Reflections in $L^2(\mathbb{T})$

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Abstract

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. For $a \in \mathbb{D}$, consider $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$ and C_a the composition operator in $L^2(\mathbb{T})$ induced by φ_a:

$$C_a f = f \circ \varphi_a.$$

Clearly C_a satisties $C_a^2 = I$, i.e., is a non-selfadjoint reflection. We also consider the following symmetries (selfadjoint reflections) related to C_a:

$$R_a = M_{\frac{|k_a|}{\|k_a\|_2}} C_a \text{ and } W_a = M_{\frac{k_a}{\|k_a\|_2}} C_a,$$

where $k_a(z) = \frac{1}{1-\bar{a}z}$ is the Szego kernel. The symmetry R_a is the unitary part in the polar decomposition of C_a. We characterize the eigenspaces $N(T_a \pm I)$ for $T_a = C_a, R_a$ or W_a, and study their relative positions when one changes the parameter a, e.g., $N(T_a \pm I) \cap N(T_b \pm I)$, $N(T_a \pm I) \cap N(T_b \pm I)^\perp$, $N(T_a \pm I)^\perp \cap N(T_b \pm I)$, etc., for $a \neq b \in \mathbb{D}$.

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1 Introduction

Let $\mathcal{H} = L^2(\mathbb{T})$ with normalized Lebesgue measure. For each $a \in \mathbb{D}$ let

$$\varphi_a(z) := \frac{a-z}{1-\bar{a}z}.$$

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Note that $\varphi_a \circ \varphi_a(z) = z$ and $|\varphi_a(z)| = 1$ for $z \in \mathbb{T}$, and therefore the composition operator

$$C_a : \mathcal{H} \to \mathcal{H}, \ C_a f = f \circ \varphi_a$$

is well defined, and is a *reflection* in $\mathcal{H}$: $C_a^2 = I$. A reflection C is a *symmetry* if additionally $C^* = C$. Clearly C_a is a symmetry only if $a = 0$. The adjoint of C_a is easily computed: $C_a^* = (1 - |a|^2) M_{\frac{1}{|1-\bar{a}z|^2}} C_a$, where M_g denotes the multiplication operator (by g).

There are two symmetries closely relaed to C_a. First the unitary part R_a in the polar decomposition $C_a = R_a (C_a^* C_a)^{1/2}$ of C_a (in [9], G. Corach, H. Porta and L. Recht noted this fact, that the unitary part of a reflection is in fact a symmetry). Next the operator W_a given by $W_a = M_{\psi_a} C_a$, where ψ_a is the normalized Szego kernel $\psi_a = \frac{(1-|a|^2)^{1/2}}{1-\bar{a}z}$.

The object of this paper is the study of eigenspaces of these symmetries, and their relative position when one changes the parameter $a \in \mathbb{D}$. By relative position of two subspaces $\mathcal{S}$, $\mathcal{T}$, we mean the computation of the intersections $\mathcal{S} \cap \mathcal{T}$, $\mathcal{S}^\perp \cap \mathcal{T}^\perp$, $\mathcal{S} \cap \mathcal{T}^\perp$ and $\mathcal{S}^\perp \cap \mathcal{T}$. Of particular interest to us are the last two, for their geometric significance: if these two intersections have the same dimention, it means that there exists a geodesic of the Grassmann manifold of $\mathcal{H}$ that joins them; if they are trivial, the geodesic is unique. When the four intersections are trivial, the subspaces $\mathcal{S}$, $\mathcal{T}$ are said to be in generic position.

This paper is a continuation of [4], where composition operators with symbols φ_a where considered in the Hardy space of the disc. Several results otained there will be useful here.

The contents of the paper are the following. In Section 2 the eigenspaces of C_a, R_a and W_a are characterized. In this characterization, an important role is played by the unique fixed point ω_a of φ_a inside $\mathbb{D}$. In Section 3 we make some brief remarks on the maps $a \mapsto C_a$ and $a \mapsto R_a$. In Section 4 we obtain partial results on the relative position of the eigenspaces of these reflections induced by different $a, b \in \mathbb{D}$. The main result is Theorem 4.6, which states that for $a \neq b$

$$N(C_a - I) \cap N(C_b - I) = \langle 1 \rangle \ \text{ and } \ N(C_a + I) \cap N(C_b + I) = \{0\}.$$

In Section 5 we consider the special case of the subspaces $N(C_a - I)$ and $N(C_a + I)$, where a complete study can be done. In Section 6, using the conjugation properties

$$R_a C_a R_a = C_a^*, \ R_a C_0 R_a = R_{\Omega_a}, \ W_a C_a W_a = M_{\frac{k_a^2}{\|k_a\|_2^2}} C_a = M_{\frac{k_a}{|k_a|}} C_a^* \ \text{ and } \ W_a C_0 W_a = W_{\Omega_a}$$

we can compute further intersections between the eigenspaces. Here $\Omega_a = \frac{2a}{1+|a|^2}$ is characterized by $\omega_{\Omega_a} = a$, i.e., a is the unique fixed point of φ_{Ω_a} in $\mathbb{D}$. In Section 7 we examine the consequence of these intersections for the geometry of the Grassmann manifold of $\mathcal{H}$: they determine wether given pairs of subspaces can be joined by a geodesic of this manifold.

