n \in \mathbb{N}$, we define

$$\begin{aligned} \Lambda(m, n) &= \{\alpha \in \mathbb{N}_0^n : |\alpha| = m\}, \\ \Lambda(\leq m, n) &= \{\alpha \in \mathbb{N}_0^n : |\alpha| \leq m\}, \end{aligned}$$

as well as

$$\Lambda(m) := \bigcup_n \Lambda(m, n) \quad \text{and} \quad \Lambda(\leq m) := \bigcup_n \Lambda(\leq m, n).$$

The following formula and estimate

$$(9) \quad |\Lambda(m, n)| = \binom{n+m-1}{m} \leq e^m \left(1 + \frac{n}{m}\right)^m$$

for the cardinality of $\Lambda(m, n)$ is crucial for our purposes.

For simplicity of notation and presentation throughout the paper, a subset of $\mathbb{N}_0^{(\mathbb{N})}$ is called an *index set*. If J is an index set and $m, n \in \mathbb{N}$, then we will frequently consider the following four building blocks of J :

$$\begin{aligned} J(m, n) &:= J \cap \Lambda(m, n) \quad \text{and} \quad J(\leq m, n) := J \cap \Lambda(\leq m, n), \\ J(m) &:= J \cap \Lambda(m) \quad \text{and} \quad J(\leq m) := J \cap \Lambda(\leq m). \end{aligned}$$

An index set J is said to have degree at most m whenever $J = J(\leq m)$, that is, $|\alpha| \leq m$ for all $\alpha \in J$. Moreover, J is called m -homogeneous if $J = J(m)$, so $|\alpha| = m$ for all $\alpha \in J$.

A multi-index $\alpha = (\alpha_i) \in \mathbb{N}_0^{(\mathbb{N})}$ is called tetrahedral if each entry α_i is either 0 or 1. We denote the index set of all tetrahedral multi-indices by Λ_T . The preceding definitions also provide precise meanings for $\Lambda_T(m, n)$, $\Lambda_T(\leq m, n)$, $\Lambda_T(m)$, and $\Lambda_T(\leq m)$.

Given $J \subset \Lambda(m, n)$, we need to consider the index set

$$J^b \subset \Lambda(m-1, n),$$

which consists of all $\alpha \in \Lambda(m-1, n)$ for which there is $1 \leq k \leq n$ and $\beta \in J$ such $\beta_i = \alpha_i$ for all $1 \leq i \neq k \leq n$ and $\beta_k = \alpha_k + 1$. We call J^b the reduced index set of J . Note that $\Lambda(m, n)^b = \Lambda(m-1, n)$.

We point out that it will sometimes be convenient to use an equivalent description of $\Lambda(m, n)$. To describe it, let us denote

$$\mathcal{M}(m, n) := \{1, \dots, n\}^m \quad \text{and} \quad \mathcal{J}(m, n) := \{\mathbf{j} = (j_1, \dots, j_m) \in \mathcal{M}(m, n) : j_1 \leq \dots \leq j_m\},$$

and observe that there is a canonical bijection between $\mathcal{J}(m, n)$ and $\Lambda(m, n)$. Indeed, assign to $\mathbf{j} \in \mathcal{J}(m, n)$ the multi-index $\alpha \in \Lambda(m, n)$ given by $\alpha_r = |\{k : \mathbf{j}_k = r\}|$, $1 \leq r \leq n$, and conversely for each $\alpha \in \Lambda(m, n)$ the index $\mathbf{j} \in \mathcal{J}(m, n)$, where

$$j_1 = \dots = j_{\alpha_1} = 1, \quad j_{\alpha_1+1} = \dots = j_{\alpha_1+\alpha_2} = 2, \dots, j_{\alpha_{n-1}+1} = \dots = j_{\alpha_{n-1}+\alpha_n} = n.$$

On $\mathcal{M}(m, n)$ we consider the following equivalence relation: $\mathbf{i} \sim \mathbf{j}$ if there is a permutation σ on $\{1, \dots, m\}$ such that $(i_1, \dots, i_k) = (i_{\sigma(1)}, \dots, i_{\sigma(m)})$. The equivalence class of $\mathbf{i} \in \mathcal{M}(m, n)$ is denoted by $[\mathbf{i}]$, and its cardinality by $||[\mathbf{i}]||$. Provided that \mathbf{j} is associated with α , we have that

$$||[\alpha]|| := ||[\mathbf{j}]|| = \frac{m!}{\alpha!}.$$

Note that, given an index set $J \subset \mathcal{J}(m, n) = \Lambda(m, n)$, the reduced set J^b in terms of the \mathbf{j} -mode is described as follows:

$$(10) \quad J^b = \{\mathbf{j} \in \mathcal{J}(m-1, n) : \exists 1 \leq k \leq n \text{ such that } (\mathbf{j}, k)_* \in J\},$$

where we associate to each $\mathbf{i} \in \mathcal{M}(m, n)$ the unique element $\mathbf{i}_* \in \mathcal{J}(m, n)$ for which $\mathbf{i} \in [\mathbf{i}_*]$.

By $\mathcal{P}(\mathbb{C}^n)$ we denote the vector space of all finite polynomials

$$P(z) = \sum_{\alpha \in \mathbb{N}_0^n} c_\alpha(P) z^\alpha, \quad z = (z_1, \dots, z_n) \in \mathbb{C}^n,$$

where $c_\alpha(P) \in \mathbb{C}$ for each $\alpha \in \mathbb{N}_0^n$. More generally, for any nonempty finite index set J , we define

$$\mathcal{P}_J(\mathbb{C}^n)$$

to be the subspace of all $P \in \mathcal{P}(\mathbb{C}^n)$ for which $c_\alpha(P) = 0$ for all $\alpha \notin J$. Clearly, $\mathcal{P}_J(\mathbb{C}^n) = \mathcal{P}_{J \cap \mathbb{N}_0^n}(\mathbb{C}^n)$.

For $m \in \mathbb{N}$, we denote by $\mathcal{P}_{\leq m}(\mathbb{C}^n)$ the space of all polynomials $P \in \mathcal{P}(\mathbb{C}^n)$ which have degree $\deg P := \max\{|\alpha| : \alpha \in J\} \leq m$, and by $\mathcal{P}_m(\mathbb{C}^n)$ the space of all m -homogeneous polynomials. Note that under the above notation $\mathcal{P}_{\leq m}(\mathbb{C}^n) = \mathcal{P}_{\Lambda(\leq m)}(\mathbb{C}^n)$ and $\mathcal{P}_m(\mathbb{C}^n) = \mathcal{P}_{\Lambda(m)}(\mathbb{C}^n)$.

Given a nonempty finite index set J and a Banach space $X_n := (\mathbb{C}^n, \|\cdot\|)$, we equip $\mathcal{P}(\mathbb{C}^n)$ with the sup norm

$$\|P\|_{B_{X_n}} := \sup_{z \in B_{X_n}} |P(z)|,$$

and denote the resulting Banach space by $\mathcal{P}_J(X_n)$. This is, in fact, the main object of our interest.

We also need a few facts on polynomials defined on arbitrary Banach spaces X (and not only $X_n := (\mathbb{C}^n, \|\cdot\|)$). For all relevant information we refer to [15, 25, 30, 32].

Let X be a Banach space X over the field \mathbb{K} , where $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, and $m \in \mathbb{N}$. A mapping $P: X \rightarrow \mathbb{K}$ is said to be a (bounded) m -homogeneous polynomial if there exists a (bounded) m -linear form $T: X \times \cdots \times X \rightarrow \mathbb{K}$ such that $P(z) = T(z, \dots, z)$ for all $z \in X$. In fact, given a (bounded) m -homogeneous polynomial P on X , by the usually called polarization formula

$$(11) \quad T(z^{(1)}, \dots, z^{(m)}) = \frac{1}{m!2^m} \sum_{\varepsilon_i = \pm 1} \varepsilon_1 \dots \varepsilon_m P\left(\sum_{i=1}^m \varepsilon_i z^{(i)}\right),$$

there is a unique symmetric (bounded) m -linear form T with the property that $P(z) = T(z, \dots, z)$ for all $z \in X$. We as usual denote this unique form by $\overset{\vee}{P} = T$. The space of all bounded m -linear forms over X is denoted by $\mathcal{L}_m(X)$, whereas we write $\mathcal{P}_m(X)$ for all bounded m -homogeneous polynomials on X . Endowed with the norms

$$\|T\| = \sup_{z^{(i)} \in B_X} |T(z^{(1)}, \dots, z^{(m)})| \quad \text{resp.}, \quad \|P\| = \sup_{z \in B_X} |P(z)|,$$

each gets a Banach space. Furthermore, we need the notion of a polynomial of degree less than or equal to m on a given Banach space X . By $\mathcal{P}_{\leq m}(X)$ we denote the linear space of all functions $P: X \rightarrow \mathbb{K}$ having the form $P = P_0 + \sum_{k=1}^m P_k$, where $P_k \in \mathcal{P}_k(X)$, $1 \leq k \leq m$ and $P_0 \in \mathbb{C}$. As before, $\mathcal{P}_{\leq m}(X)$ together with the supremum norm (of P on B_X) leads to a Banach space.

We recall the notion of usually called polarization constants, which appear naturally when relating homogeneous polynomials with their associated symmetric multilinear forms. From (11) we easily obtain the following polarization inequality

$$\|\overset{\vee}{P}\| \leq \frac{m^m}{m!} \|P\|, \quad P \in \mathcal{P}_m(X).$$

Given a finite dimensional Banach space X , we at a few occasions need to represent $\mathcal{L}_m(X)$ and $\mathcal{P}_m(X)$ as injective tensor products. Recall that in this case the canonical identities

$$\mathcal{L}_m(X) = \otimes_{\varepsilon}^m X^* \quad \text{and} \quad \mathcal{P}_m(X) = \otimes_{\varepsilon_s}^m X^*$$

hold algebraically as well as isometrically, where \otimes_{ε}^m stands for the m th full injective tensor product and $\otimes_{\varepsilon_s}^m$ for the m th symmetric injective tensor product (see, e.g., [32]). We also need the m th full projective tensor product \otimes_{π}^m (resp., the m th symmetric projective tensor product $\otimes_{\pi_s}^m$), which is dual to \otimes_{ε}^m (resp., $\otimes_{\varepsilon_s}^m$).

Important geometric concepts from the theory of Banach lattices will be used, and we will repeat some of them here. For more details, we refer to [44, 45].

A Banach function lattice X over a measure space $(\Omega, \mathcal{A}, \mu)$ is said to be p -convex, $1 \leq p \leq \infty$, respectively q -concave, $1 \leq q \leq \infty$, if there is a constant $C > 0$ such that for every choice of finitely many $x_1, \dots, x_N \in X$

$$\left\| \left(\sum_{k=1}^N |x_k|^p \right)^{1/p} \right\|_X \leq C \left(\sum_{k=1}^N \|x_k\|_X^p \right)^{1/p},$$

resp.,

$$\left(\sum_{k=1}^N \|x_k\|_X^q \right)^{1/q} \leq C \left\| \left(\sum_{k=1}^N |x_k|^q \right)^{1/q} \right\|_X$$

(with the usual modification whenever $p = \infty$ or $q = \infty$). We define the p -convexity constant $M^{(p)}(X)$ (resp., q -concavity constant $M_{(q)}(X)$) to be the least constant $C > 0$ satisfying the above inequality. In case that X is not p -convex (resp., not q -concave), then we write $M^{(p)}(X) = \infty$ (resp., $M_{(q)}(X) = \infty$). Clearly, every Banach function lattice X is 1-convex and ∞ -concave (with constants 1). We also note that, if $r < p < s$ and X is p -convex (resp., p -concave), then X is r -convex (resp., s -concave) with $M^{(r)}(X) \leq M^{(p)}(X)$ (resp., $M_{(s)}(X) \leq M_{(p)}(X)$). For details we refer to [44, Proposition 1.d.5]. If a Banach lattice is r -convex, for some $1 < r < \infty$, then it is p -convex, respectively r -concave implies q -concave, for every $1 < p < r < q < \infty$ (see [44, Theorem 1.f.7]).

(Quasi-) Banach sequence lattices X are of special interest for our purposes. We recall that a (quasi-) Banach function lattice over the counting measure space $(\Omega, 2^{\Omega}, \mu)$ is called a (quasi-) Banach sequence lattice over Ω . A Banach sequence lattice X is said to be maximal (or X has the Fatou property) whenever the closed unit ball of X is closed in the topology of pointwise convergence.

We are mainly interested in the case $\Omega := \{1, \dots, n\}$ and $\Omega := \mathbb{N}$. The fundamental function of a (quasi-) Banach sequence lattice X is defined by

$$\varphi_X(k) := \left\| \sum_{j=1}^k e_j \right\|_X, \quad k \in \Omega.$$

A (quasi-) Banach sequence lattice X is said to be symmetric whenever we have that $(x_{\sigma(n)}) \in X$ with $\|x_{\sigma}\|_X = \|x\|_X$ for every $x = (x_n) \in X$ and every permutation $\sigma : \mathbb{N} \rightarrow \mathbb{N}$.

Notice that if X is a separable Banach sequence lattice over \mathbb{N} , then the Banach dual X^* can be identified in a natural way with the Köthe dual X' of X , that is,

$$X' := \left\{ (x_k) : \sum_{k=1}^{\infty} |x_k y_k| < \infty \text{ for all } (y_k) \in X \right\}$$

equipped with the norm

$$\|(x_k)\|_{X'} = \sup \left\{ \sum_{k=1}^{\infty} |x_k y_k| : \|(y_k)\|_X \leq 1 \right\}.$$

Clearly, X' is a maximal Banach sequence lattice. We also note that a Banach sequence lattice X is maximal if and only if $X \equiv X''$.

Given two (quasi-) Banach sequence lattices X, Y , one may define the pointwise product $xy = (x_k y_k)$ of $x = (x_k) \in X$ and $y = (y_k) \in Y$. This leads to the definition of the pointwise product space

$$X \circ Y := \{xy : x \in X, y \in Y\},$$

equipped with the (quasi-) norm $\|z\|_{X \circ Y} := \inf \{ \|x\|_X \|y\|_Y : z = xy, x \in X, y \in Y \}$. By $M(X, Y)$ we denote the space of all multipliers from X to Y , that is, all sequences ξ such that $\xi x \in Y$ for all $x \in X$, equipped with the (quasi-) norm $\|\xi\|_{M(X, Y)} := \sup \{ \|\xi x\|_Y : \|x\|_X \leq 1 \}$.

If X is a (quasi-) Banach sequence lattice over \mathbb{N} and $n \in \mathbb{N}$, then

$$\|z\| := \left\| \sum_{k=1}^n |z_k| e_k \right\|_X, \quad z = (z_1, \dots, z_n) \in \mathbb{C}^n$$

defines a lattice (quasi-) norm on \mathbb{C}^n , where e_k stands for k th standard unit vectors of c_0 . This defines the (quasi-) Banach sequence lattice

$$X_n = (\mathbb{C}^n, \|\cdot\|),$$

which we call the n th section of X . Obviously, the collection $\{e_k\}_{k=1}^n$ of all unit basis vectors forms a 1-unconditional basis of X_n .

Note that the projection constant is bounded by the Banach-Mazur distance to ℓ_{∞}^n , i.e. $\lambda(X_n) \leq d(X_n, \ell_{\infty}^n)$, for any n -dimensional Banach lattice X_n , so it follows that

$$(12) \quad \lambda(X_n) \leq d(\ell_{\infty}^n, X_n) \leq \varphi_{X_n}(n), \quad n \in \mathbb{N},$$

provided that $\|\text{id}: X_n \rightarrow \ell_\infty^n\| \leq 1$, or equivalently $\|e_k\|_{X_n} \leq 1$, $1 \leq k \leq n$. Conversely, Schütt proved in [58] that

$$(13) \quad \varphi_{X_n}(n) \leq \sqrt{2} \|\text{id}: \ell_2^n \rightarrow X_n\| \lambda(X_n).$$

2. UNCONDITIONALITY, PROJECTION AND GORDON-LEWIS CONSTANTS

Given a Banach sequence lattice $X_n := (\mathbb{C}^n, \|\cdot\|)$ and a finite index set $J \subset \mathbb{N}_0^{(\mathbb{N})}$, our goal in this section is to estimate the projection and unconditionality constant of the Banach space $\mathcal{P}_J(X_n)$. These estimates will be expressed as functions of the dimension n of the space X_n and the degree $m = \max\{|\alpha| : \alpha \in J\}$ of the set J . To achieve this, we will explore the relationships between the invariants of the local theory of Banach spaces as discussed in the Introduction.

2.1. Building blocks. We start by examining how the projection constant varies when considering subspaces derived from corresponding subsets of indices.

Let $X_n = (\mathbb{C}^n, \|\cdot\|)$ be a Banach space. Then for each pair of (finite) index sets $I, J \subset \mathbb{N}_0^n$ with $I \subset J$ we define the projection

$$(14) \quad \mathbf{Q}_{J,I}: \mathcal{P}_J(X_n) \rightarrow \mathcal{P}_I(X_n), \quad P \mapsto \sum_{\alpha \in I} c_\alpha(P) z^\alpha.$$

This projection annihilates the coefficients of a polynomial in $\mathcal{P}_J(X_n)$ whose indices are in the complement of I .

Remark 2.1. Given $I, J \subset \mathbb{N}_0^n$ with $I \subset J$ and a Banach space $X_n = (\mathbb{C}^n, \|\cdot\|)$, we have

$$\lambda(\mathcal{P}_I(X_n)) \leq \|\mathbf{Q}_{J,I}\| \lambda(\mathcal{P}_J(X_n)).$$

Indeed, factorize $\text{id}_{\mathcal{P}_I(X_n)} = \mathbf{Q}_{J,I} \circ j_{I,J}$ through the canonical embedding $j_{I,J}: \mathcal{P}_I(X_n) \hookrightarrow \mathcal{P}_J(X_n)$ and the projection $\mathbf{Q}_{J,I}: \mathcal{P}_J(X_n) \hookrightarrow \mathcal{P}_I(X_n)$, and use (4).

We will start with a simple but crucial case for further discussion

Proposition 2.2. *Let $J \subset \mathbb{N}_0^n$ be an index set of degree m . Then, for each $0 \leq k \leq m$ and each Banach space $X_n = (\mathbb{C}^n, \|\cdot\|)$,*

$$\|\mathbf{Q}_{J,J(k)}: \mathcal{P}_J(X_n) \rightarrow \mathcal{P}_{J(k)}(X_n)\| = 1,$$

so in particular

$$\lambda(\mathcal{P}_{J(k)}(X_n)) \leq \lambda(\mathcal{P}_J(X_n)).$$

Proof. Given $P \in \mathcal{P}_J(X_n)$ consider $P = \sum_{k=0}^m \mathbf{Q}_{J,J(k)}(P)$, the unique Taylor expansion of P . Cauchy's inequality gives the bound $\|\mathbf{Q}_{J,J(k)}(P)\| \leq \|P\|$ for all $0 \leq k \leq m$ (see, e.g., [25, Proposition 15.33]). \square

Less standard is the result we now prove for subsets of tetrahedral indices α , i.e., $\alpha \in \Lambda_T(\leq m, n)$. The following tool is crucial for our purposes and is essentially due to Ortega-Cerdà, Ounaïes and Seip. Since this result appears in the unpublished manuscript [50] and we need a slight improvement, we provide a detailed proof for completeness. For clarity, we will first define the relevant constant

$$\kappa := \left(\prod_{k=1}^{\infty} \operatorname{sinc} \frac{\pi}{\mathfrak{p}_k} \right)^{-1} = 2.209\dots,$$

where $(\mathfrak{p}_n)_{n=1}^{\infty}$ stands for the sequence of prime numbers and $\operatorname{sinc} x := (\sin x)/x$.

Theorem 2.3. *Let $X = (\mathbb{C}^n, \|\cdot\|)$ be a Banach lattice and $J \subset \mathbb{N}_0^n$ an index set of degree at most m . Then*

$$\|\mathbf{Q}_{J, J_T} : \mathcal{P}_J(X_n) \rightarrow \mathcal{P}_{J_T}(X_n)\| \leq \kappa^m,$$

where $J_T = J \cap \Lambda_T(\leq m, n)$. In particular,

$$\lambda(\mathcal{P}_{J_T}(X_n)) \leq \kappa^m \lambda(\mathcal{P}_J(X_n)).$$

Proof. As usual, we write $\pi(x)$ for the counting function of the prime numbers. Now, given

$$t = (t_1, \dots, t_{\pi(m)}) \in \mathbf{R} := [0, 1]^{\pi(m)},$$

define

$$r_m(t) = c_m \exp \left(2\pi i \left(\frac{t_1}{2} + \frac{t_2}{3} + \dots + \frac{t_{\pi(m)}}{\mathfrak{p}_{\pi(m)}} \right) \right),$$

where

$$c_m = \prod_{k=1}^{\pi(m)} \left(\frac{\mathfrak{p}_k}{2\pi i} \left(e^{\frac{2\pi i}{\mathfrak{p}_k}} - 1 \right) \right)^{-1}.$$

Note that the function $r_m : \mathbf{R} \rightarrow \mathbb{C}$ has the following properties:

- (i) $\int_{\mathbf{R}} r_m(t) d\mu(t) = 1$,
- (ii) $\int_{\mathbf{R}} r_m^k(t) d\mu(t) = 0$ for each $2 \leq k \leq m$,
- (iii) $|r_m(t)| \leq \kappa$ for all $t \in \mathbf{R}$.

Here μ denotes the Lebesgue measure on \mathbf{R} . Indeed, (i) and (ii) are trivial and follow by the definition of the function, and (iii) holds because $|r_m(t)| = |c_m|$ and

$$|c_m|^{-2} = \prod_{k=1}^{\pi(m)} \frac{\mathfrak{p}_k^2}{(2\pi)^2} \left| e^{\frac{2\pi i}{\mathfrak{p}_k}} - 1 \right|^2 = \prod_{k=1}^{\pi(m)} \operatorname{sinc}^2 \frac{\pi}{\mathfrak{p}_k}.$$

Given a polynomial $P \in \mathcal{P}_J(X_n)$, note that by the properties (i) and (ii) we have the representation

$$\mathbf{Q}_{J, J_T} P(z) = \int_{\mathbf{R}^n} P(z_1 r_m(t^1), \dots, z_n r_m(t^n)) d\mu(t^1) \cdots d\mu(t^n), \quad z \in X_n.$$

Since X_n is a Banach lattice we by (iii) deduce that $|P(z_1 r_m(t^1), \dots, z_n r_m(t^n))| \leq \kappa^m \|P\|_{\mathcal{P}_J(X_n)}$ for every $z \in B_{X_n}$, and therefore

$$\|\mathbf{Q}_{J, J_T}(P)\|_{\mathcal{P}_{\Lambda_T}(X_n)} \leq \kappa^m \|P\|_{\mathcal{P}_J(X_n)}.$$

This proves the first statement, the second one is an immediate consequence of the observation from Remark 2.1. \square

Observe that by the Kadets-Snobar theorem from (1) we know that for any Banach space $X_n = (\mathbb{C}^n, \|\cdot\|)$ and any finite index set $J \subset \mathbb{N}_0^n$ we have

$$\lambda(\mathcal{P}_J(X_n)) \leq \sqrt{|J|} \leq \sqrt{m+1} \max_{0 \leq k \leq m} \sqrt{|J(k)|}.$$

We can now provide estimates of the projection constant of $\mathcal{P}_J(X_n)$ from above or below by the projection constants of its 'homogeneous building blocks' $\mathcal{P}_{J(k)}(X_n)$, $0 \leq k \leq m$.

