On the geometry of projections with fixed commutator

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Abstract

Let $\mathcal{C}_A$ be the set of all pairs of orthogonal projections (P, Q) on a Hilbert space such that $PQ - QP = A$, where A is a fixed bounded linear operator. The unitary group $\mathcal{U}_{\{A\}'}$ of the commutant $\{A\}'$ acts on $\mathcal{C}_A$ by conjugation. We give a characterization of when this action is locally transitive in terms of spectral properties of A, and use it to discuss the relation between orbits of the mentioned action, path components and connected components of $\mathcal{C}_A$. We then prove that the orbits are reductive homogeneous spaces. Finally, we present a sufficient condition on the orbits to get that any two pairs of projections can be joined by a geodesic of minimal length with respect to a Finsler metric. [1]

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1 Introduction

Let $\mathcal{H}$ be a complex separable Hilbert space, $\mathcal{B}(\mathcal{H})$ the algebra of bounded linear operators on $\mathcal{H}$ and $\mathcal{P}(\mathcal{H})$ the set of all orthogonal projections on $\mathcal{H}$. For a fixed $A \in \mathcal{B}(\mathcal{H})$, consider the set of all projections with fixed commutator A, namely

$$\mathcal{C}_A = \{(P, Q) \in \mathcal{P}(\mathcal{H}) \times \mathcal{P}(\mathcal{H}) : A = PQ - QP\}.$$

The analysis of this set leads to interesting problems from the operator theoretic perspective. A characterization of those operators that can be written as a commutator of two projections was given in [22]. More recently, the closure of the set of all commutators of projections in the uniform topology was described in [25]. Another point of view to understand the structure of $\mathcal{C}_A$ was pursued in [27], where a natural action of a unitary group on $\mathcal{C}_A$ was introduced, and it was shown that the orbits of this action coincide with the path components of $\mathcal{C}_A$.

[1]**2010 MSC:** 22E65, 47B02, 58B20

Keywords: projection, commutator, homogeneous space, Finsler metric, geodesic

Let us recall the above mentioned action on $\mathcal{C}_A$. Let $\mathcal{U}(\mathcal{H})$ be the full unitary group of $\mathcal{H}$, and $\{A\}' = \{X \in \mathcal{B}(\mathcal{H}) : XA = AX\}$ be the commutant of A. The unitary group of this algebra is given by $\mathcal{U}_{\{A\}'} = \mathcal{U}(\mathcal{H}) \cap \{A\}'$, and it acts by conjugation on $\mathcal{C}_A$ as follows

$$W \cdot (P, Q) = (WPW^*, WQW^*), \;\; W \in \mathcal{U}_{\{A\}'}, \, (P, Q) \in \mathcal{C}_A.$$

The orbits of this action are given by

$$\mathcal{O}_{(P,Q)} = \{(WPW^*, WQW^*) : W \in \mathcal{U}_{\{A\}'}\} \subseteq \mathcal{C}_A, \quad (P, Q) \in \mathcal{C}_A.$$

In this paper, we study aspects of the differential and metric geometry of these orbits. We will show that the above orbits are (infinite dimensional) smooth homogeneous spaces, and in dealing with them we will apply techniques from operator theory, Banach-Lie groups and their Lie algebras. For instance, we refer the reader to [9, 15, 16, 21, 28] for general results and examples of homogeneous spaces related to operator theory. More specifically, several homogeneous spaces related to orthogonal projections that motivate the present work have been studied. In this direction we can mention homogeneous spaces constructed from product of projections [4], sphere bundles related to projections [2], pencils of projections [12], projections with compact commutator [3] and differences of projections [5].

The study of orbits in $\mathcal{C}_A$ can be divided into the following two cases: A is an injective operator, or A is a non injective operator. We consider in detail the injective case, while we make remarks on the non injective case at the end of each section. It is not difficult to see that given a pair of projections $(P, Q) \in \mathcal{C}_A$, the nullspace of A can be expressed in terms of the nullspaces and ranges of the projections as follows

$$N(A) = (R(P) \cap R(Q)) \oplus (R(P) \cap N(Q)) \oplus (N(P) \cap R(Q)) \oplus (N(P) \cap N(Q)).$$

In the case of A an injective operator, Halmos' two projections theorem establishes the following unitary equivalence of the projections P and Q with two operator matrices in terms of a Hilbert space $\mathcal{L} \times \mathcal{L}$,

$$P \simeq \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad Q \simeq \begin{pmatrix} C^2 & CS \\ CS & S^2 \end{pmatrix},$$

where $C = \cos(\Gamma)$ is the *operator cosine* and $S = \sin(\Gamma)$ is the *operator sine* associated to the pair (P, Q). Here Γ is a uniquely determined self-adjoint operator acting on $\mathcal{L}$ with spectrum $\sigma(\Gamma) \subseteq [0, \frac{\pi}{2}]$, which is convenient for us to refer as the *operator angle* associated to the pair (P, Q). Throughout the paper we will link geometric properties of the orbits and spectral properties of operator angles associated to pairs of projections in $\mathcal{C}_A$.

The contents of this paper are as follows. In Section 2 we give the necessary background and notation. We recall Halmos' two projection theorem, we include from [27] a useful parametrization of pairs of projections in $\mathcal{C}_A$ and we present basic facts on infinite dimensional manifolds.

In Section 3 we study when the action of $\mathcal{U}_{\{A\}'}$ on $\mathcal{C}_A$ is locally transitive. This is motivated by the well-known fact that the conjugacy action in the set of all orthogonal projections $\mathcal{P}(\mathcal{H})$ is locally transitive: if $P, Q \in \mathcal{P}(\mathcal{H})$ and $\|P - Q\| < 1$, then there exists a unitary U such that $UPU^* = Q$. This is a crucial property that shows up in several geometric aspects of $\mathcal{P}(\mathcal{H})$. For the case where A is injective, we prove in Theorem 3.6 that the action of $\mathcal{U}_{\{A\}'}$ on $\mathcal{C}_A$ is locally transitive if and only if $\frac{i}{2}$ is not in or is an isolated point of the spectrum of A. This can be restated as saying that $\frac{\pi}{4}$ is not in or is an isolated point of the spectrum of Γ, where

Γ denotes the operator angle associated to any pair $(P,Q) \in \mathcal{C}_A$. Another equivalent condition turns out to be that connected components and path components in $\mathcal{C}_A$ coincide. In general, these characterizations do not extend to the case where A is not injective. We give an example showing that the action is not locally transitive whenever $\dim N(A)^\perp \geq 4$.

In Section 4 we prove in Theorem 4.6 that the orbits $\mathcal{O}_{(P,Q)}$, $(P,Q) \in \mathcal{C}_A$, admit the structure of reductive homogeneous spaces ([20, 24]). The group $\mathcal{U}_{\{A\}'}$ is indeed a Banach-Lie group whose Lie algebra $\mathfrak{u}_{\{A\}'}$ is identified with the anti-selfadjoint operators that commute with A. As it happens with other infinite dimensional homogeneous spaces, the main difficulty becomes to prove that the Lie algebra $\mathfrak{i}_{(P,Q)}$ of the isotropy group $\mathcal{I}_{(P,Q)}$ at a pair (P,Q), is a complemented subspace of the Lie algebra $\mathfrak{u}_{\{A\}'}$. This problem is related to the previously mentioned operators associated to the pair (P,Q): the operator angle Γ, the operator cosine $C = \cos(\Gamma)$ and operator sine $S = \sin(\Gamma)$. Notice the inclusion between commutants $\{\Gamma\}' \subseteq \{CS\}'$ that follows by elementary properties of the functional calculus. We show that there exists a continuous projection $\mathcal{E} : \{CS\}' \to \{\Gamma\}'$, which is actually a key step for finding the required supplement of $\mathfrak{i}_{(P,Q)}$. Furthermore, we then use the supplements of $\mathfrak{i}_{(P,Q)}$ obtained in this way to construct a reductive structure for the orbits.

In Section 5 we focus on the problem of minimality of geodesics of the reductive connection with respect to a Finsler metric introduced in the seminal work [15] for general homogeneous spaces of unitary groups of operator algebras. For A an injective operator we first compute the geodesics of the reductive connection in the orbits $\mathcal{O}_{(P,Q)} \subseteq \mathcal{C}_A$. Then we give a sufficient condition that guarantees the existence of geodesics of minimal length joining any pair of points in the orbits (Theorem 5.9). Given an orbit $\mathcal{O}_{(P,Q)}$ this sufficient condition can be simply stated as $\{CS\}' = \{\Gamma\}'$, where C, S and Γ are the operators cosine, sine and angle associated to the pair (P,Q). Indeed, this condition does not depend on the pair projections chosen in the orbit, and it can be interpreted also in terms of spectral properties of the operator angle Γ. Roughly speaking, it means that the function $f(t) = \cos(t)\sin(t)$ is injective on the spectrum of Γ, except possibly on sets of measure zero with respect to the spectral measure of Γ (see Proposition 5.6). We point out that the same type of results cannot be extended for orbits $\mathcal{O}_{(P,Q)} \subseteq \mathcal{C}_A$ in the case where A is not injective.

2 Preliminaries

Let $\mathcal{H}$ be a complex separable Hilbert space, $\mathcal{B}(\mathcal{H})$ the algebra of bounded linear operators, and $\mathcal{P}(\mathcal{H})$ the set of all orthogonal projections on $\mathcal{H}$. For $T \in \mathcal{B}(\mathcal{H})$, we denote by $R(T)$ and $N(T)$ the range and nullspace of T, respectively. For $\mathcal{S} \subseteq \mathcal{H}$ a closed subspace, we write $P_\mathcal{S}$ for the orthogonal projection onto $\mathcal{S}$.

Halmos' two projections theorem. We follow the presentation in the survey [10] on Halmo's two projection theorem, where the reader can also find the connection with other works about two projections (see, e.g., [13, 14]). Take two projections $P, Q \in \mathcal{P}(\mathcal{H})$ with ranges $R(P) = \mathcal{L}$ and $R(Q) = \mathcal{N}$. Then the underlying Hilbert space can be decomposed as

$$\mathcal{H} = (\mathcal{L} \cap \mathcal{N}) \oplus (\mathcal{L} \cap \mathcal{N}^\perp) \oplus (\mathcal{L}^\perp \cap \mathcal{N}) \oplus (\mathcal{L}^\perp \cap \mathcal{N}^\perp) \oplus \mathcal{H}_0,$$

where $\mathcal{H}_0$ is defined as the orthogonal complement of the first four summands. If $\mathcal{H}_0^\perp = \{0\}$, then $\mathcal{L}$ and $\mathcal{N}$ are said to be in *generic position*, and the pair of projections (P,Q) is called a *generic pair*. Let us denote $\mathcal{H}_{00} = \mathcal{L} \cap \mathcal{N}$, $\mathcal{H}_{01} = \mathcal{L} \cap \mathcal{N}^\perp$, $\mathcal{H}_{10} = \mathcal{L}^\perp \cap \mathcal{N}$ and $\mathcal{H}_{11} = \mathcal{L}^\perp \cap \mathcal{N}^\perp$.

Put also $\mathcal{M}_0 = \mathcal{L} \ominus (\mathcal{H}_{00} \oplus \mathcal{H}_{01})$ and $\mathcal{M}_1 = \mathcal{L}^\perp \ominus (\mathcal{H}_{10} \oplus \mathcal{H}_{11})$, so that $\mathcal{H}_0 = \mathcal{M}_0 \oplus \mathcal{M}_1$. Notice that the subspaces $\mathcal{H}_{ij}$, $i, j = 0, 1$, and $\mathcal{H}_0$ are invariant for both projections P, Q. Then Halmos' two projections theorem can be stated as follows.

Theorem 2.1. *If one of the spaces $\mathcal{M}_0$ and $\mathcal{M}_1$ is non trivial, then these subspaces have the same dimension and there exists a unitary operator $R : \mathcal{M}_1 \to \mathcal{M}_0$ and selfadjoint operators $S, C \in \mathcal{B}(\mathcal{M}_0)$ such that $0 \leq S \leq I$, $0 \leq C \leq I$, $S^2 + C^2 = I$, $N(S) = N(C) = \{0\}$, and*

$$P = (1,1,0,0) \oplus \begin{pmatrix} I & 0 \\ 0 & R^* \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & R \end{pmatrix}, \quad Q = (1,0,1,0) \oplus \begin{pmatrix} I & 0 \\ 0 & R^* \end{pmatrix} \begin{pmatrix} C^2 & CS \\ CS & S^2 \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & R \end{pmatrix}.$$

In the notation used in the statement, $A = (a_{00}, a_{01}, a_{10}, a_{11})$ indicates the block diagonal operator acting on $\mathcal{H}_0^\perp$ such that $A = a_{ij} I_{\mathcal{H}_{ij}}$ on $\mathcal{H}_{ij}$ and the remaining 2×2 block operators are with respect the decomposition $\mathcal{H}_0 = \mathcal{M}_0 \oplus \mathcal{M}_1$. On the other hand, for the projections $P_0 = P|_{\mathcal{H}_0}$ and $Q_0 = Q|_{\mathcal{H}_0}$, one has that

$$P_0 \simeq \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad Q_0 \simeq \begin{pmatrix} C^2 & CS \\ CS & S^2 \end{pmatrix}, \tag{1}$$

where $\simeq$ stands for unitary equivalence and these matrices are represented in the decomposition $\mathcal{M}_0 \times \mathcal{M}_0$.

For later use we observe that in the proof the operators C, S, usually known as the *operator cosine* and *sine* of the pair $(\mathcal{M}_0, \mathcal{M}_1)$, are determined by the conditions

$$C^2 = PQP|_{\mathcal{M}_0}, \quad S^2 = R(I-P)(I-Q)(I-P)|_{\mathcal{M}_1} R^*,$$

where $R : \mathcal{M}_1 \to \mathcal{M}_0$ is the unitary operator whose adjoint arises in the polar decomposition $(I-P)QP|_{\mathcal{M}_0} = R^* |(I-P)QP||_{\mathcal{M}_0}$. Indeed, R is a partial isometry, which becomes a unitary operator as a consequence of being (P_0, Q_0) a generic pair on $\mathcal{H}_0$. The *operator angle* Γ of the pair $(\mathcal{M}_0, \mathcal{M}_1)$ is defined by functional calculus as $\Gamma = \sin^{-1}(S)$ for inverse of the sine given by $\sin^{-1} : [0,1] \to [0, \frac{\pi}{2}]$, or equivalently as $\Gamma = \cos^{-1}(\Gamma)$ for the inverse of the cosine function given by $\cos^{-1} : [0,1] \to [0, \frac{\pi}{2}]$. Notice that $\Gamma \in \mathcal{B}(\mathcal{M}_0)$ is selfadjoint and $0 \leq \Gamma \leq \frac{\pi}{2} I$. We observe that for a generic pair (P, Q) the decomposition (1) is given in terms of $\mathcal{L} \times \mathcal{L}$, and C, S, Γ are operators acting on $\mathcal{L}$.

Pair of projections with fixed commutator. For a fixed $A \in \mathcal{B}(\mathcal{H})$, we consider the set

$$\mathcal{C}_A := \{(P, Q) \in \mathcal{P}(\mathcal{H}) \times \mathcal{P}(\mathcal{H}) : A = PQ - QP\}.$$

Notice that $A = PQ - QP$, for $P, Q \in \mathcal{P}(\mathcal{H})$, gives that A is anti-selfadjoint ($A = -A^*$), and thus the commutant $\{A\}' := \{X \in \mathcal{B}(\mathcal{H}) : XA = AX\}$ is a von Neumann algebra. Let $\mathcal{U}(\mathcal{H})$ be the unitary group on $\mathcal{H}$, and let $\mathcal{U}_{\{A\}'} := \mathcal{U}(\mathcal{H}) \cap \{A\}'$ be the unitary group of $\{A\}'$. This latter group plays an important role along this work, since it acts on $\mathcal{C}_A$ as follows

$$W \cdot (P, Q) = (WPW^*, WQW^*), \quad W \in \mathcal{U}_{\{A\}'}, (P, Q) \in \mathcal{C}_A. \tag{2}$$

Some characterizations of when $\mathcal{C}_A \neq \emptyset$ in terms of properties of A appeared in the literature. For instance, it was proved in [22] that A is a commutator of two orthogonal projections if and only if A is an anti-selfadjoint operator such that $\|A\| \leq \frac{1}{2}$ and A is unitarily equivalent to its adjoint. Another characterization was obtained in [27] by studying $\mathcal{C}_A$ according to the nullspace of A. For any pair of projections P, Q such that $A = PQ - QP$, one can easily verify that

$$N(A) = (R(P) \cap R(Q)) \oplus (R(P) \cap N(Q)) \oplus (N(P) \cap R(Q)) \oplus (N(P) \cap N(Q)). \tag{3}$$

Thus, A has trivial nullspace if and only if (P, Q) is a generic pair. Then the analysis of the set $\mathcal{C}_A$ can be divided into two cases according to A is an injective operator, or A is a general operator with possible non trivial nullspace. We now collect several results on the injective case.

Theorem 2.2. *(Shi-Ji [27]) Let $A \in \mathcal{B}(\mathcal{H})$ be an anti-selfadjoint operator such that $N(A) = \{0\}$ and $\|A\| \leq \frac{1}{2}$. Then following hold:*

i) A is a commutator of a pair of projections if and only if there is an injective anti-selfadjoint operator B on a Hilbert space $\mathcal{K}$ with $\sigma(B) \subseteq [0, \frac{i}{2}]$ such that A is unitarily equivalent to the form $B \oplus (-B)$ with respect to the space decomposition $\mathcal{H} = \mathcal{K} \oplus \mathcal{K}$.

ii) Suppose that $A = B \oplus (-B)$ as in the previous item. For $\mathcal{M} = \overline{R(\sqrt{I + 4B^2})}$, $U \in \mathcal{U}_{\{B\}'}$ and $V \in \{B\}'$ a partial isometry with $V^* = V$ and $V^2 = P_{\mathcal{M}}$, define the orthogonal projections

$$P_U := \frac{1}{2}\begin{pmatrix} I & U \\ U^* & I \end{pmatrix}, \quad Q_{U,V} := \frac{1}{2}\begin{pmatrix} I & (V\sqrt{I + 4B^2} - 2B)U \\ U^*(V\sqrt{I + 4B^2} + 2B) & I \end{pmatrix}. \tag{4}$$

Then $A = PQ - QP$ for some $P, Q \in \mathcal{P}(\mathcal{H})$ if and only if there exist operators U, V satisfying the above mentioned conditions such that $P = P_U$ and $Q = Q_{U,V}$.