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2 Preliminaries and notations

If $\mathcal{S}$ is a closed subspace of $\mathcal{H}$, $P_{\mathcal{S}}$ will denote the orthogonal projection onto $\mathcal{S}$. For a bounded operator T, $|T|$ will denote the modulus of T, i.e., $|T| = (T^*T)^{1/2}$; $N(T)$ and $R(T)$ will denote the nullspace and range of T, respectively.

Note that

$$C_a^*C_a = (1-|a|^2)M_{\frac{1}{|1-\bar{a}z|^2}} \quad \text{and} \quad |C_a| = (C_a^*C_a)^{1/2} = (1-|a|^2)^{1/2}M_{\frac{1}{|1-\bar{a}z|}}. \tag{1}$$

In [9], Corach, Porta and Recht, made the following remarkable observation:

Remark 2.1. Let C be a reflection in $\mathcal{H}$, and let $C = R|C|$ be its polar decomposition. Then the unitary operator R is in fact a symmetry.

Let us denote by R_a the symmetry obtained in the polar decomposition of C_a. After routine computations one obtains that

$$R_a = C_a|C_a|^{-1} = \frac{1}{(1-|a|^2)^{1/2}}C_aM_{|1-\bar{a}z|} = (1-|a|^2)^{1/2}M_{\frac{1}{|1-\bar{a}z|}}C_a. \tag{2}$$

Note that if $C^2 = I$ and $C = R|C|$ is the polar decomposition, then $C^* = |C|R$, so that $RCR = C^*$. In particular, for $C = C_a$ we have

$$R_aC_aR_a = C_a^*. \tag{3}$$

There is another symmetry in $\mathcal{H}$ associated with φ_a, known in the literature (at least when acting in the (invariant) Hardy subspace $H^2(\mathbb{T}) \subset \mathcal{H}$, [10]):

$$W_a = M_{\psi_a}C_a = (1-|a|^2)^{1/2}M_{\frac{1}{1-\bar{a}z}}C_a. \tag{4}$$

Note the similarity between R_a in (2) and W_a in (4).

We shall say that an element $f \in \mathcal{H}$ is *even* if $f(z) = f(-z)$ *p.p.* in $\mathbb{T}$. Equivalently, if the Fourier coefficients $\hat{f}(n)$ of f are 0 for n odd. We shall denote by $\mathcal{E}$ the closed subspace of even elements. Similarly, $\mathcal{O}$ will denote the subspace of *odd* elements of $\mathcal{H}$ ($g \in \mathcal{H}$ such that $g(-z) = -g(z)$ *p.p.* in $\mathbb{T}$ or $\hat{g}(n) = 0$ for n even). Note that for $a = 0$, $C_0 = R_0$, and $N(C_0 - I) = \mathcal{E}$ and $N(C_0 + I) = \mathcal{O}$.

Finally, let $\mathcal{H}^+$ be the Hardy space, as a subset of $\mathcal{H}$, $\mathcal{H}^+ := H^2(\mathbb{T}) = \{f \in \mathcal{H} : \hat{f}(n) = 0 \text{ for } n < 0\}$, identified with the Hardy space of the disk $H^2(\mathbb{D}) = \{g(z) = \sum_{k=0}^\infty a_kz^k : \sum_{k=0}^\infty |a_k|^2 < \infty\}$.

A reflection T is completely described by its two eigenspaces $N(T-I)$ and $N(T+1)$. The object of this study is the characterization of the eigenspaces $N(T-I)$ and $N(T+I)$ for $T = C_a$, R_a or W_a, and their relative positions for different $a \in \mathbb{D}$.

3 Eigenspaces of C_a and R_a

Let us first characterize the two eigenspaces $N(C_a \pm I)$ of C_a. Note that φ_a has a unique fixed point ω_a inside $\mathbb{D}$, which is given by

$$\omega_a = \frac{1}{\bar{a}}\{1 - \sqrt{1-|a|^2}\} = \frac{a}{|a|^2}\{1 - \sqrt{1-|a|^2}\} \ \text{ if } a \neq 0, \ \text{ and } \omega_0 = 0. \tag{5}$$