Theorem 2.4. *Let $X_n = (\mathbb{C}^n, \|\cdot\|)$ be a Banach space, and let $J \subset \mathbb{N}_0^n$ be an index set of degree at most m . Then*

$$\max_{0 \leq k \leq m} \lambda(\mathcal{P}_{J(k)}(X_n)) \leq \lambda(\mathcal{P}_J(X_n)) \leq (m+1) \max_{0 \leq k \leq m} \lambda(\mathcal{P}_{J(k)}(X_n)).$$

In particular, for any Banach sequence lattice X

$$\limsup_{m \rightarrow \infty} \sup_{n \in \mathbb{N}} \frac{\sqrt[m]{\lambda(\mathcal{P}_{\leq m}(X_n))}}{\sqrt[m]{\max_{0 \leq k \leq m} \lambda(\mathcal{P}_k(X_n))}} = 1.$$

Proof. Note first that by Proposition 2.2 for all $0 \leq k \leq m$ it holds $\lambda(\mathcal{P}_{J(k)}(X_n)) \leq \lambda(\mathcal{P}_J(X_n))$, so that it remains to check the second estimate. We use that each $P \in \mathcal{P}_J(X_n)$ has a unique Taylor series expansion $P = \sum_{k=0}^m P_k$ with $P_k \in \mathcal{P}_{J(k)}(X_n)$, and from the Cauchy inequality we know that $\|P_k\| \leq \|P\|$ for all $0 \leq k \leq m$. Consequently, the two operators

$$U: \mathcal{P}_J(X_n) \rightarrow \bigoplus_{\infty} \mathcal{P}_{J(k)}(X_n), \quad P \mapsto (P_k)_{k=1}^m$$

$$V: \bigoplus_1 \mathcal{P}_{J(k)}(X_n) \rightarrow \mathcal{P}_J(X_n), \quad (Q_k)_{k=1}^m \mapsto \sum_{k=1}^m Q_k,$$

both have norms less than or equal to 1. Now fix some $\varepsilon > 0$, and choose for each $1 \leq k \leq m$ an appropriate factorization

$$\begin{array}{ccc} \mathcal{P}_{J(k)}(X_n) & \xrightarrow{\text{id}} & \mathcal{P}_{J(k)}(X_n) \\ u_k \downarrow & \nearrow v_k & \\ \ell_{\infty}^{M_k} & & \end{array}$$

such that $\|u_k\| \leq 1$ and $\|v_k\| \leq (1 + \varepsilon)\lambda(\mathcal{P}_{J(k)}(X_n))$. Then we arrive at the following commutative diagram

$$\begin{array}{ccc}
 \mathcal{P}_J(X_n) & \xrightarrow{\text{id}} & \mathcal{P}_J(X_n) \\
 U \downarrow & & \uparrow V \\
 \bigoplus_{\infty} \mathcal{P}_{J(k)}(X_n) & & \bigoplus_1 \mathcal{P}_{J(k)}(X_n) \\
 \bigoplus u_k \downarrow & & \uparrow \bigoplus v_k \\
 \bigoplus_{\infty} \ell_{\infty}^{M_k} & \xrightarrow{\Phi} & \bigoplus_1 \ell_{\infty}^{M_k},
 \end{array}$$

where Φ stands for the identity map which here obviously has norm $\leq m + 1$. But

$$\left\| \bigoplus u_k \right\| \leq \max \|u_k\| \leq 1,$$

as well as

$$\left\| \bigoplus v_k \right\| \leq \max \|v_k\| \leq (1 + \varepsilon) \max_{0 \leq k \leq m} \lambda(\mathcal{P}_{J(k)}(X_n)).$$

This finally gives

$$\lambda(\mathcal{P}_J(X_n)) \leq \|U\| \left\| \bigoplus u_k \right\| \lambda\left(\bigoplus_{\infty} \ell_{\infty}^{M_k}\right) \|\Phi\| \left\| \bigoplus v_k \right\| \|V\| \leq (m + 1) \max_{0 \leq k \leq m} \lambda(\mathcal{P}_{J(k)}(X_n)),$$

the conclusion. □

2.2. The Gordon-Lewis bridge. In this subsection, we use the Gordon-Lewis constant to bridge the gap between the unconditionality and projection constants. We start with the following result which relates the Gordon-Lewis constant $\mathbf{gl}(\mathcal{P}_J(X_n))$ with the unconditional basis constant $\chi_{\text{mon}}(\mathcal{P}_J(X_n))$ of the monomial basis $(z^{\alpha})_{\alpha \in J}$ of $\mathcal{P}_J(X_n)$.

Theorem 2.5. *Let $X_n = (\mathbb{C}^n, \|\cdot\|)$ be a Banach lattice, and let $J \subset \mathbb{N}_0^n$ be a finite index set of degree at most m . Then*

$$\mathbf{gl}(\mathcal{P}_J(X_n)) \leq \chi(\mathcal{P}_J(X_n)) \leq \chi_{\text{mon}}(\mathcal{P}_J(X_n)) \leq 2^m \mathbf{gl}(\mathcal{P}_J(X_n)).$$

When $J = \Lambda(m, n)$, we recover the result stated in [16, Proposition 3.1]. Theorem 2.5 extends Theorem 21.11 from [25] to any index set J with degree at most m ; its proof can be adapted almost verbatim from the one provided there. Note that the first estimate in Theorem 2.5 follows directly from the Gordon-Lewis inequality (3), while the second estimate is straightforward.

The following result estimates the Gordon-Lewis constant of $\mathcal{P}_J(X_n)$, where J is a finite index set of degree m , in terms of the projection constants of the spaces $\mathcal{P}_{J(k)^b}(X_n)$, $1 \leq k \leq m$.

Theorem 2.6. *Let $X_n = (\mathbb{C}^n, \|\cdot\|)$ be a Banach lattice and let an index set $J \subset \Lambda(m, n)$. Then*

$$\mathbf{gl}(\mathcal{P}_J(X_n)) \leq e \|\mathbf{Q}_{\Lambda(m, n), J}\| \lambda(\mathcal{P}_J(X_n)).$$

Moreover, if $J \subset \mathbb{N}_0^n$ is an index set of degree m , then

$$\mathbf{gl}(\mathcal{P}_J(X_n)) \leq e(m+1) \max_{1 \leq k \leq m} \|\mathbf{Q}_{\Lambda(k,n),J(k)}\| \max_{1 \leq k \leq m} \boldsymbol{\lambda}(\mathcal{P}_{J(k)^c}(X_n)).$$

The case $J = \Lambda(m, n)$ was previously established in [16, Proposition 4.2], and a detailed alternative presentation of the proof for this homogeneous case is provided in [25, Proposition 22.1]. In this context, we will show how to adapt those arguments to tackle the general situation, as it involves more complex technical subtleties.

We start with an elementary observation taken from [25, Lemma 22.2], which up to polarization covers the case $J = \Lambda(2, n)$ of Theorem 2.6 (take $Y = X^*$): For every finite dimensional Banach lattice X , and every finite dimensional Banach space Y one has

$$(15) \quad \mathbf{gl}(\mathcal{L}(X, Y)) \leq \boldsymbol{\lambda}(Y).$$

We now proceed with the proof of Theorem 2.6. For the first part of the statement, we will factorize the identity of $\mathcal{P}_J(X_n)$ through $\mathcal{L}(X_n, \mathcal{P}_{J^c}(X_n))$ and apply the previous inequality. In the second part, we will decompose into homogeneous indices and use the factorization from the first part for each homogeneous piece.

Proof of Theorem 2.6. To see the first statement, we consider the following commutative diagram:

$$(16) \quad \begin{array}{ccc} \mathcal{P}_J(X_n) & \xrightarrow{\quad id \quad} & \mathcal{P}_J(X_n) \\ \downarrow U_m & & \uparrow \mathbf{Q}_{\Lambda(m,n),J} \\ \mathcal{L}(X_n, \mathcal{P}_{J^c}(X_n)) & \xrightarrow{I_m} \mathcal{L}(X_n, \mathcal{P}_{m-1}(X_n)) \xrightarrow{V_m} & \mathcal{P}_m(X_n), \end{array}$$

where I_m is the canonical inclusion map and

$$(U_m(P)x)(u) := \check{P}(u, \dots, u, x) \text{ for } P \in \mathcal{P}_J(X_n) \text{ and } x, u \in X_n,$$

$$V_m(T)(y) := (Ty)y \text{ for } T \in \mathcal{L}(X_n, \mathcal{P}_{m-1}(X_n)) \text{ and } y \in X_n.$$

We show that U_m , as an operator from $\mathcal{P}_J(X_n)$ into $\mathcal{L}(X_n, \mathcal{P}_{J^c}(X_n))$, is well-defined. Indeed, define

$$a_{\mathbf{i}}(\check{P}) = \frac{c_{\mathbf{j}}(P)}{|\mathbf{j}|} \text{ for } \mathbf{j} \in \mathcal{J}(m, n) \text{ and } \mathbf{i} \in [\mathbf{j}].$$

Then, given $P \in \mathcal{P}_J(X)$ and $x, u \in X_n$,

$$\begin{aligned}
\check{P}(u, \dots, u, x) &= \sum_{\mathbf{i} \in \mathcal{M}(m, n)} a_{\mathbf{i}}(\check{P}) u_{i_1} \dots u_{i_{m-1}} x_{i_m} \\
&= \sum_{\mathbf{i} \in \mathcal{M}(m-1, n)} \sum_{\ell=1}^n a_{(\mathbf{i}, \ell)}(\check{P}) u_{\mathbf{i}} x_{\ell} = \sum_{\mathbf{j} \in \mathcal{J}(m-1, n)} \sum_{\mathbf{i} \in [\mathbf{j}]} \sum_{\ell=1}^n a_{(\mathbf{i}, \ell)}(\check{P}) u_{\mathbf{i}} x_{\ell} \\
&= \sum_{\mathbf{j} \in \mathcal{J}(m-1, n)} \sum_{\ell=1}^n \sum_{\mathbf{i} \in [\mathbf{j}]} a_{(\mathbf{i}, \ell)}(\check{P}) u_{\mathbf{i}} x_{\ell} = \sum_{\mathbf{j} \in \mathcal{J}(m-1, n)} \sum_{\ell=1}^n \sum_{\mathbf{i} \in [\mathbf{j}]} \frac{c_{(\mathbf{i}, \ell)_*}(P)}{\|[(\mathbf{i}, \ell)_*]\|} u_{\mathbf{i}} x_{\ell} \\
&= \sum_{\mathbf{j} \in \mathcal{J}(m-1, n)} \sum_{\ell=1}^n \left[\frac{c_{(\mathbf{j}, \ell)_*}(P)}{\|[(\mathbf{j}, \ell)_*]\|} u_{\mathbf{j}} \right] \|\mathbf{j}\| x_{\ell} = \sum_{\mathbf{j} \in \mathcal{J}(m-1, n)} \left[\sum_{\ell=1}^n \frac{c_{(\mathbf{j}, \ell)_*}(P)}{\|[(\mathbf{j}, \ell)_*]\|} \|\mathbf{j}\| x_{\ell} \right] u_{\mathbf{j}} \\
&= \sum_{\mathbf{j} \in \mathcal{J}(m-1, n)} \left[\sum_{\substack{1 \leq \ell \leq n \\ (\mathbf{j}, \ell)_* \in J}} \frac{c_{(\mathbf{j}, \ell)_*}(P)}{\|[(\mathbf{j}, \ell)_*]\|} \|\mathbf{j}\| x_{\ell} \right] u_{\mathbf{j}} = \sum_{\mathbf{j} \in \mathcal{J}^b} \left[\sum_{\substack{1 \leq \ell \leq n \\ (\mathbf{j}, \ell)_* \in J}} \frac{c_{(\mathbf{j}, \ell)_*}(P)}{\|[(\mathbf{j}, \ell)_*]\|} \|\mathbf{j}\| x_{\ell} \right] u_{\mathbf{j}},
\end{aligned}$$

which shows that $U_m(P)x \in \mathcal{P}_J(X_n)$ for every $x \in X_n$. By the Harris polarization formula (see, e.g., [25, Proposition 2.34]) we have $\|U_m\| \leq e$, and moreover trivially $\|V_m\| \leq 1$. Hence by the ideal properties of the involved ideal norms we see that

$$\mathbf{gl}(\mathcal{P}_J(X_n)) \leq e \|\mathbf{Q}_{\Lambda(m, n), J}\| \mathbf{gl}(\mathcal{L}(X_n, \mathcal{P}_J(X_n))) \leq e \|\mathbf{Q}_{\Lambda(m, n), J}\| \boldsymbol{\lambda}(\mathcal{P}_J(X_n)),$$

where for the last estimate we use (15). This proves the first claim.

For the second claim we have to handle an index set J of degree m , and consider the following commutative diagram

$$\begin{array}{ccccc}
\mathcal{P}_J(X_n) & \xrightarrow{\text{id}_{\mathcal{P}_J(X_n)}} & \mathcal{P}_J(X_n) & & \\
\downarrow \mathbf{O} \oplus \mathbf{Q}_{J, J(k)} & & \uparrow \Sigma & & \\
\mathbb{C} \oplus_{\infty} \bigoplus_{\infty} \mathcal{P}_{J(k)}(X_n) & \xrightarrow{\text{id}_{\mathbb{C}} \oplus \text{id}_{\mathcal{P}_{J(k)}(X_n)}} & \mathbb{C} \oplus_1 \bigoplus_1 \mathcal{P}_{J(k)}(X_n) & \xleftarrow{\text{id}_{\mathbb{C}} \oplus \mathbf{Q}_{\Lambda(k, n), J(k)}} & \mathbb{C} \oplus_1 \bigoplus_1 \mathcal{P}_k(X_n) \\
\downarrow \text{id}_{\mathbb{C}} \oplus U_k & & & & \uparrow \text{id}_{\mathbb{C}} \oplus V_k \\
\mathbb{C} \oplus_{\infty} \bigoplus_{\infty} \mathcal{L}(X_n, \mathcal{P}_{J(k)^b}(X_n)) & \xrightarrow{\text{id}_{\mathbb{C}} \oplus I_k} & \mathbb{C} \oplus_{\infty} \bigoplus_{\infty} \mathcal{L}(X_n, \mathcal{P}_{k-1}(X_n)) & \xrightarrow{\Phi} & \mathbb{C} \oplus_1 \bigoplus_1 \mathcal{L}(X_n, \mathcal{P}_{k-1}(X_n)).
\end{array}$$

Let us explain our notation in this diagram: Here U_k , V_k and I_k for $1 \leq k \leq m$ are the operators from (16). If $P = a_0 + \sum_{k=1}^m P_k$ is the Taylor decomposition of $P \in \mathcal{P}_J(X_n)$, then $\mathbf{O}(P) = a_0$, and hence $(\mathbf{O} \oplus \mathbf{Q}_{J, J(k)})(P) = (a_0, (P_k)_{k=1}^m)$. Additionally, Σ is the mapping which assigns to every $(a_0, (P_k)_{k=1}^m)$ the polynomial $P = a_0 + \sum_{k=1}^m P_k$, and Φ stands for the identity map – whereas the notation for the rest of the maps is self-explaining. Obviously, this gives that

$$\mathbf{gl}(\mathcal{P}_J(X_n)) \leq e(m+1) \max_{1 \leq k \leq m} \|\mathbf{Q}_{\Lambda(k, n), J(k)}\| \max_{1 \leq k \leq m} \mathbf{gl}\left(\bigoplus_{\infty} \mathcal{L}(X_n, \mathcal{P}_{J(k)^b}(X_n))\right),$$

so it remains to prove the following claim

$$\mathbf{gl}\left(\bigoplus_{\infty} \mathcal{L}(X_n, \mathcal{P}_{J(k)^b}(X_n))\right) \leq \max_{1 \leq k \leq m} \boldsymbol{\lambda}(\mathcal{P}_{J(k)^b}(X_n)).$$

Indeed, using standard properties of ε - and π -tensor products, we have

$$\begin{aligned} \bigoplus_{\infty} \mathcal{L}(X_n, \mathcal{P}_{J(k)^b}(X_n)) &\hookrightarrow \bigoplus_{\infty} \mathcal{L}(X_n, \bigoplus_{\infty} \mathcal{P}_{J(k)^b}(X_n)) \\ &= \ell_{\infty}^m \otimes_{\varepsilon} [X_n^* \otimes_{\varepsilon} \bigoplus_{\infty} \mathcal{P}_{J(k)^b}(X_n)] \\ &= [\ell_{\infty}^m \otimes_{\varepsilon} X_n^*] \otimes_{\varepsilon} \bigoplus_{\infty} \mathcal{P}_{J(k)^b}(X_n) \\ &= (\ell_1^m \otimes_{\pi} X_n)^* \otimes_{\varepsilon} \bigoplus_{\infty} \mathcal{P}_{J(k)^b}(X_n) = \mathcal{L}(\ell_1^m(X_n), \bigoplus_{\infty} \mathcal{P}_{J(k)^b}(X_n)), \end{aligned}$$

where the first space in fact is 1-complemented in the second one, and all other identifications are isometries. Then we deduce from (15) that

$$\mathbf{gl}\left(\bigoplus_{\infty} \mathcal{L}(X_n, \mathcal{P}_{J(k)^b}(X_n))\right) \leq \mathbf{gl}\left(\mathcal{L}(\ell_1^m(X_n), \bigoplus_{\infty} \mathcal{P}_{J(k)^b}(X_n))\right) \leq \boldsymbol{\lambda}\left(\bigoplus_{\infty} \mathcal{P}_{J(k)^b}(X_n)\right).$$

Since obviously

$$\boldsymbol{\lambda}\left(\bigoplus_{\infty} \mathcal{P}_{J(k)^b}(X_n)\right) = \gamma_{\infty}(\text{id}_{\bigoplus_{\infty} \mathcal{P}_{J(k)^b}(X_n)}) \leq \max_{1 \leq k \leq m} \gamma_{\infty}(\text{id}_{\mathcal{P}_{J(k)^b}(X_n)}) = \max_{1 \leq k \leq m} \boldsymbol{\lambda}(\mathcal{P}_{J(k)^b}(X_n)),$$

the proof is complete. \square

From the Theorems 2.5 and 2.6, the following result follows.

Theorem 2.7. *Let $X_n = (\mathbb{C}^n, \|\cdot\|)$ be a Banach lattice, and let $J \subset \mathbb{N}_0^n$ be an index set of degree at most m . Then*

$$\boldsymbol{\chi}_{\text{mon}}(\mathcal{P}_J(X_n)) \leq e(m+1)2^m \max_{1 \leq k \leq m} \|\mathbf{Q}_{\Lambda(k,n), J(k)}\| \max_{1 \leq k \leq m} \boldsymbol{\lambda}(\mathcal{P}_{J(k)^b}(X_n)).$$

In addition,

$$\boldsymbol{\chi}_{\text{mon}}(\mathcal{P}_J(X_n)) \leq e2^m \|\mathbf{Q}_{\Lambda(m,n), J}\| \boldsymbol{\lambda}(\mathcal{P}_J(X_n)),$$

whenever $J \subset \Lambda(m, n)$.

Corollary 2.8. *Let $m \in \mathbb{N}$ with $m \geq 2$ and let $X_n = (\mathbb{C}^n, \|\cdot\|)$ be a Banach lattice. Then*

$$\boldsymbol{\chi}_{\text{mon}}(\mathcal{P}_m(X_n)) \leq \boldsymbol{\chi}_{\text{mon}}(\mathcal{P}_{\leq m}(X_n)) \leq e(m+1)2^m \max_{1 \leq k \leq m-1} \boldsymbol{\lambda}(\mathcal{P}_k(X_n)),$$

and

$$\boldsymbol{\chi}_{\text{mon}}(\mathcal{P}_{\Lambda_T(m)}(X_n)) \leq \boldsymbol{\chi}_{\text{mon}}(\mathcal{P}_{\Lambda_T(\leq m)}(X_n)) \leq e(m+1)2^m \kappa^m \max_{1 \leq k \leq m-1} \boldsymbol{\lambda}(\mathcal{P}_{\Lambda_T(k)}(X_n)).$$

The previous results, comparing Gordon-Lewis constants, unconditionality constants, and projection constants, allow us to transfer the bound provided by the Kadets-Snobar theorem (1) for the projection constant to the unconditionality constant.

Corollary 2.9. *Let X be a Banach sequence lattice, and let (J_m) be a sequence of index sets, each with degree at most m . Then the following estimates hold:*

$$(17) \quad \lambda(\mathcal{P}_{J_m}(X_n)) <_{C^m} \left(1 + \frac{n}{m}\right)^{\frac{m}{2}}$$

and

$$\chi_{\text{mon}}(\mathcal{P}_{J_m}(X_n)) <_{C^m} \left(1 + \frac{n}{m}\right)^{\frac{m-1}{2}},$$

where $C > 0$ is universal.

Proof. Using (9), a simple calculation shows that

$$|J_m| \leq |\Lambda(\leq m, n)| <_{C^m} \left(1 + \frac{n}{m}\right)^m.$$

Then estimate (17) follows from the Kadets-Snohar inequality (1). Hence, using Corollary 2.8, we conclude that

$$\chi_{\text{mon}}(\mathcal{P}_{J_m}(X_n)) \leq \chi_{\text{mon}}(\mathcal{P}_{\leq m}(X_n)) \leq e(m+1)2^m \max_{1 \leq k \leq m-1} \lambda(\mathcal{P}_k(X_n)) <_{C^m} \left(1 + \frac{n}{m}\right)^{\frac{m-1}{2}}. \quad \square$$

2.3. Convexity and concavity. Assuming that the Banach sequence lattice is 2-convex, we can demonstrate that the previous estimate provides the correct asymptotics. Before presenting and proving this result, we introduce general estimates that relate the unconditional basis constants $\chi_{\text{mon}}(\mathcal{P}_J(X_n))$ and $\chi_{\text{mon}}(\mathcal{P}_J(Y_n))$, provided that X and Y satisfy certain general lattice properties, such as convexity and concavity. These results, combined with those discussed in the previous subsection, often enable us to obtain accurate asymptotic estimates for the local constants under consideration for the space of analytic polynomials on finite-dimensional Banach lattices.

Much of what we prove below is grounded in Lozanovskii's profound factorization theorem [46]. Since we will need to reference it again later, we prefer to state it explicitly here. It states that for any maximal Banach function lattice X over a complete and σ -finite measure space $(\Omega, \mathcal{A}, \mu)$, we have $\text{supp } X = \Omega$, we have $X \circ X' \cong L_1(\mu)$. More precisely, for every $f \in L_1(\mu)$, there exist $g \in X$ and $h \in X'$ such that

$$f = gh \quad \text{and} \quad \|f\|_{L_1(\mu)} = \|g\|_X \|h\|_{X'}.$$

Clearly, g and h can be chosen positive whenever f is positive. Moreover, if $\|f\|_{L_1(\mu)} = 1$, then we can find $g \in S_X$ and $h \in S_{X'}$.

The following lemma serves as our starting point.