Remark 2.3. We assume that $A = B \oplus (-B)$ is injective, where $B \in \mathcal{B}(\mathcal{K})$ as in the previous result, and set $\mathcal{M} = \overline{R(\sqrt{I + 4B^2})}$. Let us consider the set

$$\mathcal{R} = \{(U, V) \in \mathcal{B}(\mathcal{K}) \times \mathcal{B}(\mathcal{K}) : U \in \mathcal{U}_{\{B\}'},\, V \in \{B\}',\, V = V^*,\, V^2 = P_{\mathcal{M}}\}.$$

Then there is bijection defined by

$$F : \mathcal{C}_A \to \mathcal{R}, \quad F(P_U, Q_{U,V}) = (U, V).$$

It is clearly injective by the definition of $\mathcal{R}$. The previous result ensures that F is surjective; it will be useful to recall some parts of the procedure to prove it (see the proof of [27, Thm. 2.2]). Every $(P, Q) \in \mathcal{C}_A$ has a representation in terms of $\mathcal{H} = \mathcal{K} \oplus \mathcal{K}$ as

$$P = \begin{pmatrix} P_{11} & P_{12} \\ P_{12}^* & P_{22} \end{pmatrix}, \quad Q = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{12}^* & Q_{22} \end{pmatrix}.$$

Since B is injective, one can see that $P_{11} = P_{22} = \frac{1}{2}I$ and $P_{12}^* P_{12} = \frac{1}{4}I$. Then there is a unique $U \in \mathcal{U}_{\{B\}'}$ given by the polar decomposition $P_{12} = U|P_{12}|$. Analogously, there is $W \in \mathcal{U}_{\{B\}'}$ defined by the polar decomposition $Q_{12} = W|Q_{12}|$. Thus, we have

$$P = P_U = \frac{1}{2}\begin{pmatrix} I & U \\ U^* & I \end{pmatrix}, \quad Q = Q_W = \frac{1}{2}\begin{pmatrix} I & W \\ W^* & I \end{pmatrix}.$$

From $A = P_U Q_W - Q_W P_U$, it follows that $UW^* - WU^* = 4B$. Let $UW^* = H + iK$, where H and K are the selfadjoint operators given by the real and imaginary parts of UW^*. One can further show $H^2 = I + 4B^2$, and then V is defined by the polar decomposition $H = V|H| = V\sqrt{I + 4B^2}$, where V is a partial isometry with initial space $\mathcal{M}$.

If $A = B \oplus (-B)$ as in the previous result, then note that $W \in \mathcal{U}_{\{A\}'}$ if and only $W = W_1 \oplus W_2$, where $W_i \in \mathcal{U}_{\{B\}'}$, $i = 1, 2$. Then a straightforward computation shows that the action of $\mathcal{U}_{\{A\}'}$ on $\mathcal{C}_A$ defined in (2) can also be described as

$$W \cdot (P_U, Q_{U,V}) = (P_{W_1 U W_2^*}, Q_{W_1 U W_2^*, W_1 V W_1^*}), \tag{5}$$

where $W = W_1 \oplus W_2 \in \mathcal{U}_{\{A\}'}$, and $W_i \in \mathcal{U}_{\{B\}'}$, $i = 1, 2$.

Theorem 2.4. *(Shi-Ji [27]) Let $A \in \mathcal{B}(\mathcal{H})$ be an injective operator. Then the action of $\mathcal{U}_{\{A\}'}$ on $\mathcal{C}_A$ is transitive on each path component of $\mathcal{C}_A$ in the relative topology inherited from $\mathcal{B}(\mathcal{H})$. If, in addition, $A = B \oplus (-B)$, where B satisfies the conditions given above, then $(P_{U_i}, Q_{U_i,V_i}) \in \mathcal{C}_A$, $i = 1, 2$, belong to the same path component if and only if $V_2 = W V_1 W^*$ for some $W \in \mathcal{U}_{\{B\}'}$.*

Below we state the main results on the case where A is not injective.

Remark 2.5. Given $A \in \mathcal{B}(\mathcal{H})$, notice that $N(A)$ and $N(A)^\perp$ are invariant under A. Then set $\mathcal{H}_0 = N(A)^\perp$ and $A_0 = A|_{\mathcal{H}_0} \in \mathcal{B}(\mathcal{H}_0)$. The corresponding results on the structure of $\mathcal{C}_A$ from [27] can be stated as follows.

i) Given $A = -A^*$, $\|A\| \leq 1/2$, then $\mathcal{C}_A \neq \emptyset$ if and only if $\mathcal{C}_{A_0} \neq \emptyset$. Furthermore, $(P, Q) \in \mathcal{C}_A$ if and only if $(P_0, Q_0) \in \mathcal{C}_{A_0}$, where $P_0, Q_0 \in \mathcal{P}(\mathcal{H}_0)$ are the generic parts of $P, Q \in \mathcal{P}(\mathcal{H})$.

ii) Now assume that $A_0 = B \oplus (-B)$. Then $(P, Q) \in \mathcal{C}_A$ if and only there exist $E, F \in \mathcal{P}(\mathcal{H}_0^\perp)$ such that $EF = FE$ and operators U_0, V_0 as in Theorem 2.2 *ii)* that now act on $\mathcal{H}_0$ such that $P_0 = P_{U_0}$, $Q_0 = Q_{U_0,V_0}$, satisfying

$$P = P_{E,U_0} := E \oplus P_{U_0}, \quad Q = Q_{F,U_0,V_0} := F \oplus Q_{U_0,V_0}.$$

In this case, we clearly have $E = P|_{\mathcal{H}_0^\perp}$ and $F = Q|_{\mathcal{H}_0^\perp}$.

iii) Consider unitary group $\mathcal{U}(\mathcal{H}_0^\perp)$ of the Hilbert space $\mathcal{H}_0^\perp$ and the unitary group $\mathcal{U}_{\{A_0\}'} \subseteq \mathcal{B}(\mathcal{H}_0)$ of the commutant $\{A_0\}'$. Then the group $\mathcal{U}_{\{A\}'} \simeq \mathcal{U}(\mathcal{H}_0^\perp) \times \mathcal{U}_{\{A_0\}'}$ acts on $\mathcal{C}_A$ as follows

$$(G, W) \cdot (P, Q) = (GP|_{\mathcal{H}_0^\perp} G^* \oplus W P_0 W^*, GQ|_{\mathcal{H}_0^\perp} G^* \oplus W Q_0 W^*). \tag{6}$$

where $G \in \mathcal{U}(\mathcal{H}_0^\perp)$, $W \in \mathcal{U}_{\{A_0\}'}$ and $(P, Q) \in \mathcal{C}_A$.

Banach manifolds. We refer to [9, 28] for more details and proofs of the results we recall here. We consider C^∞ manifolds modeled on Banach spaces. Given M, N manifolds and a smooth map $f : M \to N$, we denote by $T_p f : (TM)_p \to (TN)_{f(p)}$ the tangent map at $p \in M$, where $(TM)_p$ and $(TN)_{f(p)}$ are the tangent spaces of M at p and N at $f(p)$. A smooth map $f : M \to N$ is called a *submersion at* $p \in M$ if $N(T_p f)$ is a closed complemented subspace of $(TM)_p$ and $T_p f$ is surjective. If $f : M \to N$ is a *submersion* at every point $p \in M$, then f is called a submersion. A real Banach-Lie group is a Banach manifold G such that the group multiplication $G \times G \to G$, $(g, h) \mapsto gh$, and the inverse $G \to G$, $g \mapsto g^{-1}$, are real smooth maps. An action of a Banach-Lie group G on a manifold M is a map $L : G \times M \to M$, $L(g, p) = g \cdot p$, $g \in G$ and $p \in M$, such that $h \cdot (g \cdot p) = (hg) \cdot p$ and $1 \cdot p = p$, for all $h, g \in G$ and $p \in M$. The action is said to be smooth if the map L is C^∞. A *(smooth) homogeneous space of a Banach-Lie group G* is a manifold M such that G acts transitively and smoothly on M, and there exists $p \in M$ such that the map $\pi_p : G \to M$, $\pi_p(g) = g \cdot p$, is a submersion. We observe the existence of such a point $p \in M$ in the above definition is actually equivalent to the fact that every point in M have that property.

Let M be a manifold, and $N \subseteq M$. A chart $(\phi, \mathcal{V}, E)$ at $p \in M$ consists in an open neighborhood $\mathcal{V}$ of p, a Banach space E and a homeomorphism $\phi : \mathcal{V} \to \phi(\mathcal{V}) \subseteq E$. If for every $p \in N$ there exists a chart $(\phi, \mathcal{V}, E)$ at p, and a closed subspace F complemented in E satisfying $\phi(\mathcal{V} \cap N) = F \cap \phi(\mathcal{V})$, then N is called a *submanifold* of M. In this case, N turns out to be a manifold endowed with the topology inherited from M. If K is a subgroup of a Banach-Lie group G, then K is said to be a *Banach-Lie subgroup* of G when K is a submanifold of G.

One can construct homogeneous spaces from Banach-Lie subgroups. Let K be a Lie subgroup of a Lie group G. Then the quotient space $M := G/K$ carries the structure of a manifold equipped with the quotient topology such that the natural projection $\pi : G \to M$, $\pi(g) = gK$, is a submersion. Furthermore, G acts smoothly on M by the left translation action defined by $G \times M \to M$, $g \cdot hK = ghK$. The manifold structure on G/K is uniquely determined by the condition that the projection π is a submersion. Also one has the following converse. A smooth homogeneous space M of a Lie group G must be diffeomorphic to G/G_p, where G_p is the isotropy group of G at any point $p \in M$, and G_p turns out to be a Lie subgroup of G.

Let M be a homogeneous space of a Banach-Lie group G with Lie algebra $\mathfrak{g}$, take $\pi_p : G \to M$, $\pi_p(g) = g \cdot p$ the map induced by the action and set $\mathfrak{g}_p$ for the Lie algebra of the isotropy group G_p at each $p \in M$. A *reductive structure* for M is a smooth distribution of subspaces $\{\mathfrak{m}_p\}_{p \in M}$ such that $\mathfrak{g}_p \oplus \mathfrak{m}_p = \mathfrak{g}$, and $I_g(\mathfrak{m}_p) = \mathfrak{m}_p$ for all $g \in G_p$ and $p \in M$, where $I_g := T_1 Ad_g$ and $Ad_g : G \to G$, $Ad_g(h) = ghg^{-1}$. When M admits a reductive structure, it is said to be a *reductive homogeneous space*. For these types of homogeneous spaces note that $T_1 \pi_p|_{\mathfrak{m}_p} : \mathfrak{m}_p \to (TM)_p$ is an isomorphism, so one can define a 1-form with values in $\mathfrak{g}$ associated to the reductive structure by setting $K_p : (TM)_p \to \mathfrak{m}_p$, $K_p := (T_p \pi|_{\mathfrak{m}_p})^{-1}$, for $p \in M$. This in turn, induces the *reductive connection* ∇^K on the tangent bundle TM. For Y a tangent field on M and $X \in (TM)_p$, this connection is determined by

$$K_p(\nabla^K_X Y(p)) = K_p(X)(K_p(Y(p))) + [K_p(Y(p)), K_p(X)],$$

where here $A(B)$ denotes the derivative of B in the direction of A and $[\,,\,]$ is the Lie bracket of $\mathfrak{g}$. Geodesics of this connection can be computed explicitly. Indeed, the geodesic at $p \in M$ with initial velocity $X \in (TM)_p$ is given by $\delta(t) = e^{tK_p(X)} \cdot p$. These elementary facts on reductive homogeneous spaces can be found in the classical text [20] for finite-dimensional manifolds, or in [24] for the setting of Banach manifolds.

3 Orbits, path components and connected components

In this section we introduce the use of operator angles between pairs of projections to further investigate the action of the unitary group $\mathcal{U}_{\{A\}'}$ on $\mathcal{C}_A$ and the topology of $\mathcal{C}_A$. We deal with the case where A is an injective operator almost all this section (see Remark 3.8 for the case of A non injective). For A an injective operator, the action of $\mathcal{U}_{\{A\}'}$ on $\mathcal{C}_A$ defines the following orbits

$$\mathcal{O}_{(P,Q)} = \{(WPW^*, WQW^*) : W \in \mathcal{U}_{\{A\}'}\}, \quad (P, Q) \in \mathcal{C}_A.$$

Thus, the set $\mathcal{C}_A$ is the (disjoint) union of all of these orbits.

Remark 3.1. Associated to a unitary $X : \mathcal{H} \to \mathcal{H}'$ between two Hilbert spaces $\mathcal{H}$ and $\mathcal{H}'$, we consider the unitary conjugation $Ad_X : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}')$ defined by $Ad_X(T) = XTX^*$, for

$T \in \mathcal{B}(\mathcal{H})$. A unitary conjugation of $\mathcal{C}_A \subseteq \mathcal{B}(\mathcal{H}) \times \mathcal{B}(\mathcal{H})$ is given by

$$\begin{aligned} Ad_X(\mathcal{C}_A) &:= \{(Ad_X(P), Ad_X(Q)) \in \mathcal{P}(\mathcal{H}') \times \mathcal{P}(\mathcal{H}') : (P,Q) \in \mathcal{C}_A\} \\ &= \{(P,Q) \in \mathcal{P}(\mathcal{H}') \times \mathcal{P}(\mathcal{H}') : PQ - QP = Ad_X(A)\}. \end{aligned}$$

This might be regarded as a change in the presentation of the set $\mathcal{C}_A$, which can be helpful to study the topological, geometric and metric properties that are unitarily invariant. Throughout this work, we will often use the following facts about unitary conjugations of $\mathcal{C}_A$.

i) For $A \in \mathcal{B}(\mathcal{H})$ an injective operator, Theorem 2.2 can be rephrased as saying that $\mathcal{C}_A \neq \emptyset$ if and only if there exists a unitary conjugation of $\mathcal{C}_A$ satisfying $Ad_X(A) = B \oplus (-B)$, $B^* = -B$ with $\sigma(B) \subseteq [0, \frac{i}{2}]$. Then every $(P,Q) \in \mathcal{C}_A$ can be uniquely written as $Ad_X(P) = P_U$, $Ad_X(Q) = Q_{U,V}$, where U and V satisfy the conditions explained before.

ii) Let $A \in \mathcal{B}(\mathcal{H})$ be an injective operator and $(P,Q) \in \mathcal{C}_A$. Since A is injective, $(P,Q) \in \mathcal{C}_A$ is a generic pair. Then there exists an operator X' from $\mathcal{H} = \mathcal{L} \oplus \mathcal{L}^\perp$ onto $\mathcal{H}' = \mathcal{L} \times \mathcal{L}$ such that

$$Ad_{X'}(P) = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad Ad_{X'}(Q) = \begin{pmatrix} C^2 & CS \\ CS & S^2 \end{pmatrix}, \quad X' := \begin{pmatrix} I & 0 \\ 0 & R \end{pmatrix},$$

where the first two block operator matrices are in terms of the decomposition $\mathcal{L} \times \mathcal{L}$, and the operator $R : \mathcal{L}^\perp \to \mathcal{L}$ is the unitary defined by Halmos' two projection theorem. Also note

$$Ad_{X'}(A) = Ad_{X'}([P,Q]) = [Ad_{X'}(P), Ad_{X'}(Q)] = \begin{pmatrix} 0 & CS \\ -CS & 0 \end{pmatrix}.$$

Now define

$$N := \frac{1}{\sqrt{2}} \begin{pmatrix} I & -iI \\ I & iI \end{pmatrix}, \quad X := NX'.$$

Then X is a unitary operator from $\mathcal{H} = \mathcal{L} \oplus \mathcal{L}^\perp$ onto $\mathcal{H}' = \mathcal{L} \times \mathcal{L}$ satisfying

$$\begin{aligned} Ad_X(A) &= Ad_N(\begin{pmatrix} 0 & CS \\ -CS & 0 \end{pmatrix}) = \begin{pmatrix} iCS & 0 \\ 0 & -iCS \end{pmatrix}, \\ Ad_X(P) &= Ad_N(\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}) = \frac{1}{2} \begin{pmatrix} I & I \\ I & I \end{pmatrix}, \\ Ad_X(Q) &= Ad_N(\begin{pmatrix} C^2 & CS \\ CS & S^2 \end{pmatrix}) = \frac{1}{2} \begin{pmatrix} I & (C - iS)^2 \\ (C + iS)^2 & I \end{pmatrix} = \frac{1}{2} \begin{pmatrix} I & e^{-2i\Gamma} \\ e^{2i\Gamma} & I \end{pmatrix}. \end{aligned} \tag{7}$$

Here the block operator matrices are in terms of the decomposition $\mathcal{L} \times \mathcal{L}$, $\mathcal{L} = R(P)$, and Γ, C and S are the operators associated the angle, cosine and sine of the pair (P,Q). We remark that these three operators are not associated to the pair $(Ad_X(P), Ad_X(Q))$, which has $Ad_X(\Gamma)$, $Ad_X(C)$ and $Ad_X(S)$ as the corresponding angle, cosine and sine operators.

iii) Notice that $B = iCS$ in the previous item satisfies $Ad_X(A) = B \oplus (-B)$, $B^* = -B$ and $\sigma(B) \subseteq [0, \frac{i}{2}]$ as in Theorem 2.2. One can further express the projections $Ad_X(P) = P_U$ and $Ad_X(Q) = Q_{U,V}$ for some operators U and V as described in (4). Clearly, we have $U = I$. Recall from Remark 2.3 that V is the partial isometry in the polar decomposition of $\cos(2\Gamma) = \Re(e^{-2i\Gamma}) = V|\cos(2\Gamma)| = V\sqrt{I + 4(iCS)^2}$. Using Borel functional calculus, we find that $V = g(\Gamma)$, where $g(t) = \operatorname{sgn}(\cos(2t))$, $t \in [0, \frac{\pi}{2}]$, and sgn denotes the sign function. Then, we have that the projection $Ad_X(Q)$ in (3) is also given by

$$Ad_X(Q) = \frac{1}{2}\begin{pmatrix} I & g(\Gamma)\sqrt{I + 4(iCS)^2} - 2iCS \\ g(\Gamma)\sqrt{I + 4(iCS)^2} + 2iCS & I \end{pmatrix}. \quad (8)$$

The natural problem of determining conditions for unitary equivalence of pairs of projections, which do not necessarily belong to $\mathcal{C}_A$, was already considered in the literature. Raeburn and Sinclair proved in [26] that given (P_0, Q_0), (P_1, Q_1) two pairs of projections and $\lambda > 1$, then there exists a unitary W such that $WP_0W^* = P_1$ and $WQ_0W^* = Q_1$ if and only if the operators $\lambda P_0 + Q_0$ and $\lambda P_1 + Q_1$ are unitarily equivalent. This was indeed motivated by an analogous result previously proved by Dixmier that holds for the case of generic pairs of projections with $\lambda = 1$. Now we give an immediate consequence of Halmos' two projection theorem, which will be useful later to characterize orbits in $\mathcal{C}_A$ in terms of operator angles.

Proposition 3.2. *Let Γ_i be the operator angle associated to the generic pair (P_i, Q_i), $i = 0, 1$. Then there exists $W \in \mathcal{U}(\mathcal{H})$ such that $W \cdot (P_0, Q_0) = (P_1, Q_1)$, if and only if Γ_0 and Γ_1 are unitarily equivalent.*

Proof. Let $W \in \mathcal{U}(\mathcal{H})$ such that $WP_0W^* = P_1$ and $WQ_0W^* = Q_1$. As we have observed in Section 2 the operator cosine C_0 of the generic pair (P_0, Q_0) is given by $C_0^2 = P_0Q_0P_0|_{\mathcal{L}_0}$, where $\mathcal{L}_0 = R(P_0)$. By the same formula for the cosine C_1 relative to (P_1, Q_1), one sees that $C_1^2 = P_1Q_1P_1|_{R(P_1)} = WP_0Q_0P_0W^*|_{W(\mathcal{L}_0)}$, and consequently, $C_1^2 = WC_0^2W^*$ as operators acting on $\mathcal{L}_1 = R(P_1)$. Since C_0, C_1 are positive, it follows that $C_1 = WC_0W^*$, and by functional calculus $\Gamma_1 = W\Gamma_0W^*$, where Γ_0 and Γ_1 are the operator angles of (P_0, Q_0) and (P_1, Q_1), respectively, and W is considered as a unitary operator from $\mathcal{L}_0$ to $\mathcal{L}_1$.