Accordingly, for $a \in \mathbb{D}$ there is a unique $\Omega_a \in \mathbb{D}$ such that the unique fixed point of φ_{Ω_a} is a, i.e., $\omega_{\Omega_a} = a$. It is given by

$$\Omega_a = \frac{2a}{1+|a|^2}. \tag{6}$$

Remark 3.1. Note the following (straightforward) formulas:

$$\varphi_{\omega_a} \circ \varphi_a = -\varphi_{\omega_a}. \tag{7}$$

and

$$\varphi_a \circ \varphi_{\omega_a} = -\varphi_{-\omega_a}. \tag{8}$$

Theorem 3.2. *For $a \in \mathbb{D}$, we have that*

$$N(C_a - I) = \{f \in \mathcal{H} : C_{\omega_a} f \in \mathcal{E}\} = C_{\omega_a}\mathcal{E}$$

and

$$N(C_a + I) = \{g \in \mathcal{H} : C_{\omega_a} g \in \mathcal{O}\} = C_{\omega_a}\mathcal{O}.$$

Proof. The formula (7) implies that if f is even, then

$$f(\varphi_{\omega_a}) = f(-\varphi_{\omega_a}) = f(\varphi_{\omega_a} \circ \varphi_a) = C_a f(\varphi_{\omega_a}),$$

i.e., $C_{\omega_a} f \in N(C_a - I)$. The computation for odd elements is similar. Thus $C_{\omega_a}\mathcal{E} \subset N(C_a - I)$ and $C_{\omega_a}\mathcal{O} \subset N(C_a + I)$.

Conversely, if $f \in N(C_a - I)$, then $f = C_{\omega_a} h$ for a unique h (equal to $C_{\omega_a} f$). If $h = h_e + h_o$, for $h_e \in \mathcal{E}$ and $h_o \in \mathcal{O}$. Then $f = C_{\omega_a}(h_e) + C_{\omega_a}(h_o)$. Since $C_{\omega_a} h_o \in N(C_a + I)$, it must be $h_o = 0$, and $f \in C_{\omega_a}\mathcal{E}$. The assertion for $N(C_a + I)$ is proved analogously. □

Note that the idempotents (or non orthogonal projections) corresponding to the direct sum decomposition $\mathcal{H} = N(C_a - I) \dot{+} N(C_a + I)$ are

$$P_{N(C_a - I)\|N(C_a+I)} = \frac{1}{2}(C_a + I) \ \text{ and } \ P_{N(C_a+I)\|N(C_a-I)} = \frac{1}{2}(I - C_a). \tag{9}$$

The above result implies the following:

Corollary 3.3.

$$\frac{1}{2}(C_a + I) = C_{\omega_a} P_{\mathcal{E}} C_{\omega_a} \quad \textit{and} \quad \frac{1}{2}(I - C_a) = C_{\omega_a} P_{\mathcal{O}} C_{\omega_a},$$

i.e., $C_a = C_{\omega_a} C_0 C_{\omega_a}$.

Proof. $C_{\omega_a} P_{\mathcal{E}} C_{\omega_a}$ is an idempotent operator whose range is

$$R(C_{\omega_a} P_{\mathcal{E}} C_{\omega_a}) = C_{\omega_a} R(P_{\mathcal{E}}) = C_{\omega_a}\mathcal{E} = N(C_a - I),$$

and whose nullspace is

$$N(C_{\omega_a} P_{\mathcal{E}} C_{\omega_a}) = C_{\omega_a} N(P_{\mathcal{E}}) = C_{\omega_a}\mathcal{O} = N(C_a + I).$$

The other verification is similar. $\frac{1}{2}(I + C_a) = C_{\omega_a} P_{\mathcal{E}} C_{\omega_a}$ means that

$$C_a = 2C_{\omega_a} P_{\mathcal{E}} C_{\omega_a} - I = C_{\omega_a}(2P_{\mathcal{E}} - I)C_{\omega_a} = C_{\omega_a} C_0 C_{\omega_a}.$$

□

Let us relate the eigenspaces of C_a with those of R_a. First note the following:

Lemma 3.4.

$$C_a|1 - \bar{a}z|^{1/2} = \frac{(1 - |a|^2)^{1/2}}{|1 - \bar{a}z|^{1/2}}$$

and

$$C_a \frac{1}{|1 - |a|^2|^{1/2}} = \frac{|1 - \bar{a}z|^{1/2}}{(1 - |a|^2)^{1/2}}.$$

Proof. Direct computation. □

Proposition 3.5.