Lemma 2.10. *Let $X_n = (\mathbb{C}^n, \|\cdot\|)$ and $Y_n = (\mathbb{C}^n, \|\cdot\|)$ be two Banach lattices such that*

$$\|\text{id}: X_n \rightarrow Y_n \circ M(Y_n, X_n)\| \leq 1.$$

Then, for any index set $J \subset \mathbb{N}_0^n$, we have

$$\chi_{\text{mon}}(\mathcal{P}_J(X_n)) \leq \chi_{\text{mon}}(\mathcal{P}_J(Y_n)).$$

Proof. Fix some $z \in B_{X_n}$. Then by assumption there exist $\xi, v \in \mathbb{C}^n$ such that $z = v\xi$, $\|v\|_{Y_n} \leq 1$, and $\|D_\xi: Y_n \rightarrow X_n\| \leq 1$. Hence, for every $P \in \mathcal{P}_J(X_n)$, we have

$$\begin{aligned} \left| \sum_{\alpha \in J} |c_\alpha(P)| z^\alpha \right| &= \left| \sum_{\alpha \in J} |c_\alpha(P \circ D_\xi)| v^\alpha \right| \leq \chi_{\text{mon}}(\mathcal{P}_J(Y_n)) \|P \circ D_\xi\|_{\mathcal{P}_J(Y_n)} \\ &\leq \chi_{\text{mon}}(\mathcal{P}_J(Y_n)) \|P\|_{\mathcal{P}_J(X_n)}, \end{aligned}$$

which completes the proof. \square

An immediate consequence is the following result, which shows that for any set $J \subset \mathbb{N}_0^n$, the minimum (resp., maximum) value of the unconditional basis constant $\chi_{\text{mon}}(\mathcal{P}_J(X))$ for the class of all n -dimensional Banach lattices is attained by ℓ_1^n (resp., ℓ_∞^n).

Corollary 2.11. *Let $X_n = (\mathbb{C}^n, \|\cdot\|)$ be a Banach lattice. Then, for any index set $J \subset \mathbb{N}_0^n$, the following estimates holds:*

$$\chi_{\text{mon}}(\mathcal{P}_J(\ell_1^n)) \leq \chi_{\text{mon}}(\mathcal{P}_J(X_n)) \leq \chi_{\text{mon}}(\mathcal{P}_J(\ell_\infty^n)).$$

Proof. We apply Lemma 2.10. Clearly, $\ell_\infty^n \circ M(\ell_\infty^n, X_n) \equiv X_n$, which gives the second inequality. On the other hand, from Lozanowski's theorem, we obtain

$$X_n \circ M(X_n, \ell_1^n) \equiv X_n \circ X_n' \equiv \ell_1^n,$$

which leads to the first estimate. \square

Under the assumptions of convexity and concavity of the underlying Banach lattice, the previous results can be further improved.

Theorem 2.12. *Let $X_n = (\mathbb{C}^n, \|\cdot\|)$ and $Y_n = (\mathbb{C}^n, \|\cdot\|)$ be two Banach lattices such that $M_{(r)}(X_n) = M^{(r)}(Y_n) = 1$, where $1 < r < \infty$. Then, for any index set $J \subset \mathbb{N}_0^n$, we have*

$$\chi_{\text{mon}}(\mathcal{P}_J(X_n)) \leq \chi_{\text{mon}}(\mathcal{P}_J(Y_n)).$$

In view of Lemma 2.10, the proof is an consequence of [57, Theorem 3.8,(i)] which implies that $X_n \equiv Y_n \circ M(Y_n, X_n)$.

Corollary 2.13. *Let $X_n = (\mathbb{C}^n, \|\cdot\|)$ be a Banach lattice such that $M_{(r)}(X_n) = 1$, where $1 < r < \infty$. Then, for any index set $J \subset \mathbb{N}_0^n$, we have*

$$\chi_{\text{mon}}(\mathcal{P}_J(X_n)) \leq \chi_{\text{mon}}(\mathcal{P}_J(\ell_r^n)).$$

As already shown in Corollary 2.11, this result also holds for $r = \infty$ (note that $M_{(\infty)}(X_n) = 1$ for any X_n).

Corollary 2.14. *Let $Y_n = (\mathbb{C}^n, \|\cdot\|)$ be a Banach lattice such that $M^{(r)}(Y_n) = 1$, where $1 < r < \infty$. Then, for any index set $J \subset \mathbb{N}_0^n$, we have*

$$\chi_{\text{mon}}(\mathcal{P}_J(\ell_r^n)) \leq \chi_{\text{mon}}(\mathcal{P}_J(Y_n)).$$

Since $M^{(1)}(Y_n) = 1$ holds for any Y_n , the case $r = 1$ is again covered by Corollary 2.11.

Finally, we show that, if in the above corollaries the concavity constant of X_n and the convexity constant of Y_n differ from 1, then, at least in the homogeneous case (i.e., $J \subset \Lambda(m, n)$ for some m), the estimates hold up to a constant C^m , where C depends on the r -concavity and the r -convexity constants.

To understand this, we recall the usually called renorming theorem of Figiel and Johnson (see, e.g., [45, Proposition 1.d.8]): If a Banach function lattice X on $(\Omega, \mathcal{A}, \mu)$ is p -convex and q -concave for some $1 \leq p \leq q \leq \infty$, then on X there is an equivalent lattice norm $\|\cdot\|_0$ such that both the p -convexity and q -concavity constants are equal to 1, and furthermore the following inequality holds

$$\|x\|_X \leq \|x\|_0 \leq M^{(p)}(X)M^{(q)}(X)\|x\|_X, \quad x \in X.$$

Corollary 2.15. *Let X and Y be Banach sequence lattices such that X is r -concave and Y is r -convex for $1 < r < \infty$. Then, for any index set $J \subset \Lambda(m, n)$, we have*

$$\chi_{\text{mon}}(\mathcal{P}_J(X_n)) \leq (M_{(r)}(X)M^{(r)}(Y))^m \chi_{\text{mon}}(\mathcal{P}_J(Y_n)).$$

Proof. Observe first that if $E := (\mathbb{C}^n, \|\cdot\|_E)$ and $F := (\mathbb{C}^n, \|\cdot\|_F)$ are n -dimensional Banach lattices such that for some $\gamma \geq 1$

$$\|z\|_E \leq \|z\|_F \leq \gamma \|z\|_E, \quad z \in \mathbb{C}^n,$$

then for any polynomial $P \in \mathcal{P}_J(\mathbb{C}^n)$, we have $\gamma^{-m} \|P\|_{\mathcal{P}_J(E)} \leq \|P\|_{\mathcal{P}_J(F)} \leq \|P\|_{\mathcal{P}_J(E)}$. Clearly, this yields

$$\gamma^{-m} \chi_{\text{mon}}(\mathcal{P}_J(E)) \leq \chi_{\text{mon}}(\mathcal{P}_J(F)) \leq \gamma^m \chi_{\text{mon}}(\mathcal{P}_J(E)).$$

Applying the renorming theorem mentioned above, we conclude that there exist equivalent lattice norms $\|\cdot\|_{\tilde{X}}$ and $\|\cdot\|_{\tilde{Y}}$ on X and Y , respectively, such that the Banach lattices $\tilde{X} := (X, \|\cdot\|_{\tilde{X}})$ and $\tilde{Y} := (Y, \|\cdot\|_{\tilde{Y}})$ satisfy $M_{(r)}(\tilde{X}) = M^{(r)}(\tilde{Y}) = 1$ as well as

$$\|x\|_X \leq \|x\|_{\tilde{X}} \leq M_{(r)}(X)\|x\|_X, \quad x \in X, \quad \text{and} \quad \|y\|_Y \leq \|y\|_{\tilde{Y}} \leq M^{(r)}(Y)\|y\|_Y, \quad y \in Y.$$

The above facts combined with Proposition 2.12 yield

$$M_{(r)}(X)^{-m} \chi_{\text{mon}}(\mathcal{P}_J(X_n)) \leq \chi_{\text{mon}}(\mathcal{P}_J(\tilde{X}_n)) \leq \chi_{\text{mon}}(\mathcal{P}_J(\tilde{Y}_n)) \leq M^{(r)}(Y)^m \chi_{\text{mon}}(\mathcal{P}_J(Y_n))$$

and this completes the proof of the claim. \square

We are now prepared to prove the result concerning the correct asymptotic estimates stated in Corollary 2.9.

Theorem 2.16. *Let X be a 2-convex Banach sequence lattice, and let (J_m) be a sequence of index sets, each with degree at most m . Then, for all m such that $\Lambda_T(m) \subset J_m$, we have*

$$(18) \quad \lambda(\mathcal{P}_{J_m}(X_n)) \sim_{C^m} \left(1 + \frac{n}{m}\right)^{\frac{m}{2}}$$

and

$$(19) \quad \chi_{\text{mon}}(\mathcal{P}_{J_m}(X_n)) \sim_{C^m} \left(1 + \frac{n}{m}\right)^{\frac{m-1}{2}},$$

where $C > 0$ only depends on the 2-convexity constant of X .

Proof. The upper bounds are a special case of Corollary 2.9. To prove the lower bounds, we start with that of (19). From Corollary 2.15 with $r = 2$, it follows that for all m with $\Lambda_T(m) \subset J_m$,

$$\chi_{\text{mon}}(\mathcal{P}_{\Lambda_T(m)}(\ell_2^n)) \leq M^{(2)}(X)^m \chi_{\text{mon}}(\mathcal{P}_{\Lambda_T(m)}(X_n)) \leq M^{(2)}(X)^m \chi_{\text{mon}}(\mathcal{P}_{J_m}(X_n)),$$

Consequently by Proposition 2.26 (in the Hilbert space case) for $m \leq n$, we obtain

$$\left(1 + \frac{n}{m}\right)^{\frac{m-1}{2}} \leq 2^{\frac{m-1}{2}} \left(\frac{n}{m}\right)^{\frac{m-1}{2}} <_{C^m} \chi_{\text{mon}}(\mathcal{P}_{\Lambda_T(m)}(\ell_2^n)) <_{C^m} \chi_{\text{mon}}(\mathcal{P}_{J_m}(X_n)).$$

Clearly, for all $m \geq n$, we have

$$(20) \quad \left(1 + \frac{n}{m}\right)^{\frac{m-1}{2}} \leq 2^{\frac{m-1}{2}} <_{C^m} \chi_{\text{mon}}(\mathcal{P}_{J_m}(X_n)),$$

so the lower bound in (19) follows.

Finally, it remains to check the lower bound in (18). We apply Theorem 2.7, Theorem 2.3 (twice), and Proposition 2.2, which all together prove that for m with $\Lambda_T(m) \subset J_m$, we get

$$(21) \quad \begin{aligned} \chi_{\text{mon}}(\mathcal{P}_{\Lambda_T(m+1)}(X_n)) &\leq e2^{m+1} \|\mathbf{Q}_{\Lambda(m+1,n), \Lambda_T(m+1,n)}\| \lambda(\mathcal{P}_{\Lambda_T(m)}(X_n)) \\ &\leq e2^{m+1} \kappa^{m+1} \lambda(\mathcal{P}_{\Lambda_T(m)}(X_n)) \\ &\leq e2^{m+1} \kappa^{2m+1} \lambda(\mathcal{P}_{J_m(m)}(X_n)) \leq e2^{m+1} \kappa^{2m+1} \lambda(\mathcal{P}_{J_m}(X_n)); \end{aligned}$$

for the third estimate note that that $\Lambda_T(m) = J_m(m) \cap \Lambda_T(\leq m)$. Then for m with $\Lambda_T(m) \subset J_m$, we obtain the lower bound in (18) from (19). \square

2.4. Estimates by norms of coefficient functionals. In this subsection we introduce another useful technique to estimate the projection constant of spaces $\mathcal{P}_J(X_n)$ of multivariate polynomials on finite dimensional Banach lattices X_n supported on a finite index set J .

Given an index set $J \subset \mathbb{N}_0^n$ and Banach space $X_n = (\mathbb{C}^n, \|\cdot\|)$, we define the isometrical embedding $I: \mathcal{P}_J(X_n) \rightarrow \ell_\infty(B_{X_n})$ by

$$I(P) := P|_{B_{X_n}}, \quad P \in \mathcal{P}_J(X_n).$$

For each $\alpha \in J$ let $c_\alpha^*: \mathcal{P}_J(X_n) \rightarrow \mathbb{C}$ be the linear functional, given, for any polynomial $P = \sum_{\alpha \in J} c_\alpha(P) e_\alpha$ by $c_\alpha^*(P) := c_\alpha(P)$. From the Hahn-Banach theorem, it follows that for any c_α^* , we can find a norm-preserving extension

$$\tilde{c}_\alpha^* \in \ell_\infty(B_{X_n})^*,$$

that is, for all $P \in \mathcal{P}_J(X_n)$, we have $\|\tilde{c}_\alpha^*\| = \|c_\alpha^*\|$ and $\tilde{c}_\alpha^*(P) = c_\alpha(P)$.

Consider the following obvious factorization of the identity map $\text{id}: \mathcal{P}_J(X_n) \rightarrow \mathcal{P}_J(X_n)$,

$$\text{id}: \mathcal{P}_J(X_n) \xrightarrow{I} \ell_\infty(B_{X_n}) \xrightarrow{Q} \mathcal{P}_J(X_n),$$

where the finite dimensional operator Q is given by the formula

$$(22) \quad Qf := \sum_{\alpha \in J} \tilde{c}_\alpha^*(f) e_\alpha, \quad f \in \ell_\infty(B_{X_n}).$$

Clearly, this implies that

$$\lambda(\mathcal{P}_J(X_n)) \leq \|Q: \ell_\infty(B_{X_n}) \rightarrow \ell_\infty(B_{X_n})\|.$$

Although determining the norm of this projection Q is generally a challenging task, we find upper estimates which, under certain restrictive assumptions on the index set J and the Banach space X_n , lead to asymptotically optimal bounds for $\lambda(\mathcal{P}_J(X_n))$. More precisely, we introduce what we call the polynomial projection constant:

$$\hat{\lambda}(\mathcal{P}_J(X_n)) := \sup_{z \in B_{X_n}} \sum_{\alpha \in J} c_{X_n}(\alpha) |z^\alpha|,$$

where $c_{X_n}(\alpha)$, the characteristic of the multi-index α , is the reciprocal of the norm of the monomial z^α in $\mathcal{P}_J(X_n)$, that is,

$$c_{X_n}(\alpha) := \frac{1}{\sup_{z \in B_{X_n}} |z^\alpha|}.$$

An interesting fact is that we can actually bound the projection constant of the space $\mathcal{P}_J(X_n)$ by this quantity, as shown in Theorem 2.18. Therefore, having accurate upper bounds for the characteristic $c_{X_n}(\alpha)$ is crucial for our purposes. In fact, it turns out that the polynomial projection $\hat{\lambda}(\mathcal{P}_J(X_n))$, facilitates the systematic and comfortable extension of estimates such as (7) to wider ranges of index sets J and spaces X_n . At the same time, it adheres to a reasonable abstract theory. In many concrete situations, this estimate is sufficient; the advantage here is that this new parameter is more manageable compared to $\lambda(\mathcal{P}_J(X_n))$.

The norm of the functional c_α^* can be bounded by the characteristic of the index α . Indeed, the following lemma is just a rephrasing of the Cauchy inequalities.

Lemma 2.17. *Let $X_n = (\mathbb{C}^n, \|\cdot\|)$ be a Banach lattice and $J \subset \mathbb{N}_0^n$. Then, for every $P \in \mathcal{P}_J(X_n)$ and $\alpha \in J$, we have*

$$|c_\alpha(P)| \leq c_{X_n}(\alpha) \|P\|_{\mathcal{P}_J(X_n)};$$

in other terms, $\|c_\alpha^\|_{\mathcal{P}_J(X_n)^*} \leq c_{X_n}(\alpha)$.*

Coming back to the discussion from the introduction of this chapter, we get an estimate of the projection constant by the polynomial projection constant.

Theorem 2.18. *Let $X_n = (\mathbb{C}^n, \|\cdot\|)$ be a Banach lattice and $J \subset \mathbb{N}_0^n$. Then*

$$\lambda(\mathcal{P}_J(X_n)) \leq \widehat{\lambda}(\mathcal{P}_J(X_n)).$$

Proof. As noted above, the mapping $Q: \ell_\infty(B_{X_n}) \rightarrow \ell_\infty(B_{X_n})$ defined by the formula (22) is the projection of $\ell_\infty(B_{X_n})$ onto $I(\mathcal{P}_J(X_n))$. Then by Lemma 2.17, it follows that

$$\begin{aligned} \|Q\| &= \sup_{\|f\|_{\ell_\infty(B_{X_n})} \leq 1} \left\| \sum_{\alpha \in J} \tilde{c}_\alpha^*(f) z^\alpha \right\|_{\mathcal{P}_J(X_n)} \leq \sup_{z \in B_{X_n}} \sum_{\alpha \in J} \|\tilde{c}_\alpha^*\|_{\ell_\infty(B_{X_n})^*} |z^\alpha| \\ &= \sup_{z \in B_{X_n}} \sum_{\alpha \in J} \|c_\alpha^*\|_{\mathcal{P}_J(X_n)^*} |z^\alpha| \leq \sup_{z \in B_{X_n}} \sum_{\alpha \in J} c_{X_n}(\alpha) |z^\alpha| \\ &= \widehat{\lambda}(\mathcal{P}_J(X_n)), \end{aligned}$$

and since $\lambda(\mathcal{P}_J(X_n)) \leq \|Q\|$, the desired estimate follows. \square

It is well-known that for every $\alpha \in \Lambda(m, n)$ and $1 \leq r \leq \infty$, we have

$$(23) \quad c_{\ell_r^n}(\alpha) = \left(\frac{m^m}{\alpha^\alpha} \right)^{1/r};$$

for an elementary proof see [30, Lemma 1.38]. Observe as an immediate consequence that for tetrahedral indices $\alpha \in \Lambda_T(m, n)$ with $m \leq n$, we get

$$(24) \quad c_{\ell_r^n}(\alpha) = m^{\frac{m}{r}}.$$

By Theorem 2.18 accurate upper bounds for $c_{X_n}(\alpha)$ provide good upper bounds for the projection constant of $\mathcal{P}_J(X_n)$.

We will explore how to derive general variants of the estimates mentioned above. As in the previous subsection, we will apply Lozanovskii's theorem discussed earlier.

Theorem 2.19. *The following statements are true for any Banach lattice $X_n = (\mathbb{C}^n, \|\cdot\|_{X_n})$, any $J \subset \mathbb{N}_0^n$, and any $\alpha \in \Lambda(m, n)$:*

- (i) $c_{X_n}(\alpha) c_{X_n'}(\alpha) \leq \frac{m^m}{\alpha^\alpha}$.
- (ii) $c_{X_n}(\alpha) \leq \frac{\|\alpha\|_{X_n}^m}{\alpha^\alpha}$.

Proof. (i) Clearly, X_n has the Fatou property because it is finite-dimensional. Lozanovskii's factorization theorem (summarized in Equation (2.3)) implies that for every $z \in S_{\ell_1^n}$, we can select $u \in S_{X_n}$ and $v \in S_{X'_n}$ such that $|u||v| = |z|$. Thus, we have:

$$c_{X_n}(\alpha) c_{X'_n}(\alpha) \leq \frac{1}{\sup_{u \in S_{X_n}} |u^\alpha|} \frac{1}{\sup_{v \in S_{X'_n}} |v^\alpha|} \leq \frac{1}{\sup_{z \in S_{\ell_1^n}} |z^\alpha|} \leq \frac{m^m}{\alpha^\alpha},$$

where the last inequality holds taking $z = \frac{\alpha}{m}$.

(ii) For $z \in S_{\ell_1^n}$ let $u \in S_{X_n}$, $v \in S_{X'_n}$ be positive and such that $|z| = uv$. Since

$$(v^\alpha)^{1/m} = (v_1^{\alpha_1} \cdots v_n^{\alpha_n})^{1/m} \leq \frac{1}{m}(\alpha_1 v_1 + \cdots + \alpha_n v_n),$$

it follows that

$$(v^\alpha)^{1/m} \leq \frac{1}{m} \|(\alpha_1, \dots, \alpha_n)\|_{X_n} \|v_1, \dots, v_n\|_{X'_n} = \frac{1}{m} \|(\alpha_1, \dots, \alpha_n)\|_{X_n},$$

and hence

$$c_{X'_n}(\alpha) \leq \frac{1}{|u^\alpha|} = \frac{|v^\alpha|}{|z^\alpha|} \leq \frac{1}{m^m} \frac{\|(\alpha_1, \dots, \alpha_n)\|_{X_n}^m}{|z|^\alpha}.$$

Then taking the supremum in $z \in S_{\ell_1^n}$, we obtain

$$c_{X'_n}(\alpha) \leq \frac{1}{m^m} \|(\alpha_1, \dots, \alpha_n)\|_{X_n}^m \frac{m^m}{\alpha^\alpha} = \frac{\|(\alpha_1, \dots, \alpha_n)\|_{X_n}^m}{\alpha^\alpha}. \quad \square$$

Corollary 2.20. *Let $X_n = (\mathbb{C}^n, \|\cdot\|)$ be a symmetric Banach lattice and $J \subset \Lambda_T(m, n)$ some index set, where $m \leq n$. Then*

$$\widehat{\lambda}(\mathcal{P}_J(X_n)) \leq e^m \left(\frac{\varphi_{X'_n}(n)}{\varphi_{X'_n}(m)} \right)^m.$$

Proof. In case α is tetrahedral, $\|(\alpha_1, \dots, \alpha_n)\|_{X_n} = \varphi_X(m) = m/\varphi_{X'_n}(m)$ and thus by Theorem 2.19,

$$c_{X_n}(\alpha) \leq \frac{\|\alpha\|_{X_n}^m}{\alpha^\alpha} \leq \frac{m^m}{\alpha^\alpha} \frac{1}{\varphi_{X'_n}(m)^m} \leq e^m \frac{m!}{\alpha!} \frac{1}{\varphi_{X'_n}(m)^m}.$$

Thus, we obtain that

$$\begin{aligned} \widehat{\lambda}(\mathcal{P}_J(X_n)) &\leq e^m \frac{1}{\varphi_{X'_n}(m)^m} \sup_{z \in B_{X_n}} \sum_{\alpha \in J} \frac{m!}{\alpha!} |z^\alpha| \\ &\leq e^m \frac{1}{\varphi_{X'_n}(m)^m} \sup_{z \in B_{X_n}} (|z_1| + \cdots + |z_n|)^m = e^m \left(\frac{\varphi_{X'_n}(n)}{\varphi_{X'_n}(m)} \right)^m. \end{aligned} \quad \square$$

We conclude this subsection with a result that is interesting in its own right. It is motivated by the surprising fact noted in [56] (as a simple consequence of (6)) that

$$\lambda(\mathcal{P}_m(\ell_2^n)) \leq 2^{n-1}, \quad m, n \in \mathbb{N}.$$

In fact, the sequence $(X_m)_{m \geq 1}$ with $X_m := \mathcal{P}_m(\ell_2^2)$ was the first known nontrivial example of a sequence of finite-dimensional Banach spaces for which $\lim_{m \rightarrow \infty} \dim X_m = \infty$, even though

$$\sup_m \lambda(X_m) < \infty.$$

It is worth mentioning that Bourgain [13] gave an affirmative solution to a problem considered in [56], showing that the sequence $(d(X_m, \ell_\infty^{\dim X_m}(\mathbb{C})))_m$ of Banach-Mazur distances is bounded.

Theorem 2.21. *Let X be a Banach sequence lattice. Then the following statements are equivalent:*

- (i) $\sup_{m,n} \lambda(\mathcal{P}_{\leq m}(X_n))^{\frac{1}{m}} < \infty$.
- (ii) $\sup_{m,n} \lambda(\mathcal{P}_m(X_n))^{\frac{1}{m}} < \infty$.
- (iii) $X_n = \ell_1^n$ uniformly.
- (iv) $X = \ell_1$ whenever X is separable.