Conversely, assume that $\Gamma_1 = W\Gamma_0W^*$ for some unitary $W : \mathcal{L}_0 \to \mathcal{L}_1$. Then, take the unitary $\tilde{W} : \mathcal{L}_0 \times \mathcal{L}_0 \to \mathcal{L}_1 \times \mathcal{L}_1$ defined by $\tilde{W} = W \oplus W$. Let C_i and S_i be the operator cosine and sine of (P_i, Q_i), respectively $(i = 0, 1)$. Then we have $C_1 = WC_0W^*$ and $S_1 = WS_0W^*$. Therefore,

$$\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} = \tilde{W}\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}\tilde{W}^*, \quad \begin{pmatrix} C_1^2 & C_1S_1 \\ C_1S_1 & S_1^2 \end{pmatrix} = \tilde{W}\begin{pmatrix} C_0^2 & C_0S_0 \\ C_0S_0 & S_0^2 \end{pmatrix}\tilde{W}^*,$$

where the projections on the left-hand sides of these equations are with respect $\mathcal{L}_1 \times \mathcal{L}_1$, meanwhile the projections on the second factors of the right-hand sides are with respect $\mathcal{L}_0 \times \mathcal{L}_0$. We obtain that $P_1 = (\tilde{R}_1^*\tilde{W}\tilde{R}_0)P_0(\tilde{R}_1^*\tilde{W}\tilde{R}_0)^*$ and $Q_1 = (\tilde{R}_1^*\tilde{W}\tilde{R}_0)Q_0(\tilde{R}_1^*\tilde{W}\tilde{R}_0)^*$, where $\tilde{R}_i = I \oplus R_i$, $i = 0, 1$, are the unitaries defined in Halmos' two projections theorem. □

Corollary 3.3. *Let A be an injective operator, $(P_i, Q_i) \in \mathcal{C}_A$, $i = 0, 1$, and Γ_i the operator angle associated to the pair (P_i, Q_i). Then $\mathcal{O}_{(P_0,Q_0)} = \mathcal{O}_{(P_1,Q_1)}$ if and only if Γ_0 and Γ_1 are unitarily equivalent.*

Proof. First note that since A is injective, (P_0, Q_0) and (P_1, Q_1) are generic pairs by Eq. (3). Also observe that $\mathcal{O}_{(P_0,Q_0)} = \mathcal{O}_{(P_1,Q_1)}$ if and only if there exists $W \in \mathcal{U}_{\{A\}'}$ such that $W \cdot (P_0, Q_0) = (P_1, Q_1)$. Then the equivalence follows at once from Proposition 3.2. We only remark that for the converse, we get in this way a unitary W such that $W \cdot (P_0, Q_0) = (P_1, Q_1)$. By assumption $(P_i, Q_i) \in \mathcal{C}_A$, $i = 0, 1$, so that $A = P_1Q_1 - Q_1P_1 = W(P_0Q_0 - Q_0P_0)W^* = WAW^*$, and thus $W \in \mathcal{U}_{\{A\}'}$. □

To understand how the orbits sit inside $\mathcal{C}_A$ we now discuss the local transitivity of the action in the relative topology inherited from $\mathcal{B}(\mathcal{H})\times\mathcal{B}(\mathcal{H})$. Recall that this property means that for every $(P_0, Q_0) \in \mathcal{C}_A$, there exists $r = r_{(P_0,Q_0)} > 0$ such that if $(P_1, Q_1) \in \mathcal{C}_A$ and $\max\{\|P_1 - P_0\|, \|Q_1 - Q_0\|\} < r$, then there is a unitary $W \in \mathcal{U}_{\{A\}'}$ such that $W \cdot (P_0, Q_0) = (WP_0W^*, WQ_0W^*) = (P_1, Q_1)$. Our main result is stated in Theorem 3.6, which gives a characterization of the local transitivity of the action in terms of the spectrum of A, the relation between path and connected components, and the continuity of the map F defined in Remark 2.3. Before we get to this characterization, we prove the following lemma.

Lemma 3.4. *Let $A \in \mathcal{B}(\mathcal{H})$ be an injective operator and $(P, Q) \in \mathcal{C}_A$. If $\frac{i}{2}$ is not an isolated point of $\sigma(A)$, then there is a sequence $\{(P_n, Q_n)\}_{n\geq 1}$ in $\mathcal{C}_A$ such that $\|P_n - P\| \to 0$, $\|Q_n - Q\| \to 0$ and $\mathcal{O}_{(P_n,Q_n)} \cap \mathcal{O}_{(P_m,Q_m)} = \emptyset$ for all $n, m \geq 1$, $n \neq m$.*

Proof. Using (7) and (8) in Remark 3.1 we can assume that, after unitary conjugation, A, P, and Q have the form

$$A = \begin{pmatrix} iCS & 0 \\ 0 & -iCS \end{pmatrix}, \quad P = \frac{1}{2}\begin{pmatrix} I & I \\ I & I \end{pmatrix}, \quad Q = \frac{1}{2}\begin{pmatrix} I & g(\Gamma)D - 2iCS \\ g(\Gamma)D + 2iCS & I \end{pmatrix},$$

where $D = \sqrt{I + 4(iCS)^2}$ and the above operator matrices are defined in $\mathcal{L} \times \mathcal{L}$. Since $C = \cos(\Gamma)$, $S = \sin(\Gamma)$ and $\sigma(\Gamma) \subseteq [0, \frac{\pi}{2}]$, the hypothesis on the spectrum of A is equivalent to have that $\frac{\pi}{4}$ is not an isolated point of $\sigma(\Gamma)$. Then either there exists a sequence $\{\lambda_k^+\}_{k\geq 1}$ in $\sigma(\Gamma) \cap [0, \frac{\pi}{4})$ such that $\lambda_k^+ \to \frac{\pi}{4}$, or there exists a sequence $\{\lambda_k^-\}_{k\geq 1}$ in $\sigma(\Gamma) \cap (\frac{\pi}{4}, \frac{\pi}{2}]$ such that $\lambda_k^- \to \frac{\pi}{4}$. Furthermore, we can take in each case the sequence satisfying $\lambda_k^+ < \lambda_{k+1}^+$ or $\lambda_{k+1}^- < \lambda_k^-$, for all $k \geq 1$.

We only consider the first case, the other case being similar. Take $\{\delta_k\}_{k\geq 1}$ an increasing sequence in $\mathbb{R}^+$ such that $\lambda_k^+ \in (\delta_k, \delta_{k+1})$, $\delta_1 = 0$ and $[0, \frac{\pi}{4}) = \cup_{k\geq 1}[\delta_k, \delta_{k+1})$. Let E_Γ be the spectral measure of Γ. Since $(\delta_k, \delta_{k+1}) \cap \sigma(\Gamma)$ is nonempty and a relatively open subset of $\sigma(\Gamma)$, it follows that $E_\Gamma((\delta_k, \delta_{k+1}) \cap \sigma(\Gamma)) > 0$ (see [11, Sec. IX.2.2]). Thus, $E_k^+ := E_\Gamma([\delta_k, \delta_{k+1})) \geq E_\Gamma((\delta_k, \delta_{k+1})) = E_\Gamma((\delta_k, \delta_{k+1}) \cap \sigma(\Gamma)) > 0$. Then take $E_k^- := E_\Gamma((\frac{\pi}{2} - \delta_{k+1}, \frac{\pi}{2} - \delta_k])$. In contrast to the projections $E_k^+ > 0$, notice that E_k^- may be trivial projections. The property $E_k^+ > 0$ will be used at the end of the proof to show that orbits are disjoint.

Next define

$$V_n := \sum_{k=1}^{n} E_k^+ - \sum_{k>n} E_k^+ + \sum_{k=1}^{n} E_k^- - \sum_{k>n} E_k^- .$$

The convergence of the above series is understood in the SOT topology. Now we use the representation of pairs in $\mathcal{C}_A$ given in Theorem 2.2 with $B = iCS = f(\Gamma)$, where $f(t) = i\cos(t)\sin(t)$, $t \in [0, \frac{\pi}{2}]$. Let E_B denotes the spectral measure of B. Recall that $E_B(\Omega) = E_\Gamma(f^{-1}(\Omega))$, for every Borel measurable set $\Omega \subseteq \sigma(B)$. Then, set $P_0 := E_B(\{\frac{i}{2}\}) = P_{N(D)}$, and for $k \geq 1$,

$$\begin{aligned} P_k &:= E_B(f([\delta_k, \delta_{k+1}) \cup (\frac{\pi}{2} - \delta_{k+1}, \frac{\pi}{2} - \delta_k])) \\ &= E_\Gamma([\delta_k, \delta_{k+1}) \cup (\frac{\pi}{2} - \delta_{k+1}, \frac{\pi}{2} - \delta_k]) = E_k^+ + E_k^- . \end{aligned}$$

Now we can define the required sequence $\{(P_n, Q_n)\}_{n\geq 1}$. For we let $U_n = I$, and

$$P_n := P_{U_n} = \frac{1}{2}\begin{pmatrix} I & I \\ I & I \end{pmatrix} = P,$$
$$Q_n := Q_{U_n,V_n} = \frac{1}{2}\begin{pmatrix} I & V_nD - 2B \\ V_nD + 2B & I \end{pmatrix}.$$

Since $N(D)^\perp = \overline{R(D)} = \mathcal{M}$ and $\sum_{k\geq 0} P_k = I$, we get $V_n^*V_n = P_{\mathcal{M}}$. Clearly, V_n are partial isometries such that $V_n^* = V_n$ and $V_n \in \{B\}'$ since V_n are given in terms of spectral projections of Γ and $B = f(\Gamma)$.

To prove that $\|Q_n - Q\| \to 0$, or equivalently $\|V_nD - g(\Gamma)D\| \to 0$, we first observe that $g(t) = \operatorname{sgn}(\cos(2t)) = \chi_{[0,\frac{\pi}{4})}(t) - \chi_{(\frac{\pi}{4},\frac{\pi}{2}]}(t) = \sum_{k\geq 1} \chi_{[\delta_k,\delta_{k+1})}(t) - \chi_{(\frac{\pi}{2}-\delta_{k+1},\frac{\pi}{2}-\delta_k]}(t)$, for $t \in [0, \frac{\pi}{2}]$. The partial sums $S_N = \sum_{k=1}^N \chi_{[\delta_k,\delta_{k+1})} - \chi_{(\frac{\pi}{2}-\delta_{k+1},\frac{\pi}{2}-\delta_k]}$, are Borel measurable functions satisfying $\sup_{N\geq 1}\|S_N\|_\infty = 1$ and $S_N \to g$ pointwise. Then we obtain the following convergence in the SOT topology (see, e.g. [29, Sec. 2.18]):

$$g(\Gamma) = \operatorname{sgn}(\cos(2\Gamma)) = \sum_{k\geq 1} E_k^+ - \sum_{k\geq 1} E_k^-.$$

Any unit vector $\xi \in \mathcal{L}$ can be uniquely written as a convergent series $\xi = \sum_{k\geq 0} a_k\xi_k$, where $\sum_{k\geq 0} |a_k|^2 = 1$, $\xi_k \in R(P_k)$, $\|\xi_k\| = 1$ and $k \geq 0$. Indeed, the term corresponding to P_0 is included or not according to whether $P_0 > 0$ or $P_0 = 0$. On the other hand, observe that $P_k = E_k^+ + E_k^- \geq E_k^+ > 0$ for all $k \geq 1$. Since V_n, $g(\Gamma)$ and D are functions of the operator Γ, it follows that these are commuting operators. Then, note $(V_n - g(\Gamma))D\xi_k = D(V_n - g(\Gamma))\xi_k = 0$ for $0 \leq k \leq n$, and $(V_n - g(\Gamma))D\xi_k = D(V_n - g(\Gamma))\xi_k = -2DE_k^+\xi_k$ for $k > n$. Let E_{Γ,ξ_k} be the real valued measure defined by $E_{\Gamma,\xi_k}(\Omega) = \|E_\Gamma(\Omega)\xi_k\|^2$, for Ω measurable subset of $\sigma(\Gamma)$. Then we have the following estimate for $k > n$,

$$\begin{aligned}\|(V_n - g(\Gamma))D\xi_k\|^2 = \|2DE_k^+\xi_k\|^2 &= 4\int_{[\delta_k,\delta_{k+1})} 1 + 4(i\cos(t)\sin(t))^2 dE_{\Gamma,\xi_k}(t) \\ &= 4\int_{[\delta_k,\delta_{k+1})} \cos^2(2t) dE_{\Gamma,\xi_k}(t) \leq 4\cos^2(2\delta_k)E_{\Gamma,\xi_k}([\delta_k,\delta_{k+1})) \\ &= 4\cos^2(2\delta_k)\|E_k^+\xi_k\|^2 \leq 4\cos^2(2\delta_n).\end{aligned}$$

Since $R(E_k^+)$ and $R(E_j^+)$ are orthogonal subspaces for $k \neq j$, we use the previous expression of a unit vector $\xi \in \mathcal{L}$ and the above estimate to obtain

$$\begin{aligned}\|(V_n - g(\Gamma))D\xi\|^2 = \left\|\sum_{k>n} 2a_kDE_k^+\xi_k\right\|^2 &= \sum_{k>n} \|2a_kDE_k^+\xi_k\|^2 \\ &\leq 4\cos^2(2\delta_n)\left(\sum_{k>n} |a_k|^2\right) \leq 4\cos^2(2\delta_n) \to 0.\end{aligned}$$

We thus get $\|Q_n - Q\| \to 0$.

It remains to show that $\mathcal{O}_{(P_n,Q_n)} \cap \mathcal{O}_{(P_m,Q_m)} = \emptyset$ for $n \neq m$. To this end, note that, for instance when $n > m$, we have $E_n^+(V_n - V_m) = 2E_n^+ > 0$. Hence $V_n \neq V_m$ for $n \neq m$. Now observe that V_n can be rewritten as

$$V_n = \sum_{k=1}^{n} P_k - \sum_{k>n} P_k,$$

which means that V_n are given by Borel functional calculus of B. This implies $WV_nW^* = V_n$ for all $W \in \mathcal{U}_{\{B\}'}$ and $n \geq 1$. Then from the expression of the action in (5) we find that pairs in $\mathcal{O}_{(P_n,Q_n)}$ have the form $(W_1 \oplus W_2) \cdot (P_n, Q_n) = (P_{W_1W_2^*}, Q_{W_1W_2^*, W_1V_nW_1^*}) = (P_{W_1W_2^*}, Q_{W_1W_2^*, V_n})$ for $W_1, W_2 \in \mathcal{U}_{\{B\}'}$. Similarly, pairs in $\mathcal{O}_{(P_m,Q_m)}$ have the form $(P_{W_1W_2^*}, Q_{W_1W_2^*, V_m})$. By using the bijection in Remark 2.3 and the fact that $V_n \neq V_m$ for $n \neq m$, we conclude $\mathcal{O}_{(P_n,Q_n)} \cap \mathcal{O}_{(P_m,Q_m)} = \emptyset$. □

We denote by $\mathcal{P}_{(P,Q)}$ and $\mathcal{C}_{(P,Q)}$ the path component and the connected component of a pair $(P, Q) \in \mathcal{C}_A$, respectively.

Remark 3.5. *i)* Let $A \in \mathcal{B}(\mathcal{H})$ be an injective operator. Then $\mathcal{O}_{(P,Q)} = \mathcal{P}_{(P,Q)}$ for every pair $(P, Q) \in \mathcal{C}_A$. Indeed, note that Theorem 2.4 gives that the action of $\mathcal{U}_{\{A\}'}$ is transitive on $\mathcal{P}_{(P,Q)}$. Thus, the inclusion $\mathcal{P}_{(P,Q)} \subseteq \mathcal{O}_{(P,Q)}$ follows. For the reversed inclusion, take $(P_0, Q_0) \in \mathcal{O}_{(P,Q)}$. Then there is a unitary $W \in \mathcal{U}_{\{A\}'}$ such that $W \cdot (P, Q) = (P_0, Q_0)$. Since $\{A\}'$ is a von Neumann algebra, there exists an anti-selfadjoint operator $Z \in \{A\}'$ such that $e^Z = W$. Thus, $\delta(t) = e^{tZ} \cdot (P, Q)$, $t \in [0, 1]$, is a continuous path in $\mathcal{C}_A$ joining (P, Q) and (P_0, Q_0). This yields $(P_0, Q_0) \in \mathcal{P}_{(P,Q)}$, and hence $\mathcal{P}_{(P,Q)} \supseteq \mathcal{O}_{(P,Q)}$, as desired.

ii) Let $A = B \oplus (-B)$ be an injective operator, where B is an anti-selfadjoint operator such that $\sigma(B) \subseteq [0, \frac{i}{2}]$. We observe that, with this assumption on A, and the notation given in Remark 2.3, we can consider again the map $F : \mathcal{C}_A \to \mathcal{R}$, defined by $F(P_U, Q_{U,V}) = (U, V)$.

Now we can state and proof the main result of this section.

Theorem 3.6. *Let $A \in \mathcal{B}(\mathcal{H})$ be an injective operator such that $\mathcal{C}_A \neq \emptyset$. Then the following conditions are equivalent:*

i) The action of $\mathcal{U}_{\{A\}'}$ on $\mathcal{C}_A$ is locally transitive.

ii) $\frac{i}{2}$ is not in or is an isolated point of $\sigma(A)$.

iii) $\frac{\pi}{4}$ is not in or is an isolated point of $\sigma(\Gamma)$, where Γ is the operator angle associated to any $(P, Q) \in \mathcal{C}_A$.

iv) $\mathcal{P}_{(P,Q)} = \mathcal{C}_{(P,Q)}$, for every $(P, Q) \in \mathcal{C}_A$

If, in addition, $A = B \oplus (-B)$ for some anti-selfadjoint operator B with $\sigma(B) \subseteq [0, \frac{i}{2}]$, the previous items are equivalent to the following:

v) The map $F : \mathcal{C}_A \to \mathcal{R}$ defined by $F(P_U, Q_{U,V}) = (U, V)$, is continuous.

Proof. *i)* $\to$ *ii)* If $\frac{i}{2}$ is not an isolated point of $\sigma(A)$, then the sequence $\{(P_n, Q_n)\}_{n\geq 1}$ constructed in Lemma 3.4 converges to (P, Q) and it is contained in infinitely many distinct orbits. Hence the action is not locally transitive.

ii) $\to$ *i)* Suppose that $\frac{i}{2}$ is not in or is an isolated point of $\sigma(A)$. Again, since local transitivity is an invariant property by unitary conjugation of $\mathcal{C}_A$, we may assume $A = B \oplus (-B)$ for some anti-selfadjoint operator B such that $\sigma(B) \subseteq [0, \frac{i}{2}]$. Set $D = \sqrt{I + 4B^2}$. The assumption on the spectrum of A means that 0 is not in or is an isolated point of $\sigma(D)$. Equivalently, D has closed range (see [8, Lemma A.1]). Take two pairs of projections $(P_i, Q_i) \in \mathcal{C}_A$, $i = 0, 1$, such that $\max\{\|P_0 - P_1\|, \|Q_0 - Q_1\|\} < r$, where $r > 0$ will be determined below. According to Theorem 2.2 we can write

$$P_i = P_{U_i} := \frac{1}{2}\begin{pmatrix} I & U_i \\ U_i^* & I \end{pmatrix}, \quad Q_i = Q_{U_i,V_i} := \frac{1}{2}\begin{pmatrix} I & (V_iD - 2B)U_i \\ U_i^*(V_iD + 2B) & I \end{pmatrix},$$

for $U_i \in \mathcal{U}_{\{B\}'}$ and $V_i \in \{B\}'$, $V_i^* = V_i$ and $V_i^2 = P_{\mathcal{M}}$, $i = 0, 1$, where $\mathcal{M} = \overline{R(D)}$. Then, we have $\|U_1 - U_0\| < 2r$ and $\|(V_1D - 2B)U_1 - (V_0D - 2B)U_0\| < 2r$. Therefore,

$$\begin{aligned} \|(V_1 - V_0)D\| &= \|(V_1D - 2B)U_0 - (V_0D - 2B)U_0\| \\ &\leq \|(V_1D - 2B)(U_0 - U_1)\| + \|(V_1D - 2B)U_1 - (V_0D - 2B)U_0\| \\ &< (\|D\| + 2\|B\|)2r + 2r = 2r(\|D\| + 2\|B\| + 1). \end{aligned}$$

Denote by $D^\dagger$ the Moore-Penrose inverse of D. Since D has closed range, $D^\dagger$ is a bounded linear operator, and it satisfies $DD^\dagger = P_{R(D)} = P_{\mathcal{M}}$. Therefore,

$$\|V_1 - V_0\| = \|(V_1 - V_0)DD^\dagger\| \leq \|(V_1 - V_0)D\|\|D^\dagger\| < 2r(\|D\| + 2\|B\| + 1)\|D^\dagger\|.$$

Taking $r := (2(\|D\| + 2\|B\| + 1)\|D^\dagger\|)^{-1}$, we find that $\|V_1 - V_0\| < 2$.