1. *$f \in N(R_a - I)$ if and only if $|1 - \bar{a}z|^{1/2} f \in N(C_a - I)$.*
2. *$g \in N(R_a + I)$ if and only if $|1 - \bar{a}z|^{1/2} g \in N(C_a + I)$.*

Proof. We only prove assertion 1., assertion 2. is similar. Let $f \in \mathcal{H}$ such that $R_a f = f$. i.e.,

$$(1 - |a|^2)^{1/2} \frac{1}{|1 - \bar{a}z|} C_a f = f.$$

As in the above Lemma, $C_a|1 - \bar{a}z|^{1/2} = \frac{(1-|a|^2)^{1/2}}{|1-\bar{a}z|^{1/2}}$. Then

$$f = (1 - |a|^2)^{1/2} \frac{1}{|1 - \bar{a}z|} C_a f = \frac{1}{|1 - \bar{a}z|^{1/2}} C_a(|1 - \bar{a}z|^{1/2}) C_a f.$$

Note the elementary multiplicative property of C_a: if $g, h \in \mathcal{H}$ such that $gh \in \mathcal{H}$, then $C_a(gh) = C_a g C_a h$. Then we have

$$f = \frac{1}{|1 - \bar{a}z|^{1/2}} C_a(|1 - \bar{a}z|^{1/2} f),$$

i.e., $C_a(|1 - \bar{a}z|^{1/2} f) = |1 - \bar{a}z|^{1/2} f$.

The other inclusion: suppose that $C_a f = f$. Then (using Lemma 3.4, and the multiplicative property of C_a)

$$R_a(\frac{1}{|1-\bar{a}z|^{1/2}}f) = \frac{(1-|a|^2)^{1/2}}{|1-\bar{a}z|}C_a(\frac{1}{|1-\bar{a}z|^{1/2}}f) = \frac{(1-|a|^2)^{1/2}}{|1-\bar{a}z|}C_a(\frac{1}{|1-\bar{a}z|^{1/2}})C_a f$$
$$= \frac{1}{|1-\bar{a}z|^{1/2}}f,$$

which completes the proof. □

We may thus summarize these facts:

Corollary 3.6.

1. *$N(R_a - I) = \{f \in \mathcal{H} : C_{\omega_a}(|1-\bar{a}z|^{1/2}f) \in \mathcal{E}\} = M_{\frac{1}{|1-\bar{a}z|^{1/2}}}C_{\omega_a}\mathcal{E}$.*

2. *$N(R_a + I) = \{g \in \mathcal{H} : C_{\omega_a}(|1-\bar{a}z|^{1/2}g) \in \mathcal{O}\} = M_{\frac{1}{|1-\bar{a}z|^{1/2}}}C_{\omega_a}\mathcal{O}$.*

Remark 3.7. Note that

$$C_{\omega_a}(|1-\bar{a}z|) = (1-|a|^2)^{1/2}\frac{|1+\bar{\omega}_a z|}{|1-\bar{\omega}_a z|}. \tag{10}$$

Thus $C_{\omega_a}(|1-\bar{a}z|^{1/2}) = (1-|a|^2)^{1/4}\frac{|1+\bar{\omega}_a z|^{1/2}}{|1-\bar{\omega}_a z|^{1/2}}$, and

$$N(R_a - I) = \{f \in \mathcal{H} : \frac{|1+\bar{\omega}_a z|^{1/2}}{|1-\bar{\omega}_a z|^{1/2}}f(\varphi_{\omega_a}) \in \mathcal{E}\}. \tag{11}$$

Indeed,

$$C_{\omega_a}(|1-\bar{a}z|) = |1-\bar{a}\frac{\omega_a - z}{1-\bar{\omega}_a z}| = \frac{|1-\bar{a}\omega_a + z(\bar{a}-\bar{\omega}_a)|}{|1-\bar{\omega}_a z|}.$$

Note that if $r = (1-|a|^2)^{1/2}$ then

$$\bar{a}\omega_a = 1 - r \quad \text{and} \quad \bar{a} - \bar{\omega}_a = \frac{1}{a}(|a|^2 - (1-r)) = \frac{1}{a}(r - r^2).$$

Then, in the above computation we have

$$\frac{|r + \frac{1}{a}(r-r^2)z)|}{|1-\bar{\omega}_a z|} = r\frac{|1+\frac{1}{a}(1-r)z)|}{|1-\bar{\omega}_a z|} = r\frac{|1+\bar{\omega}_a z|}{|1-\bar{\omega}_a z|}.$$

Similarly $C_{\omega_a}(\frac{1}{|1-\bar{a}z|}) = \frac{1}{(1-|a|^2)^{1/2}}\frac{|1-\bar{\omega}_a z|}{|1+\bar{\omega}_a z|}$, and

$$N(C_{\omega_a} + I) = \{g \in \mathcal{H} : \frac{|1-\bar{\omega}_a z|^{1/2}}{|1+\bar{\omega}_a z|^{1/2}}g(\varphi_{\omega_a}) \in \mathcal{O}\}. \tag{12}$$

Remark 3.8. We shall consider other symmetries which interact with C_a:

1. Let V be the symmetry given by $Vf(z) = f(\bar{z})$. The eigenspaces of V are

$$N(V - I) = \{f = \sum_{m \in \mathbb{Z}} a_m z^m : a_m = a_{-m}\}$$

and

$$N(V + I) = \{f = \sum_{m \in \mathbb{Z}} a_m z^m : a_m = -a_{-m}\}.$$

This symmetry has the following commutation relation with C_a:

$$VC_a = C_{\bar{a}}V, \tag{13}$$

and in particular, if $a \in (-1, 1)$, V and C_a commute. Indeed,

$$VC_a f(z) = Vf(\frac{a - z}{1 - \bar{a}z}) = f(\frac{a - \bar{z}}{1 - \bar{a}\bar{z}}) = f(\frac{a - \frac{1}{z}}{1 - \bar{a}\frac{1}{z}}) = f(\frac{1 - \bar{a}z}{a - z}) = f\left(\overline{\varphi_{\bar{a}}(z)}\right)$$
$$= C_{\bar{a}}Vf(z).$$

If one identifies $L^2(\mathbb{T})$ with $L^2(-\pi, \pi)$, by means of $z = e^{it}$, then $N(V - I)$ consists of elements f such that $f(t) = f(-t)$ *pp*, i.e., are even in the parameter t. Similarly $N(V + I)$ consists of elements which are odd in the parameter t.

2. Recall the symmetry W_a from (4), defined in $\mathcal{H}$ as

$$W_a = M_{\frac{\sqrt{1-|a|^2}}{1-\bar{a}z}} C_a, \text{ i.e., } W_a f = \frac{\sqrt{1 - |a|^2}}{1 - \bar{a}z} f(\varphi_a).$$

Note that W_a leaves $\mathcal{H}^+$ invariant, in fact $W_a\mathcal{H}^+ = \mathcal{H}^+$. The restriction $\mathbf{W}_a = W_a\Big|_{\mathcal{H}^+}$ was considered before, see for instance [10] (Exercise 2.1.9), where it was shown that it acts isometrically in $H^p(\mathbb{D})$ for $1 < p < \infty$. It is easy to see that W_a is a symmetry in $\mathcal{H}$:

$$W_a^2 f = (1 - |a|^2)^{1/2} W_a(\frac{1}{1 - \bar{a}z} f(\varphi_a)) = \frac{1 - |a|^2}{1 - \bar{a}z} \frac{1}{1 - \bar{a}\frac{a-z}{1-\bar{a}z}} f = f$$

and

$$W_a^* f = C_a^* M^*_{\frac{\sqrt{1-|a|^2}}{1-\bar{a}z}} f = M_{\frac{1-|a|^2}{|1-\bar{a}z|^2}} C_a M_{\frac{1-|a|^2}{1-a\bar{z}}} f = \frac{(1 - |a|^2)^{3/2}}{|1 - \bar{a}z|^2} \frac{1}{1 - a\frac{\bar{a}-\bar{z}}{1-a\bar{z}}} f(\varphi_a)$$
$$= \frac{(1 - |a|^2)^{1/2}}{1 - \bar{a}z} f(\varphi_a) = W_a f.$$

Note that $W_a = M_{\frac{\sqrt{1-|a|^2}}{1-\bar{a}z}} C_a$ and $R_a = M_{\frac{\sqrt{1-|a|^2}}{|1-\bar{a}z|}} C_a = C_a M_{\frac{1-\bar{a}z}{\sqrt{1-|a|^2}}}$ satisfy

$$W_a R_a = M_{\frac{|1-\bar{a}z|}{1-\bar{a}z}} \text{ and } R_a W_a = M_{\frac{1-\bar{a}z}{|1-\bar{a}z|}}.$$

For $a \in \mathbb{D}$, denote by $(1-\bar{a}z)^{1/2}$ the (non continuous) bounded Borel function

$$(1-\bar{a}z)^{1/2} := \exp\left(\frac{1}{2}\log(1-\bar{a}z)\right), \quad \text{for } z \in \mathbb{T}$$

where log is the usual determination of the lograrithm: $\log(z) = \ln|z| + i\arg(z)$, with $\arg(z) \in [-\pi, \pi)$. Similarly as with the eigenspaces of R_a, we have:

Proposition 3.9. *Let $a \in \mathbb{D}$. Then*

$$N(W_a - I) = \{f \in \mathcal{H} : \frac{f}{(1-\bar{a}z)^{1/2}} \in N(C_a - I)\},$$

and

$$N(W_a + I) = \{g \in \mathcal{H} : \frac{g}{(1-\bar{a}z)^{1/2}} \in N(C_a + I)\},$$

Proof. The proof is similar as in Proposition 3.5. We sketch the argument. If $C_a f = f$, then

$$W_a \frac{f}{(1-\bar{a}z)^{1/2}} = \frac{(1-|a|^2)^{1/2}}{1-\bar{a}z} f(\varphi_a)\left(\frac{1}{1-\bar{a}\frac{a-z}{1-\bar{a}z}}\right)^{1/2} = \frac{(1-|a|^2)^{1/2}}{1-\bar{a}z}\left(\frac{1-\bar{a}z}{1-|a|^2}\right)^{1/2} f = f.$$