Proof. By Theorem 2.4 the first two statements are equivalent. Again Theorem 2.4 combined with the well known estimate $\lambda(\mathcal{P}_m(\ell_1^n)) \leq e^m$ yields $\lambda(\mathcal{P}_{\leq m}(\ell_1^n)) \leq (m+1)e^m$. Thus we need only show that (ii) implies (iii). Under the assumption (ii) we have that

$$\sup_n \gamma_1(X_n) = \sup_n \gamma_\infty(X_n^*) = \sup_n \lambda(X_n^*) \leq C := \sup_{m,n} \lambda(\mathcal{P}_m(X_n))^{\frac{1}{m}} < \infty.$$

We prove that there is $K > 0$ such that for any diagonal operator $D_\lambda : X_n \rightarrow \ell_2^n$, $(e_i) \mapsto (\lambda_i e_i)$ we have

$$\sup_n \pi_1(D_\lambda : X_n \rightarrow \ell_2^n) \leq K \|D_\lambda\|.$$

Indeed, take a factorization $\text{id} = uv$, where $u : X_n \rightarrow L_1(\mu)$ and $v : L_1(\mu) \rightarrow X_n$ for some measure μ and $\|u\| \|v\| \leq 2\gamma_1(X_n)$. From the classical Grothendieck's theorem, it follows that

$$\pi_1(D_\lambda \circ v) \leq K_G \|D_\lambda \circ v\|,$$

and hence

$$\begin{aligned} \pi_1(D_\lambda : X_n \rightarrow \ell_2^n) &= \pi_1(D_\lambda \circ v \circ u) \leq \pi_1(D_\lambda \circ v) \|u\| \\ &\leq K_G \|D_\lambda\| \|v\| \|u\| \leq K_G \|D_\lambda\| 2\gamma_1(X_n) \leq 2K_G C \|D_\lambda\|. \end{aligned}$$

Then we deduce from [59, Proposition 33.6] (a result which essentially goes back to Lindenstrauss and Pełczyński) that

$$\|\text{id} : X_n \rightarrow \ell_1^n\| \leq K = (2K_G C)^2,$$

and this completes the proof. \square

Determining whether, in Theorem 2.21, the projection constant of $\mathcal{P}_m(X_n)$ can be replaced by the unconditional basis constant of the monomial basis remains an open problem, known as 'Lempert's problem' in [25]. Specifically, it is unknown if $X \equiv \ell_1$ if and only if $\sup_{m,n} \chi_{\mathbf{mon}}(\mathcal{P}_m(X_n))^{\frac{1}{m}} < \infty$.

We recall a significant partial solution provided in [3] (see also [25, Theorem 20.26]), which essentially shows that in this case X is very similar to ℓ_1 : there is $D = D(\varepsilon) > 0$ such that $\|\text{id}: X_n \rightarrow \ell_1^n\| \leq D(\log \log n)^\varepsilon$ for all n .

As we have observed, the projection constant and the unconditional basis constant are closely related. This indicates that, in a certain sense, the results of Theorem 2.21 can be viewed as a solution to a weaker version of Lempert's problem.

2.5. Probabilistic estimates. For future reference, we establish several probabilistic lower bounds for the unconditional basis constant of the monomial basis $(z^\alpha)_{\alpha \in J}$ in the space $\mathcal{P}_J(X_n)$ of multivariate polynomials supported on J . These bounds are derived under specific conditions on the underlying Banach sequence lattice X_n and the index set $J \subset \mathbb{N}_0^n$.

We closely follow the methods from [3], [9], [24], [26], and [48], which primarily address the homogeneous case $J = \Lambda(m, n)$. In particular, we will rely on two key theorems from Bayart's work in [3], which we enclose below for completeness. It is important to note that the subtle proofs of these results utilize different techniques: the "coverage method" and the "entropy method." For all the necessary preliminary tools and proofs of these results, we refer to [25], specifically Corollary 17.5 and Corollary 17.22.

Theorem 2.22. *Given $1 \leq r \leq 2$, there is a constant $C > 0$ such that for each $m \geq 2$, for every Banach space $X_n = (\mathbb{C}^n, \|\cdot\|)$, and for every choice of scalars $(c_\alpha)_{\alpha \in \Lambda(m,n)}$ there exists a choice of signs $\varepsilon_\alpha = \pm 1$, $\alpha \in \Lambda(m, n)$ such that*

$$\sup_{z \in B_{X_n}} \left| \sum_{\alpha \in \Lambda(m,n)} \varepsilon_\alpha c_\alpha z^\alpha \right| \leq C(n \log m)^{\frac{1}{r}} \sup_{\alpha} \left(|c_\alpha| \left(\frac{\alpha!}{m!} \right)^{\frac{1}{r}} \right) \sup_{z \in B_{X_n}} \left(\sum_{k=1}^n |z_k|^r \right)^{\frac{m}{r}}.$$

Theorem 2.23. *Given $1 \leq r \leq 2$, there is a constant $C > 0$ such that for each $m \geq 2$, for every Banach space $X_n = (\mathbb{C}^n, \|\cdot\|)$, and for every choice of scalars $(c_\alpha)_{\alpha \in \Lambda(m,n)}$ there exists a choice of signs $\varepsilon_\alpha = \pm 1$, $\alpha \in \Lambda(m, n)$ such that*

$$\sup_{z \in B_{X_n}} \left| \sum_{\alpha \in \Lambda(m,n)} \varepsilon_\alpha c_\alpha z^\alpha \right| \leq C m (\log n)^{1 + \frac{1}{r}} \sup_{\alpha} \left(|c_\alpha| \left(\frac{\alpha!}{m!} \right)^{\frac{1}{r}} \right) \sup_{z \in B_{X_n}} \left(\sum_{k=1}^n |z_k|^r \right)^{\frac{m-1}{r}} \sup_{z \in B_{X_n}} \sum_{k=1}^n |z_k|.$$

We present some lower bounds for $\chi_{\mathbf{mon}}(\mathcal{P}_J(X_n))$, where X_n is the n th section of a Banach sequence lattice X and the index set $J \subset \mathbb{N}_0^{(\mathbb{N})}$ contains all m -homogeneous tetrahedral multi-indices

of length $m \leq n$. The proofs of these estimates are based on Theorem 2.22, Theorem 2.23 and the following lemma.

Lemma 2.24. *For each $1 \leq r \leq 2$, there is a constant $C > 0$ such that for every Banach lattice $X_n = (\mathbb{C}^n, \|\cdot\|)$ and for each $m \leq n$ one has*

$$\frac{1}{\|\text{id}: X_n \rightarrow \ell_r^n\|^m} \frac{|\Lambda_T(m, n)|}{\varphi_{X_n}(n)^m n^{\frac{1}{r'}} m^{-\frac{m}{r}}} \leq C e^{\frac{m}{r}} (\log m)^{1/r'} \chi_{\text{mon}}(\mathcal{P}_{\Lambda_T(m)}(X_n)).$$

Proof. Clearly, taking $z = (\varphi_{X_n}(n)^{-1}, \dots, \varphi_{X_n}(n)^{-1}) \in B_{X_n}$, we get

$$\frac{|\Lambda_T(m, n)|}{\varphi_{X_n}(n)^m} \leq \sup_{z \in B_{X_n}} \left| \sum_{\alpha \in \Lambda_T(m, n)} z^\alpha \right|.$$

Then, it follows from Theorem 2.22 that we find signs $\varepsilon_\alpha = \pm 1, \alpha \in \Lambda_T(m, n)$ for which for all $m \leq n$

$$\begin{aligned} \sup_{z \in B_{X_n}} \left| \sum_{\alpha \in \Lambda_T(m, n)} \varepsilon_\alpha z^\alpha \right| &\leq C (n \log m)^{1/r'} \sup_{\alpha \in \Lambda_T(m, n)} \left(\frac{\alpha!}{m!} \right)^{1/r} \sup_{z \in B_X} \left(\sum_{k=1}^n |z_k|^r \right)^{m/r} \\ &\leq C (\log m)^{1/r'} n^{1/r'} m!^{-1/r} \sup_{z \in B_X} \left(\sum_{k=1}^n |z_k|^r \right)^{m/r} \\ &= C (\log m)^{1/r'} n^{1/r'} m!^{-1/r} \|\text{id}: X_n \rightarrow \ell_r^n\|^m, \end{aligned}$$

where $C > 0$ is a constant only depending on r . Using that $m^m \leq e^m m!$, we finally arrive at

$$\begin{aligned} \frac{|\Lambda_T(m, n)|}{\varphi_{X_n}(n)^m} &\leq \sup_{z \in B_{X_n}} \left| \sum_{\alpha \in \Lambda_T(m, n)} \varepsilon_\alpha \varepsilon_\alpha z^\alpha \right| \\ &\leq \chi_{\text{mon}}(\mathcal{P}_{\Lambda_T(m, n)}) \sup_{z \in B_{X_n}} \left| \sum_{\alpha \in \Lambda_T(m, n)} \varepsilon_\alpha z^\alpha \right| \\ &\leq \chi_{\text{mon}}(\mathcal{P}_{\Lambda_T(m, n)}) C (\log m)^{1/r'} n^{1/r'} (m!)^{-1/r} \|\text{id}: X_n \rightarrow \ell_r^n\|^m \\ &\leq \chi_{\text{mon}}(\mathcal{P}_{\Lambda_T(m, n)}) C e^{\frac{m}{r}} (\log m)^{1/r'} n^{1/r'} (m!)^{-m/r} \|\text{id}: X_n \rightarrow \ell_r^n\|^m. \quad \square \end{aligned}$$

We are now ready to prove the aforementioned lower bounds.

Proposition 2.25. *Let $1 \leq r \leq 2$, and let X be a Banach sequence lattice such that $\varphi_X(n) < n^{1/r}$. Then, for any sequence (J_m) of index sets, each with degree at most m and $\Lambda_T(m) \subset J_m$, the following estimate holds:*

$$\frac{1}{\|\text{id}: X_n \rightarrow \ell_r^n\|^m} \left(\frac{n}{m} \right)^{\frac{m-1}{r'}} < C^m \chi_{\text{mon}}(\mathcal{P}_{J_m}(X_n)),$$

where $C \geq 1$ is a constant depending only on X .

Proof. We only have to consider the case $J = \Lambda_T(m, n)$. Note first that

$$(25) \quad \left(\frac{n}{m} \right)^m \leq \binom{n}{m} = |\Lambda_T(m, n)|.$$

Combining this with the assumption $\varphi_X(n) \leq Cn^{1/r}$, we get

$$\frac{|\Lambda_T(m, n)|}{\varphi_{X_n}(n)^m n^{\frac{1}{r'}} m^{-\frac{m}{r}}} \geq \frac{n^m}{C^m n^{\frac{m}{r}} n^{\frac{1}{r'}} m^{\frac{m}{r}}} = \left(\frac{n}{m}\right)^{\frac{m-1}{r'}} \frac{1}{C^m m^{\frac{1}{r}}},$$

and hence the claim is an immediate consequence of Lemma 2.24. \square

Proposition 2.26. *Let X be a Banach sequence lattice such that*

$$(26) \quad \varphi_{X_n}(n) \varphi_{X'_n}(n) < n \quad \text{and} \quad \|\text{id}: X_n \rightarrow \ell_2^n\| < \frac{1}{\sqrt{n}} \|\text{id}: X_n \rightarrow \ell_1^n\|.$$

Then, for any sequence (J_m) of index sets, each with degree at most m and $\Lambda_T(m) \subset J_m$, we have:

$$\left(\frac{n}{m}\right)^{\frac{m-1}{2}} <_{C^m} \chi_{\text{mon}}(\mathcal{P}_{J_m}(X_n)),$$

where $C \geq 1$ is a constant depending only on X .

Proof. Since for $x = \frac{e_1}{\|e_1\|_X}$ one has $\|x\|_X = 1$, it follows that

$$\gamma := \frac{1}{\|e_1\|_X} \leq \|\text{id}: X_n \rightarrow \ell_2^n\| < \frac{1}{\sqrt{n}} \|\text{id}: X_n \rightarrow \ell_1^n\|.$$

Using the estimate $\varphi_{X_n}(n) \varphi_{X'_n}(n) < n$, we get

$$\gamma < \frac{1}{\sqrt{n}} \|\text{id}: X_n \rightarrow \ell_1^n\| = \frac{1}{\sqrt{n}} \varphi_{X'_n}(n) < \frac{\sqrt{n}}{\varphi_{X_n}(n)},$$

and hence $\varphi_{X_n}(n) < \sqrt{n}$. This combined with the above estimate (25) yields

$$\frac{1}{\|\text{id}: X_n \rightarrow \ell_2^n\|^m} \frac{|\Lambda_T(m, n)|}{\varphi_{X_n}(n)^m n^{\frac{1}{2}} m^{-\frac{m}{2}}} > \frac{n^{\frac{m}{2}}}{\varphi_{X'_n}(n)^m} \left(\frac{n}{m}\right)^m \frac{1}{n^{\frac{1}{2}} \varphi_{X_n}(n)^m m^{-\frac{m}{2}}} \sim \frac{n^{\frac{m-1}{2}}}{m^{\frac{m}{2}}} = \frac{1}{m^{\frac{1}{2}}} \left(\frac{n}{m}\right)^{\frac{m-1}{2}},$$

and hence the conclusion follows (as in the preceding proof) from Lemma 2.24. \square

Applying Theorem 2.23, we obtain an additional lower bound that will be crucial later on.

Proposition 2.27. *Let $1 \leq r \leq 2$, and let X be a Banach sequence lattice X such that $\varphi_{X_n}(n) \varphi_{X'_n}(n) < n$.*

Then, there exists a constant $C = C(r, X)$ such that for each $m \leq n$, the following estimate holds:

$$\frac{\left(\frac{n}{n-m}\right)^{n-m} \sqrt{\frac{n}{m(n-m)}}}{Cm (\log n)^{1+1/r'} e^{\frac{m}{r}} m^{m/r'}} \left(\frac{\|\text{id}: X_n \rightarrow \ell_1^n\|}{\|\text{id}: X_n \rightarrow \ell_r^n\|} \right)^{m-1} \leq \chi_{\text{mon}}(\mathcal{P}_{\Lambda_T(m)}(X_n)).$$

Proof. Note first that for each $m, n \in \mathbb{N}$

$$(27) \quad \frac{m^{1/r}}{C_1 m (\log n)^{1+1/r'}} \frac{|\Lambda_T(m, n)|}{\|\text{id}: X_n \rightarrow \ell_r^n\|^{m-1} \varphi_{X_n}(n)^m \varphi_{X'_n}(n)} \leq \chi_{\text{mon}}(\mathcal{P}_{\Lambda_T(m)}(X_n)),$$

where $C_1 = C_1(r)$ only depends on r . Indeed, choose signs $\varepsilon_\alpha = \pm 1$, $\alpha \in \Lambda_T(m, n)$ such that

$$\sup_{z \in B_{X_n}} \left| \sum_{\alpha \in \Lambda_T(m, n)} \varepsilon_\alpha z^\alpha \right| \leq C_1 m (\log n)^{1+1/r'} m!^{-1/r} \|\text{id}: X_n \rightarrow \ell_r^n\|^{m-1} \|\text{id}: X_n \rightarrow \ell_1^n\|,$$

where $C_1 = C_1(r)$ is the constant from Theorem 2.23. Since $\|\text{id}: X_n \rightarrow \ell_1^n\| = \varphi_{X_n'}(n)$, we may now proceed exactly as in the proof of Lemma 2.24 to get (27). We use now Stirling's formula,

$$\sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n} \leq n! \leq \frac{12}{11} \sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n}, \quad n \in \mathbb{N},$$

in (27) to conclude that there is a uniform constant $\gamma \geq 1$ such that for each $m \leq n$

$$|\Lambda_T(m, n)| = \binom{n}{m} \geq \gamma \left(\frac{n}{m}\right)^m \left(\frac{n}{n-m}\right)^{n-m} \sqrt{\frac{n}{m(n-m)}}.$$

Estimating now (27), we get with $C_2 = C_1/\gamma$ that for all $m \leq n$

$$\chi_{\text{mon}}(\mathcal{P}_{\Lambda_T(m)}(X_n)) \geq \frac{m!^{1/r}}{C_2 m (\log n)^{1+1/r'}} \frac{n^m \left(\frac{n}{n-m}\right)^{n-m} \sqrt{\frac{n}{m(n-m)}}}{m^m \|\text{id}: X_n \rightarrow \ell_r^n\|^{m-1} \varphi_{X_n}(n)^m \varphi_{X_n'}(n)}.$$

Finally, since

$$\frac{n^{m-1}}{\|\text{id}: X_n \rightarrow \ell_r^n\|^{m-1} \varphi_{X_n}(n)^{m-1}} = \left(\frac{\|\text{id}: X_n \rightarrow \ell_1^n\|}{\|\text{id}: X_n \rightarrow \ell_r^n\|} \right)^{m-1}$$

and $1 < \frac{n}{\varphi_{X_n}(n) \varphi_{X_n'}(n)}$, we have

$$\chi_{\text{mon}}(\mathcal{P}_{\Lambda_T(m)}(X_n)) \geq \frac{\left(\frac{n}{n-m}\right)^{n-m} \sqrt{\frac{n}{m(n-m)}}}{C m (\log n)^{1+1/r'} m^m m!^{-1/r}} \left(\frac{\|\text{id}: X_n \rightarrow \ell_1^n\|}{\|\text{id}: X_n \rightarrow \ell_r^n\|} \right)^{m-1},$$

where now $C \geq 1$ depends on r and X . Since $m^m \leq e^m m!$, we have that $(m!)^{-\frac{1}{r}} \leq e^{\frac{m}{r}} m^{-\frac{m}{r}}$, and so the proof is complete. \square

2.6. Impact on Bohr radii. In this section, we focus in the study of multivariate Bohr radii. As already defined in the introduction in Equation (8), $K(B_{X_n}, J)$ stands for the Bohr radius of the open unit ball B_{X_n} of a Banach lattice $X_n = (\mathbb{C}^n, \|\cdot\|)$ with respect to an index set $J \subset \mathbb{N}_0^{(\mathbb{N})}$.

It is immediate that $K(B_{X_n}, J) = K(B_{X_n}, J \cap \mathbb{N}_0^n)$ for any index sets J , and for $J, J' \subset \mathbb{N}_0^n$ satisfying $J \subset J'$, the following monotonicity estimate holds:

$$(28) \quad K(B_{X_n}, J') \leq K(B_{X_n}, J).$$

Moreover, for any Banach lattice $X_n = (\mathbb{C}^n, \|\cdot\|)$ and index set J , we have

$$(29) \quad K(B_{\ell_\infty^n}, J) \leq K(B_{X_n}, J).$$

To see the proof of this inequality, one only needs to follow the argument in [25, Proposition 19.2].

We write $K_m(B_{X_n}) := K(B_{X_n}, \Lambda(m))$, and call this number the m -homogeneous Bohr radius of B_{X_n} . Note that in this case $J = \Lambda(m)$, so in (8), we only consider m -homogeneous polynomials $f \in \mathcal{P}_m(X_n) = H_\infty^{\Lambda(m)}(B_{X_n})$.

Typically, $X_n = (\mathbb{C}^n, \|\cdot\|)$ will be the n -th section of a Banach sequence lattice X with a specific geometrical structure (e.g., a 2-convex space), or a concrete spaces X (e.g., a Lorentz spaces $\ell_{r,s}$). Additionally, we focus on various index sets J in with particular structures (e.g., index sets which consist of indices of degree at most m , or index sets formed by tetrahedral indices, or sets of multi indices α generated by the prime number decompositions $n = \mathfrak{p}^\alpha$ of certain subsets of natural numbers).

To illustrate our objectives, we recall a couple of results that serve as motivation for what follows.

From [16] (see also [25, Theorem 19.1]) we know that for each $1 \leq r \leq \infty$

$$K(B_{\ell_r^n}) \sim_C \left(\frac{\log n}{n} \right)^{\min\{\frac{1}{2}, \frac{1}{r}\}}.$$

This result is intimately connected with the topic of the previous section – the asymptotic determination of $\lambda(\mathcal{P}_J(X_n))$ and $\chi_{\text{mon}}(\mathcal{P}_J(X_n))$.

In fact, in Theorem 3.22, we will integrate many of the results obtained so far to characterize the asymptotic decay of $K(B_{\ell_{r,s}^n}, J)$ as n tends to infinity. This will cover (almost) all possible values of r and s , and as well as a broad class of index sets J .

2.6.1. Bohr radii vs unconditionality and projection constants. We begin this subsection by presenting two results that generalize the cases for the entire index set $J = \mathbb{N}_0^n$, first noted in [23] (see also [25, Proposition 19.4 and Lemma 19.5]). The proofs of these results are straightforward extensions. They are crucial for studying multidimensional Bohr radii in Banach spaces and demonstrate a close connection with the study of unconditionality in spaces of multivariate polynomials on Banach spaces.

Proposition 2.28. *Let $X_n = (\mathbb{C}^n, \|\cdot\|)$ be a Banach lattice and $J \subset \mathbb{N}_0^n$ an index set. Then for each m , we have*

$$K_m(B_{X_n}, J) = \frac{1}{\chi_{\text{mon}}(\mathcal{P}_{J(m)}(X_n))^{1/m}}.$$

Theorem 2.29. *Let $X_n = (\mathbb{C}^n, \|\cdot\|)$ be a Banach lattice and $J \subset \mathbb{N}_0^n$ an index set. Then*

$$\frac{1}{3} \inf_{m \in \mathbb{N}} K_m(B_{X_n}, J) \leq K(B_{X_n}, J) \leq \inf_{m \in \mathbb{N}} K_m(B_{X_n}, J).$$

We now present a direct relationship between Bohr radii and the projection constants of spaces of homogeneous polynomials, which will be crucial for the proof of Theorem 2.32 (and subsequently for Theorem 3.22).

Given $I, J \subset \mathbb{N}_0^n$ with $I \subset J$, recall the definition of the projection $\mathbf{Q}_{J,I}: \mathcal{P}_J(X_n) \rightarrow \mathcal{P}_I(X_n)$ from (14), and from (10) the definition of the reduced index sets J^b .

Theorem 2.30. *Let X be a Banach sequence lattice. Then, for every n and each index set $J \subset \mathbb{N}_0^n$, it holds that*

$$K(B_{X_n}, J) \geq \frac{1}{6} \inf_{m \in \mathbb{N}} \left(\frac{1}{\sqrt[m]{e \|\mathbf{Q}_{\Lambda(m), J(m)}\|}} \frac{1}{\sqrt[m]{\lambda(\mathcal{P}_{J(m)^b}(X_n))}} \right).$$

Observe that on the right side of this inequality we take the m -root, even though $J(m)^b$ is by definition contained in $\Lambda(m-1, n)$.

Proof. From Theorem 2.29 we deduce that for all n

$$\frac{1}{3} \inf_{m \in \mathbb{N}} K_m(B_{X_n}, J) \leq K(B_{X_n}, J).$$

On the other hand, by Theorem 2.7 for all m and n

$$\chi_{\text{mon}}(\mathcal{P}_{J(m)}(X_n)) \leq e2^m \|\mathbf{Q}_{\Lambda(m,n), J(m)}\| \lambda(\mathcal{P}_{J(m)^b}(X_n)).$$

The proof is then completed by taking the m -th root and using Proposition 2.28. \square

As an application, we establish both upper and lower bounds for multivariate Bohr radii in the tetrahedral case. First, let us note the following fact, which will be repeatedly used in the subsequent results.