Now we adapt a well-known technique to conjugate two orthogonal projections (see, e.g., [8, 18, 23]). Indeed, for $E_i := \frac{V_i + I}{2}$, $i = 0, 1$, note that $\mathcal{M}$ and $\mathcal{M}^\perp$ are invariant subspaces for E_i, $E_i|_{\mathcal{M}}$ is a projection because $V_i|_{\mathcal{M}}$ is a symmetry on $\mathcal{M}$, and $\|E_0|_{\mathcal{M}} - E_1|_{\mathcal{M}}\| < 1$ by the estimate $\|V_1 - V_0\| < 2$. Then, we take $S := E_1E_0 + (I - E_1)(I - E_0)$. Observe that $\mathcal{M}$ and $\mathcal{M}^\perp$ are invariant subspaces of S. By standard computations, the estimate $\|E_0|_{\mathcal{M}} - E_1|_{\mathcal{M}}\| < 1$ implies that $\|S|_{\mathcal{M}} - I_{\mathcal{M}}\| < 1$, so that $S|_{\mathcal{M}}$ is invertible. Since $E_i|_{\mathcal{M}^\perp} = \frac{1}{2}I_{\mathcal{M}^\perp}$, then $S|_{\mathcal{M}^\perp} = \frac{1}{2}I_{\mathcal{M}^\perp}$, and hence S is invertible. Then take the unitary $W := S|S|^{-1}$. Since $SE_0 = E_1S = E_1E_0$, it follows that $|S|^2E_0 = S^*SE_0 = S^*E_1S = E_0S^*S = E_0|S|^2$. Thus, $|S|E_0 = E_0|S|$. This gives $WE_0W^* = S|S|^{-1}E_0|S|^{-1}S^* = S|S|^{-2}S^*E_1 = E_1$, or equivalently, $WV_0W^* = V_1$. Also note that $V_i \in \{B\}'$, $i = 0, 1$, which implies $S \in \{B\}'$, and consequently, $W \in \{B\}'$ by functional calculus.

We write $W_1 = W$ and $W_2 = U_1^*WU_0$. Note that $W_1, W_2 \in \mathcal{U}_{\{B\}'}$. Next we use the expression of the action in (5) to get

$$(W_1 \oplus W_2) \cdot (P_{U_0}, Q_{U_0,V_0}) = (P_{W_1U_0W_2^*}, Q_{W_1V_0W_1^*}) = (P_{U_1}, Q_{U_1,V_1}).$$

ii) $\leftrightarrow$ *iii)* Take $(P, Q) \in \mathcal{C}_A$. After a unitary conjugation we may assume that A, P and Q have the form (7) and (8). In particular, we take $A = iCS \oplus (-iCS)$, where $C = \cos(\Gamma')$ and $S = \sin(\Gamma')$. We have $\Gamma' = X^*\Gamma X$, where Γ the operator angle corresponding to (P, Q) and X is a unitary operator (see Remark 3.1 *ii)*). By the spectral mapping theorem,

$$\sigma(A) = \{i\cos(\lambda)\sin(\lambda) : \lambda \in \sigma(\Gamma)\} \cup \{-i\cos(\lambda)\sin(\lambda) : \lambda \in \sigma(\Gamma)\},$$

which clearly gives the equivalence between *ii)* and *iii)*.

i) $\to$ *iv)* Assume that the action is locally transitive. We consider the set $\mathcal{X}$ consisting of all pairs of projections (P_0, Q_0) in $\mathcal{C}_{(P,Q)}$ that can be joined to (P, Q) by a path contained in $\mathcal{C}_{(P,Q)}$.

We will prove that $\mathcal{X}$ is a nonempty subset of $\mathcal{C}_{(P,Q)}$ which is closed and open relatively to $\mathcal{C}_{(P,Q)}$. Since $\mathcal{C}_{(P,Q)}$ is connected, we will get $\mathcal{X} = \mathcal{C}_{(P,Q)}$, and hence $\mathcal{P}_{(P,Q)} = \mathcal{C}_{(P,Q)}$.

First note that $\mathcal{X} \neq \emptyset$ because $(P,Q) \in \mathcal{X}$ by taking in the definition of $\mathcal{X}$ a constant path. To see that $\mathcal{X}$ is open, take $(P_0,Q_0) \in \mathcal{X}$, and let $r > 0$ be the radius given by the local transitivity of the action. If $\max\{\|P_0 - P_1\|, \|Q_0 - Q_1\|\} < r$, then there exists $W \in \mathcal{U}_{\{A\}'}$ such that $W \cdot (P_0,Q_0) = (P_1,Q_1)$. Given $Z \in \{A\}'$ such that $W = e^Z$, the path $\alpha(t) = e^{tZ} \cdot (P_0,Q_0)$ joins (P_0,Q_0) and (P_1,Q_1) in $\mathcal{C}_A$. By definition of $\mathcal{X}$ there is another path β joining (P,Q) and (P_0,Q_0) contained in $\mathcal{C}_{(P,Q)}$. Hence the concatenation of the paths $\gamma := \beta\#\alpha$ is a path joining (P,Q) and (P_1,Q_1). Observe that γ is contained in $\mathcal{C}_{(P,Q)}$ since $\mathcal{P}_{(P,Q)} \subseteq \mathcal{C}_{(P,Q)}$. This shows that $(P_1,Q_1) \in \mathcal{X}$. Thus, $\mathcal{X}$ is an open subset.

We now prove that $\mathcal{X}$ is closed. Take $(P_n,Q_n) \in \mathcal{X}$ such that $(P_n,Q_n) \to (P_0,Q_0)$, for some $(P_0,Q_0) \in \mathcal{C}_{(P,Q)}$. We take $r > 0$ again given by the local transitivity, then there exists $n_0 \geq 1$ such that $\max\{\|P_0 - P_n\|, \|Q_0 - Q_n\|\} < r$ for all $n \geq n_0$. Fix any of these integers $n \geq n_0$. As in the previous paragraph we can construct a path α joining (P_0,Q_0) and (P_n,Q_n). Using that $(P_n,Q_n) \in \mathcal{X}$, we have another path β joining (P_n,Q_n) and (P,Q). Hence the path $\gamma := \beta\#\alpha$ joins (P_0,Q_0) and (P,Q) in $\mathcal{C}_{(P,Q)}$, which means that $(P_0,Q_0) \in \mathcal{C}_{(P,Q)}$.

iv) $\to$ *i)* Assume that $\mathcal{C}_{(P,Q)} = \mathcal{P}_{(P,Q)}$. By Remark 3.5 *i)*, we have $\mathcal{C}_{(P,Q)} = \mathcal{O}_{(P,Q)}$. Therefore there are nonempty open sets $\mathcal{V}_1, \mathcal{V}_2 \subseteq \mathcal{C}_A$ such that $\mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset$, $\mathcal{O}_{(P,Q)} \subseteq \mathcal{V}_1$ and $\mathcal{C}_A \setminus \mathcal{O}_{(P,Q)} \subseteq \mathcal{V}_2$. Suppose that the action is not locally transitive. Then we can construct a sequence $\{(P_n,Q_n)\}_{n\geq1}$ as in Lemma 3.4 converging to (P,Q). On one hand, note that $(P_n,Q_n) \in \mathcal{V}_1$ for all $n \geq n_0$, since $(P_n,Q_n) \to (P,Q)$. On the other hand, $\mathcal{O}_{(P_n,Q_n)} \cap \mathcal{O}_{(P_m,Q_m)} = \emptyset$, $n \geq m$, implies that $\mathcal{O}_{(P_n,Q_n)} \subseteq \mathcal{C}_A \setminus \mathcal{O}_{(P,Q)} \subseteq \mathcal{V}_2$ for infinitely many values of $n \geq 1$. Thus, we get $(P_n,Q_n) \in \mathcal{V}_1 \cap \mathcal{V}_2$ for infinitely many values of $n \geq 1$, which is a contradiction.

i) $\leftrightarrow$ *v)* If the action is locally transitive, then $\frac{i}{2}$ is not in or is an isolated point of $\sigma(A)$ by the equivalence we have proved. Equivalently, $R(D)$ is closed, where $D = \sqrt{1+4B^2}$. Then F is continuous by [27, Prop. 2.8].

Now suppose that the action is not locally transitive. For a fixed $(P,Q) \in \mathcal{C}_A$, Lemma 3.4 gives a sequence $\{(P_n,Q_n)\}_{n\geq1}$ such that $\|P_n - P\| \to 0$, $\|Q_n - Q\| \to 0$ and $\mathcal{O}_{(P_n,Q_n)} \cap \mathcal{O}_{(P_m,Q_m)} = \emptyset$, for all $m \neq m$. Set $P_n = P_{U_n}$ and $Q_n = Q_{U_n,V_n}$, where $(U_n,V_n) \in \mathcal{R}$. Then note that $\|V_n - V_m\| = 2$, for all $m \neq n$. Otherwise, if $\|V_n - V_m\| < 2$, then the same argument as in the implication *ii)* $\to$ *i)* would give a unitary $W = W_1 \oplus W_2 \in \mathcal{U}_{\{A\}'}$ such that $W \cdot (P_n,Q_n) = (P_m,Q_m)$, which contradicts the condition $\mathcal{O}_{(P_n,Q_n)} \cap \mathcal{O}_{(P_m,Q_m)} = \emptyset$. Hence $\{V_n\}_{n\geq1}$ cannot be convergent, and F is not continuous at (P,Q). □

Remark 3.7. We give the following elementary remarks about the previous proof.

i) From Theorem 2.2 we know that $\mathcal{C}_A \neq \emptyset$ if and only if A is unitarily equivalent to $B \oplus (-B)$, where $B = -B^*$ and $\sigma(B) \subseteq [0, \frac{i}{2}]$. Hence $\sigma(A) = \sigma(B) \cup (-\sigma(B))$. Then we could have also written $-\frac{i}{2}$ instead of $\frac{i}{2}$ in the statement of our previous result to characterize the local transitivity of the action.

ii) The radius $r > 0$ in the previous proof that guarantees that two pairs are in the same orbit does not depend on the pairs of projections in $\mathcal{C}_A$. Indeed, recall that $r = (2(\|D\| + 2\|B\| + 1)\|D^\dagger\|)^{-1}$, so it can be written only in terms of the spectrum of A as follows:

$$r = \frac{1}{2}\left(\max_{\lambda\in\sigma(A)} \sqrt{1+4\lambda^2} + 2\|A\| + 1\right)^{-1}\left(\min_{\lambda\in\sigma(A)\setminus\{\pm\frac{i}{2}\}} \sqrt{1+4\lambda^2}\right).$$

In the second factor we have used that $\|D^\dagger\| = \min_{\mu\in\sigma(D)\setminus\{0\}} \mu$ since $D \geq 0$ has closed range.

iii) Under the assumption that the action is locally transitive, the distance between two disjoint orbits in $\mathcal{C}_A$ corresponding to pairs (P_0, Q_0) and (P_1, Q_1) must satisfy

$$d(\mathcal{O}_{(P_0,Q_0)}, \mathcal{O}_{(P_1,Q_1)}) = \inf\{\max\{\|P-P'\|, \|Q-Q'\|\} : (P,Q) \in \mathcal{O}_{(P_0,Q_0)},\ (P',Q') \in \mathcal{O}_{(P_1,Q_1)}\} \geq r.$$

iv) As we quote in the proof, it was shown that F is continuous if $R(D)$ is closed, $D = \sqrt{1+4B^2}$ ([27, Prop. 2.8]). Also it was given an example showing that F can be discontinuous if $R(D)$ is not closed ([27, Remark 2.3]). Our proof above regarding the continuity of F is inspired in that example. Furthermore, note that when the action is not locally transitive, the above idea to prove that F is discontinuous can be carried out at any pair $(P,Q) \in \mathcal{C}_A$. Hence F turns out be discontinuous at every point of $\mathcal{C}_A$.

Remark 3.8 (Local transitivity for A non injective). In the case where A is not injective the action defined in Remark 2.5 *iii*) fails to be locally transitive whenever $\dim \mathcal{H}_0^\perp \geq 4$. To show this take $r > 0$ and define the projections $P_i = E_i \oplus P_0$ and $Q_i = F_i \oplus Q_0$, $i = 1, 2$, where E_i, F_i acts on $\mathcal{H}_0^\perp$ and P_0 acts on $\mathcal{H}_0$, and

$$E_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad F_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Here we have assumed that $\dim \mathcal{H}_0^\perp = 4$, otherwise in what follows one can replace E_i and F_i by $E_i \oplus 0$ and $F_i \oplus 0$, respectively. Pick $t > 0$ such that $c = \cos(t)$, $s = \sin(t)$, and

$$E_2 = \begin{pmatrix} c^2 & cs & 0 & 0 \\ cs & s^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad F_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & c^2 & cs \\ 0 & 0 & cs & s^2 \end{pmatrix},$$

satisfy $\|E_1 - E_2\| < r$ and $\|F_1 - F_2\| < r$. Observe that $E_iF_i = F_iE_i$, $i = 1, 2$. Setting $[P_0, Q_0] = A_0 = A|_{\mathcal{H}_0}$, we have$(P_1, Q_1), (P_2, Q_2) \in \mathcal{C}_A$, and consequently $\max\{\|P_1 - P_2\|, \|Q_1 - Q_2\|\} < r$. However, the action fails to be locally transitive. In fact, there is no operator $(G, W) \in \mathcal{U}(\mathcal{H}_0^\perp) \times \mathcal{U}_{\{A_0\}'}$ such that

$$(G, W) \cdot (P_1, Q_1) = (GE_1G^* \oplus WP_0W^*, GF_1G^* \oplus WQ_0W^*) = (E_2 \oplus P_0, F_2 \oplus Q_0) = (P_2, Q_2).$$

Indeed, if this holds true, $E_1 \leq F_1$ would give $GE_1G^* \leq GF_1G^*$, or in other words, $E_2 \leq F_2$, which is clearly false.

4 Differentiable structure

In this section we treat the homogeneous space structure of orbits $\mathcal{O}_{(P,Q)} \subseteq \mathcal{C}_A$ when A an injective operator. The results are then extended in Remark 4.8 to the case of a non injective operator A.

Let A be an injective operator. The orbits can be considered as a quotient space $\mathcal{O}_{(P,Q)} \simeq \mathcal{U}_{\{A\}'}/\mathcal{I}_{(P,Q)}$, where $\mathcal{I}_{(P,Q)}$ is the isotropy group at $(P,Q) \in \mathcal{C}_A$, i.e.

$$\mathcal{I}_{(P,Q)} = \{W \in \mathcal{U}_{\{A\}'} : W \cdot (P,Q) = (P,Q)\}.$$

The following characterization of $\mathcal{I}_{(P,Q)}$ will be useful.

Lemma 4.1. *Assume that A is injective and $(P,Q) \in \mathcal{C}_A$ have the form (7) for some operator Γ such that $\sigma(\Gamma) \subseteq [0, \frac{\pi}{2}]$. Then the isotropy group can be rewritten as*

$$\mathcal{I}_{(P,Q)} = \left\{ \begin{pmatrix} W & 0 \\ 0 & W \end{pmatrix} : W \in \mathcal{U}_{\{\Gamma\}'} \right\}. \tag{9}$$

Proof. Under our assumption recall that $\mathcal{U}_{\{A\}'} \subseteq \mathcal{B}(\mathcal{L} \times \mathcal{L})$ is given by

$$\mathcal{U}_{\{A\}'} = \left\{ \begin{pmatrix} W_1 & 0 \\ 0 & W_2 \end{pmatrix} : W_1, W_2 \in \mathcal{U}_{\{CS\}'} \right\},$$

where $C = \cos(\Gamma)$ and $S = \sin(\Gamma)$. Then it follows that

$$\begin{pmatrix} W_1 & 0 \\ 0 & W_2 \end{pmatrix} \begin{pmatrix} I & I \\ I & I \end{pmatrix} = \begin{pmatrix} I & I \\ I & I \end{pmatrix} \begin{pmatrix} W_1 & 0 \\ 0 & W_2 \end{pmatrix}$$

if and only if $W_1 = W_2 := W$. Also note

$$\begin{pmatrix} W & 0 \\ 0 & W \end{pmatrix} \begin{pmatrix} I & e^{-2i\Gamma} \\ e^{2i\Gamma} & I \end{pmatrix} = \begin{pmatrix} I & e^{-2i\Gamma} \\ e^{2i\Gamma} & I \end{pmatrix} \begin{pmatrix} W & 0 \\ 0 & W \end{pmatrix}$$

if and only if $We^{2i\Gamma} = e^{2i\Gamma}W$. Since $\sigma(\Gamma) \subseteq [0, \frac{\pi}{2}]$, there exists a unique analytic logarithm, and hence $W\Gamma = \Gamma W$. ☐

Remark 4.2. Curiously, the simultaneous commutator of two generic projections was already computed in [17] as follows

$$\left\{ \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} C^2 & CS \\ CS & S^2 \end{pmatrix} \right\}' = \left\{ \begin{pmatrix} X & 0 \\ 0 & X \end{pmatrix} \in \mathcal{B}(\mathcal{L} \times \mathcal{L}) : X \in \{C\}' \right\}.$$

We have proved the above lemma for a more self-contained exposition. Notice that the isotropy group consists of unitary operators, but both characterizations coincide for arbitrary operators on $\mathcal{L} \times \mathcal{L}$. Indeed, using a conjugation of the projections by the operator N as in (7), and noting that $C = \cos(\Gamma)$ with $\sigma(\Gamma) \subseteq [0, \frac{\pi}{2}]$, where the cosine is injective, we obtain $CX = XC$ if and only if $\Gamma X = X\Gamma$.

Remark 4.3. Take Γ a selfadjoint operator with $\sigma(\Gamma) \subseteq [0, \frac{\pi}{2}]$, $C = \cos(\Gamma)$ and $S = \sin(\Gamma)$. From the previous characterization of the isotropy group, it is of interest to understand the relation between $\{\Gamma\}'$ and $\{CS\}'$. It is evident that $\{\Gamma\}' \subseteq \{CS\}'$, though the reversed inclusion may not hold true. This happens because the function $f(t) = \cos(t)\sin(t)$ is not injective in $[0, \frac{\pi}{2}]$. The simplest example can be given with 2×2 matrices. Take $a \in [0, \frac{\pi}{4})$ and $b = \frac{\pi}{2} - a \in (\frac{\pi}{4}, \frac{\pi}{2}]$. Consider the diagonal matrix $\Gamma = \text{diag}(a, b)$. Then, we have $CS = \cos(a)\sin(a)I$. Thus, $X\Gamma = \Gamma X$ if and only if X is diagonal, and $XCS = CSX$ for all X. We will give equivalent conditions to have $\{CS\}' = \{\Gamma\}'$ in Proposition 5.6. Furthermore, we will see in the next section that this condition on the commutants is linked to the existence of geodesics of minimal length in the orbits.