For the reverse inclusion, note that $C_a(1-\bar{a}z)^{1/2} = \frac{(1-|a|^2)^{1/2}}{(1-\bar{a}z)^{1/2}}$. Therefore if $W_a f = f$, then

$$f = \frac{(1-|a|^2)^{1/2}}{1-\bar{a}z} C_a f = \frac{1}{(1-\bar{a}z)^{1/2}} \frac{(1-|a|^2)^{1/2}}{(1-\bar{a}z)^{1/2}} C_a f = \frac{1}{(1-\bar{a}z)^{1/2}} C_a (1-\bar{a}z)^{1/2} C_a f$$

$$= \frac{1}{(1-\bar{a}z)^{1/2}} C_a\left((1-\bar{a}z)^{1/2} f\right),$$

i.e.,

$$(1-\bar{a}z)^{1/2} f = C_a\left((1-\bar{a}z)^{1/2} f\right).$$

□

4 The maps $a \mapsto C_a$ and $a \mapsto R_a$

Note that C_a restricts to the Hardy space $\mathcal{H}^+ = H^2(\mathbb{D})$. Let us denote by

$$\mathbf{C}_a := C_a\Big|_{\mathcal{H}^+} \in \mathcal{B}(\mathcal{H}^+).$$

In [6], E. Berkson proved that in the Hardy space $\mathcal{H}^+$, if $a \neq b$, one has

$$\|\mathbf{C}_a - \mathbf{C}_b\| \geq \frac{1}{\sqrt{2}}.$$

Therefore, $\|C_a - C_b\| \geq \frac{1}{\sqrt{2}}$. However, note that

$$\||C_a| - |C_b|\| = \|(1-|a|^2)^{1/2} M_{\frac{1}{|1-\bar{a}z|}} - (1-|b|^2)^{1/2} M_{\frac{1}{|1-\bar{b}z|}}\| = \|M_{\frac{(1-|a|^2)^{1/2}}{|1-\bar{a}z|} - \frac{(1-|b|^2)^{1/2}}{|1-\bar{b}z|}}\|$$

$$= \| \frac{(1-|a|^2)^{1/2}}{|1-\bar{a}z|} - \frac{(1-|b|^2)^{1/2}}{|1-\bar{b}z|} \|_\infty = \sup_{|z|=1} \left| \frac{(1-|a|^2)^{1/2}}{|1-\bar{a}z|} - \frac{(1-|b|^2)^{1/2}}{|1-\bar{b}z|} \right| := \gamma_{a,b}.$$

It is clear that $\gamma_{a,b} \to 0$ if $b \to a$, and therefore the map

$$\mathbb{D} \ni a \mapsto |C_a| \in \mathcal{B}(\mathcal{H})$$

is continuous.

Note the following elementary estimation

$$\frac{1}{\sqrt{2}} \le \|C_a - C_b\| = \|R_a M_{\frac{(1-|a|^2)^{1/2}}{|1-\bar{a}z|}} - R_b M_{\frac{(1-|b|^2)^{1/2}}{|1-\bar{b}z|}}\|$$

$$\le \|R_a M_{\frac{(1-|a|^2)^{1/2}}{|1-\bar{a}z|}} - R_a M_{\frac{(1-|b|^2)^{1/2}}{|1-\bar{b}z|}}\| + \|R_a M_{\frac{(1-|b|^2)^{1/2}}{|1-\bar{b}z|}} - R_b M_{\frac{(1-|b|^2)^{1/2}}{|1-\bar{b}z|}}\|$$

$$\le \|R_a - R_b\| \| \frac{(1-|a|^2)^{1/2}}{|1-\bar{a}z|} \|_\infty + \| \frac{(1-|a|^2)^{1/2}}{|1-\bar{a}z|} - \frac{(1-|b|^2)^{1/2}}{|1-\bar{b}z|} \|_\infty = \|R_a - R_b\| \left(\frac{1+|a|}{1-|a|} \right)^{1/2} + \gamma_{a,b}.$$

By symmetry,

$$\frac{1}{\sqrt{2}} \le \|R_a - R_b\| \min\left\{ \frac{1+|a|}{1-|a|}, \frac{1+|b|}{1-|b|} \right\}^{1/2} + \gamma_{a,b}.$$

In particular, since $\gamma_{a,b} \to 0$ if $b \to a$, the map

$$\mathbb{D} \ni a \mapsto R_a \in \mathcal{B}(\mathcal{H})$$

is not continuous.