Remark 2.31. Given $n \in \mathbb{N}$, the function f_n defined by $f_n(t) = \frac{1}{tn^{1/t}}$ for all $t > 0$ attains its maximum at $t = \log n$. In particular,

$$\max_{m \in \mathbb{N}} f_n(m) \sim_C \frac{1}{\log n}.$$

Theorem 2.32. *Let X be a symmetric Banach sequence lattice. Then*

$$\inf_{m \leq n} \left(\frac{\varphi_{X'}(m-1)}{\varphi_{X'}(n)} \right)^{\frac{m-1}{m}} <_C K(B_{X_n}, \Lambda_T),$$

and for $1 \leq r \leq 2$,

$$K(B_{X_n}, \Lambda_T) <_C \frac{(\log n)^{\frac{1}{r}}}{\varphi_{X'}(n)} \|\text{id}: X_n \rightarrow \ell_r^n\|.$$

Proof. Let us start with the proof of the first claim. From Theorem 2.3 we know that for all m, n

$$\|\mathbf{Q}_{\Lambda(m,n), \Lambda_T(m,n)}: \mathcal{P}_{\Lambda(m,n)}(X_n) \rightarrow \mathcal{P}_{\Lambda_T(m,n)}(X_n)\| \leq \kappa^m.$$

Moreover, by Theorem 2.18 and Corollary 2.20, we have

$$\lambda(\mathcal{P}_{\Lambda_T(m,n)^b}(X_n)) \leq \widehat{\lambda}(\mathcal{P}_{\Lambda_T(m,n)^b}(X_n)) \leq e^{m-1} \left(\frac{\varphi_{X'_n}(n)}{\varphi_{X'_n}(m-1)} \right)^{m-1},$$

where the last equality holds as $m \leq n$ (recall that $\Lambda_T(m, n) = \emptyset$ when $m > n$). Implementing the preceding two inequalities into Theorem 2.30, yields the assertion.

For the proof of the upper estimate, note first that by Theorem 2.29 and Proposition 2.28

$$K(B_{X_n}, \Lambda_T) \leq \inf_{m \in \mathbb{N}} K(B_{X_n}, \Lambda_T(m, n)) = \frac{1}{\sup_{m \in \mathbb{N}} \chi_{\text{mon}}(\mathcal{P}_{\Lambda_T(m,n)}(X_n))^{1/m}}.$$

But applying Lemma 2.24, the fact that $\varphi_X(n)\varphi_{X'}(n) = n$ by the symmetry of X and (25), we see that

$$\begin{aligned} \sup_{m \in \mathbb{N}} \chi_{\text{mon}}(\mathcal{P}_{\Lambda_T(m,n)}(X_n))^{1/m} &\leq \frac{1}{\|\text{id}: X_n \rightarrow \ell_r^n\|} \sup_{m \in \mathbb{N}} \frac{|\Lambda_T(m, n)|^{1/m}}{\varphi_X(n) n^{\frac{1}{mr'}} m^{-\frac{1}{r}}} \\ &\leq \frac{\varphi_{X'}(n)}{n \|\text{id}: X_n \rightarrow \ell_r^n\|} \sup_{m \in \mathbb{N}} \frac{n}{n^{\frac{1}{mr'}} m^{\frac{1}{r}}} \leq \frac{\varphi_{X'}(n)}{\|\text{id}: X_n \rightarrow \ell_r^n\|} \sup_{m \in \mathbb{N}} \frac{1}{n^{\frac{1}{mr'}} m^{\frac{1}{r}}}, \end{aligned}$$

so that the conclusion again follows using Remark 2.31. \square

2.6.2. Bohr radii and convexity. We now use the geometric concept of convexity in Banach sequence spaces to provide estimates of Bohr radii. For 2-convex spaces, we present an explicit characterization of the asymptotic growth of the Bohr radius for a broad class of index sets.

Theorem 2.33. *Let X be a 2-convex Banach sequence space and J an index set such that $\Lambda_T \subset J$. Then, with a constant $C \geq 1$ only depending on X ,*

$$K_m(B_{X_n}, J) \sim_C \left(\frac{m}{n+m} \right)^{\frac{m-1}{2m}}.$$

Moreover, with a constant $C \geq 1$ only depending on X ,

$$K(B_{X_n}, J) \sim_C \sqrt{\frac{\log n}{n}}.$$

Proof. The first claim follows using Proposition 2.28 and the fact that, by Theorem 2.16 for all m, n

$$\left(1 + \frac{n}{m} \right)^{\frac{m-1}{2}} \sim_C^m \chi_{\text{mon}}(\mathcal{P}_{J(m)}(X_n)).$$

Note that this in particular means that

$$K_m(B_{X_n}, J) \sim_C \left(\frac{m}{n} \right)^{\frac{m-1}{2m}} \quad \text{for } m \leq n \quad \text{and} \quad K_m(B_{X_n}, J) \sim_C 1 \quad \text{for } m \geq n.$$

As a consequence, we deduce from Theorem 2.29 that for some constant $C \geq 1$

$$\begin{aligned} \frac{1}{3} \min \left\{ \frac{1}{C}, \inf_{m \leq n} \left(\frac{m}{n} \right)^{\frac{m-1}{2m}} \right\} &\leq \frac{1}{3} \min \left\{ \inf_{n \leq m} K_m(B_{X_n}, J), \inf_{m \leq n} K_m(B_{X_n}, J) \right\} \\ &= \frac{1}{3} \inf_{m \in \mathbb{N}} K_m(B_{X_n}, J) \leq K(B_{X_n}, J) \\ &\leq \inf_{m \in \mathbb{N}} K_m(B_{X_n}, J) \leq C \inf_{m \leq n} \left(\frac{m}{n} \right)^{\frac{m-1}{2m}}. \end{aligned}$$

Using the fact that $\left(\frac{m}{n} \right)^{\frac{m-1}{2m}} \sim_C \left(\frac{mn^{\frac{1}{m}}}{n} \right)^{\frac{1}{2}}$, we conclude from Remark 2.31 and the first statement of the theorem (already proved) that

$$K(B_{X_n}, J) \sim_C K_{[\log n]}(B_{X_n}, J) \sim_C \sqrt{\frac{\log n}{n}}. \quad \square$$

For the rest of this section the aim is to establish a far reaching extension of the remarkable result by Bayart, Pellegrino, and Seoane [5] which states that

$$(30) \quad \lim_{n \rightarrow \infty} \frac{K(B_{\ell_\infty^n})}{\sqrt{\frac{\log n}{n}}} = 1.$$

We prove that this limit formula is valid for a class of Banach lattices X_n distinct from ℓ_∞^n . Moreover, we consider more general sets of indices beyond the full set $J = \mathbb{N}_0^{(\mathbb{N})}$, including all tetrahedral indices. In particular, we will demonstrate that a limit analogous to (30) holds for a class of Lorentz sequence spaces (see Theorem 3.22).

Theorem 2.34. *Let X be a Banach sequence lattice such that*

$$\varphi_X(n) \varphi_{X'}(n) < n \quad \text{and} \quad \|\text{id}: X_n \rightarrow \ell_2^n\| \leq \frac{1}{\sqrt{n}} \|\text{id}: X_n \rightarrow \ell_1^n\|.$$

Then, for every index set J for which $\Lambda_T \subset J$, one has

$$\lim_{n \rightarrow \infty} \frac{K(B_{X_n}, J)}{\sqrt{\frac{\log n}{n}}} = 1.$$

Proof. Combining (29) and (30) with the monotonicity of Bohr radii in the index set, yields

$$1 = \lim_{n \rightarrow \infty} \frac{K(B_{\ell_\infty^n})}{\sqrt{\frac{\log n}{n}}} = \lim_{n \rightarrow \infty} \inf \frac{K(B_{X_n})}{\sqrt{\frac{\log n}{n}}} \leq \lim_{n \rightarrow \infty} \inf \frac{K(B_{X_n}, J)}{\sqrt{\frac{\log n}{n}}} \leq \limsup_{n \rightarrow \infty} \frac{K(B_{X_n}, J)}{\sqrt{\frac{\log n}{n}}}$$

and this is less than or equal to $\limsup_{n \rightarrow \infty} \frac{K(B_{X_n, \Lambda_T})}{\sqrt{\frac{\log n}{n}}}$. Thus to complete the proof, it remains to show that the last term is ≤ 1 . To do so we apply Proposition 2.27 with $r = 2$, and get for each $m \leq n$

$$\frac{1}{\chi_{\text{mon}}(\mathcal{P}_{\Lambda_T(m,n)}(X_n))^{\frac{1}{m}}} \leq \left(\frac{\|\text{id} : X_n \rightarrow \ell_2^n\|}{\|\text{id} : X_n \rightarrow \ell_1^n\|} \right)^{\frac{m-1}{m}} \frac{(Cm(\log n)^{\frac{3}{2}})^{\frac{1}{m}} (e^{\frac{m}{2}} m^{m/2})^{\frac{1}{m}}}{\left(\left(\frac{n}{n-m} \right)^{n-m} \right)^{\frac{1}{m}} \left(\sqrt{\frac{n}{m(n-m)}} \right)^{\frac{1}{m}}},$$

and consequently for each $m \leq n$

$$\begin{aligned} \frac{1}{\chi_{\text{mon}}(\mathcal{P}_{\Lambda_T(m,n)}(X_n))^{\frac{1}{m}}} &\leq n^{-\frac{m-1}{2m}} \frac{(Cm(\log n)^{\frac{3}{2}})^{\frac{1}{m}} (e^{\frac{m}{2}} m^{m/2})^{\frac{1}{m}}}{\left(\left(\frac{n}{n-m} \right)^{n-m} \right)^{\frac{1}{m}} \left(\sqrt{\frac{n}{m(n-m)}} \right)^{\frac{1}{m}}} \\ &= \left(\frac{m}{n} \right)^{\frac{1}{2}} n^{\frac{1}{2m}} m^{-\frac{1}{2}} \frac{(Cm(\log n)^{\frac{3}{2}})^{\frac{1}{m}} (e^{\frac{m}{2}} m^{m/2})^{\frac{1}{m}}}{\left(\left(\frac{n}{n-m} \right)^{n-m} \right)^{\frac{1}{m}} \left(\sqrt{\frac{n}{m(n-m)}} \right)^{\frac{1}{m}}} \\ &= \left(\frac{m}{n} \right)^{\frac{1}{2}} n^{\frac{1}{2m}} e^{\frac{1}{2}} \left(\frac{n-m}{n} \right)^{\frac{n-m}{m}} \frac{(Cm(\log n)^{\frac{3}{2}})^{\frac{1}{m}}}{\left(\sqrt{\frac{n}{m(n-m)}} \right)^{\frac{1}{m}}} \\ &= \left(\frac{m}{n} \right)^{\frac{1}{2}} \left[n^{\frac{1}{2m}} e^{\frac{1}{2}} \left(1 - \frac{m}{n} \right)^{\frac{n}{m}} \right] \left[\frac{n}{n-m} \frac{(Cm(\log n)^{\frac{3}{2}})^{\frac{1}{m}}}{\left(\sqrt{\frac{n}{m(n-m)}} \right)^{\frac{1}{m}}} \right]. \end{aligned}$$

Taking $m = \lceil \log n \rceil$, the last factor in the previous chain of inequalities tends to 1 as $n \rightarrow \infty$. Since

$$\lim_{n \rightarrow \infty} n^{\frac{1}{2\lceil \log n \rceil}} e^{\frac{1}{2}} \left(1 - \frac{\lceil \log n \rceil}{n} \right)^{\frac{n}{\lceil \log n \rceil}} = e^{\frac{1}{2}} e^{\frac{1}{2}} e^{-1} = 1,$$

we from Theorem 2.29 and Proposition 2.28 deduce

$$\limsup_{n \rightarrow \infty} \frac{K(B_{X_n})}{\sqrt{\frac{\log n}{n}}} = \limsup_{n \rightarrow \infty} \frac{\sqrt{\frac{n}{\log n}}}{\sup_{m \in \mathbb{N}} \chi_{\text{mon}}(\mathcal{P}_m(X_n))^{\frac{1}{m}}} \leq \limsup_{n \rightarrow \infty} \frac{\sqrt{\frac{n}{\log n}}}{\chi_{\text{mon}}(\mathcal{P}_{\lceil \log n \rceil}(X_n))^{\frac{1}{\lceil \log n \rceil}}} \leq 1,$$

which completes the proof. \square

Note that with a similar proof, but applying Proposition 2.26 instead of Proposition 2.27, we can establish the following, more general result for the entire index set.

Theorem 2.35. *Let X be a Banach sequence lattice such that $\frac{\|\text{id} : X_n \rightarrow \ell_2^n\|}{\|\text{id} : X_n \rightarrow \ell_1^n\|} \leq \frac{1}{\sqrt{n}}$ for each n . Then*

$$\lim_{n \rightarrow \infty} \frac{K(B_{X_n})}{\sqrt{\frac{\log n}{n}}} = 1.$$

We conclude by presenting a class of Banach sequence lattices X for which (30) holds when ℓ_∞ is replaced by X , and the entire index set is replaced by any arbitrary index set that includes all tetrahedral indices. To achieve this, we show how to construct Banach sequence lattices that satisfy the assumptions of Theorem 2.35.

First, we note that the 2-convexification $X := E^{(2)}$ of any Banach sequence lattice E is 2-convex with $M^{(2)}(X) = 1$. If, in addition, X satisfies

$$\varphi_X(n) \varphi_{X'}(n) < n,$$

then X is indeed an example we seek. This estimate clearly holds whenever E is symmetric. However, it is important to point out that it is straightforward to show that this estimate for $X := E^{(2)}$ is valid even when E is an arbitrary Banach sequence lattice satisfying $\varphi_E(n) \varphi_{E'}(n) < n$, which includes non-symmetric spaces E .

3. APPLICATIONS FOR POLYNOMIALS ON LORENTZ SPACES $\ell_{r,s}^n$

In this section, we concentrate on polynomials equipped with supremum norms on unit spheres within a significant class of symmetric finite-dimensional Banach spaces, specifically Lorentz sequence spaces. We provide some essential definitions and properties that will be used throughout without further reference. For a more in-depth exploration of Lorentz sequence spaces, we refer to [14, 37, 44, 54].

For $1 \leq p, q \leq \infty$ the space $\ell_{p,q}$ consists of those (complex) sequences z for which (we use the convention $\frac{1}{\infty} := 0$)

$$\|z\|_{\ell_{p,q}} := \left\| \left(z_n^* n^{\frac{1}{p} - \frac{1}{q}} \right)_{n=1}^\infty \right\|_{\ell_q} < \infty,$$

where $z^* = (z_k^*)$ as usual denotes the decreasing rearrangement of $|z|$. Observe that, in general, this is a quasi-norm and only defines a norm for $1 \leq q \leq p \leq \infty$. For $z \in \ell_{p,q}$ we define

$$\|z\|_{\ell_{p,q}}^* := \left(\sum_{n=1}^\infty n^{\frac{q}{p}-1} \left(\frac{1}{n} \sum_{k=1}^n z_k^* \right)^q \right)^{1/q}.$$

It should be noted (see [7, Lemma 4.5]) that for $1 \leq p, q \leq \infty$ and $z \in \ell_{p,q}$, it holds

$$\|z\|_{\ell_{p,q}} \leq \|z\|_{\ell_{p,q}}^* \leq p' \|z\|_{\ell_{p,q}}.$$

Thus, we can consistently work with the quasi-norm $\|\cdot\|_{\ell_{p,q}}$ and treat $(\ell_{p,q}, \|\cdot\|_{\ell_{p,q}})$ as a Banach sequence space, albeit with the trade-off of p' (the conjugate exponent of p) as a cost each time we do so. Henceforth, if $X = \ell_{p,q}$, then X_n is denoted by $\ell_{p,q}^n$.

The fundamental function for $\ell_{p,q}$ satisfies the equivalence $\varphi_{\ell_{p,q}}(n) \sim_C n^{\frac{1}{p}}$. Note that if $1 \leq q < p$, then $\ell_{p,q}$ is a symmetric Banach space.

We heavily use the fact that the spaces $\ell_{p,q}$ are ordered lexicographically:

$$\begin{aligned} \ell_{p,q} &\hookrightarrow \ell_{r,s}, & \text{for } p < r \\ \ell_{p,q} &\hookrightarrow \ell_{r,s}, & \text{for } p = r \text{ and } q < s. \end{aligned}$$

It is easy to check (see, e.g., [28, Lemma 22]) that, for $1 \leq r, s, t \leq \infty$

$$(31) \quad \|\text{id}: \ell_{r,t}^n \rightarrow \ell_{r,s}^n\| <_C \max\{1, (1 + \log n)^{\frac{1}{s} - \frac{1}{t}}\}.$$

Moreover, notice that for $1 \leq q \leq p \leq \infty$ it holds that

$$(32) \quad \|\text{id}: \ell_{p,q} \rightarrow \ell_p\| = 1.$$

Indeed, since $(k^{\frac{q}{p}-1})_k$ is a decreasing sequence, then

$$z_n^* n^{1/p} = \left(n(z_n^*)^q n^{\frac{q}{p}} \right)^{\frac{1}{q}} \leq \left(\sum_{k=1}^n (z_k^*)^q k^{\frac{q}{p}-1} \right)^{1/q} = \|z\|_{\ell_{p,q}},$$

hence for $z \in B_{\ell_{p,q}}$,

$$\|z\|_{\ell_p}^p = \sum_{n \geq 1} (z_n^*)^p = \sum_{n \geq 1} (z_n^*)^q (z_n^*)^{p-q} \leq \sum_{n \geq 1} (z_n^*)^q n^{\frac{q}{p}-1} \leq 1.$$

Also we will need the fact that $\ell_{p,q}$ is 2-convex whenever $2 < p < \infty$ and $2 \leq q < \infty$, and $\ell_{p,q}$ is 2-concave whenever $1 \leq p < 2$ and $1 \leq q \leq 2$. We refer to [37], where explicit formulae for the q -concavity constant of the Lorentz spaces.

Focusing on the projection constant for polynomials on Lorentz spaces $\ell_{r,s}^n$, it is essential to first address the case of 1-homogeneous polynomials. This case corresponds to the projection constant of the scale of finite-dimensional Lorentz spaces themselves. The following result encapsulates the current state of knowledge on this topic.

Proposition 3.1. *For all $1 < r < \infty$ and $1 \leq s \leq \infty$ one has*

$$\lambda(\ell_{r,s}^n) \sim \begin{cases} n^{\min\{\frac{1}{2}, \frac{1}{r}\}}, & r \neq 2 \\ n^{\frac{1}{2}}, & r = 2, 2 \leq s \leq \infty \\ \left(\frac{n}{\log(e+\log n)} \right)^{\frac{1}{2}}, & r = 2, s = 1. \end{cases}$$

Proof. As explained above, $\ell_{r,s}$ is 2-concave (then its dual is 2-convex) for $r < 2$ and $1 \leq s \leq 2$. Thus the statement follows from Theorem 2.16. For the proof of the case $r < 2$ and $2 \leq s \leq \infty$ see Theorem 3.3.

The case $r = 2$, $s \geq 2$ and the case $r > 2$ follow from (12) and (13). The last equivalence is a remarkable result due to Kwapien and Schütt [43], which states that $\lambda(\ell_{2,1}^n) \sim (n/\log \log n)^{\frac{1}{2}}$, $n \geq 5$. \square

It should be noted that, to the best of our knowledge, the optimal estimate for the projection constant of $\ell_{r,s}^n$ with respect to the parameters r and s remains an open problem in general. Even in the specific case where $r = 2$ and $1 < s < 2$ —a scenario not covered by Proposition 3.1—the asymptotic behavior of $\lambda(\ell_{2,s}^n)$ as a function of the dimension n remains unknown. This presents a challenge, particularly when extending the problem to the more complex issue of determining the optimal estimates for the projection constant of polynomials on Lorentz spaces $\ell_{r,s}^n$ in a general setting.

We conclude with the following observation, which is deduced from combining standard estimates involving Banach-Mazur distances.

Lemma 3.2. *For every $1 < s < 2$, it holds that*

$$\lambda(\ell_{2,s}^n) > \begin{cases} \frac{1}{(1+\log n)^{1-\frac{1}{s}}} \left(\frac{n}{\log(e+\log n)} \right)^{\frac{1}{2}}, & 1 < s < \frac{4}{3} \\ \frac{\sqrt{n}}{(1+\log n)^{\frac{1}{s}-\frac{1}{2}}}, & \frac{4}{3} \leq s < 2. \end{cases}$$

Proof. Recall that for any pair of isomorphic Banach spaces E and F one has

$$\lambda(E) \leq d(E, F) \lambda(F).$$

Thus, for each $n \in \mathbb{N}$, we have the estimates

$$\lambda(\ell_2^n) \leq d(\ell_2^n, \ell_{2,s}^n) \lambda(\ell_{2,s}^n), \quad \lambda(\ell_{2,1}^n) \leq d(\ell_{2,1}^n, \ell_{2,s}^n) \lambda(\ell_{2,s}^n),$$

and hence

$$\lambda(\ell_{2,s}^n) \geq \max \left\{ \frac{\lambda(\ell_2^n)}{d(\ell_2^n, \ell_{2,s}^n)}, \frac{\lambda(\ell_{2,1}^n)}{d(\ell_{2,1}^n, \ell_{2,s}^n)} \right\}.$$

By (31), it follows that

$$d(\ell_2^n, \ell_{2,s}^n) < (1+\log n)^{\frac{1}{s}-\frac{1}{2}}, \quad d(\ell_{2,1}^n, \ell_{2,s}^n) < (1+\log n)^{1-\frac{1}{s}}.$$

The above inequalities, combined with the asymptotic behavior $\lambda(\ell_2^n) \sim \sqrt{n}$ and the asymptotic for $\lambda(\ell_{2,1}^n)$ from Proposition 3.1, provide the necessary estimates. \square

3.1. Tetrahedral index sets of multi indices. As previously noted, understanding the unconditional basis constants and projection constants for the space $\mathcal{P}_J(\ell_{r,s}^n)$, where J is a tetrahedral index set, is in fact crucial.

We will first show the upper estimates in the tetrahedral case.

Theorem 3.3. *Let $1 < r < \infty$ and $1 \leq s \leq \infty$. Then, there exists a constant $C = C(r, s) > 0$ such that, for any sequence (J_m) of tetrahedral index sets, each with degree at most m , and every n the following estimates hold:*

$$\chi_{\text{mon}}(\mathcal{P}_{J_m}(\ell_{r,s}^n)) \leq C^m \left(\frac{n}{m}\right)^{(m-1)\min\{\frac{1}{2}, \frac{1}{r'}\}} \quad \text{and} \quad \lambda(\mathcal{P}_{J_m}(\ell_{r,s}^n)) \leq C^m \left(\frac{n}{m}\right)^{m\min\{\frac{1}{2}, \frac{1}{r'}\}}.$$

Proof. For $2 \leq r < \infty$ both claims follow from Corollary 2.9.

For $1 < r \leq 2$, we begin with the upper bound for the projection constant. Using Theorem 2.18 and Corollary 2.20, we obtain

$$\begin{aligned} \lambda(\mathcal{P}_{J_m}(\ell_{r,s}^n)) &\leq \widehat{\lambda}(\mathcal{P}_{J_m}(\ell_{r,s}^n)) \\ &\leq \sup_{z \in B_{\ell_{r,s}^n}^n} \sum_{\alpha \in J_m} c_{\ell_{r,s}^n}(\alpha) |z^\alpha| \leq \sup_{z \in B_{\ell_{r,s}^n}^n} \sum_{\alpha \in \Lambda_T(\leq m)} c_{\ell_{r,s}^n}(\alpha) |z^\alpha| \\ &\leq \sum_{k=0}^m \sup_{z \in B_{\ell_{r,s}^n}^n} \sum_{\alpha \in \Lambda_T(k)} c_{\ell_{r,s}^n}(\alpha) |z^\alpha| \leq \sum_{k=0}^m e^k \left(\frac{\varphi_{\ell_{r',s'}(n)}}{\varphi_{\ell_{r',s'}(k)}} \right)^k. \end{aligned}$$

However, since $\varphi_{\ell_{r',s'}(n)} \sim n^{1/r'}$, we need to provide a bound for the sum $\sum_{0 \leq k \leq m} \left(\frac{n}{k}\right)^{\frac{k}{r'}}$. Notice that $t \mapsto \left(\frac{n}{t}\right)^t$ is increasing in the interval $(1, \frac{n}{e})$ and has a maximum at $t = \frac{n}{e}$, hence

$$(33) \quad \sum_{0 \leq k \leq m} \left(\frac{n}{k}\right)^{\frac{k}{r'}} \leq (m+1) \left(\max_{0 \leq k \leq m} \left(\frac{n}{k}\right)^k \right)^{\frac{1}{r'}} \leq (m+1) \max \left\{ e^{\frac{m}{r'}}; \left(\frac{n}{m}\right)^{\frac{m}{r'}} \right\},$$

and the claim follows. To prove the upper bound for the unconditional basis constant, note that by Corollary 2.8 (and by our previous results), it follows that

$$\begin{aligned} \chi_{\text{mon}}(\mathcal{P}_{J_m}(\ell_{r,s}^n)) &\leq \chi_{\text{mon}}(\mathcal{P}_{\Lambda_T(\leq m)}(\ell_{r,s}^n)) \\ &\leq C^m \max_{1 \leq k \leq m-1} \lambda(\mathcal{P}_{\Lambda_T(k)}(X_n)) \leq C^m \max_{1 \leq k \leq m-1} \left(\frac{n}{k}\right)^{\frac{k}{r'}}, \end{aligned}$$

hence the claim is again a consequence of the elementary estimate from (33). \square

Proposition 3.1 shows that the upper estimates mentioned above, specifically for $r = 2$ and $s = \infty$, do not provide the correct asymptotic behavior. We now turn our attention to the lower estimates.