Before giving our main result we recall basic facts about partitions and a convergence result that will be useful to construct the differentiable structure.

Remark 4.4. For $\mathcal{X} \subseteq \mathbb{C}$ we say that $\Pi = \{\Omega_1, \dots, \Omega_n\}$ is a partition of $\mathcal{X}$ if the following conditions are satisfied: Ω_j is a Borel subset of $\mathcal{X}$, $\Omega_i \cap \Omega_j = \emptyset$ if $i \neq j$, for all $i, j = 1, \dots, n$, and $\cup_{i=j}^n \Omega_j = \mathcal{X}$. A partition $\Pi' = \{\Omega'_1, \dots, \Omega'_m\}$ is a refinement of another partition $\Pi = \{\Omega_1, \dots, \Omega_n\}$ if for every $\Omega \in \Pi$, there are $\Omega'_{i_1}, \dots, \Omega'_{i_k} \in \Pi'$, $1 \leq k \leq m$, such that $\Omega = \cup_{j=1}^k \Omega'_{i_j}$. In this case we write $\Pi \preceq \Pi'$. In this way the set of all partitions of $\mathcal{X}$ becomes an ordered set. Given two partitions $\Pi = \{\Omega_1, \dots, \Omega_n\}$, $\Pi' = \{\Omega'_1, \dots, \Omega'_m\}$, then $\Pi'' := \{\Omega_i \cap \Omega'_j : i = 1, \dots, n,\, j = 1, \dots, m\}$ is also a partition, which satisfies $\Pi \preceq \Pi''$ and $\Pi' \preceq \Pi''$. Thus, the set of all partitions of $\mathcal{X}$ is also a directed set.

Lemma 4.5. *([7, Lemma 3.3]) Let $\mathcal{Y}$ a Banach space, $\mathcal{A}$ a von Neumann algebra in $\mathcal{B}(\mathcal{H})$, and $\mathcal{B}(\mathcal{Y}, \mathcal{A})$ the Banach space of bounded linear operators $T : \mathcal{Y} \to \mathcal{A}$. Let $\{T_i\}_{i\in I}$ be a bounded net in $\mathcal{B}(\mathcal{Y}, \mathcal{A})$. Then there exists a subnet $\{T_j\}_{j\in J}$ and $T_0 \in \mathcal{B}(\mathcal{Y}, \mathcal{A})$, $\|T_0\| \leq \sup_{i\in I} \|T_i\|$ such that for each $y \in \mathcal{Y}$, the net $\{T_j(y)\}_{j\in J}$ converges to $T_0(y)$ in the WOT (weak operator topology).*

Theorem 4.6. *Let $A \in \mathcal{B}(\mathcal{H})$ be an injective operator and $(P, Q) \in \mathcal{C}_A$. Then $\mathcal{I}_{(P,Q)}$ is a Banach-Lie subgroup of $\mathcal{U}_{\{A\}'}$, and $\mathcal{O}_{(P,Q)} \simeq \mathcal{U}_{\{A\}'}/\mathcal{I}_{(P,Q)}$ is a reductive homogeneous space.*

Proof. The proof is divided into the following two steps.

First step: We first show that $\mathcal{O}_{(P,Q)}$ is a smooth homogeneous space. After a unitary conjugation, we may assume that A and (P, Q) have the form (7). We consider the exponential map of the Banach-Lie group $\mathcal{U}_{\{A\}'}$ given by $\exp : \mathfrak{u}_{\{A\}'} \to \mathcal{U}_{\{A\}'}$, $\exp(Z) = e^Z$ (usual exponential of operators). The Lie algebra of $\mathcal{U}_{\{A\}'}$ can be identified with

$$\mathfrak{u}_{\{A\}'} = \left\{ \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} : X, Y \in \{CS\}',\, X^* = -X,\, Y^* = -Y \right\}.$$

By Lemma 4.1 we know that $\mathcal{I}_{(P,Q)}$ can be expressed as in (9), so its Lie algebra is given by

$$\begin{aligned} \mathfrak{i}_{(P,Q)} &= \left\{ \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \in \mathfrak{u}_{\{A\}'} : \exp\left(t \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \right) \in \mathcal{I}_{(P,Q)},\, \forall t \in \mathbb{R} \right\} \\ &= \left\{ \begin{pmatrix} Z & 0 \\ 0 & Z \end{pmatrix} : Z \in \{\Gamma\}',\, Z^* = -Z \right\}. \end{aligned}$$

Consider the restriction of the exponential map to arbitrary open neighbourhoods $\mathcal{V}$ and $\mathcal{W}$ of $0 \in \mathfrak{u}_{\{A\}'}$ and $1 \in \mathcal{U}_{\{A\}'}$, respectively, such that $\exp : \mathcal{V} \to \mathcal{W}$ is a diffeomorphism. To show that $\mathcal{I}_{(P,Q)}$ is a Banach-Lie subgroup of $\mathcal{U}_{\{A\}'}$ is equivalent to prove that $\exp(\mathcal{V} \cap \mathfrak{i}_{(P,Q)}) = \mathcal{W} \cap \mathcal{I}_{(P,Q)}$, for all sufficiently small open neighbourhoods $\mathcal{V}$, $\mathcal{W}$ as mentioned, and $\mathfrak{i}_{(P,Q)}$ is a (real) closed and complemented subspace of $\mathfrak{u}_{\{A\}'}$ (see [28, Prop. 8.13]).

For the first condition, we have to see that every $U \in \mathcal{W} \cap \mathcal{I}_{(P,Q)}$ can be obtained as $e^Z = U$, where $Z \in \mathcal{V} \cap \mathfrak{i}_{(P,Q)}$. We may further assume that $\mathcal{W} \subseteq \{U \in \mathcal{U}_{\{A\}'} : \|U - I\| < 1\}$. The logarithm is given by the series $\log(U) = \sum_{n\geq 1} (-1)^{n+1} \frac{(U-I)^n}{n}$. From this expression and using that $U \in \mathcal{I}_{(P,Q)}$, it is clear that $Z = \log(U) \in \mathfrak{i}_{(P,Q)}$. Also it is straightforward to see that $\mathfrak{i}_{(P,Q)}$ is closed in the operator norm topology. Next to prove that $\mathfrak{i}_{(P,Q)}$ is complemented in $\mathfrak{u}_{\{A\}'}$ we first construct a continuous projection from $\mathcal{B}(\mathcal{L})$ onto $\{\Gamma\}'$. Consider the function

$f(t) = i\cos(t)\sin(t)$, $t \in [0, \frac{\pi}{2}]$. For each $\Pi = \{\Omega_1, \dots, \Omega_n\}$ partition of $\sigma(iCS)$ (see Remark 4.4), let $\mathcal{E}_\Pi : \{CS\}' \to \{CS\}'$ be the map defined by

$$\mathcal{E}_\Pi(X) = \sum_{j=1}^{n} E_\Gamma\left(f^{-1}(\Omega_j) \cap \left[0, \frac{\pi}{4}\right]\right) X E_\Gamma\left(f^{-1}(\Omega_j) \cap \left[0, \frac{\pi}{4}\right]\right) + \\ + E_\Gamma\left(f^{-1}(\Omega_j) \cap \left(\frac{\pi}{4}, \frac{\pi}{2}\right]\right) X E_\Gamma\left(f^{-1}(\Omega_j) \cap \left(\frac{\pi}{4}, \frac{\pi}{2}\right]\right).$$

The projections $E_\Gamma(\Delta)$, where Δ is a Borel subset of $\sigma(\Gamma)$, commute with CS, so that $\mathcal{E}_\Pi(X)CS = CS\mathcal{E}_\Pi(X)$, for all $X \in \{CS\}'$. The map $\mathcal{E}_\Pi$ is a projection ($\mathcal{E}_\Pi^2 = \mathcal{E}_\Pi$) because the $2n$ projections $E_\Gamma\left(f^{-1}(\Omega_i) \cap \left[0, \frac{\pi}{4}\right]\right)$, $E_\Gamma\left(f^{-1}(\Omega_j) \cap \left(\frac{\pi}{4}, \frac{\pi}{2}\right]\right)$, $i, j = 1, \dots, n$, are mutually orthogonal. Also note that $\mathcal{E}_\Pi$ is clearly continuous in the topology defined by the operator norm. Furthermore, using again that the projections are mutually orthogonal it is not difficult to check that $\|\mathcal{E}_\Pi\| := \sup_{\|X\|=1} \|\mathcal{E}_\Pi(X)\| \leq 1$. Let for I be the set of all partitions of $\sigma(iCS)$. Therefore we can apply Lemma 4.5 with $\mathcal{Y} = \mathcal{A} = \{CS\}'$ and the net $\{\mathcal{E}_\Pi\}_{\Pi \in I}$ to find a subnet $\{\mathcal{E}_\Pi\}_{\Pi \in J}$ and $\mathcal{E} \in \mathcal{B}(\{CS\}') = \mathcal{B}(\{CS\}', \{CS\}')$ such that $\|\mathcal{E}\| \leq 1$, and $\{\mathcal{E}_\Pi(X)\}_{\Pi \in J}$ converges in the WOT to $\mathcal{E}(X)$, for all $X \in \{CS\}'$.

Next we claim that $\mathcal{E}(X) = X$ for all $X \in \{\Gamma\}'$. For we take a partition $\Pi = \{\Omega_1, \dots, \Omega_n\}$ of $\sigma(iCS)$ and note that for $X \in \{\Gamma\}'$,

$$\begin{aligned}\mathcal{E}_\Pi(X) &= \sum_{j=1}^{n} E_\Gamma\left(f^{-1}(\Omega_j) \cap \left[0, \frac{\pi}{4}\right]\right) X + E_\Gamma\left(f^{-1}(\Omega_j) \cap \left(\frac{\pi}{4}, \frac{\pi}{2}\right]\right) X \\ &= \sum_{j=1}^{n} E_\Gamma(f^{-1}(\Omega_j))X = X.\end{aligned}$$

Since Π is arbitrary, this implies $\mathcal{E}(X) = X$ for all $X \in \{\Gamma\}'$, and our claim follows. In particular, this gives $R(\mathcal{E}) := \{\mathcal{E}(X) : X \in \{CS\}'\} \supseteq \{\Gamma\}'$.

We now prove that $R(\mathcal{E}) \subseteq \{\Gamma\}'$. This is equivalent to show that $\mathcal{E}(X)$ commutes with $E_\Gamma(\Delta)$, for $X \in \{CS\}'$ and an arbitrary Borel set $\Delta \subseteq \sigma(\Gamma)$. We now consider in detail the case where $\Delta \subseteq [0, \frac{\pi}{4})$. By the condition $X \in \{CS\}'$, we have $E_{iCS}(f(\Delta))X = XE_{iCS}(f(\Delta))$. Recall that $E_{iCS}(f(\Delta)) = E_\Gamma(f^{-1}(f(\Delta))) = E_\Gamma(\Delta) + E_\Gamma(\frac{\pi}{2} - \Delta) := E_\Gamma^+ + E_\Gamma^-$, where we use the additivity of the spectral measure since $\Delta \cap (\frac{\pi}{2} - \Delta) = \emptyset$ for $\Delta \subseteq [0, \frac{\pi}{4})$. Thus, $(E_\Gamma^+ + E_\Gamma^-)X = X(E_\Gamma^+ + E_\Gamma^-)$. For every $\Pi = \{\Omega_1, \dots, \Omega_n\}$ partition of $\sigma(iCS)$, note that $E_\Gamma^+ E_\Gamma\left(f^{-1}(\Omega_j) \cap \left(\frac{\pi}{4}, \frac{\pi}{2}\right]\right) = 0$. Using this together with $E_\Gamma^+ X = -E_\Gamma^- X + X(E_\Gamma^+ + E_\Gamma^-)$, we get

$$\begin{aligned}E_\Gamma^+ \mathcal{E}_\Pi(X) &= \sum_{j=1}^{n} E_\Gamma\left(f^{-1}(\Omega_j) \cap \left[0, \frac{\pi}{4}\right]\right) E_\Gamma^+ X E_\Gamma\left(f^{-1}(\Omega_j) \cap \left[0, \frac{\pi}{4}\right]\right) \\ &= \sum_{j=1}^{n} E_\Gamma\left(f^{-1}(\Omega_j) \cap \left[0, \frac{\pi}{4}\right]\right) (-E_\Gamma^- X + X(E_\Gamma^+ + E_\Gamma^-)) E_\Gamma\left(f^{-1}(\Omega_j) \cap \left[0, \frac{\pi}{4}\right]\right) \\ &= \sum_{j=1}^{n} E_\Gamma\left(f^{-1}(\Omega_j) \cap \left[0, \frac{\pi}{4}\right]\right) X E_\Gamma^+ E_\Gamma\left(f^{-1}(\Omega_j) \cap \left[0, \frac{\pi}{4}\right]\right) \\ &= \mathcal{E}_\Pi(X) E_\Gamma^+.\end{aligned}$$

Taking the limit we obtain $E_\Gamma(\Delta)\mathcal{E}(X) = \mathcal{E}(X)E_\Gamma(\Delta)$. Next observe that the case where $\Delta \subseteq (\frac{\pi}{4}, \frac{\pi}{2}]$ is similar, and for the point $\lambda = \frac{\pi}{4}$ we have $E_\Gamma(\{\frac{\pi}{4}\}) = E_{iCS}(\{\frac{i}{2}\})$, which commutes with every $X \in \{CS\}'$. The general case follows by noting $E_\Gamma(\Delta) = E_\Gamma(\Delta \cap [0, \frac{\pi}{4})) + E_\Gamma(\Delta \cap \{\frac{\pi}{4}\}) + E_\Gamma(\Delta \cap (\frac{\pi}{4}, \frac{\pi}{2}])$.

We have proved that $\mathcal{E} : \{CS\}' \to \{CS\}'$ is a bounded linear operator such that $R(\mathcal{E}) = \{\Gamma\}'$ and $\mathcal{E}(X) = X$ for all $X \in \{\Gamma\}'$. In particular, the last two conditions imply that $\mathcal{E}^2 = \mathcal{E}$. Since taking the operator adjoint is continuous in the WOT, we note that $\mathcal{E}(X)^* = \mathcal{E}(X^*)$. This gives $\mathcal{E}(\{CS\}'_{as}) = \{\Gamma\}'_{as}$, where $\{CS\}'_{as}$ and $\{\Gamma\}'_{as}$ are the anti-selfadjoint parts of $\{CS\}'$ and $\{\Gamma\}'$, respectively. Finally, we set

$$\mathcal{F} : \mathfrak{u}_{\{A\}'} \to \mathfrak{u}_{\{A\}'}, \quad \mathcal{F}\left(\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}\right) := \begin{pmatrix} \frac{\mathcal{E}(X)+\mathcal{E}(Y)}{2} & 0 \\ 0 & \frac{\mathcal{E}(X)+\mathcal{E}(Y)}{2} \end{pmatrix}.$$

This defines a continuous projection such that $R(\mathcal{F}) = \mathfrak{i}_{(P,Q)}$. Thus, $\mathfrak{i}_{(P,Q)}$ is complemented in $\mathfrak{u}_{\{A\}'}$. Hence $\mathcal{I}_{(P,Q)}$ is a Banach-Lie subgroup of $\mathcal{U}_{\{A\}'}$, and this implies that $\mathcal{O}_{(P,Q)} \simeq \mathcal{U}_{\{A\}'}/\mathcal{I}_{(P,Q)}$ is a smooth homogeneous space (see [9, 28]).

Second step. We now give a reductive structure $\{\mathfrak{m}_{(P_1,Q_1)}\}_{(P_1,Q_1)\in\mathcal{O}_{(P,Q)}}$. The idea is to use the previously constructed supplement of the Lie algebra of the isotropy group at every point $(P_1, Q_1) \in \mathcal{O}_{(P,Q)}$, but now we make explicit the unitary conjugation to understand better how it depends on the pair of projections. Denote by Γ_1 (resp. Γ) the operator angle associated to (P_1, Q_1) (resp. (P, Q)), and C_1, S_1 (resp. C, S) the corresponding operators cosine and sine. According to Corollary 3.3, Γ_1 and Γ are unitarily equivalent for $(P_1, Q_1) \in \mathcal{O}_{(P,Q)}$. Then C_1S_1 and CS are also unitarily equivalent, and hence $\sigma(iC_1S_1) = \sigma(iCS)$, whenever $(P_1, Q_1) \in \mathcal{O}_{(P,Q)}$. This leads us to only use the set $\sigma(iCS)$ when we consider the following projections, which are indeed the same as above defined, but now depending on the pairs $(P_1, Q_1) \in \mathcal{O}_{(P,Q)}$. That is, for $\Pi = \{\Omega_1, \dots, \Omega_n\}$ partition of $\sigma(iCS)$, let $\mathcal{E}_{\Gamma_1,\Pi} : \{C_1S_1\}' \to \{C_1S_1\}'$ be the map defined by

$$\mathcal{E}_{\Gamma_1,\Pi}(X) = \sum_{j=1}^{n} E_{\Gamma_1}\left(f^{-1}(\Omega_j) \cap \left[0, \frac{\pi}{4}\right]\right) X E_{\Gamma_1}\left(f^{-1}(\Omega_j) \cap \left[0, \frac{\pi}{4}\right]\right) + \\ + E_{\Gamma_1}\left(f^{-1}(\Omega_j) \cap \left(\frac{\pi}{4}, \frac{\pi}{2}\right]\right) X E_{\Gamma_1}\left(f^{-1}(\Omega_j) \cap \left(\frac{\pi}{4}, \frac{\pi}{2}\right]\right).$$

As we have seen in the first step the net $\{\mathcal{E}_{\Gamma,\Pi}\}_{\Pi\in I}$ has a subnet $\{\mathcal{E}_{\Gamma,\Pi}\}_{\Pi\in J}$ such that $\{\mathcal{E}_{\Gamma,\Pi}(Y)\}_{\Pi\in J}$ converges in the WOT to $\mathcal{E}_\Gamma(Y)$, for all $Y \in \{CS\}'$, where $\mathcal{E}_\Gamma = \mathcal{E}$ is a projection of norm one onto $\{\Gamma\}'$. Pick $(P_1, Q_1) \in \mathcal{O}_{(P,Q)}$. Thus, $W \cdot (P, Q) = (P_1, Q_1)$ for some $W \in \mathcal{U}_{\{A\}'}$. Again by Corollary 3.3, the restriction $W : \mathcal{L} \to \mathcal{L}_1$ satisfies $W\Gamma W^* = \Gamma_1$, so that $E_{\Gamma_1} = Ad_W \circ E_\Gamma$ and $\{C_1S_1\}' = Ad_W(\{CS\}')$. Hence the net $\{\mathcal{E}_{\Gamma_1,\Pi}\}_{\Pi\in J}$ also satisfies that $\{\mathcal{E}_{\Gamma_1,\Pi}(Y)\}_{\Pi\in J}$ converges in the WOT to $\mathcal{E}_{\Gamma_1}(Y)$, for all $Y \in \{C_1S_1\}'$, and $\mathcal{E}_{\Gamma_1} = Ad_W \circ \mathcal{E}_\Gamma \circ Ad_{W^*}$. Let $X_1 : \mathcal{H} \to \mathcal{L}_1 \times \mathcal{L}_1$ unitary as in Remark 3.1 such that $Ad_{X_1}(P_1)$, $Ad_{X_1}(Q_1)$ and $Ad_{X_1}(A)$ have the form (7). Set

$$\mathcal{F}_{\Gamma_1} : \mathfrak{u}_{\{Ad_{X_1}(A)\}'} \to \mathfrak{u}_{\{Ad_{X_1}(A)\}'}, \quad \mathcal{F}_{\Gamma_1}\left(\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}\right) := \begin{pmatrix} \frac{\mathcal{E}_{\Gamma_1}(X)+\mathcal{E}_{\Gamma_1}(Y)}{2} & 0 \\ 0 & \frac{\mathcal{E}_{\Gamma_1}(X)+\mathcal{E}_{\Gamma_1}(Y)}{2} \end{pmatrix}.$$

In particular, $\mathcal{F}_\Gamma = \mathcal{F}$ is the projection defined in the first step. Then define the reductive structure by the following distribution of subspaces

$$\mathfrak{m}_{(P_1,Q_1)} := R(I - Ad_{X_1^*} \circ \mathcal{F}_{\Gamma_1} \circ Ad_{X_1}), \quad (P_1, Q_1) \in \mathcal{O}_{(P,Q)}.$$

As in the first step, $\mathcal{F}_{\Gamma_1}$ is continuous projection onto $\{\Gamma_1\}'$, and using that $\mathcal{F}_{\Gamma_1}$ projects onto $Ad_{X_1}(\mathfrak{i}_{(P_1,Q_1)}) = \mathfrak{i}_{(Ad_{X_1}(P_1),Ad_{X_1}(Q_1))} \subseteq \mathfrak{u}_{\{Ad_{X_1}(A)\}'}$, we find that that $\mathfrak{i}_{(P_1,Q_1)} \oplus \mathfrak{m}_{(P_1,Q_1)} = \mathfrak{u}_{\{A\}'}$.