5 Relative position of the eigenspaces

Denote by P^+ the orthogonal projection onto $\mathcal{H}^+ \subset \mathcal{H}$. We already noted that $C_a\mathcal{H}^+ \subset \mathcal{H}^+$ (in fact, $C_a\mathcal{H}^+ = \mathcal{H}^+$). Note that $\mathcal{H}^+$ is not invariant for C_a^*, but for a one dimensional space. Denote by $\mathcal{H}^- := (\mathcal{H}^+)^\perp$.

If $g \in \mathcal{H}^-$, then $\bar{g} \in \mathcal{H}^+$. Denote by $\mathbf{g}$ the extension of $\bar{g}$ to an analytic function in $\mathbb{D}$.

Lemma 5.1.

$$C_a\mathcal{H}^- = \langle 1 \rangle \oplus \mathcal{H}^-.$$

More precisely, if $g \in \mathcal{H}^-$, then

$$C_a g = \overline{\mathbf{g}(a)} + h, \quad \textit{for} \ \ h \in \mathcal{H}^-.$$

Proof. Let $g \in \mathcal{H}^-$ and $n \ge 1$. Then

$$\langle C_a g, z^n \rangle = \langle g, C_a^* z^n \rangle = (1-|a|^2) \langle g, \frac{1}{|1-\bar{a}z|^2} \left(\frac{a-z}{1-\bar{a}z} \right)^n \rangle.$$

Note that if $|z| = 1$, then $\frac{1}{|1-\bar{a}z|^2} = \frac{z}{(1-\bar{a}z)(z-a)}$. Then

$$\langle C_a g, z^n \rangle = -(1-|a|^2)\langle g, \frac{z(a-z)^{n-1}}{(1-\bar{a}z)^{n+1}} \rangle = 0.$$

Moreover,

$$\langle C_a g, 1 \rangle = (1-|a|^2)\langle g, \frac{z}{(1-\bar{a}z)(z-a)} \rangle = (1-|a|^2)\langle \frac{\bar{z}}{\bar{z}-\bar{a}}, \frac{\bar{g}}{1-\bar{a}z} \rangle.$$

Note that (for $|z| = 1$), $\frac{\bar{z}}{\bar{z}-\bar{a}} = \frac{\frac{1}{z}}{\frac{1}{z}-\bar{a}} = \frac{1}{1-\bar{a}z} = k_a(z)$. Then, since $\bar{g} \in \mathcal{H}^+$,

$$\langle C_a g, 1 \rangle = (1-|a|^2)\langle k_a, \frac{\bar{g}}{1-\bar{a}z} \rangle = (1-|a|^2)\overline{\frac{\bar{\mathbf{g}}(a)}{1-|a|^2}} = \overline{\bar{\mathbf{g}}(a)},$$

which completes the proof. □

As a consequence, we have the following:

Proposition 5.2.

1. *Let $f \in L^2(\mathbb{T})$ such that $f = f \in N(C_a - I)$, and $f = f_+ + f_-$, with $f_+ \in H^2$ and $f_- \in H^2_-$. Then*
$$f_+, f_- \in N(C_a - I).$$
2. *Let $f \in N(C_a + I)$, and $f = f_+ + f_-$ with $f_+ \in \mathcal{H}^+$ and $f_- \in \mathcal{H}^-$. Then*
$$C_a f_+ = -f_+ - \overline{\bar{\mathbf{f}}_-(a)} \quad \textit{and} \quad C_a f_- = -f_- + \overline{\bar{\mathbf{f}}_-(a)}.$$

Proof. Suppose that $C_a f = f$. Then $f_+ + f_- = C_a f_+ + C_a f_-$. Note that $C_a f_+ \in \mathcal{H}^+$. By Lemma 5.1, $C_a f_- = g + k1$, for $g \in \mathcal{H}^-$ and $k \in \mathbb{C}$. Thus

$$C_a f_+ + g = f_+ - k1 + f_- \implies C_a f_+ = f_+ - k1.$$

Applying C_a to this equality, using that constant functions belong to $N(C_a - I)$, we get $f_+ = C_a f_+ - k1$, and thus $k = 0$. Therefore $C_a f_+ = f_+$ and $C_a f_- = f_-$.

For the case of $N(C_a + I)$, we can proceed as above, using Lemma 5.1: $f \in N(C_a + I)$, $f = f_+ + f_-$, and thus $C_a f_+ + C_a f_- = -f_+ - f_-$. Then $C_a f_- = g + \overline{\bar{\mathbf{f}}_-(a)}$, and therefore $C_a f_+ = -f_+ - \overline{\bar{\mathbf{f}}_-(a)}$ and $C_a f_- = g = -f_- + \overline{\bar{\mathbf{f}}_-(a)}$. Only this time we cannot deduce that $\overline{\bar{\mathbf{f}}_-(a)}$ is zero. □