Theorem 3.4. *Assume that*

$$\mathbf{A}: 2 < r < \infty, 2 \leq s \leq \infty \quad \text{or} \quad \mathbf{B}: 2 \leq r < \infty, 1 \leq s \leq 2 \quad \text{or} \quad \mathbf{C}: 1 < r < 2, r \leq s.$$

Then, there is a constant $C = C(r, s) > 0$ such that for every sequence (J_m) of index sets, each with degree at most m and such that $\Lambda_T(m) \subset J_m$, and for each n the following estimate holds:

$$\left(\frac{n}{m}\right)^{(m-1)\min\{\frac{1}{2}, \frac{1}{r'}\}} \leq C^m \chi_{\mathbf{mon}}(\mathcal{P}_{J_m}(\ell_{r,s}^n)) \quad \text{and} \quad \left(\frac{n}{m}\right)^{m\min\{\frac{1}{2}, \frac{1}{r'}\}} \leq C^m \lambda(\mathcal{P}_{J_m}(\ell_{r,s}^n)).$$

Moreover, in the case $2 \leq r < \infty$ and $r \leq s$, we (only) know that for all n and m with $\Lambda_T(m) \subset J_m$

$$\frac{1}{(\log n)^{\frac{m}{r} - \frac{m}{s}}} \left(\frac{n}{m}\right)^{\frac{m-1}{r'}} <_{C^m} \chi_{\mathbf{mon}}(\mathcal{P}_{J_m}(\ell_{r,s}^n))$$

and

$$\frac{1}{(\log n)^{\frac{m}{r} - \frac{m}{s}}} \left(\frac{n}{m}\right)^{\frac{m}{r'}} <_{C^m} \lambda(\mathcal{P}_{J_m}(\ell_{r,s}^n)).$$

Proof. In case **A** both claims are consequences of Theorem 2.16 as $\ell_{r,s}$ is 2-convex.

In case **B**, we first address the unconditional basis constant using Proposition 2.26. To proceed, we need to verify that the hypotheses are satisfied. First, recall that

$$(34) \quad \varphi_{\ell_{r,s}^n}(n) \sim n^{\frac{1}{r}} \quad \text{and} \quad \varphi_{(\ell_{r,s}^n)'}(n) \sim \varphi_{\ell_{r',s'}}(n) \sim n^{\frac{1}{r'}},$$

hence the first assumption is satisfied. For the second assumption of this proposition note that, using equation (32), we obtain

$$\begin{aligned} \|\text{id}: \ell_{r,s}^n \rightarrow \ell_2^n\| &\leq \|\text{id}: \ell_{r,s}^n \rightarrow \ell_r^n\| \cdot \|\text{id}: \ell_r^n \rightarrow \ell_2^n\| \\ &\leq \|\text{id}: \ell_r^n \rightarrow \ell_2^n\| = n^{\frac{1}{2} - \frac{1}{r}} = \frac{n^{1 - \frac{1}{r}}}{n^{\frac{1}{2}}} = \frac{\|\text{id}: \ell_{r,s}^n \rightarrow \ell_1^n\|}{\sqrt{n}}. \end{aligned}$$

Consequently, we deduce from Proposition 2.26 that, for $m \leq n$, it holds

$$\left(\frac{n}{m}\right)^{\frac{m-1}{2}} <_{C^m} \chi_{\mathbf{mon}}(\mathcal{P}_{\Lambda_T(m)}(\ell_{r,s}^n)).$$

Since the case $n \leq m$ is anyway obvious, as the index set would be empty, the claim for the lower bound of the unconditional basis constant is proved.

Let us consider case **C**, so $1 \leq s \leq r < 2$: For the unconditional basis constant, this is an immediate consequence of Proposition 2.25 and equation (32):

$$\left(\frac{n}{m}\right)^{\frac{m-1}{r'}} = \frac{1}{\|\text{id}: \ell_{r,s}^n \rightarrow \ell_r^n\|^m} \left(\frac{n}{m}\right)^{\frac{m-1}{r'}} <_{C^m} \chi_{\mathbf{mon}}(\mathcal{P}_{\Lambda_T(m)}(\ell_{r,s}^n)).$$

It remains to show the claims for the projection constant in the cases **B** and **C**: We apply Theorem 2.7 in conjunction with Theorem 2.3 (note that $\Lambda_T(m) = J_m(m) \cap \Lambda_T(\leq m)$) and Proposition 2.2. This shows that

$$(35) \quad \chi_{\mathbf{mon}}(\mathcal{P}_{\Lambda_T(m+1)}(\ell_{r,s}^n)) <_{C^m} \lambda(\mathcal{P}_{\Lambda_T(m)}(\ell_{r,s}^n)) <_{C^m} \lambda(\mathcal{P}_{J_m(m)}(\ell_{r,s}^n)) \leq \lambda(\mathcal{P}_{J_m}(\ell_{r,s}^n)).$$

This provides the lower bound for the projection constant, using the bounds obtained for the unconditional basis constant.

Finally, it remains to prove the last statement of the theorem. The estimate for the unconditional basis constant follows from Proposition 2.25, and for the estimate on the projection constant we again repeat the argument from (35). \square

3.2. Arbitrary index sets of multi indices. Having discussed the tetrahedral case, we now turn to more general index sets. A natural question that arises is whether variants of the estimates established for the spaces ℓ_r^n in [4, 16] remain valid for Lorentz spaces $\ell_{r,s}^n$, regardless of the parameter s . Theorems 3.3 and 3.4 indicate that this is indeed the case when we restrict ourselves to tetrahedral indices, at least for $r > 2$. However, there are subtle differences in more complex scenarios. It turns out that the case of $r > 2$ can be treated with methods similar to those used for tetrahedral index sets. In contrast, the case where $1 < r \leq 2$ is significantly different and technically much more challenging.

3.2.1. The case $2 \leq r < \infty$ and $1 \leq s \leq \infty$. We present a theorem that summarizes our results for the case of general index sets.

Theorem 3.5. *Let $2 \leq r < \infty$ and $1 \leq s \leq \infty$. Then, there exists a constant $C = C(r, s) > 0$ such that for any sequence (J_m) of index sets, each with degree at most m , and for any n , the following estimate holds:*

$$\chi_{\mathbf{mon}}(\mathcal{P}_{J_m}(\ell_{r,s}^n)) \leq C^m \left(1 + \frac{n}{m}\right)^{\frac{m-1}{2}} \quad \text{and} \quad \lambda(\mathcal{P}_{J_m}(\ell_{r,s}^n)) \leq C^m \left(1 + \frac{n}{m}\right)^{\frac{m}{2}}.$$

Moreover, in each of the cases **A**: $2 < r < \infty$, $1 \leq s \leq \infty$ or **B**: $r = 2$, $1 \leq s \leq 2$ the preceding estimates are optimal for all n and all m for which $\Lambda_T(m) \subset J_m$ (in the sense that under this assumption that the inequality is replaced equivalence \sim_{C^m}).

Proof. The upper estimates are special cases from Theorem 2.16. The lower one for the unconditional basis constants follows from Theorem 3.4, since for all m with $\Lambda_T(m) \subset J_m$ clearly

$$\chi_{\mathbf{mon}}(\mathcal{P}_{\Lambda_T(m)}(\ell_{r,s}^n)) \leq \chi_{\mathbf{mon}}(\mathcal{P}_{J(m)}(\ell_{r,s}^n)).$$

It remains to show the lower bound for the projection constant: By what was proved in (21), we then for all m with $\Lambda_T(m) \subset J_m$ have

$$\chi_{\mathbf{mon}}(\mathcal{P}_{\Lambda_T(m+1)}(\ell_{r,s}^n)) <_{C^m} \lambda(\mathcal{P}_{J_m}(\ell_{r,s}^n)),$$

hence the claim follows from the result on the unconditional basis constant. \square

We already indicated that the case $r = 2$ is somewhat special – at least for $s = \infty$ (see again Proposition 3.1). For a better overview, we collect our knowledge for $r = 2$ in the following corollary. The statements are all covered by Theorem 3.5 and Theorem 3.4

Corollary 3.6. *Let $1 \leq s \leq \infty$. Then, there exists a constant $C = C(s) > 0$ such that for any sequence (J_m) of index sets, each with degree at most m , the following estimates hold:*

$$\chi_{\text{mon}}(\mathcal{P}_{J_m}(\ell_{2,s}^n)) \leq C^m \left(1 + \frac{n}{m}\right)^{\frac{m-1}{2}} \quad \text{and} \quad \lambda(\mathcal{P}_{J_m}(\ell_{2,s}^n)) \leq C^m \left(1 + \frac{n}{m}\right)^{\frac{m}{2}},$$

and for $1 \leq s \leq 2$ these estimates are optimal for all n and m with $\Lambda_T(m) \subset J_m$.

For $2 \leq s \leq \infty$ and for all n and m with $\Lambda_T(m) \subset J_m$ we have that

$$\frac{1}{(\log n)^{(m-1)(\frac{1}{2}-\frac{1}{s})}} \left(1 + \frac{n}{m}\right)^{\frac{m-1}{2}} <_{C^m} \chi_{\text{mon}}(\mathcal{P}_{J_m}(\ell_{2,s}^n))$$

and

$$(36) \quad \frac{1}{(\log n)^{m(\frac{1}{2}-\frac{1}{s})}} \left(1 + \frac{n}{m}\right)^{\frac{m}{2}} <_{C^m} \lambda(\mathcal{P}_{J_m}(\ell_{2,s}^n)).$$

3.2.2. The case $1 < r \leq 2$ and $s \leq r$. Notice that estimates of the characteristic $c_{X_n}(\alpha)$ for all α in the given index set J will play an essential role in the estimates of the polynomial projection. The following remark regarding estimates of characteristics will be useful in what follows.

Remark 3.7. Let $X_n = (\mathbb{C}^n, \|\cdot\|_{X_n})$ and $Y_n = (\mathbb{C}^n, \|\cdot\|_{Y_n})$ be two Banach spaces, and $\alpha \in \mathbb{N}_0^{(\mathbb{N})}$. Then the following estimate holds:

$$c_{Y_n}(\alpha) \leq \|\text{id}: X_n \rightarrow Y_n\|^m c_{X_n}(\alpha).$$

Moreover, for every $1 \leq r \leq \infty$, we have

$$c_{X_n}(\alpha) \leq \|m^{-1/r}(\alpha_1^{1/r}, \dots, \alpha_n^{1/r})\|_{X_n}^m \left(\frac{m^m}{\alpha^\alpha}\right)^{1/r}.$$

The first estimate is immediate from the given definitions. The second one follows by normalizing the vector $m^{-1/r} \alpha^{1/r}$ in X_n and using it to estimate $c_{X_n}(\alpha)$.

In the study of the estimates of polynomial projection and unconditional basis constants of $\mathcal{P}_J(\ell_{r,s}^n)$, we consider two subcases, each requiring different techniques: the first subcase is for $1 \leq s \leq r \leq 2$, and the second subcase is for $1 < r \leq 2$ with $r \leq s$.

The subcase $1 \leq s \leq r \leq 2$:

The following result describes the asymptotic behavior of the polynomial projection constant of $\mathcal{P}_J(\ell_{r,s}^n)$ for a broad range of index sets J , subject to mild restrictions on m and n .

Theorem 3.8. *Let $1 \leq s \leq r \leq 2$. Then, there exists a constant $C = C(r, s) > 0$ such that for all $n \geq m$ with*

$$(37) \quad \log m + \frac{r'}{r} (\log m)^{r(\frac{1}{s} - \frac{1}{r})} \leq \log n,$$

and for all sequences (J_m) of index sets, each with degree at most m , the following estimates hold:

$$\chi_{\text{mon}}(\mathcal{P}_{J_m}(\ell_{r,s}^n)) \leq C^m \left(\frac{n}{m}\right)^{\frac{m-1}{r'}} \quad \text{and} \quad \lambda(\mathcal{P}_{J_m}(\ell_{r,s}^n)) \leq C^m \left(\frac{n}{m}\right)^{\frac{m}{r'}}.$$

Moreover, these estimates are optimal for all n and m with $\Lambda_T(m) \subset J_m$ (without restricting to (37)); in particular, whenever $m^{1+\delta} <_C n$ for some $\delta > 0$ in the case $(r, s) \neq (2, 1)$, or $m^2 <_C n$ in the case $(r, s) = (2, 1)$.

For the homogeneous case $J_m = \Lambda(m)$ the following theorem presents an upper bound that applies to the entire range, which will be essential in deriving asymptotically optimal bounds for the Bohr radius in Lorentz spaces.

Theorem 3.9. *Let $1 \leq s \leq r \leq 2$ and $0 < \kappa < 1$. Then, there exists a constant $C = C(r, s) > 0$ such that for all $n \geq m$, we have*

$$\widehat{\lambda}(\mathcal{P}_m(\ell_{r,s}^n)) \leq C^m \left(\frac{n}{m}\right)^{\frac{m}{r'}} \max \left\{ 1, \frac{\log(m)^{m(\frac{1}{s} - \frac{1}{r})}}{n^{\frac{m\kappa}{2r'}}} \right\}.$$

To establish the upper estimates in the preceding theorems, we will isolate the following upper estimates for the polynomial projection constants of $\mathcal{P}_{J_m}(\ell_{r,s}^n)$, in light of Theorem 2.18.

Theorem 3.10. *Let $1 \leq s \leq r \leq 2$. Then there exists a constant $C = C(r, s)$ such that the following results hold true:*

- (a) *For any sequence (J_m) of index sets, each with degree at most m , and for each $n \geq m$ that satisfies the condition in (37), we have*

$$\widehat{\lambda}(\mathcal{P}_{J_m}(\ell_{r,s}^n)) \leq C^m \left(\frac{n}{m}\right)^{\frac{m}{r'}}.$$

- (b) *For all $n \leq m$*

$$\widehat{\lambda}(\mathcal{P}_m(\ell_{r,s}^n)) \leq C^m \left(\frac{n}{m}\right)^{\frac{m}{r'}} \max \left\{ 1, \frac{\log(m)^{m(\frac{1}{s} - \frac{1}{r})}}{n^{\frac{m\kappa}{2r'}}} \right\}.$$

Before we proceed to the proof of Theorem 3.10 (a), we want to ensure that it, along with our findings in Section 2, actually proves Theorem 3.8. Additionally, note that Theorem 3.10 (a), when combined with Theorem 2.18, leads to Theorem 3.9.

Proof of Theorem 3.8. We distinguish the cases $n \leq m$ and $n \geq m$. In the first case, $n \leq m$, we can immediately conclude from Corollary 2.9

$$\chi_{\text{mon}}(\mathcal{P}_{J_m}(\ell_{r,s}^n)) \sim_{C^m} 1 \quad \text{and} \quad \lambda(\mathcal{P}_{J_m}(\ell_{r,s}^n)) \sim_{C^m} 1.$$

For $n \geq m$, the lower bound for the unconditional basis constant follows directly from Proposition 2.25. Meanwhile, the lower estimate for the projection constant relies on Theorem 3.4, Theorem 2.3 (note that we by the assumption on J_m have $\Lambda_T(m) = J_m(m) \cap \Lambda_T(\leq m)$) and finally Proposition 2.2, which all together shows that

$$\left(\frac{n}{m}\right)^{\frac{m}{r'}} <_{C^m} \lambda(\mathcal{P}_{\Lambda_T(m)}(\ell_{r,s}^n)) <_{C^m} \lambda(\mathcal{P}_{J_m(m)}(\ell_{r,s}^n)) <_{C^m} \lambda(\mathcal{P}_{J_m}(\ell_{r,s}^n)).$$

Obviously, the upper bound for the projection constant is a consequence of Theorem 2.18 and Theorem 3.10 (which remains to be proved). For the upper bound of the unconditional basis constant we may apply Corollary 2.8 and the upper bound for the projection constant (already shown) to conclude that

$$\begin{aligned} \chi_{\text{mon}}(\mathcal{P}_{J_m}(\ell_{r,s}^n)) &\leq \chi_{\text{mon}}(\mathcal{P}_{\leq m}(\ell_{r,s}^n)) <_{C^m} \chi_{\text{mon}}(\mathcal{P}_m(\ell_{r,s}^n)) \\ &<_{C^m} \max_{k \leq m-1} \lambda(\mathcal{P}_{m-1}(\ell_{r,s}^n)) <_{C^m} \left(\frac{n}{m}\right)^{\frac{m-1}{r'}}. \end{aligned} \quad \square$$

We begin the proof of Theorem 3.10 and first note that, without loss of generality, we may assume that $J_m = \Lambda(m)$. Indeed, if we establish this homogeneous case, we can then consider an arbitrary J_m of degree at most m , and observe that

$$(38) \quad \widehat{\lambda}(\mathcal{P}_{J_m}(\ell_{r,s}^n)) \leq 1 + \sum_{k=1}^m \widehat{\lambda}(\mathcal{P}_{\Lambda(k)}(\ell_{r,s}^n)).$$

The hypothesis on n and m from (37) implies that $\log k + \frac{r'}{r}(\log k)^{r(\frac{1}{s}-\frac{1}{r})} \leq \log n$ for all $1 \leq k \leq m$, and hence

$$(39) \quad \widehat{\lambda}(\mathcal{P}_{J_m}(\ell_{r,s}^n)) <_{C^m} \sum_{k=1}^m \left(\frac{n}{k}\right)^{\frac{k}{r'}} <_{C^m} \left(\frac{n}{m}\right)^{\frac{m}{r'}},$$

where the last estimate is a straightforward consequence of the following technical remark, which will also be utilized later.

Remark 3.11. Given $a > e$, define $f(t) := \left(\frac{a}{t}\right)^t$ for all $t \geq 1$. Then f is increasing on $[1, a/e]$ and decreasing on $[a/e, \infty)$. Moreover, f reaches its maximum value at $t_0 = a/e$, where $f(t_0) = e^{\frac{a}{e}}$.

After this reduction of the proof of Theorem 3.10 to $J = \bigcup_m \Lambda(m)$, the idea is to leverage the flexibility in the definition of $\widehat{\lambda}(\mathcal{P}_{\Lambda(m)}(\ell_{r,s}^n))$. Specifically, we break down the sum

$$\sum_{\alpha \in J_m} c_{\ell_{r,s}^n}(\alpha) |z^\alpha|$$

into smaller components. This decomposition relates to the number of variables involved in each monomial. The key insight is that if an index α involves a large number of variables, then $c_{\ell_{r,s}^n}(\alpha)$ closely resembles $c_{\ell_r^n}(\alpha)$, placing us in a classical scenario. Conversely, if the number of variables in α is limited, we encounter a troublesome logarithmic term in $c_{\ell_{r,s}^n}(\alpha)$ compared to $c_{\ell_r^n}(\alpha)$. To address this, we analyze the number of monomials associated with a fixed number of variables. The underlying philosophy is that the indices leading to poor estimates are relatively few, allowing for a compensatory effect. However, managing all these elements simultaneously introduces significant technical challenges, requiring a great deal of subtlety. The primary difficulty lies in balancing the number of indices with the estimates obtained and ensuring that all components can be cohesively integrated.

To accomplish this, we need to consider specific sets of multi-indices that define subspaces of polynomials based on the number of variables involved in each monomial. We define, for $1 \leq L \leq m \leq n$, the set of indices

$$\Lambda^L(m, n) := \{\alpha \in \Lambda(m, n) : |\{i : \alpha_i \neq 0\}| = L\}.$$

In other words, $\alpha \in \Lambda^L(m, n)$ whenever the monomial z^α involves exactly L variables. Given a Banach space $X_n = (\mathbb{C}^n, \|\cdot\|)$, we denote

$$\mathcal{P}_{m,L}(X_n) := \mathcal{P}_{\Lambda^L(m,n)}(X_n).$$

With this notation, the set of tetrahedral m -homogeneous polynomials can be denoted by $\mathcal{P}_{m,m}(X_n)$. The following estimate for the cardinality of the sets $\Lambda^L(m, n)$ is crucial.

Lemma 3.12. *For $1 \leq L \leq m \leq n$, we have*

$$|\Lambda^L(m, n)| \sim_{C^m} \left(\frac{n}{L}\right)^L \sim_{C^m} \binom{n}{L}.$$

Proof. Any multi-index $\alpha \in \Lambda^L(m, n)$ can be expressed as the sum of a tetrahedral index $\beta \in \Lambda_T(L, n)$ and another multi-index whose support is contained within the support of β . Furthermore, if α has $L - k$ coordinates equal to 1, then the remaining k non-zero coordinates of α are at least 2, and so $L - k + 2k \leq m$, or, $k \leq m - L$. Thus, the decreasing reordering of α can be expressed as

$$\alpha^* = \beta^* + (\alpha^* - \beta^*) = (1, \dots, 1, 0, \dots) + (\alpha_1^* - 1, \dots, \alpha_k^* - 1, 0, \dots).$$

Therefore, as any α^* can be decomposed as a sum of $\beta^* \in \Lambda_T(L, n)$ and $\alpha^* - \beta^* \in \Lambda(m-L, m-L)$, we have

$$|\Lambda^L(m, n)| \leq |\Lambda_T(L, n)| \cdot |\Lambda(m-L, m-L)| \leq \binom{n}{L} \left(1 + \frac{m-L}{m-L}\right)^{m-L} \prec_{C^m} \left(\frac{n}{L}\right)^L.$$

The lower bound follows from the fact that we can define an injection from

$$\Lambda_T(L, n) \rightarrow \Lambda^L(m, n), \quad \alpha \mapsto (\alpha_1 + m - L, \alpha_2, \dots, \alpha_n).$$

Therefore, $|\Lambda^L(m, n)| \geq |\Lambda_T(L, n)| \sim_{C^m} \left(\frac{n}{L}\right)^L$, which completes the argument. \square

As already mentioned, to prove Theorem 3.10 (a), we may concentrate on the index set $J = \bigcup_m \Lambda(m)$. Now, observe that for $1 \leq L \leq m \leq n$,

$$\Lambda^L(m, n) \subset \bigcup_{k=0}^k \Lambda^L(k, n),$$

implying

$$(40) \quad \widehat{\lambda}(\mathcal{P}_m(\ell_{r,s}^n)) \leq \sum_{L=0}^m \widehat{\lambda}(\mathcal{P}_{m,L}(\ell_{r,s}^n)).$$

So, given $1 \leq L \leq m$, our next goal is to provide upper estimates for

$$\widehat{\lambda}(\mathcal{P}_{m,L}(\ell_{r,s}^n)), \quad 1 \leq L \leq m.$$

From Lemma 3.12 we obtain a first simple bound.