Next we take $W \in \mathcal{I}_{(P_1,Q_1)}$, note that $Ad_{X_1}(W) := W_0$ satisfies $Ad_{W_0} \circ \mathcal{E}_{\Gamma_1} = \mathcal{E}_{\Gamma_1} \circ Ad_{W_0}$, which one can check first for $\mathcal{E}_{\Gamma_1,\Pi}$, and then take the limit. We omit the details, but the latter property of $\mathcal{E}_{\Gamma_1}$ implies that $Ad_W(\mathfrak{m}_{(P_1,Q_1)}) = \mathfrak{m}_{(P_1,Q_1)}$. Now we show that the distribution is smooth. This amounts to proving that the map $\Phi : \mathcal{O}_{(P,Q)} \to \mathcal{B}(\mathfrak{u}_{\{A\}'})$, $\Phi((P_1, Q_1)) = I - Ad_{X_1^*} \circ \mathcal{F}_{\Gamma_1} \circ Ad_{X_1}$, is smooth. For recall that X_1 depends on the pair $(P_1, Q_1) = W \cdot (P, Q)$ as is indicated in Remark 3.1: take R_1 the unitary defined by the polar decomposition $(I - P_1)Q_1P_1|_{\mathcal{L}_1} = R_1^*|(I - P_1)Q_1P_1||_{\mathcal{L}_1}$, $R(P_1) = \mathcal{L}_1$, and

$$X_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} I & -iI \\ I & iI \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & R_1 \end{pmatrix}.$$

This can be written in terms of W, P, and Q by noting that $(I - P)QP|_{\mathcal{L}} = R^*|(I - P)QP||_{\mathcal{L}}$ implies that $WRW^* = R_1$. Then, we obtain $X_1 = (W \oplus W)X(W^* \oplus W^*)$. As we noted above $\mathcal{E}_{\Gamma_1} = Ad_W \circ \mathcal{E}_\Gamma \circ Ad_{W^*}$. Hence $\mathcal{F}_1 = Ad_{(W \oplus W)} \circ \mathcal{F}_\Gamma \circ Ad_{(W^* \oplus W^*)}$, and $\Phi((P_1, Q_1)) = \Phi(W \cdot (P, Q)) = Ad_{(W \oplus W)} \circ (I - \mathcal{F}_\Gamma) \circ Ad_{(W^* \oplus W^*)}$, which is a smooth function of $W \in \mathcal{U}_{\{A\}'}$. Since $\mathcal{O}_{(P,Q)}$ is a smooth homogeneous space, the map $\pi := \pi_{(P,Q)} : \mathcal{U}_{\{A\}'} \to \mathcal{O}_{(P,Q)}$, $\pi(W) = W \cdot (P, Q)$ is a submersion. So it admits at every $(P_1, Q_1) \in \mathcal{O}_{(P,Q)}$ a smooth local cross section $s_1 : \mathcal{W}_1 \to \mathcal{U}_{\{A\}'}$, where $\mathcal{W}_1$ is an open neighbourhood of (P_1, Q_1). We have shown that $\Phi \circ \pi$ is smooth, then it follows that $\Phi|_{\mathcal{W}_1} = \Phi \circ id|_{\mathcal{W}_1} = \Phi \circ \pi \circ s_1|_{\mathcal{W}_1}$ is smooth. Hence Φ is smooth. $\square$

Remark 4.7. We make here a few comments on the projection $\mathcal{F}$ and the reductive structure defined in the previous proof.

i) The algebra $\{\Gamma\}'$ is known to be an injective von Neumann algebra, so there exists a conditional expectation $\tilde{\mathcal{E}}$ from $\mathcal{B}(\mathcal{L})$ onto $\{\Gamma\}'$. For an explanation of this in the context of homogeneous spaces we refer to [9, Thm. 4.3]. As a conditional expectation, it holds $\tilde{\mathcal{E}}(\mathcal{B}(\mathcal{L})_{as}) = \{\Gamma\}'_{as}$, where $\mathcal{B}(\mathcal{L})_{as}$ and $\{\Gamma\}'_{as}$ denote the anti-selfadjoint parts of these algebras. Then, using this fact, one can construct a projection from $\mathfrak{u}_{\{A\}'}$ onto $\mathfrak{i}_{(P,Q)}$ as in the first step of the previous proof.

ii) The proof of the existence of $\mathcal{E} := \mathcal{E}_\Gamma$ in our proof is adapted from [7]. We observe that $\mathcal{E}$ is indeed a conditional expectation from $\{CS\}'$ onto $\{\Gamma\}'$. This follows by using Lemma 4.5, which yields $\|\mathcal{E}\| \leq 1$. Since $\mathcal{E}$ is a projection, it must be $\|\mathcal{E}\| = 1$. Hence $\mathcal{E}$ is a conditional expectation by a well-known result by Tomiyama [30]. Instead of using the results mentioned in *i*) we prefer to explicitly construct the projection $\mathcal{E}$ because it shows the relation between $\{\Gamma\}'$ and $\{CS\}'$ that also appears later in other aspects of the geometry of $\mathcal{C}_A$.

iii) A simple case is following. If $\{CS\}' = \{\Gamma\}'$, then $\mathcal{E} = I$ and

$$\mathcal{F} : \mathfrak{u}_{\{A\}'} \to \mathfrak{u}_{\{A\}'}, \quad \mathcal{F}\left(\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}\right) = \begin{pmatrix} \frac{X+Y}{2} & 0 \\ 0 & \frac{X+Y}{2} \end{pmatrix}.$$

iv) Notice that the definition of $\mathcal{E}$ depends on a convergent subnet given by Lemma 4.5, but the fact that $\mathcal{E}$ projects onto $\{\Gamma\}'$ is independent of the subnet. The convergent subnet might not be unique (see [7] and the references therein). As it is observed in the proof, once a convergent subnet $\{\mathcal{E}_{\Gamma,\Pi}\}_{\Pi \in J}$ is chosen at a point (P, Q), then the corresponding subnet $\{\mathcal{E}_{\Gamma_1,\Pi}\}_{\Pi \in J}$ at every $(P_1, Q_1) \in \mathcal{O}_{(P,Q)}$ is also convergent. Thus each convergent subnet defines a reductive structure $\{\mathfrak{m}_{(P_1,Q_1)}\}_{(P_1,Q_1) \in \mathcal{O}_{(P,Q)}}$.

Remark 4.8 (Homogeneous space structure of orbits for a non injective operator A). Without covering all details, let us generalize Theorem 4.6 to the case where A is a non injective operator. In such case recall that $\mathcal{H}_0^\perp \neq \{0\}$, and any pair $(P, Q) \in \mathcal{C}_A$ has the form $P = E \oplus P_0$, $Q = F \oplus Q_0$ for projections E, F, P_0 and Q_0 as in Remark 2.5. Accordingly, the unitary group $\mathcal{U}_{\{A\}'}$ decomposes as $\mathcal{U}_{\{A\}'} \simeq \mathcal{U}(\mathcal{H}_0^\perp) \times \mathcal{U}_{\{A_0\}'}$, and it acts on $\mathcal{C}_A$ as defined in (6). As a consequence, the orbit of $(P, Q) \in \mathcal{C}_A$ can be identified as $\mathcal{O}_{(P,Q)} \simeq \mathcal{O}_{(E,F)} \times \mathcal{O}_{(P_0,Q_0)}$, where

$$\mathcal{O}_{(E,F)} = \{(GEG^*, GFG^*) : G \in \mathcal{U}(\mathcal{H}_0^\perp)\}$$

and $\mathcal{O}_{(P_0,Q_0)}$ is the orbit corresponding to the injective operator $A_0 := A|_{\mathcal{H}_0}$ on $\mathcal{H}_0$.

Then the isotropy group at a fixed (P, Q) of the action can be identified with $\mathcal{I}_{(P,Q)} \simeq \mathcal{I}_{(E,F)} \times \mathcal{I}_{(P_0,Q_0)}$, where $\mathcal{I}_{(P_0,Q_0)}$ is the isotropy group at (P_0, Q_0) determined by the operator A_0, and

$$\mathcal{I}_{(E,F)} := \{G \in \mathcal{U}(\mathcal{H}_0^\perp) : GE = EG,\, GF = FG\}.$$

Since $EF = FE$, it follows by taking $A = 0$ in (3) that there is no generic part of the pair (E, F) in $\mathcal{H}_0^\perp$. Thus, we have the representations in terms of $\mathcal{H}_0^\perp = \mathcal{H}_{00} \oplus \mathcal{H}_{01} \oplus \mathcal{H}_{10} \oplus \mathcal{H}_{11}$ given by

$$E = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{10}$$

where $\mathcal{H}_{00} = R(E) \cap R(F)$, $\mathcal{H}_{01} = R(E) \cap N(F)$, $\mathcal{H}_{10} = N(E) \cap R(F)$ and $\mathcal{H}_{11} = N(E) \cap N(F)$. Therefore,

$$\mathcal{I}_{(E,F)} = \{G \in \mathcal{U}(\mathcal{H}_0^\perp) : GP_{\mathcal{H}_{ij}} = P_{\mathcal{H}_{ij}}G,\, i,j = 0,1\}.$$

Then the Lie algebra of $\mathcal{I}_{(P,Q)}$ is expressed as $\mathfrak{i}_{(P,Q)} \simeq \mathfrak{i}_{(E,F)} \times \mathfrak{i}_{(P_0,Q_0)}$, where the first factor is given by

$$\mathfrak{i}_{(E,F)} := \{X \in \mathcal{B}(\mathcal{H}_0^\perp)_{as} : XP_{\mathcal{H}_{ij}} = P_{\mathcal{H}_{ij}}X,\, i,j = 0,1\}.$$

and the second factor $\mathfrak{i}_{(P_0,Q_0)}$ is the Lie algebra of operators on $\mathcal{H}_0$ already considered in the injective case. We denote by $\mathcal{B}(\mathcal{H}_0^\perp)_{as}$ the set of all anti-selfadjoint operators in $\mathcal{B}(\mathcal{H}_0^\perp)$. The continuous projection

$$\mathcal{G}_{(E,F)} : \mathcal{B}(\mathcal{H}_0^\perp)_{as} \to \mathcal{B}(\mathcal{H}_0^\perp)_{as}, \quad \mathcal{G}_{(E,F)}(X) = \sum_{i,j=0,1} P_{\mathcal{H}_{ij}} X P_{\mathcal{H}_{ij}},$$

clearly gives a supplement $\mathfrak{m}_{(E,F)} := R(I - \mathcal{G}_{(E,F)})$ of $\mathfrak{i}_{(E,F)}$ in $\mathcal{B}(\mathcal{H}_0^\perp)$. Translating these supplements on $\mathcal{H}_0^\perp$ and using the supplements $\{\mathfrak{m}_{(P_1,Q_1)}\}_{(P_1,Q_1)\in\mathcal{O}_{(P_0,Q_0)}}$ acting on $\mathcal{H}_0$ for the injective case as in the above theorem, we get that $\{\mathfrak{m}_{(E,F)} \times \mathfrak{m}_{(P_1,Q_1)}\}_{(E\oplus P_1, F\oplus Q_1)\in\mathcal{O}_{(P,Q)}}$ is a reductive structure for $\mathcal{O}_{(P,Q)}$. We conclude that $\mathcal{O}_{(P,Q)} \simeq (\mathcal{U}(\mathcal{H}_0^\perp) \times \mathcal{U}_{\{A_0\}'})/\mathcal{I}_{(P,Q)}$ is a reductive homogeneous space.

5 Geodesics of minimal length

In this section we give the explicit form of the geodesics of the reductive connection in the orbits. Then we discuss their minimality with respect to a Finsler metric that is naturally defined using the structure of homogeneous spaces of a unitary group. As before we only consider in detail the case of A an injective operator and orbits $\mathcal{O}_{(P,Q)} \subseteq \mathcal{C}_A$. We end with some remarks for the case of a non injective operator A.

Since the orbit $\mathcal{O}_{(P,Q)} \simeq \mathcal{U}_{\{A\}'}/\mathcal{I}_{(P_1,Q_1)}$ is a smooth homogeneous space, tangent vectors at $(P_1, Q_1) \in \mathcal{O}_{(P,Q)}$ are given by the range of the differential at the identity of the submersion $\pi_{(P_1,Q_1)} : \mathcal{U}_{\{A\}'} \to \mathcal{O}_{(P,Q)}$, $\pi_{(P_1,Q_1)}(W) = (WP_1W^*, WQ_1W^*)$. Thus, we have that tangent spaces have the form

$$\begin{aligned}(T\mathcal{O}_{(P,Q)})_{(P_1,Q_1)} &= R(T_1\pi_{(P_1,Q_1)}) \\ &= \{(ZP_1 - P_1Z, ZQ_1 - Q_1Z) : Z \in \mathfrak{u}_{\{A\}'}\} \simeq \mathfrak{u}_{\{A\}'}/\mathfrak{i}_{(P_1,Q_1)}.\end{aligned}$$

We use the notation $V_Z := (ZP_1 - P_1Z, ZQ_1 - Q_1Z) = (T_1\pi_{(P_1,Q_1)})(Z)$, where $Z \in \mathfrak{u}_{\{A\}'}$, for tangent vectors as above. We assume that a reductive structure $\{\mathfrak{m}_{(P_1,Q_1)}\}_{(P_1,Q_1)\in\mathcal{O}_{(P,Q)}}$ is fixed on the orbit $\mathcal{O}_{(P,Q)}$. Each subspace $\mathfrak{m}_{(P_1,Q_1)}$ is constructed as in the previous subsection in terms of projections $\mathcal{E}_{\Gamma_1}$ and $\mathcal{F}_{\Gamma_1}$, where Γ_1 is the operator angle of the (generic) pair (P_1, Q_1).

Proposition 5.1. *Let $A \in \mathcal{B}(\mathcal{H})$ an injective operator and $(P, Q) \in \mathcal{C}_A$. Then the geodesic of the reductive connection at $(P_1, Q_1) \in \mathcal{O}_{(P,Q)}$ with initial velocity $V_Z \in (T\mathcal{O}_{(P,Q)})_{(P_1,Q_1)}$ is given by*

$$\delta(t) = (e^{tK_1(V_Z)}P_1e^{-tK_1(V_Z)}, e^{tK_1(V_Z)}Q_1e^{-tK_1(V_Z)}), \ \ t \in \mathbb{R}$$

where

$$K_1(V_Z) = \begin{pmatrix} M - \mathcal{E}_{\Gamma_1}(M) & NR_1 \\ -R_1^*N & R_1^*(M - \mathcal{E}_{\Gamma_1}(M))R_1 \end{pmatrix}, \ \ Z = \begin{pmatrix} M & NR_1 \\ -R_1^*N & R_1^*MR_1 \end{pmatrix} \in \mathfrak{u}_{\{A\}'}.$$

These operator matrices are with respect to the decomposition $\mathcal{H} = \mathcal{L}_1 \oplus \mathcal{L}_1^\perp$, $\mathcal{L}_1 = R(P_1)$ and $M, N \in \{C_1S_1\}' \subseteq \mathcal{B}(\mathcal{L}_1)$, $M^* = -M$, $N^* = N$. As before, Γ_1 is the operator angle associated to (P_1, Q_1), $C_1 = \cos(\Gamma_1)$ and $S_1 = \sin(\Gamma_1)$. The unitary operator $R_1 : \mathcal{L}_1^\perp \to \mathcal{L}_1$ is defined by $R_1 = |(I - P_1)Q_1P_1|^{-1}P_1Q_1(I - P_1)|_{\mathcal{L}_1^\perp}$.