Remark 3.13. For all $1 \leq L \leq m$

$$\widehat{\lambda}(\mathcal{P}_{m,L}(\ell_{r,s}^n)) \prec_{C^m} \left(\frac{n}{L}\right)^{\frac{1}{r'}} (\log m)^{m(\frac{1}{s}-\frac{1}{r})}.$$

Proof. Observe that for a given $\alpha \in \Lambda^L(m, n)$, using Remark 3.7, (31) and Lemma 3.12, we obtain

$$c_{\ell_{r,s}^n}(\alpha) \leq \|m^{-1/r}(\alpha_1^{1/r}, \dots, \alpha_n^{1/r})\|_{r,s}^m \left(\frac{m^m}{\alpha^\alpha}\right)^{1/r} \prec_{C^m} (\log L)^{m(\frac{1}{s}-\frac{1}{r})} \|\alpha\|^{1/r}.$$

Thus, applying Hölder's inequality along with the binomial formula, we have

$$\begin{aligned} \widehat{\lambda}(\mathcal{P}_{m,L}(\ell_{r,s}^n)) &\prec_{C^m} \log(m)^{m(\frac{1}{s}-\frac{1}{r})} \left(\sup_{z \in B_{\ell_{r,s}^n}} \sum_{\alpha \in \Lambda^L(m,n)} \|\alpha\| \|z^{\alpha r}\| \right)^{\frac{1}{r}} |\Lambda^L(m, n)|^{1/r'} \\ &\prec_{C^m} (\log m)^{m(\frac{1}{s}-\frac{1}{r})} \sup_{z \in B_{\ell_{r,s}^n}} \|z\|_{\ell_r}^m |\Lambda^L(m, n)|^{1/r'} \prec_{C^m} \left(\frac{n}{L}\right)^{\frac{1}{r'}} (\log m)^{m(\frac{1}{s}-\frac{1}{r})}. \quad \square \end{aligned}$$

Unfortunately, the bound given in the previous remark is inadequate for proving the two bounds from Theorem 3.10. A more precise estimation is required - in fact, we for all index sets need precisely

the estimate given in Theorem 3.10, (a), that is: Let $1 \leq s \leq r \leq 2$, and $1 \leq m \leq n$ such that $\log m + \frac{r'}{r}(\log m)^{r(\frac{1}{s}-\frac{1}{r})} \leq \log n$. Then, for any $0 \leq L \leq m$

$$(41) \quad \widehat{\lambda}(\mathcal{P}_{m,L}(\ell_{r,s}^n)) <_{C^m} \left(\frac{n}{m}\right)^{\frac{m}{r'}}.$$

To achieve this, we divide the interval of all possible L 's into three subintervals:

$$\left[1, \frac{m}{2}\right], \quad \left[\frac{m}{2}, s_m\right], \quad [s_m, m], \quad \text{where } s_m = \left(1 - \frac{1}{(\log m)^{r(\frac{1}{s}-\frac{1}{r})}}\right)$$

and handle each of these three cases separately.

We now proceed to the proof of (41), which requires some additional preparation. In fact, this lemma directly follows from Lemma 3.15, Lemma 3.17, and Lemma 3.18, which are presented below. We start by examining the case when L is considered 'big'—that is, when the polynomials are 'almost tetrahedral'.

Lemma 3.14. *Let $1 \leq s \leq r$ and $1 \leq L < m$ with $\frac{m}{2} \leq L$. Then, there exists a constant $C = C(r, s) > 0$ such that, for each $\alpha \in \Lambda_L(m, n)$, the following estimate holds:*

$$c_{\ell_{r,s}^n}(\alpha) <_{C^m} \left(\left(\frac{m-L}{m}\right)^{1/r} \log(m-L)^{\frac{1}{s}-\frac{1}{r}} + 1 \right)^m \|\alpha\|^{1/r}.$$

Proof. Before starting, let us recall that $\|M^{-1/r} \beta^{1/r}\|_{\ell_r} = 1$ for any $\beta \in \Lambda(M, N)$, and if β is tetrahedral, then by (34)

$$(42) \quad \|M^{-1/r} \alpha^{1/r}\|_{r,s} = \|M^{-1/r} \underbrace{(1, \dots, 1, 0, \dots)}_M\|_{r,s} \sim_C 1.$$

Given $\alpha \in \Lambda^L(m, n)$, for $1 \leq L < m$ with $\frac{m}{2} \leq L$, note α has at least $2L - m$ coordinates which are equal to 1. Indeed, if k denotes the numbers 1's then α has $L - k$ coordinates which are greater than or equal to 2, so that $m \geq 2(L - k) + k$. Thus,

$$\alpha^* = (\alpha_1^*, \dots, \alpha_{m-L}^*, \underbrace{1, \dots, 1}_{2L-m}, 0, \dots).$$

Since the degree of $(\alpha_1^*, \dots, \alpha_{m-L}^*)$ equals $2(m - L)$, it follows from (42) and (31) that

$$\begin{aligned} \|m^{-1/r}(\alpha_1^{1/r}, \dots, \alpha_n^{1/r})\|_{r,s} &\leq \|m^{-1/r}((\alpha_1^*)^{1/r}, \dots, (\alpha_{m-L}^*)^{1/r})\|_{r,s} + \|m^{-1/r} \underbrace{(1, \dots, 1, 0, \dots)}_{2L-m}\|_{r,s} \\ &\leq \|(2(m-L))^{-1/r}((\alpha_1^*)^{1/r}, \dots, (\alpha_{m-L}^*)^{1/r})\|_r \left(\frac{2(m-L)}{m}\right)^{1/r} \log(m-L)^{\frac{1}{s}-\frac{1}{r}} \\ &\quad + \|(2L-m)^{-1/r} \underbrace{(1, \dots, 1, 0, \dots)}_{2L-m}\|_{r,s} \\ &<_C \left(\frac{2(m-L)}{m}\right)^{1/r} \log(m-L)^{\frac{1}{s}-\frac{1}{r}} + 1. \end{aligned}$$

Using that $z = \frac{m^{-1/r} \alpha^{1/r}}{\|m^{-1/r} \alpha^{1/r}\|_{\ell_{r,s}^n}} \in B_{\ell_{r,s}^n}$, we obtain

$$c_{\ell_{r,s}^n}(\alpha) \leq \|m^{-1/r}(\alpha_1^{1/r}, \dots, \alpha_n^{1/r})\|_{r,s} \left(\frac{m}{\alpha}\right)^{1/r} <_{C^m} \left(\frac{2(m-L)}{m}\right)^{1/r} \log(m-L)^{\frac{1}{s}-\frac{1}{r}} + 1)^m e^{m/r} |[\alpha]|^{1/r}. \quad \square$$

Lemma 3.15. *Let $1 \leq s \leq r \leq 2$, and assume $2 \leq m \leq n$ and $L \geq m \left(1 - \frac{1}{(\log m)^{r(\frac{1}{s}-\frac{1}{r})}}\right)$. Then, there exists a constant $C = C(r, s) > 0$ such that, the following estimate holds:*

$$\widehat{\lambda}(\mathcal{P}_{m,L}(\ell_{r,s}^n)) <_{C^m} \left(\frac{n}{L}\right)^{\frac{L}{r'}} <_{C^m} \left(\frac{n}{m}\right)^{\frac{m}{r'}}.$$

Proof. The case $m = L$ is the tetrahedral case which is contained in (the proof of) Theorem 3.3. We may thus assume $1 \leq L < m$. Since $m - L \leq \frac{m}{(\log m)^{r(\frac{1}{s}-\frac{1}{r})}}$, Lemma 3.14 yields

$$c_{\ell_{r,s}^n}(\alpha) <_{C^m} |[\alpha]|^{1/r}, \quad \alpha \in \Lambda^L(m, n).$$

Therefore with an application of Hölder's inequality and Lemma 3.12 we obtain,

$$\begin{aligned} \sup_{z \in B_{\ell_{r,s}^n}} \sum_{\alpha \in \Lambda^L(m, n)} c_{\ell_{r,s}^n}(\alpha) |z^\alpha| &<_{C^m} \sup_{z \in B_{\ell_{r,s}^n}} \sum_{\alpha \in \Lambda^L(m, n)} |z^\alpha| |[\alpha]|^{1/r} \\ &\leq \sup_{z \in B_{\ell_{r,s}^n}} \left(\sum_{\alpha \in \Lambda^L(m, n)} |z^\alpha|^r |[\alpha]| \right)^{1/r} |\Lambda^L(m, n)|^{1/r'} \\ &\leq \sup_{z \in B_{\ell_{r,s}^n}} \|z\|_{\ell_r}^m |\Lambda^L(m, n)|^{1/r'} <_{C^m} \left(\frac{n}{L}\right)^{\frac{L}{r'}} <_{C^m} \left(\frac{n}{m}\right)^{\frac{m}{r'}}. \quad \square \end{aligned}$$

Lemma 3.15 provides a bound for $\widehat{\lambda}(\mathcal{P}_{m,L}(\ell_{r,s}^n))$ when L is 'big', meaning that L is comparable to m . The following two lemmas offer bounds for the 'middle' values of L .

Lemma 3.16. *Let $1 \leq s \leq r \leq 2$ and $1 < \frac{m}{2} < L \leq m \left(1 - \frac{1}{(\log m)^{r(\frac{1}{s}-\frac{1}{r})}}\right)$. Then, there exists a constant $C = C(r, s) > 0$ such that, the following estimate holds:*

$$\widehat{\lambda}(\mathcal{P}_{m,L}(\ell_{r,s}^n)) <_{C^m} \left(\frac{n}{m}\right)^{m/r'} \log(m-L)^{m(\frac{1}{s}-\frac{1}{r})} \left(\frac{m}{n}\right)^{\frac{m-L}{r'}} \left(\frac{m-L}{m}\right)^{\frac{m}{r}}.$$

Proof. Note first that our hypothesis on L implies that $\left(\frac{m-L}{m}\right)^{1/r} \log(m-L)^{\frac{1}{s}-\frac{1}{r}} >_C 1$. By Lemma 3.14, for any $\alpha \in \Lambda^L(m, n)$

$$c_{\ell_{r,s}^n}(\alpha) <_{C^m} \left(\frac{m-L}{m}\right)^{m/r} \log(m-L)^{m(\frac{1}{s}-\frac{1}{r})} |[\alpha]|^{1/r}.$$

Combining Hölder's inequality with the binomial formula and Lemma 3.12, we obtain

$$\begin{aligned}
\widehat{\lambda}(\mathcal{P}_{m,L}(\ell_{r,s}^n)) &<_{C^m} \left(\frac{m-L}{m}\right)^{m/r} \log(m-L)^{m(\frac{1}{s}-\frac{1}{r})} \sup_{z \in B_{\ell_{r,s}^n}} \sum_{\alpha \in \Lambda^L(m,n)} |[\alpha]|^{1/r} |z^\alpha| \\
&<_{C^m} \left(\left(\frac{m-L}{m}\right)^{m/r} \log(m-L)^{m(\frac{1}{s}-\frac{1}{r})} |\Lambda^L(m,n)|^{1/r'}\right) \sup_{z \in B_{\ell_{r,s}^n}} \|z\|_{\ell_r}^m \\
&<_{C^m} \left(\frac{m-L}{m}\right)^{m/r} \log(m-L)^{m(\frac{1}{s}-\frac{1}{r})} \left(\frac{n}{L}\right)^{L/r'} \sup_{z \in B_{\ell_{r,s}^n}} \|z\|_{\ell_r}^m \\
&<_{C^m} \left(\frac{m-L}{m}\right)^{m/r} \log(m-L)^{m(\frac{1}{s}-\frac{1}{r})} \left(\frac{n}{m}\right)^{L/r'} \\
&= \left(\frac{n}{m}\right)^{m/r'} \log(m-L)^{m(\frac{1}{s}-\frac{1}{r})} \left(\frac{m}{n}\right)^{\frac{m-L}{r'}} \left(\frac{m-L}{m}\right)^{\frac{m}{r}}. \quad \square
\end{aligned}$$

The next lemma consists of two parts. The first part will be instrumental in proving (41) (which, as mentioned above, is necessary to complete the proof of Theorem 3.8), while the second part will be used to prove Theorem 3.9.

Lemma 3.17. *Let $1 \leq s \leq r \leq 2$ and $1 < \frac{m}{2} < L < m \left(1 - \frac{1}{(\log m)^{r(\frac{1}{s}-\frac{1}{r})}}\right)$. Then there exists a constant $C = C(r, s) > 0$ such that the following statements are true:*

(i) For $\log n \geq \log m + \frac{r'}{r}(\log m)^{r(\frac{1}{s}-\frac{1}{r})}$,

$$\widehat{\lambda}(\mathcal{P}_{m,L}(\ell_{r,s}^n)) <_{C^m} \left(\frac{n}{m}\right)^{\frac{m}{r'}} \left(1 - \frac{1}{(\log m)^{r(\frac{1}{s}-\frac{1}{r})}}\right).$$

In particular, this holds for $n \geq m^{1+\delta}$, provided $\delta \geq \frac{r'}{r(\log m)^{2-\frac{r}{s}}}$.

(ii) For $\log n \leq \log m + \frac{r'}{r}(\log m)^{r(\frac{1}{s}-\frac{1}{r})}$ and $m \leq n$, $0 < \kappa < 1$,

$$\widehat{\lambda}(\mathcal{P}_{m,L}(\ell_{r,s}^n)) <_{C^m} \left(\frac{n}{m}\right)^{\frac{m}{r'}} \max \left\{ \frac{(\log m)^{m(\frac{1}{s}-\frac{1}{r})}}{n^{\frac{m^\kappa}{2r'}}}; 1 \right\}.$$

Proof. By Lemma 3.16,

$$\widehat{\lambda}(\mathcal{P}_{m,L}(\ell_{r,s}^n)) <_{C^m} \left(\frac{n}{m}\right)^{m/r'} \log(m-L)^{m(\frac{1}{s}-\frac{1}{r})} \left(\frac{m}{n}\right)^{\frac{m-L}{r'}} \left(\frac{m-L}{m}\right)^{\frac{m}{r}}.$$

Denote now $m-L = tm$, with $(\log m)^{-r(\frac{1}{s}-\frac{1}{r})} \leq t \leq \frac{1}{2}$. Then, taking the m -th root and rearranging,

$$(43) \quad \widehat{\lambda}(\mathcal{P}_{m,L}(\ell_{r,s}^n))^{1/m} \left(\frac{m}{n}\right)^{1/r'} (\log m)^{\frac{1}{r}-\frac{1}{s}} <_C t^{\frac{1}{r}} \left(\frac{m}{n}\right)^{t/r'}.$$

Let $f(t) := \frac{1}{r} \log(t) - \frac{t}{r'} \log \frac{n}{m}$ for all $t > 0$. Then, f has a global maximum at $t_0 = \left(\frac{r}{r'} \log \frac{n}{m}\right)^{-1}$. Now, since we are looking for the maximum of f for $(\log m)^{-r(\frac{1}{s}-\frac{1}{r})} \leq t \leq \frac{1}{2}$, we have three possibilities:

- (a) $t_0 > \frac{1}{2}$, in which case we consider $t^* := \frac{1}{2}$. This is the case when $n < e^{\frac{2r'}{r}} m$.
- (b) $\log(m)^{-r(\frac{1}{s}-\frac{1}{r})} \leq t_0 \leq \frac{1}{2}$, in which case we consider $t^* := t_0$. This is the case when $me^{\frac{2r'}{r}} \leq n$ and $\log n \leq \log m + \frac{r'}{r}(\log m)^{r(\frac{1}{s}-\frac{1}{r})}$.
- (c) $t_0 < \log(m)^{-r(\frac{1}{s}-\frac{1}{r})}$, in which case we consider $t^* := (\log m)^{-r(\frac{1}{s}-\frac{1}{r})}$. This is the case when $\log n \leq \log m + \frac{r'}{r}(\log m)^{r(\frac{1}{s}-\frac{1}{r})}$.

Replacing t by t^* in (43), (c) gives us the bound stated in (i). In the other two cases, replacing t by t^* in (43), gives us:

- (a) for $e^{-\frac{2r}{r'}} n \leq m \leq n$,

$$\widehat{\lambda}(\mathcal{P}_{m,L}(\ell_{r,s}^n)) <_{C^m} (\log m)^{m(\frac{1}{s}-\frac{1}{r})}.$$

- (b) for $m \leq e^{-\frac{2r}{r'}} n$ and $\log n \leq \log m + \frac{r'}{r}(\log m)^{r(\frac{1}{s}-\frac{1}{r})}$,

$$\widehat{\lambda}(\mathcal{P}_{m,L}(\ell_{r,s}^n)) <_{C^m} \left(\frac{n}{m}\right)^{\frac{m}{r'}} \frac{(\log m)^{m(\frac{1}{s}-\frac{1}{r})}}{(\log \frac{n}{m})^{\frac{m}{r}}}.$$

Let $0 < \kappa < 1$. Then, for both cases (a) and (b) above, we have $n <_{C^m} m^2$, which implies $n^{\frac{m\kappa}{2r'}} <_{C^m} 1$. Therefore, we deduce the bound stated in (ii) for both cases. \square

It remains to prove the case $L < \frac{m}{2}$. The key point in this scenario is that, although we do not have a strong bound for $c_\alpha(\ell_{r,s}^n)$, the cardinality of $\Lambda^L(m, n)$ is sufficiently small to compensate.

Lemma 3.18. *Let $1 \leq s \leq r \leq 2$ and $L \leq \frac{m}{2}$. Then there exists a constant $C = C(r, s) > 0$ such that for each $n \geq m \log(m)^{2r'(\frac{1}{s}-\frac{1}{r})}$, we have*

$$\widehat{\lambda}(\mathcal{P}_{m,L}(\ell_{r,s}^n)) <_{C^m} \left(\frac{n}{m}\right)^{\frac{m}{r'}}.$$

Proof. Since $s \leq r$, it follows from the Remark 3.13 that

$$\widehat{\lambda}(\mathcal{P}_{m,L}(\ell_{r,s}^n)) <_{C^m} \log(L)^{m(\frac{1}{s}-\frac{1}{r})} \left(\frac{n}{L}\right)^{L/r'} = \left(\frac{n}{m}\right)^{m/r'} (\log L)^{m(\frac{1}{s}-\frac{1}{r})} \left(\frac{m^m}{n^{m-L}L}\right)^{\frac{1}{r'}}.$$

For $n \geq m(\log m)^{2r'(\frac{1}{s}-\frac{1}{r})}$ and $L \leq \frac{m}{2}$ (and hence $2(m-L) \geq m$), so

$$\begin{aligned} \widehat{\lambda}(\mathcal{P}_{m,L}(\ell_{r,s}^n)) &<_{C^m} \left(\frac{n}{m}\right)^{m/r'} (\log L)^{m(\frac{1}{s}-\frac{1}{r})} \left(\frac{m^m}{n^{m-L}L}\right)^{\frac{1}{r'}} \\ &\leq \left(\frac{n}{m}\right)^{m/r'} \log(m)^{m(\frac{1}{s}-\frac{1}{r})} \left(\frac{m^m}{m^{m-L}L}\right)^{\frac{1}{r'}} (\log m)^{-2(m-L)(\frac{1}{s}-\frac{1}{r})} \\ &\leq \left(\frac{n}{m}\right)^{m/r'} \left(\frac{m}{L}\right)^{\frac{1}{r'}} <_{C^m} \left(\frac{n}{m}\right)^{m/r'}. \end{aligned}$$

\square

We are now in position to prove (41) (still needed to complete the proof of Theorem 3.8).

Proof of Equation (41). For $m = 1$ (and thus $L = 1$) we have $c_{\ell_{r,s}^n}(\alpha) = 1$, then

$$\widehat{\lambda}(\mathcal{P}_{m,L}(\ell_{r,s}^n)) \leq \sup_{z \in B_{\ell_{r,s}^n}^n} \sum_{j=1}^n |z_j| = \varphi_{\ell_{r,s}}(n) \prec_C n^{\frac{1}{r'}} \prec_{C^m} \left(\frac{n}{m}\right)^{\frac{m}{r'}}.$$

In the case $2 \leq m$ and $1 \leq L \leq m$, all desired estimates follow from Lemma 3.15, Lemma 3.17(i), and Lemma 3.18. \square

As explained above, this completes the proof of Theorem 3.8. We conclude this subsection by presenting the

Proof of Theorem 3.10. , second estimate] Returning to (40), we observe that it suffices to show that $\widehat{\lambda}(\mathcal{P}_{m,L}(\ell_{r,s}^n))$ satisfies the bound in Theorem 3.10 (b) for all $L \leq m$.

Lemma 3.15 implies the bound for $L \geq m(1 - 1/(\log m)^{r(\frac{1}{s} - \frac{1}{r})})$. For $\frac{m}{2} \leq L \leq m(1 - 1/(\log m)^{r(\frac{1}{s} - \frac{1}{r})})$ we use Lemma 3.17(ii). Finally, for $L \leq \frac{m}{2}$, note that since $L \leq \frac{m}{2} \leq m - \frac{m^\kappa}{2} \leq n$, it holds

$$\binom{n}{L} \leq \binom{2n}{L} \leq \binom{2n}{[m - \frac{m^\kappa}{2}]}$$

Thus, by Remark 3.2.2, for every $z \in \mathbb{C}^n$, we have

$$\begin{aligned} \sum_{\alpha \in \Lambda^L(m,n)} c_{\ell_{r,s}^n}(\alpha) |z^\alpha| &\prec_{C^m} (\log m)^{m(\frac{1}{s} - \frac{1}{r})} \binom{2n}{[m - m^\kappa/2]}^{1/r'} \|z\|_{\ell_r}^m \\ &\prec_{C^m} (\log m)^{m(\frac{1}{s} - \frac{1}{r})} \left(\frac{n}{m - m^\kappa/2} \right)^{m - m^\kappa/2} \|z\|_{\ell_r}^m \\ &= \|z\|_{\ell_r}^m \left(\frac{n}{m} \right)^{m/r'} (\log m)^{m(\frac{1}{s} - \frac{1}{r})} \left(\frac{m - m^\kappa/2}{n} \right)^{\frac{m^\kappa}{2r'}} \left(\frac{m}{m - m^\kappa/2} \right)^{\frac{m}{r'}} \\ &\prec_{C^m} \|z\|_{\ell_{r,s}}^m \left(\frac{n}{m} \right)^{m/r'} \left(\frac{(\log m)^{m(\frac{1}{s} - \frac{1}{r})}}{n^{\frac{m^\kappa}{2r'}}} \right). \end{aligned} \quad \square$$

To achieve our goal in this subsection, only one subcase remains.

The subcase $1 < r \leq 2$ and $r \leq s$:

We follow a different method. Instead of partitioning the Banach spaces $\mathcal{P}_J(\ell_{r,s}^n)$ into smaller pieces as previously, we use a specific decomposition of multi-indices. This approach allows us to control the expression $\widehat{\lambda}(\mathcal{P}_J(\ell_{r,s}^n))$ expressing it as a product of terms involving monomials of lower degrees, for which we can establish suitable bounds.

Theorem 3.19. *Let $1 < r \leq 2$ and $r \leq s$. Then, there exists a constant $C = C(r, s) > 0$ such that for any sequence (J_m) of index sets, each with degree at most m , and any n the following estimate holds:*

$$\widehat{\lambda}(\mathcal{P}_{J_m}(\ell_{r,s}^n)) \leq C^m \left(\frac{n}{m}\right)^{\frac{m}{r}},$$

provided that m and n satisfy the condition

$$\frac{n}{e(\log n)^{r'(\frac{1}{r}-\frac{1}{s})}} \geq m.$$

In particular, we have:

$$\chi_{\text{mon}}(\mathcal{P}_{J_m}(\ell_{r,s}^n)) < C^m \left(\frac{n}{m}\right)^{\frac{m-1}{r'}} \quad \text{and} \quad \lambda(\mathcal{P}_{J_m}(\ell_{r,s}^n)) < C^m \left(\frac{n}{m}\right)^{\frac{m}{r'}}.$$

We start by noting that, regarding the first estimate on the polynomial projection constant, we may assume without loss of generality that $J_m = \Lambda(m)$. This follows as in the first subcase of Section 3.2.2; indeed, the hypothesis on m and n implies that $\frac{n}{e(\log n)^{r'(\frac{1}{r}-\frac{1}{s})}} \geq k$ for all $1 \leq k \leq m$. Hence the argument again follows like in (38) and (39) using Remark 3.11.

Since for all $s \geq r$ and $\alpha \in J$, we have $c_{\ell_{r,s}^n}(\alpha) \leq c_{\ell_r^n}(\alpha) < C^m \|\alpha\|^{1/r}$. Thus to prove Theorem 3.19 we need to show proper bounds for the sum

$$\sum_{\alpha \in \Lambda(m,n)} |z|^\alpha \|\alpha\|^{1/r}.$$

To do this, we will analyse smaller components of this sum: the *tetrahedral* and the *even* part, and use the bounds obtained for each of these parts to conclude something about sums which involve general monomials. As mentioned, this technique was developed by Galicer, Mansilla, Muro, and Sevilla-Peris [33] (see also [47]) for studying 'sets of monomial convergence'. Additionally, this decomposition method was employed in [18] to investigate the projection constant of Boolean cube function spaces.

Recall that for each m, n the set of even multi indices is given by

$$\Lambda_E(m, n) = \{\alpha \in \Lambda(m, n) : \alpha_i \text{ is even for } 1 \leq i \leq n\}.$$

Clearly, m is even whenever $\alpha \in \Lambda_E(m, n)$. Then for every $\alpha \in \Lambda_E(m, n)$ there is a unique $\beta \in \Lambda(m/2, n)$ such that $\alpha = 2\beta$. Given $\alpha \in \Lambda(M, N)$, the tetrahedral part and the even part are defined as,

$$(\alpha_T)_i = \begin{cases} 1 & \text{if } \alpha_i \text{ is odd} \\ 0 & \text{if } \alpha_i \text{ is even} \end{cases} \quad \text{and} \quad \alpha_E = \alpha - \alpha_T.$$

If $0 \leq k \leq M$ denotes the number of odd entries in α , then $\alpha_T \in \Lambda_T(k, N)$ and $\alpha_E \in \Lambda_E(M - k, N)$. Since $(\alpha_E)_i \leq \alpha_i$ for every i , it follows that $\alpha_E! \leq \alpha!$. On the other hand, because $\alpha_T! = 1$, we have

$\alpha_T! \alpha_E! \leq \alpha!$. Therefore,

$$(44) \quad |[\alpha]| = \frac{M!}{\alpha!} \leq \frac{M!}{\alpha_T! \alpha_E!} = \frac{M!}{(M-k)! k!} \frac{k!}{\alpha_T!} \frac{(M-k)!}{\alpha_E!} = \binom{M}{k} |[\alpha_T]| |[\alpha_E]| \leq 2^M |[\alpha_T]| |[\alpha_E]|.$$

We are now ready to prove a key lemma which play a key role in the proof of Theorem 3.19.

Lemma 3.20. *Let $1 < r \leq 2$ and $r \leq s$. Then, there exists a constant $C = C(r, s) > 0$ such that for all m, n and all $z \in \mathbb{C}^n$, the following estimates hold:*

$$(45) \quad \sum_{\alpha \in \Lambda_T(m, n)} |z^\alpha| |[\alpha]|^{\frac{1}{r}} \leq C^m \|z\|_{\ell_{r,s}}^m \left(\frac{n^m}{m!} \right)^{\frac{1}{r}},$$

$$(46) \quad \sum_{\alpha \in \Lambda_E(m, n)} |z^\alpha| |[\alpha]|^{\frac{1}{r}} \leq 2^{\frac{m}{2r}} \|z\|_{\ell_r}^m \leq C^m \|z\|_{\ell_{r,s}}^m (\log n)^{(\frac{1}{r} - \frac{1}{s})m},$$

$$(47) \quad \sum_{\alpha \in \Lambda_E(m, n)} |z^\alpha| |[\alpha]|^{\frac{1}{r}} \leq C^m \begin{cases} \left(\frac{n}{m} \right)^{m/r'} & \text{for } m \leq \frac{n}{e(\log n)^{r'(\frac{1}{r} - \frac{1}{s})}} \\ (\log n)^{m(\frac{1}{r} - \frac{1}{s})} & \text{else.} \end{cases}$$

Proof. We start with the first estimate, which is straightforward when $n = 1$. Thus, we can assume $n \geq 2$. Given $\alpha \in \Lambda_T(m, n)$, note that $\alpha! = 1$ and $|\alpha| = m!$. Then

$$\sum_{\alpha \in \Lambda_T(m, n)} |z^\alpha| |[\alpha]|^{\frac{1}{r}} = \sum_{\alpha \in \Lambda_T(m, n)} |z^\alpha| |[\alpha]| \frac{1}{|[\alpha]|^{\frac{1}{r}}} = \left(\sum_{k=1}^n |z_k| \right)^m \frac{1}{m!^{\frac{1}{r}}} < C^m n^{m/r'} \|z\|_{\ell_{r,s}}^m \frac{1}{m!^{\frac{1}{r}}},$$

where in the last estimate (34) is used. For the proof of the second inequality let us recall first that for each $\alpha \in \Lambda_E(m, n)$ there is a unique $\beta \in \Lambda(m/2, n)$ such that $\alpha = 2\beta$. Applying an obvious inequality $2^k \leq \frac{(2k)!}{k!^2} \leq 2^{2k}$ true for each positive integer k , we obtain

$$|[\alpha]| = \frac{m!}{\alpha_1! \cdots \alpha_n!} = \left(\frac{(m/2)!}{\beta_1! \cdots \beta_n!} \right)^2 \frac{m!}{(m/2)!(m/2)!} \prod_{i=1}^n \frac{\beta_i! \beta_i!}{(2\beta_i)!} \leq 2^{m/2} |[\beta]|^2.$$

This implies that

$$\frac{m!}{(m/2)!(m/2)!} \prod_{i=1}^n \frac{\beta_i! \beta_i!}{(2\beta_i)!} \leq 2^m \prod_{i=1}^n \frac{1}{2^{\beta_i}} = 2^{m/2}.$$

Since $2/r \geq 1$, the ℓ_1 -norm bounds the $\ell_{2/r}$ -norm, so we get

$$\begin{aligned} \sum_{\alpha \in \Lambda_E(m, n)} |z^\alpha| |[\alpha]|^{\frac{1}{r}} &\leq 2^{\frac{m}{2r}} \sum_{\beta \in \Lambda(m/2, n)} |(z^2)^\beta| |[\beta]|^{2/r} = 2^{\frac{m}{2r}} \sum_{\beta \in \Lambda(m/2, n)} \left(|(z^r)^\beta| |[\beta]| \right)^{2/r} \\ &\leq 2^{\frac{m}{2r}} \left(\sum_{\beta \in \Lambda(m/2, n)} |(z^r)^\beta| |[\beta]| \right)^{2/r} = 2^{\frac{m}{2r}} \left(\sum_{l=1}^n |z_l|^r \right)^{m/r} \\ &< C^m 2^{\frac{m}{2r}} \|z\|_{\ell_{r,s}}^m (\log n)^{(\frac{1}{r} - \frac{1}{s})m}, \end{aligned}$$

where we for the last inequality used (31). This concludes the proof of the first two inequalities.

To prove the third estimate we apply (44) and estimates (45) and (46) to get

$$\begin{aligned}
\sum_{\alpha \in \Lambda(m,n)} |z^\alpha| |[\alpha]|^{\frac{1}{r}} &= \sum_{k=0}^m \sum_{\alpha_T \in \Lambda_T(k,n)} \sum_{\alpha_E \in \Lambda_E(m-k,n)} |z^{(\alpha_T + \alpha_E)}| |[\alpha_T + \alpha_E]|^{\frac{1}{r}} \\
&\leq 2^{\frac{m}{r}} \sum_{k=0}^m \left(\sum_{\alpha_T \in \Lambda_T(k,n)} |z_T^\alpha| |[\alpha_T]|^{\frac{1}{r}} \right) \left(\sum_{\alpha_E \in \Lambda_E(m-k,n)} |z_E^\alpha| |[\alpha_E]|^{\frac{1}{r}} \right) \\
&\leq 2^{\frac{3m}{2r}} \sum_{k=0}^m \left(n^{k/r'} \|z\|_{\ell_{r,s}}^k \frac{1}{k!^{\frac{1}{r'}}} \right) \left((\log n)^{(m-k)(\frac{1}{r}-\frac{1}{s})} \|z\|_{\ell_{r,s}}^{m-k} \right) \\
&\leq 2^{\frac{3m}{2r}} \|z\|_{\ell_{r,s}}^m m \max_{k=1,\dots,m} n^{k/r'} \frac{1}{k^{\frac{k}{r'}}} (\log n)^{(m-k)(\frac{1}{r}-\frac{1}{s})}.
\end{aligned}$$

Using Remark 3.11 with $a = \frac{n}{(\log n)^{r'(\frac{1}{r}-\frac{1}{s})}}$, we have that for $m \leq a/e$, the maximum value in the previous estimate is attained when $k = m$. Therefore,

$$\sum_{\alpha \in \Lambda(m,n)} |z^\alpha| |[\alpha]|^{\frac{1}{r}} < C^m \|z\|_{\ell_{r,s}}^m n^{m/r'} \frac{1}{m^{\frac{m}{r'}}}.$$

For $m > a/e$ we have that $\max_{k=1,\dots,m} \left(\frac{a}{k}\right)^k \leq \left(\frac{a}{a/e}\right)^{a/e} \leq e^{a/e} \leq e^m$, and consequently

$$\begin{aligned}
\sum_{\alpha \in \Lambda(m,n)} |z^\alpha| |[\alpha]|^{\frac{1}{r}} &< C^m 2^{\frac{m}{r}} \|z\|_{\ell_{r,s}}^m m \max_{k=1,\dots,m} n^{k/r'} \frac{1}{k^{\frac{k}{r'}}} (\log n)^{(m-k)(\frac{1}{r}-\frac{1}{s})} \\
&< C^m \|z\|_{\ell_{r,s}}^m (\log n)^{m(\frac{1}{r}-\frac{1}{s})} \left(\max_{k=1,\dots,m} \left(\frac{n}{k(\log n)^{r'(\frac{1}{r}-\frac{1}{s})}} \right)^k \right)^{1/r'} < C^m \|z\|_{\ell_{r,s}}^m (\log n)^{m(\frac{1}{r}-\frac{1}{s})}.
\end{aligned}$$

This completes the proof. \square

Proof of Theorem 3.19. We start proving the first estimate of the theorem. Since $s \geq r$, it follows from the second part Remark 3.7 and (31) that

$$c_{\ell_{r,s}^n}(\alpha) \leq \|m^{-1/r} \alpha^{1/r}\|_{\ell_{r,s}^n} \left(\frac{m^m}{\alpha^\alpha} \right)^{1/r} \leq C \|m^{-1/r} \alpha^{1/r}\|_{\ell_r^n} \left(\frac{m^m}{\alpha^\alpha} \right)^{1/r} = C \left(\frac{m^m}{\alpha^\alpha} \right)^{1/r} \leq C e^{m/r} |[\alpha]|^{1/r}.$$

Additionally, for each $\alpha \in \Lambda(m,n)$ there is a pair (\mathbf{i}, j_m) , with $\mathbf{i} \in \mathcal{J}(m-1, j_m)$ and $j_m \in \{1, \dots, n\}$, so that $z^\alpha = z_{j_m} z_{\mathbf{i}}$. Therefore,

$$\begin{aligned}
\sum_{\alpha \in \Lambda(m,n)} c_{\ell_{r,s}^n}(\alpha) |z^\alpha| &< C^m \sum_{\alpha \in \Lambda(m,n)} |z^\alpha| |[\alpha]|^{1/r} = C^m \sum_{j \in \mathcal{J}(m,n)} |z_j| |[\mathbf{j}]|^{1/r} \\
&= C^m \sum_{j_m=1}^n |z_{j_m}| \sum_{\mathbf{i} \in \mathcal{J}(m-1, j_m)} |z_{\mathbf{i}}| |[(\mathbf{i}, j_m)]|^{\frac{1}{r}} \\
&< C^m \sum_{j_m=1}^n |z_{j_m}| \sum_{\mathbf{i} \in \mathcal{J}(m-1, j_m)} |z_{\mathbf{i}}| |[\mathbf{i}]|^{\frac{1}{r}},
\end{aligned}$$

where the last inequality is due to the fact that $|\mathbf{i}| \leq m|\mathbf{j}|$ for every $\mathbf{i} \in \mathcal{J}(m-1, j_m)$. Recall that $\varphi_{\ell_{r,s}}(k) \equiv k^{1/r'}$ and so

$$\sum_{j_m=1}^n |z_{j_m}| < \|z\|_{\ell_{r,s}^n} n^{\frac{1}{r'}},$$

Combining with Lemma 3.20, in the range $\frac{n}{e(\log n)^{r'(\frac{1}{r}-\frac{1}{s})}} \geq m$, we obtain

$$\begin{aligned} \sum_{j_m=1}^n |z_{j_m}| \sum_{\mathbf{i} \in \mathcal{J}(m-1, j_m)} |z_{\mathbf{i}}| |\mathbf{i}|^{\frac{1}{r}} &< C^m \|z\|_{\ell_{r,s}}^{m-1} \sum_{j_m=1}^n |z_{j_m}| \left(\frac{n}{m}\right)^{\frac{m-1}{r'}} < m \|z\|_{\ell_{r,s}}^{m-1} \left(\frac{n}{m}\right)^{\frac{m}{r'}} \\ &< C^m m^{\frac{1}{r'}} \|z\|_{\ell_{r,s}}^m \left(\frac{n}{m}\right)^{\frac{m}{r'}} \left(\frac{n}{m}\right)^{\frac{m-1}{r'}} < C^m \|z\|_{\ell_{r,s}}^m \left(\frac{n}{m}\right)^{\frac{m}{r'}}. \end{aligned}$$

The final assertions follows from Theorem 2.18 regarding the projection constant, and Corollary 2.8 concerning the unconditional basis constant. \square

In similar manner, we obtain the following result.

Theorem 3.21. *Let $1 < r \leq 2$ and $r \leq s$. Then, there exists a constant $C = C(r, s) > 0$ such that for any sequence (J_m) of index sets, each with degree at most m , and for any n , the following estimate holds:*

$$\widehat{\lambda}(\mathcal{P}_{J_m}(\ell_{r,s}^n)) \leq C^m (\log n)^{m(\frac{1}{r}-\frac{1}{s})},$$

provided that m and n satisfy the condition

$$\frac{n}{e(\log n)^{r'(\frac{1}{r}-\frac{1}{s})}} \leq m.$$

In particular, we have:

$$\chi_{\text{mon}}(\mathcal{P}_{J_m}(\ell_{r,s}^n)) < C^m (\log n)^{m(\frac{1}{r}-\frac{1}{s})} \quad \text{and} \quad \lambda(\mathcal{P}_{J_m}(\ell_{r,s}^n)) < C^m (\log n)^{m(\frac{1}{r}-\frac{1}{s})}.$$

3.3. Bohr radii. Finally, we apply our previous results to derive asymptotic estimates for the Bohr radii of the space of multivariate analytic polynomials on finite-dimensional Lorentz spaces $\ell_{r,s}^n$. The following theorem provides a strong improvement over [27, Corollary 10].

Theorem 3.22. *Let $1 < r < \infty$, $1 \leq s \leq \infty$, and let J be an index set for which $\Lambda_T(m) \subset J$ for all m . Then the following statements are true:*

(i) *If $2 < r < \infty$ and $1 \leq s \leq \infty$, then $K(B_{\ell_{r,s}^n}, J) \sim_C \sqrt{\frac{\log n}{n}}$. Moreover, if $2 < r < \infty$ and $r \leq s$, then*

$$\lim_{n \rightarrow \infty} \frac{K(B_{\ell_{r,s}^n}, J)}{\sqrt{\frac{\log n}{n}}} = 1.$$

(ii) *If $1 < r \leq 2$ and $1 \leq s \leq r$, then $K(B_{\ell_{r,s}^n}, J) \sim_C \left(\frac{\log n}{n}\right)^{1-\frac{1}{r}}$.*

(iii) If $1 < r \leq 2$ and $r < s$, then $\left(\frac{\log n}{n}\right)^{1-\frac{1}{r}} <_C K(B_{\ell_{r,s}^n}, J) <_C \frac{(\log n)^{1-\frac{1}{s}}}{n^{1-\frac{1}{r}}}$.

For the full index set $J = \mathbb{N}_0^{(\mathbb{N})}$ statement (i) of Theorem 3.22 was first presented in [27, Corollary 10, (i)], though without the assertion about the limit. The other two cases in Theorem 3.22 improve the results in [27, Corollary 10, (ii) and (iii)]. Notably, (ii) of Theorem 3.22 is asymptotically optimal, which contrasts with [27, Corollary 10, (ii)].

By the monotonicity property as described in (28) it suffices to prove the upper bounds in the 'tetrahedral case', and the lower bounds in the 'full case'. However, the lower estimate in the tetrahedral case appears to be technically less complex than the lower estimate in the full case. This motivates us to provide, up to a certain extent, two separate proofs for these two situations.

Proof. Proof of (i). Recall that $\ell_{r,s}$ is 2-convex whenever $2 < r < \infty$, $2 \leq s \leq \infty$. Hence in this situation the result is a special case of Theorem 2.33. But in the case $2 < r < \infty$, $1 \leq s \leq 2$ we also know from Theorem 2.16 that $\left(1 + \frac{n}{m}\right)^{\frac{m-1}{2}} \sim_{C^m} \chi_{\text{mon}}(\mathcal{P}_{J_m}(X_n))$, for all m, n , and so exactly the same arguments as in the proof of Theorem 2.33 work. To get the the limit formula, it suffices to note that the assumptions of Theorem 2.34 are satisfied in the case of $2 < r \leq s$.

Proof of (ii) and (iii) (the upper bounds). We show them for the tetrahedral case only, the general case follows by monotonicity from (28). In fact, both cases are immediate from Theorem 2.32, since under the assumption of (ii), we have $\|\text{id}: \ell_{r,s}^n \rightarrow \ell_r^n\| = 1$, and on the other hand assuming (iii) it follows $\|\text{id}: \ell_{r,s}^n \rightarrow \ell_r^n\| <_C (\log n)^{\frac{1}{r}-\frac{1}{s}}$.

Proof of (ii) and (iii) (the lower bounds in the tetrahedral case): By Theorem 2.32 we have that

$$\begin{aligned} K(B_{\ell_{r,s}^n}, \Lambda_T) &>_C \inf_{m \leq n} \left(\frac{\varphi_{\ell_{r',s'}^n(m-1)}}{\varphi_{\ell_{r',s'}^n(n)} \right)^{\frac{m-1}{m}} \sim_C \inf_{m \leq n} \frac{(m-1)^{\frac{1}{r'} \frac{m-1}{m}}}{n^{\frac{1}{r'} \frac{m-1}{m}}} \\ &\sim_C \inf_{m \leq n} \left(\frac{mn^{\frac{1}{m}}}{n} \right)^{\frac{1}{r'}} = \frac{1}{n^{\frac{1}{r'}}} \inf_{m \leq n} \left(mn^{\frac{1}{m}} \right)^{\frac{1}{r'}}, \end{aligned}$$

hence minimizing the last infimum (using Remark 2.31) leads to the claim.

Finally, given r and s like in (ii) or (iii), it remains to prove the lower bounds for $K(B_{\ell_{r,s}^n}, J)$, for any index set $J \subset \mathbb{N}_0^{(\mathbb{N})}$ for which $\Lambda_T(m) \subset J$ for all m . But, again looking at (28), we may assume that $J = \mathbb{N}_0^{(\mathbb{N})}$. To achieve the bounds we are looking for, it is central to use Theorem 2.29 and good bounds for the unconditional basis constant for the space of m -homogeneous polynomials in each of these cases.

$1 < r \leq 2$ and $1 \leq s \leq r$: By Theorem 3.9 with $\kappa = 1/2$ and Theorem 2.7, since $\frac{1}{s} - \frac{1}{r} \leq \frac{1}{r'}$, we know that,

$$\begin{aligned} \chi_{\text{mon}}(\mathcal{P}_m(\ell_{r,s}^n)) &<_{C^m} \left(\frac{n}{m}\right)^{(m-1)/r'} \max \left\{ \left(\frac{\log(m)^m}{n\sqrt{m-1}}\right)^{1/r'}, 1 \right\} \\ &<_{C^m} \left(n^{1/r'} \left(\frac{1}{mn^{1/m}}\right)^{1/r'} \max \left\{ \left(\frac{\log m}{n(\sqrt{m-1}/m)}\right)^{1/r'}, 1 \right\} \right)^m \\ &= \left(n^{1/r'} \max \left\{ \left(\frac{\log m}{mn^{1/\sqrt{m}}}\right)^{1/r'}, \left(\frac{1}{mn^{1/m}}\right)^{1/r'} \right\} \right)^m \\ &<_{C^m} \left(n^{1/r'} \max \left\{ \left(\frac{1}{\sqrt{mn}n^{1/\sqrt{m}}}\right)^{1/r'}, \left(\frac{1}{mn^{1/m}}\right)^{1/r'} \right\} \right)^m. \end{aligned}$$

Optimizing with Remark 2.31 shows $\sup_m \chi_{\text{mon}}(\mathcal{P}_m(\ell_{r,s}^n))^{1/m} <_C \left(\frac{n}{\log n}\right)^{1/r'}$, and hence Theorem 2.29

gives $\left(\frac{n}{\log n}\right)^{1/r'} <_C K(B_{X_n})$, as desired.

$1 < r \leq 2$ and $r < s$: By Theorems 3.19 and 3.21, for $m \leq n$, we have

$$\chi_{\text{mon}}(\mathcal{P}_m(\ell_{r,s}^n)) <_{C^m} \begin{cases} \left(\frac{n}{mn^{1/m}}\right)^{m/r'} & m \leq \frac{n}{e(\log n)^{r'(\frac{1}{r}-\frac{1}{s})}}, \\ (\log n)^{m(\frac{1}{r}-\frac{1}{s})} & \text{otherwise.} \end{cases}$$

Choose now some $n_0 = n_0(r, s)$ such that $\log n \leq \frac{n}{e(\log n)^{r'(\frac{1}{r}-\frac{1}{s})}}$ for all $n \geq n_0$. Then by Remark 2.31, for each $n \geq n_0$ and $m \leq \frac{n}{e(\log n)^{r'(\frac{1}{r}-\frac{1}{s})}}$, we have

$$\chi_{\text{mon}}(\mathcal{P}_m(\ell_{r,s}^n)) <_{C^m} \left(\frac{n}{\log n}\right)^{m(1-\frac{1}{r})}.$$

On the other hand, for all n and every $m \geq \frac{n}{e(\log n)^{r'(\frac{1}{r}-\frac{1}{s})}}$, it holds

$$\chi_{\text{mon}}(\mathcal{P}_m(\ell_{r,s}^n)) <_{C^m} (\log n)^{m(\frac{1}{r}-\frac{1}{s})} <_{C^m} \left(\frac{n}{\log n}\right)^{m(1-\frac{1}{r})}.$$

Combining both estimates, Proposition 2.28 and Theorem 2.29 complete the proof. \square

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