Proof. The projection given in the Theorem 4.6 onto the subspaces $\{\mathfrak{m}_{(P_1,Q_1)}\}_{(P_1,Q_1)\in\mathcal{O}_{(P,Q)}}$ can also be described as follows. Consider the unitary operator $X_1 : \mathcal{H} \to \mathcal{L}_1 \times \mathcal{L}_1$, $\mathcal{L}_1 = R(P_1)$, as in the proof of Theorem 4.6 (or Remark 3.1) such that $Ad_{X_1}(P_1)$, $Ad_{X_1}(Q_1)$ and $Ad_{X_1}(A)$ have the form (7). Given $Z \in \mathfrak{u}_A$ there are $X, Y \in \{C_1S_1\}' \subseteq \mathcal{B}(\mathcal{L}_1)$, $X^* = -X$, $Y^* = -Y$ such that

$$Ad_{X_1}(Z) = \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}.$$

Equivalently,

$$Z = Ad_{X_1^*}\left(\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}\right) = \frac{1}{2}\begin{pmatrix} X + Y & i(Y - X)R_1 \\ iR_1^*(X - Y) & R_1^*(X + Y)R_1 \end{pmatrix},$$

where the last matrix represents operators in the decomposition $\mathcal{H} = \mathcal{L}_1 \oplus \mathcal{L}_1^\perp$.Therefore,

$$\begin{aligned}(Ad_{X_1^*} \circ \mathcal{F}_{\Gamma_1} \circ Ad_{X_1})(Z) &= (Ad_{X_1^*} \circ \mathcal{F}_{\Gamma_1}) \left(\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \right) \\ &= Ad_{X_1^*} \left(\begin{pmatrix} \frac{\mathcal{E}_{\Gamma_1}(X)+\mathcal{E}_{\Gamma_1}(Y)}{2} & 0 \\ 0 & \frac{\mathcal{E}_{\Gamma_1}(X)+\mathcal{E}_{\Gamma_1}(Y)}{2} \end{pmatrix} \right) \\ &= \begin{pmatrix} \frac{\mathcal{E}_{\Gamma_1}(X)+\mathcal{E}_{\Gamma_1}(Y)}{2} & 0 \\ 0 & R_1^* \left(\frac{\mathcal{E}_{\Gamma_1}(X)+\mathcal{E}_{\Gamma_1}(Y)}{2} \right) R_1 \end{pmatrix}.\end{aligned}$$

Set $M = \frac{1}{2}(X+Y)$ and $N = \frac{i}{2}(Y-X)$. Hence,

$$(I - Ad_{X_1^*} \circ \mathcal{F}_{\Gamma_1} \circ Ad_{X_1}) \left(\begin{pmatrix} M & NR_1 \\ -R_1^*N & R_1^*MR_1 \end{pmatrix} \right) = \begin{pmatrix} M - \mathcal{E}_{\Gamma_1}(M) & NR_1 \\ -R_1^*N & R_1^*(M - \mathcal{E}_{\Gamma_1}(M))R_1 \end{pmatrix},$$

As we have observed at the end of Section 2, geodesics of the reductive connection are given in terms of the 1-from with values in $\mathfrak{u}_{\{A\}'}$ defined by $K_1(V_Z) = Z_1$ if and only if $T_I\pi_{(P_1,Q_1)}(Z_1) = V_Z$, for $Z \in \mathfrak{u}_{\{A\}'}$ and $Z_1 \in \mathfrak{m}_{(P_1,Q_1)}$. Recall that $Z = Z_0 + Z_1$, where $Z_0 \in \mathfrak{i}_{(P_1,Q_1)}$, $Z_1 \in \mathfrak{m}_{(P_1,Q_1)}$, $(T_I\pi_{(P_1,Q_1)})(Z) = V_Z$ and $N(T_I\pi_{(P_1,Q_1)}) = \mathfrak{i}_{(P_1,Q_1)}$. Then using the expression of the projection $I - Ad_{X_1^*} \circ \mathcal{F}_{\Gamma_1} \circ Ad_{X_1}$ given above, we obtain that $Z_1 = (I - Ad_{X_1^*} \circ \mathcal{F}_{\Gamma_1} \circ Ad_{X_1})(Z)$ has the desired form. $\square$

Curves of minimal length in the set of all orthogonal projections using the operator norm on tangent spaces were already studied (see, e.g. [1]). Since in this work we deal with pairs of projections, it is appropriate to endow the orbits with the Finsler metric introduced by Durán, Mata-Lorenzo and Recht [15] for general homogeneous spaces of unitary groups in operator algebras. Fix $(P,Q) \in \mathcal{C}_A$, and take the orbit $\mathcal{O}_{(P,Q)}$. The aforementioned Finsler metric can be defined on $\mathcal{O}_{(P,Q)}$ as follows. Given $(P_1,Q_1) \in \mathcal{O}_{(P,Q)} \cong \mathcal{U}_{\{A\}'}/\mathcal{I}_{(P_1,Q_1)}$ and $V_Z = (ZP_1 - P_1Z, ZQ_1 - Q_1Z) \in (T\mathcal{O}_{(P,Q)})_{(P_1,Q_1)}$, for some $Z \in \mathfrak{u}_{\{A\}'}$, set

$$\begin{aligned}\|V_Z\|_{(P_1,Q_1)} &:= \inf\{\|Z + Z_0\| : Z_0 \in \mathfrak{i}_{(P_1,Q_1)}\} \\ &= \inf\left\{ \left\| \begin{pmatrix} M+L & NR_1 \\ -R_1^*N & R_1^*(M+L)R_1 \end{pmatrix} \right\| : L \in \{\Gamma_1\}'_{as} \right\},\end{aligned} \tag{11}$$

where

$$Z = \begin{pmatrix} M & NR_1 \\ -R_1^*N & R_1^*MR_1 \end{pmatrix} \tag{12}$$

is given with respect to the decomposition $\mathcal{H} = \mathcal{L}_1 \oplus \mathcal{L}_1^\perp$, $\mathcal{L}_1 = R(P_1)$ and $M, N \in \{C_1S_1\}'$, $M^* = -M$, $N^* = N$. Here Γ_1 indicates the operator angle of (P_1,Q_1), $\{\Gamma_1\}'_{as}$ denotes the anti-selfadjoint operators in $\{\Gamma_1\}'$, $C_1 = \cos(\Gamma_1)$ and $S_1 = \sin(\Gamma_1)$. The unitary operator $R_1 : \mathcal{L}_1^\perp \to \mathcal{L}_1$ is given by $R_1 = |(I-P_1)Q_1P_1|^{-1}P_1Q_1(I-P_1)|_{\mathcal{L}_1^\perp}$. In other words, the norm of V_Z is computed as the quotient norm of the class $[Z] \in \mathfrak{u}_{\{A\}'}/\mathfrak{i}_{(P_1,Q_1)}$. Any operator of the form $Z' = Z + Z_0$ for $Z \in \mathfrak{u}_{\{A\}'}$ and $Z_0 \in \mathfrak{i}_{(P_1,Q_1)}$ will be called a *lifting* of a tangent vector V_Z. If, in addition, it satisfies $\|V_Z\|_{(P_1,Q_1)} = \|Z'\|$, then we say that Z' is a *minimal lifting.*

Remark 5.2. Take Z an operator as in (12) and $V_Z \in (T\mathcal{O}_{(P,Q)})_{(P_1,Q_1)}$. Since the operator norm is unitarily invariant, we can use the conjugation $Ad_{I\oplus R_1}$ to rewrite (11) without using the unitary R_1. Thus, we can compute the metric at each tangent space as follows

$$\|V_Z\|_{(P_1,Q_1)} = \inf\left\{ \left\| \begin{pmatrix} M+L & N \\ -N & M+L \end{pmatrix} \right\| : L \in \{\Gamma_1\}'_{as} \right\}.$$

Also note that using the conjugation

$$\sqrt{2}^{-1}\begin{pmatrix} I & -iI \\ I & iI \end{pmatrix}\begin{pmatrix} M+L & N \\ -N & M+L \end{pmatrix}\sqrt{2}^{-1}\begin{pmatrix} I & I \\ iI & -iI \end{pmatrix} = \begin{pmatrix} M+L+iN & 0 \\ 0 & M+L-iN \end{pmatrix}$$

Letting $X = M+iN$, $Y = M-iN$, we have

$$\begin{aligned} \|V_Z\|_{(P_1,Q_1)} &= \inf\left\{ \left\| \begin{pmatrix} X+L & 0 \\ 0 & Y+L \end{pmatrix} \right\| : L \in \{\Gamma_1\}'_{as} \right\} \\ &= \inf\left\{ \max\{\|X+L\|, \|Y+L\|\} : L \in \{\Gamma_1\}'_{as} \right\} \end{aligned}$$

Thus, minimal liftings can be interpreted as solutions $C \in \{\Gamma_1\}'_{as}$ to the above *best simultaneous approximation problem.*

As usual, the Finsler metric defines a length functional: for $\gamma : [0,1] \to \mathcal{O}_{(P,Q)}$ a C^1 piecewise curve set

$$L(\gamma) = \int_0^1 \|\dot{\gamma}(t)\|_{\gamma(t)} dt.$$

We are interested in finding curves of minimal length in $\mathcal{O}_{(P,Q)}$ with respect to the above length functional. That is, geodesics of these orbits in a *metric sense.* We point out that in general curves of minimal length do not necessarily coincide with geodesics of the reductive connection. A computation we will omit shows that the Finsler metric is isometric for the action of $\mathcal{U}_{\{A\}'}$. This means that for every $W \in \mathcal{U}_{\{A\}'}$ the smooth map $L_W : \mathcal{O}_{(P,Q)} \to \mathcal{O}_{(P,Q)}$, $L_W(P_1,Q_1) = W \cdot (P_1,Q_1)$, satisfies $\|(T_{(P_1,Q_1)}L_W)(V_Z)\|_{W\cdot(P_1,Q_1)} = \|V_Z\|_{(P_1,Q_1)}$ for all $(P_1,Q_1) \in \mathcal{O}_{(P,Q)}$ and $V_Z \in (T\mathcal{O}_{(P,Q)})_{(P_1,Q_1)}$. Another useful property is that $L(\delta) = \|Z'\|$, where $\delta(t) = e^{tZ'} \cdot (P_1,Q_1)$, $t \in [0,1]$ and Z' is a minimal lifting of $V_Z \in (T\mathcal{O}_{(P,Q)})_{(P_1,Q_1)}$.

Remark 5.3. More generally, one can consider the above Finsler metric defined as a quotient norm in the tangent spaces for homogeneous spaces of the form $\mathcal{U}_\mathcal{A}/\mathcal{U}_\mathcal{B}$, which are the quotient of unitary groups of unital C^*-algebras $\mathcal{B} \subset \mathcal{A}$. We give some considerations about minimal liftings and curves of minimal length.

i) Minimal liftings give solutions to the initial value problem (IVP). The IVP can be stated as follows: given $[U] \in \mathcal{U}_\mathcal{A}/\mathcal{U}_\mathcal{B}$ and $[Z] \in (T\mathcal{U}_\mathcal{A}/\mathcal{U}_\mathcal{B})_{[U]}$, find a C^1 piecewise curve δ such that $\delta(0) = [U]$, $\dot{\delta}(0) = [Z]$ and δ has minimal length for all sufficiently small t. Regarding this problem the following remarkable result was proved ([15, Thm. I]): if $[U] \in \mathcal{U}_\mathcal{A}/\mathcal{U}_\mathcal{B}$ and Z' is a minimal lifting of a tangent vector $[Z]$, then the uniparametric curve $\delta(t) = [\exp(tZ') \cdot U]$ has minimal length along its path with respect to the Finsler metric up to $|t| \leq \frac{\pi}{2\|Z'\|}$.

ii) A more difficult related problem in these types of homogeneous spaces is known as the fixed endpoints problem (FEP): given $[U_1], [U_2] \in \mathcal{U}_\mathcal{A}/\mathcal{U}_\mathcal{B}$, find a C^1 piecewise curve of minimal length inside $\mathcal{U}_\mathcal{A}/\mathcal{U}_\mathcal{B}$ joining $[U_1]$ and $[U_2]$. A sufficient condition for solving the FEP for general homogeneous spaces of unitary groups was proved in [16]. In the context of homogeneous spaces related to projections, the FEP was treated in homogeneous spaces arising from product of projections [4], sphere bundles [2], pencils of projections [12] and differences of projections [5].

Proposition 5.4. *Let $A \in \mathcal{B}(\mathcal{H})$ an injective operator and $(P,Q) \in \mathcal{C}_A$. Then the IVP admits a solution in $\mathcal{O}_{(P,Q)}$. That is, given $(P_1,Q_1) \in \mathcal{C}_A$, and $V_Z \in (T\mathcal{O}_{(P,Q)})_{(P_1,Q_1)}$, $Z \in \mathfrak{u}_{\{A\}'}$, then there exists $Z' \in \mathfrak{u}_{\{A\}'}$ such that $V_Z = V_{Z'}$ and the curve $\delta(t) = (e^{tZ'}P_1e^{-tZ'}, e^{tZ'}Q_1e^{-tZ'})$ has minimal length for all sufficiently small t.*

Proof. Minimal liftings always exist in $\mathcal{O}_{(P,Q)}$, and consequently, the IVP admits a solution in $\mathcal{O}_{(P,Q)}$. This follows immediately from the fact that $\mathfrak{u}_{\{A\}'}$ and $\mathfrak{i}_{(P_1,Q_1)}$ are the anti-selfadjoint parts of a von Neumann algebra and a von Neumann subalgebra, respectively. In general, the quotient norm in $\mathcal{A}_{sa}/\mathcal{B}_{sa}$ is always attained for the anti-selfadjoint parts of von Neumann algebras $\mathcal{B} \subset \mathcal{A}$ (see [15, Thm. 6.1]). Then there exists a minimal lifting Z' of V_Z, and the result is a consequence of the statement in Remark 5.3 *i*). □

Thus, minimal liftings always exist for tangent vectors in $\mathcal{O}_{(P,Q)}$, and the explicit form of curves solving the IVP is related to them. The problem of computing minimal liftings can be difficult, though we will present a condition under which the problem can be solved. This condition can be stated in terms of operator angles of pairs in the orbit as follows (see Remark 4.3): $\{\Gamma\}' = \{CS\}'$, where Γ denotes the operator angle associated to (P,Q), $C = \cos(\Gamma)$ and $S = \sin(\Gamma)$. Equivalently, we can state $\{\Gamma\}'' = \{CS\}''$ by the Borel functional calculus. Notice that the condition $\{\Gamma\}' = \{CS\}'$ is actually a property of the whole orbit $\mathcal{O}_{(P,Q)}$. Indeed, as a consequence of Corollary 3.3, for any $(P_1,Q_1) \in \mathcal{O}_{(P,Q)}$, the operator angle Γ_1 associated to (P_1,Q_1) is unitarily equivalent to Γ, so it follows that $\{\Gamma\}' = \{CS\}'$ if and only if $\{\Gamma_1\}' = \{C_1S_1\}'$, where $C_1 = \cos(\Gamma_1)$ and $S_1 = \sin(\Gamma_1)$. In what follows we discuss this condition in operator theoretic terms, and we then turn to analyze its geometric consequences on the orbits.

Remark 5.5. The condition $\{CS\}' = \{\Gamma\}'$ can be understood by means of spectral properties of Γ. As a motivation we consider here the set M_n of $n \times n$ complex matrices. Take $\Gamma \in M_n$, $\Gamma = \Gamma^*$ such that $\sigma(\Gamma) = \{\lambda_1, \dots, \lambda_n\} \subseteq [0, \frac{\pi}{2}]$. Then, $\sigma(CS) = \{\cos(\lambda_1)\sin(\lambda_1), \dots, \cos(\lambda_n)\sin(\lambda_n)\} \subseteq [0, \frac{1}{2}]$, and the spectral projections satisfy $E_{CS}(\{\cos(\lambda_k)\sin(\lambda_k)\}) = E_\Gamma(\{\lambda_k\}) + E_\Gamma(\{\frac{\pi}{2} - \lambda_k\})$ if $\lambda_k \neq \frac{\pi}{4}$, and $E_{CS}(\{\frac{1}{2}\}) = E_\Gamma(\{\frac{\pi}{4}\})$. Now recall that $\{CS\}' = \{\Gamma\}'$ if and only if $\{CS\}'' = \{\Gamma\}''$. This is equivalent to have that every spectral projection of CS can be written as a spectral projection of Γ, which in turn means that $E_\Gamma(\{\lambda_k\}) = 0$ or $E_\Gamma(\{\frac{\pi}{2} - \lambda_k\}) = 0$ whenever $\frac{\pi}{2} - \lambda_k \in \sigma(\Gamma)$, $k = 1, \dots, n$ and $\lambda_k \neq \frac{\pi}{4}$. This can be reformulated by saying that the function $f : \sigma(\Gamma) \to \mathbb{R}$, $f(t) = \cos(t)\sin(t)$, is injective.

An extension of the above characterizations of when $\{CS\}' = \{\Gamma\}'$ for a separable infinite-dimensional complex Hilbert can be stated by using the following measure theoretic notions. Recall that a scalar-valued spectral measure of Γ is a positive Borel measure μ on $\sigma(\Gamma)$ such that $\mu(\Delta) = 0$ if and only if $E_\Gamma(\Delta) = 0$, for every Borel set $\Delta \subseteq \sigma(\Gamma)$. Consider again the function $f : \sigma(\Gamma) \to \mathbb{R}$, $f(t) = \cos(t)\sin(t)$. We write $f_*\mu$ for the measure defined by $(f_*\mu)(\Omega) = \mu(f^{-1}(\Omega))$, for all Borel sets $\Omega \subseteq \sigma(CS)$. It is not difficult to check that if μ is a scalar-valued spectral measure for Γ, then $f_*\mu$ is a scalar-valued spectral measure for CS. Given a compact set $\mathcal{X} \subseteq \mathbb{C}$ and a Borel measure ν on $\mathcal{X}$, we denote $L^\infty(\nu, \mathcal{X})$ the space of all complex-valued functions that are Borel measurable and essentially bounded defined on $\mathcal{X}$.

Proposition 5.6. *Let Γ be a selfadjoint operator such that $\sigma(\Gamma) \subseteq [0, \frac{\pi}{2}]$, $C = \cos(\Gamma)$ and $S = \sin(\Gamma)$. Consider the function $f : \sigma(\Gamma) \to \mathbb{R}$ defined by $f(t) = \cos(t)\sin(t)$. The following conditions are equivalent:*

i) $\{CS\}' = \{\Gamma\}'$.

ii) Let μ be a scalar-valued spectral measure for Γ. Then there exists $g \in L^\infty(f_*\mu, \sigma(CS))$ such that $g \circ f = id_{\sigma(\Gamma)}$ μ-a.e.

iii) If Δ and $\frac{\pi}{2} - \Delta$ are Borel subsets of $\sigma(\Gamma)$ such that $\Delta \subseteq [0, \frac{\pi}{4})$, then $E_\Gamma(\Delta) = 0$ or $E_\Gamma(\frac{\pi}{2} - \Delta) = 0$.

Proof. *i)* $\to$ *ii)* Recall that $\{CS\}' = \{\Gamma\}'$ if and only if $\{CS\}'' = \{\Gamma\}''$. There are isomorphisms $L^\infty(f_*\mu, \sigma(CS)) \to \{CS\}''$, $g \mapsto g(CS)$, and $L^\infty(\mu, \sigma(\Gamma)) \to \{\Gamma\}''$, $g \mapsto g(\Gamma)$ (see [11, Thm. 8.10]). Since f is continuous on $\sigma(\Gamma)$, it follows that $\sigma(CS) = f(\sigma(\Gamma))$ and $g(CS) = g(f(\Gamma)) = (g \circ f)(\Gamma)$ (see [29, E.2.18]) for every $g \in L^\infty(f_*\mu, \sigma(CS))$. Using this property and $\{CS\}'' = \{\Gamma\}''$, we get the isomorphism $L^\infty(f_*\mu, \sigma(CS)) \to L^\infty(\mu, \sigma(\Gamma))$, $g \mapsto g \circ f$. Hence there exists $g \in L^\infty(f_*\mu, \sigma(CS))$ such that $g \circ f = id_{\sigma(\Gamma)}$.

ii) $\to$ *iii)* Assume for the sake of contradiction that $E_\Gamma(\Delta) \neq 0$ and $E_\Gamma(\frac{\pi}{2} - \Delta) \neq 0$ for some Borel subset $\Delta \subseteq \sigma(\Gamma) \cap [0, \frac{\pi}{4})$, where $\frac{\pi}{2} - \Delta \subseteq \sigma(\Gamma)$. Thus, note $\mu(\Delta) > 0$ and $\mu(\frac{\pi}{2} - \Delta) > 0$. Take $g \in L^\infty(f_*\mu, \sigma(CS))$ such that $g \circ f = id_{\sigma(\Gamma)}$ μ-a.e. Since Δ and $\frac{\pi}{2} - \Delta$ are contained in $\sigma(\Gamma)$, we have that $g \circ f$ is defined in both sets. Noting that $f(t) = f(\frac{\pi}{2} - t)$ for all t, we have $g(f(t)) = g(f(\frac{\pi}{2} - t)) = \frac{\pi}{2} - t$ μ-a.e. for $t \in \Delta$, where in the last equality we use that $\mu(\frac{\pi}{2} - \Delta) > 0$. But also note $g(f(t)) = t$ μ-a.e. for $t \in \Delta$, which holds since $\mu(\Delta) > 0$. This is a contradiction because $t \neq \frac{\pi}{2} - t$ for all $t \in \Delta \subseteq [0, \frac{\pi}{4})$.

iii) $\to$ *i)* Recall that $\{CS\}' \supseteq \{\Gamma\}'$ always holds true, so we only have to show that $\{CS\}' \subseteq \{\Gamma\}'$. For pick X such that $XCS = CSX$. Then, $X\Gamma = \Gamma X$ if and only if $E_\Gamma(\Delta)X = XE_\Gamma(\Delta)$ for every Borel $\Delta \subseteq \sigma(\Gamma)$. To prove the latter condition, set $\Delta_1 := \Delta \cap [0, \frac{\pi}{4})$, $\Delta_2 := \Delta \cap (\frac{\pi}{4}, \frac{\pi}{2}]$ and $\Delta_3 := \{\frac{\pi}{4}\} \cap \Delta$. Consider the sets $\tilde{\Delta}_i := \{\lambda \in \Delta_i : \frac{\pi}{2} - \Delta_i \in \sigma(\Gamma)\}$, $i = 1, 2$. Observe that $\tilde{\Delta}_i$ and $\frac{\pi}{2} - \tilde{\Delta}_i$ are Borel subsets of $\sigma(\Gamma)$. On the one hand, $f^{-1}(f(\Delta_1 \setminus \tilde{\Delta}_1)) \cap \sigma(\Gamma) = \Delta_1 \setminus \tilde{\Delta}_1$, so it follows that $E_{CS}(f(\Delta_1 \setminus \tilde{\Delta}_1)) = E_\Gamma(\Delta_1 \setminus \tilde{\Delta}_1)$. On the other hand, $f^{-1}(f(\tilde{\Delta}_1)) \cap \sigma(\Gamma) = \tilde{\Delta}_1 \cup (\frac{\pi}{2} - \tilde{\Delta}_1)$, which gives $E_{CS}(f(\tilde{\Delta}_1)) = E_\Gamma(\tilde{\Delta}_1) + E_\Gamma(\frac{\pi}{2} - \tilde{\Delta}_1)$. By hypothesis, since $\tilde{\Delta}_i \subseteq \sigma(\Gamma)$ and $\frac{\pi}{2} - \tilde{\Delta}_i \subseteq \sigma(\Gamma)$, it must be $E_\Gamma(\tilde{\Delta}_1) = 0$ or $E_\Gamma(\frac{\pi}{2} - \tilde{\Delta}_1) = 0$. Therefore $E_\Gamma(\Delta_1) = E_\Gamma(\tilde{\Delta}_1) + E_\Gamma(\Delta_1 \setminus \tilde{\Delta}_1)$ is a spectral projection of CS. Similarly, one can show that $E_\Gamma(\Delta_2)$ is also a spectral projection of CS. Also note $E_\Gamma(\Delta_3) = E_\Gamma(\{\frac{\pi}{4}\}) = E_{CS}(\{\frac{1}{2}\})$. Hence $E_\Gamma(\Delta) = \sum_{i=1}^3 E_\Gamma(\Delta_i)$, where each of these terms is a spectral projection of CS. Since $X \in \{CS\}'$, it follows that X commutes with every spectral projection of CS, and thus X commutes with $E_\Gamma(\Delta)$. □

Lemma 5.7. *Let $A \in \mathcal{B}(\mathcal{H})$ an injective operator and $(P, Q) \in \mathcal{C}_A$. Assume that the operator angle Γ associated to (P, Q) satisfies $\{\Gamma\}' = \{CS\}'$. The following assertions hold:*

i) Take $(P_1, Q_1) \in \mathcal{O}_{(P,Q)}$, $V_Z \in (T\mathcal{O}_{(P,Q)})_{(P_1,Q_1)}$, and assume that Z is represented as in (12). Then,

$$Z' = \begin{pmatrix} 0 & NR_1 \\ -R_1^* N & 0 \end{pmatrix}$$

is a minimal lifting of V_Z.

ii) The curve $\delta(t) = \exp(tZ') \cdot (P_1, Q_1) = (e^{tZ'} P_1 e^{-tZ'}, e^{tZ'} Q_1 e^{-tZ'})$, $t \in \mathbb{R}$, which solves the IVP at (P_1, Q_1) with initial velocity V_Z, turns out to be a geodesic of the reductive *connection, and it can be expressed as* $\delta(t) = (\delta_1(t), \delta_2(t))$*, where*

$$\delta_1(t) = \begin{pmatrix} \cos^2(tN) & \cos(tN)\sin(tN)R_1 \\ -R_1^*\cos(tN)\sin(tN) & R_1^*\sin^2(tN)R_1 \end{pmatrix}, \tag{13}$$

$$\delta_2(t) = \begin{pmatrix} \cos^2(\Gamma_1 - tN) & \cos(\Gamma_1 - tN)\sin(\Gamma_1 - tN)R_1 \\ R_1^*\cos(\Gamma_1 - tN)\sin(\Gamma_1 - tN) & R_1^*\sin^2(\Gamma_1 - tN)R_1 \end{pmatrix}. \tag{14}$$

Proof. *i*) Let Γ_1 be the operator angle of $(P_1, Q_1) \in \mathcal{O}_{(P,Q)}$, $C_1 = \cos(\Gamma_1)$ and $S_1 = \sin(\Gamma_1)$. Recall that $\{C_1S_1\}' = \{\Gamma_1\}'$ because Γ and Γ_1 are unitarily equivalent by Corollary 3.3 and $\{\Gamma\}' = \{CS\}'$. Then the expression of the metric in (11) reduces to the following

$$\|V_Z\|_{(P_1,Q_1)} = \inf\left\{ \left\| \begin{pmatrix} M+L & NR_1 \\ -R_1^*N & R_1^*(M+L)R_1 \end{pmatrix} \right\| : L \in \{C_1S_1\}'_{as} \right\}.$$

An operator Z' as in the statement is a minimal lifting if and only if the above quotient norm is attained at $L = -M \in \{C_1S_1\}'_{as}$. The latter follows by the well-known inequality (see, e.g., [15, Lemma 3.2]): if $\mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2$ (direct sum) for $\mathcal{H}_1$, $\mathcal{H}_2$ closed subspaces of $\mathcal{H}$, and

$$X = \begin{pmatrix} X_{11} & X_{12} \\ -X_{21}^* & X_{22} \end{pmatrix}, \quad X' = \begin{pmatrix} 0 & X_{12} \\ -X_{12}^* & 0 \end{pmatrix},$$

are bounded linear operators in terms of that decomposition, then $\|X'\| \leq \|X\|$.

ii) The curve $\delta(t) = (e^{tZ'}P_1e^{-tZ'}, e^{tZ'}Q_1e^{-tZ'})$, $t \in \mathbb{R}$, solves the IVP by Proposition 5.4. The minimal lifting satisfies $Z' \in \mathfrak{m}_{(P_1,Q_1)}$, because the by Remark 4.7 *iii*) the projection $\mathcal{E}_{\Gamma_1} = I$ for every Γ_1 associated to $(P_1, Q_1) \in \mathcal{O}_{(P,Q)}$ whenever $\{C_1S_1\}' = \{\Gamma_1\}'$. Then by Proposition 5.1 the curve δ turns out to be a geodesic of the reductive connection. The last expression for the coordinates of δ in the statement follows by Halmos' two projection theorem and the computation of the exponential of a codiagonal operator. Indeed, note the expression in (13) follows by

$$\begin{aligned}\delta_1(t) &= e^{tZ'}P_1e^{-tZ'} \\ &= \begin{pmatrix} I & 0 \\ 0 & R_1^* \end{pmatrix} \begin{pmatrix} \cos(tN) & \sin(tN) \\ -\sin(tN) & \cos(tN) \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \cos(tN) & -\sin(tN) \\ \sin(tN) & \cos(tN) \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & R_1 \end{pmatrix}\end{aligned}$$

On the other hand, (14) can be derived from

$$\begin{aligned}\delta_2(t) &= \exp(tZ')Q_1\exp(-tZ') \\ &= \begin{pmatrix} I & 0 \\ 0 & R_1^* \end{pmatrix} \begin{pmatrix} \cos(tN) & \sin(tN) \\ -\sin(tN) & \cos(tN) \end{pmatrix} \begin{pmatrix} C_1^2 & C_1S_1 \\ C_1S_1 & S_1^2 \end{pmatrix} \begin{pmatrix} \cos(tN) & -\sin(tN) \\ \sin(tN) & \cos(tN) \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & R_1 \end{pmatrix},\end{aligned}$$

and using that $N\Gamma_1 = \Gamma_1 N$ since $\{C_1S_1\}' = \{\Gamma_1\}'$. □

Remark 5.8. Minimal liftings might not lie in the supplement given by the reductive structure when $\{CS\}' \neq \{\Gamma\}'$. Let M_n be the complex $n \times n$ matrices. Take $\Gamma = \mathrm{diag}(a, b) \in M_2$ as in Remark 4.3, and the projections in M_4 defined following (7), namely

$$P = \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \quad Q = \frac{1}{2}\begin{pmatrix} 1 & 0 & e^{-2ia} & 0 \\ 0 & 1 & 0 & e^{-2ib} \\ e^{2ia} & 0 & 1 & 0 \\ 0 & e^{2ib} & 0 & 1 \end{pmatrix}. \tag{15}$$

Recall that $\{\Gamma\}' = \{\mathrm{diag}(x,y) : x, y \in \mathbb{C}\}$ and $\{CS\}' = M_2$. Then, Γ has two distinct spectral projections related to a and b, so it has associated the conditional expectation

$$\mathcal{E}_\Gamma : M_2 \to M_2,\ \mathcal{E}_\Gamma\left(\begin{pmatrix} x_{11} & x_{21} \\ x_{21} & x_{22} \end{pmatrix}\right) = \begin{pmatrix} x_{11} & 0 \\ 0 & x_{22} \end{pmatrix}.$$

This, in turn, by Proposition 5.1 defines the natural reductive structure as follows:

$$\mathfrak{m}_{(P,Q)} = \left\{ \begin{pmatrix} M - \mathcal{E}_{\Gamma_1}(M) & NR \\ -R^*N & R^*(M - \mathcal{E}_{\Gamma_1}(M))R \end{pmatrix} : M^* = -M \in M_2,\, N^* = N \in M_2 \right\},$$

where $R \in M_2$ is the unitary defined by $R = |(I-P)QP|^{-1}PQ(I-P)|_{R(P)}$. In particular, matrices in the set $Ad_{I\oplus R}(\mathfrak{m}_{(P,Q)})$ must have zero diagonal. Now take

$$Z = \begin{pmatrix} I & 0 \\ 0 & R^* \end{pmatrix} \begin{pmatrix} 0 & 2+i & 1 & 1 \\ -2+i & 0 & 1 & 1 \\ -1 & -1 & 0 & 2+i \\ -1 & -1 & -2+i & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & R \end{pmatrix} \in \mathfrak{m}_{(P,Q)}.$$

Then Z is not a minimal lifting since

$$3.414 \simeq \left\| \begin{pmatrix} i(1-\sqrt{2}) & 2+i & 1 & 1 \\ -2+i & i(1-\sqrt{2}) & 1 & 1 \\ -1 & -1 & i(1-\sqrt{2}) & 2+i \\ -1 & -1 & -2+i & i(1-\sqrt{2}) \end{pmatrix} \right\| < \|Z\| \simeq 3.828.$$

Now we are in position to give our main result on minimality of geodesics under the condition of the commutants.

Theorem 5.9. *Let $A \in \mathcal{B}(\mathcal{H})$ an injective operator and $(P,Q) \in \mathcal{C}_A$. Assume that the operator angle Γ associated to (P,Q) satisfies $\{\Gamma\}' = \{CS\}'$. Then any pairs $(P_1,Q_1), (P_2,Q_2) \in \mathcal{O}_{(P,Q)}$ can be joined by a geodesic δ of the reductive connection which has minimal length. Moreover, the geodesic $\delta = (\delta_1, \delta_2)$ has coordinates of the form derived in* (13) *and* (14)*, where the operator N is given by*

$$N = -\frac{i}{2}\log\left((P_2')_{12}|(P_2')_{12}|^{-1}\right),$$

and $(P_2')_{12} \in \mathcal{B}(\mathcal{L}_1)$ is defined for the matrix representation in terms of $\mathcal{L}_1 \times \mathcal{L}_1$, $R(P_1) = \mathcal{L}_1$, of the following projection $Ad_{X_1}(P_2)$ given by

$$Ad_{X_1}(P_2) = \begin{pmatrix} (P_2')_{11} & (P_2')_{12} \\ (P_2')^*_{12} & (P_2')_{22} \end{pmatrix}, \quad X_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} I & -iI \\ I & iI \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & R_1 \end{pmatrix},$$

for the unitary operator $R_1 : \mathcal{L}_1^\perp \to \mathcal{L}_1$ defined by $R_1 = |(I-P_1)Q_1P_1|^{-1}P_1Q_1(I-P_1)|_{\mathcal{L}_1^\perp}$.

Proof. Recall that the unitary operator $X_1 : \mathcal{H} \to \mathcal{L}_1 \times \mathcal{L}_1$ is given in Remark 3.1, and it satisfies that $Ad_{X_1}(A) = iC_1S_1 \oplus (-iC_1S_1)$,

$$Ad_{X_1}(P_1) = \frac{1}{2}\begin{pmatrix} I & I \\ I & I \end{pmatrix}, \quad Ad_{X_1}(Q_1) = \frac{1}{2}\begin{pmatrix} I & g(\Gamma_1)D_1 - 2iC_1S_1 \\ g(\Gamma_1)D_1 + 2iC_1S_1 & I \end{pmatrix},$$

where $D_1 = \sqrt{I + 4(iC_1S_1)^2}$ and the above operator matrices are defined in $\mathcal{L}_1 \times \mathcal{L}_1$. We use the notation $P_i' := Ad_{X_1}(P_i)$, $Q_i' := Ad_{X_1}(Q_i)$, $i = 1, 2$ and $A' = Ad_{X_1}(A) = iC_1S_1 \oplus (-iC_1S_1)$. In terms of the expression in (4), we can write $P_1' = P_I$, $Q_1' = Q_{I,g(\Gamma_1)}$ and $P_2' = P_{U_2}$, $Q_2' = Q_{U_2,V_2}$ for some $U_2 \in \mathcal{U}_{\{C_1S_1\}'}$ and $V_2 \in \{C_1S_1\}'$ a selfadjoint partial isometry such that $V_2^2 = P_{\overline{R(D_1)}}$.

Since (P_1', Q_1') and (P_2', Q_2') belong to the same orbit, there exists a unitary $W = W_1 \oplus W_2$, $W_i \in \mathcal{U}_{\{C_1S_1\}'}$ such that $W \cdot (P_1', Q_1') = (P_2', Q_2')$. This can be rephrased using the expression of the action in (5), and it is equivalent to have $W_1W_2^* = U_2$ and $W_1g(\Gamma_1)W_1^* = V_2$. Notice that every $W_1 \in \{C_1S_1\}'$ satisfies $W_1\Gamma_1 = \Gamma_1W_1$, since $\{C_1S_1\}' = \{\Gamma_1\}'$ by our assumption. Therefore, $W_1g(\Gamma_1) = g(\Gamma_1)W_1$, so we must have $V_2 = W_1g(\Gamma_1)W_1^* = g(\Gamma_1)$ for $W_1 \in \{C_1S_1\}'$. Thus, to construct a minimal geodesic we first find unitaries $W_1, W_2 \in \mathcal{U}_{\{C_1S_1\}'}$ such that $W_1W_2^* = U_2$. For take $X := \frac{\log(U_2)}{2}$, where we consider the Borel measurable logarithm $\log : \mathbb{T} \to [-\pi, \pi)$, $\mathbb{T}$ being the unit circumference. Then X has the following properties: $\|X\| \le \frac{\pi}{2}$ and $X \in \{C_1S_1\}'_{as}$. Take $W_1 = e^X$ and $W_2 = e^{-X}$, which clearly satisfy $W_1W_2^* = U_2$. Then the smooth curve defined by

$$\delta'(t) = \exp\left(t\begin{pmatrix} X & 0 \\ 0 & -X \end{pmatrix}\right) \cdot (P_1', Q_1')$$

satisfies $\delta'(0) = (P_1', Q_1')$ and $\delta'(1) = (P_2', Q_2')$. Therefore $\delta(t) = Ad_{X_1^*}(\delta'(t))$ joins $\delta(0) = (P_1, Q_1)$ and $\delta(1) = (P_2, Q_2)$. Also note

$$\begin{aligned}\delta(t) &= \exp\left(tAd_{X_1^*}\begin{pmatrix} X & 0 \\ 0 & -X \end{pmatrix}\right) \cdot (Ad_{X_1^*}(P_1), Ad_{X_1^*}(Q_1)) \\ &= \exp\left(t\begin{pmatrix} 0 & iXR_1 \\ R_1^*iX & 0 \end{pmatrix}\right) \cdot (P_1, Q_1),\end{aligned}$$

which is another way to express (13) and (14) with $N = iX$. In particular, the operator

$$Z' = \begin{pmatrix} 0 & iXR_1 \\ R_1^*iX & 0 \end{pmatrix}$$

is a minimal lifting of $V_{Z'}$. Thus, from Lemma 5.7 we find that δ is a geodesic of the reductive connection solving the IVP. Furthermore, δ has minimal length up to $|t| \le \frac{\pi}{2\|Z'\|}$ by Remark 5.3 *i*). Using that $\|Z'\| = \|X\| \le \frac{\pi}{2}$ we get that δ has minimal length between $\delta(0) = (P_1, Q_1)$ and $\delta(1) = (P_2, Q_2)$. Finally, observe that

$$P_2' = \begin{pmatrix} (P_2')_{11} & (P_2')_{12} \\ (P_2')_{12}^* & (P_2')_{22} \end{pmatrix}$$

is the matrix representation in terms of $\mathcal{L}_1 \times \mathcal{L}_1$. Then the unitary $U_2 \in \{C_1S_1\}'$ such that $P_2' = P_{U_2}$ is defined by $U_2 = (P_2')_{12}|(P_2')_{12}|^{-1}$ as it is explained in Remark 2.3. This completes the proof. □

Remark 5.10 (Minimal curves for the case A non injective). In the case where A is not injective the explicit computation of curves of minimal length cannot be established with same sufficient condition on the generic part. As we have observed in Remark 4.8 the orbit of a pair $(P, Q) \in \mathcal{C}_A$ can be identified as $\mathcal{O}_{(P,Q)} \simeq \mathcal{O}_{(E,F)} \times \mathcal{O}_{(P_0,Q_0)}$, which admits a reductive homogeneous space structure. For the second orbit $\mathcal{O}_{(P_0,Q_0)}$ one may impose a sufficient condition as in the injective case. Nevertheless the first orbit $\mathcal{O}_{(E,F)} \simeq \mathcal{U}(\mathcal{H}_0^\perp)/\mathcal{I}_{(E,F)}$ carries the corresponding Finsler metric defined at each $(E_1, F_1) \in \mathcal{O}_{(E,F)}$ as follows. For a tangent vector $V_Z = (ZE_1 - E_1Z, ZF_1 - F_1Z) \in (T\mathcal{O}_{(E,F)})_{(E_1,F_1)}$, for some $Z = -Z^* \in \mathcal{B}(\mathcal{H}_0^\perp)$, set

$$\begin{aligned}\|V_Z\|_{(E_1,F_1)} &:= \inf\{\|Z + Z_0\| : Z_0 \in \mathfrak{i}_{(E_1,F_1)}\} \\ &= \inf\left\{\|Z + Z_0\| : Z_0^* = -Z_0^*,\, Z_0 P_{\mathcal{H}_{ij}} = P_{\mathcal{H}_{ij}} Z_0,\, i,j = 0,1\right\},\end{aligned}$$

where $\mathcal{H}_{00} = R(E_1) \cap R(F_1)$, $\mathcal{H}_{01} = R(E_1) \cap N(F_1)$, $\mathcal{H}_{10} = N(E_1) \cap R(F_1)$ and $\mathcal{H}_{11} = N(E_1) \cap N(F_1)$. When all of these subspaces are non trivial this becomes a minimization problem involving 4×4 operator matrices. One can follow the same idea as in Proposition 5.4 to guarantee that the infimum is attained. However, the explicit computation of minimal liftings in 4×4 operator matrices is an open problem even in the case where $\dim \mathcal{H}_{ij} = 1$ for all $i, j = 0, 1$ (see [6, 19]).

Acknowledgment

This work was supported by the grants PIP 2021/2023 11220200103209CO and FCE-UNLP 11X974.

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