Remark 5.3. There is an easy example of $f \in N(C_a + I)$ that shows that $C_a f_-$ can have a constant coefficient (and therefore does not belong to $\mathcal{H}^-$). Pick $f = \frac{1}{\varphi_{\omega_a}} = \frac{1-\bar{\omega}_a z}{\omega_a - z}$. Since $f = C_{\omega_a}(\frac{1}{z})$, and $\frac{1}{z}$ is an odd element of $\mathcal{H}$, it follows that $f \in N(C_a + I)$. Also note that

$$f = \bar{\omega}_a + \frac{\bar{\omega}_a^2 - 1}{z - \omega_a},$$

with $f_- = \frac{\bar{\omega}_a^2 - 1}{z - \omega_a}$:

$$\bar{\mathbf{f}}_- = \frac{\omega_a^2 - 1}{\frac{1}{z} - \bar{\omega}_a} = \frac{\omega_a^2 - 1}{1 - \bar{\omega}_a z} z \in \mathcal{H}^+, \quad \text{and vanishes at } z = 0.$$

Clearly $\overline{\bar{\mathbf{f}}_-(a)} = \frac{\omega_a^2 - 1}{a - \bar{\omega}_a} a \neq 0$.

Note that $V\mathcal{H}^- \subset \mathcal{H}^+ \ominus \langle 1 \rangle$ and $V\mathcal{H}_+ = \langle 1 \rangle \oplus \mathcal{H}^-$.

Remark 5.4. In [4] the eigenspaces of the restriction $\mathbf{C}_a := C_a\big|_{\mathcal{H}^+}$ of C_a to $\mathcal{H}^+$ were considered. It was shown that if $a \neq b$ in $\mathbb{D}$, then

$$N(\mathbf{C}_a - I) \cap N(\mathbf{C}_b - I) = \mathbb{C}1 \;\text{ and }\; N(\mathbf{C}_a + I) \cap N(\mathbf{C}_b + I) = \{0\}.$$

([4] Theorem 5.6).

Theorem 5.5. *If $a \neq b$ in $\mathbb{C}$,*

1. $N(C_a - I) \cap N(C_b - I) = \mathbb{C}1$.
2. $N(C_a + I) \cap N(C_b + I) = \{0\}$.

Proof. The first assertion: clearly $\mathbb{C}1 \subset N(C_a - I) \cap N(C_b - I)$. Let $f \in N(C_a - I) \cap N(C_b - I)$, and $f = f_+ + f_-$ with $f_+ \in \mathcal{H}^+$ and $f_- \in \mathcal{H}^-$. Then by Proposition 5.2.1, $f_+, f_- \in N(C_a - I) \cap N(C_b - I)$. Then (restricting to $\mathcal{H}^+$) $f_+ \in N(\mathbf{C}_a - I) \cap N(\mathbf{C}_b - I) = \mathbb{C}1$. Also note that $Vf_- \in \mathcal{H}^+$. It is clear that

$$VN(C_a - I) = N(VC_aV - I) = N(C_{\bar{a}} - I),$$

and similarly for b. Then $Vf_- \in N(C_{\bar{a}} - I) \cap N(C_{\bar{b}} - I) = N(\mathbf{C}_{\bar{a}} - I) \cap N(\mathbf{C}_{\bar{b}} - I) = \mathbb{C}1$. However $Vf_- \perp 1$, and thus $f_- = 0$.

Let now $f \in N(C_a + I) \cap N(C_b + I)$. To treat this case, we shall make two reductions. First, that we can consider the case $b = 0$. To this effect, note the following elementary fact, which follows from direct computations. If $d, b \in \mathbb{D}$, then

$$C_d C_b C_d = C_{d \bullet b}\ , \quad \text{where } \; d \bullet b := \frac{2d - b - d^2\bar{b}}{|d|^2 - d\bar{b} - \bar{d}b + 1}. \tag{14}$$

Therefore, if $b \neq 0$, and we choose $d = 1 - \sqrt{1 - |b|^2}$, which is the unique element in $\mathbb{D}$ such that $d \bullet b = 0$, we have that the statement

$$N(C_a + I) \cap N(C_b + I) = \{0\}$$

is equivalent to

$$\{0\} = C_d\left(N(C_a + I) \cap N(C_b + I)\right) = N(C_d C_a C_d + I) \cap N(C_d C_b C_d + I)$$

$$= N(C_{d\bullet a} + I) \cap N(C_0 + I).$$

Next, we can further reduce to the case $a = r \in (0, 1)$. Indeed, if $a = re^{i\theta}$, consider the unitary operator U_θ in $\mathcal{H}$ given by $U_\theta f(z) = f(e^{-i\theta}z)$. Then it is elementary to verify that

$$U_\theta C_r U_\theta^* = C_{e^{i\theta}r} = C_a. \tag{15}$$

Thus

$$N(C_a + I) \cap N(C_0 + I) = \{0\}$$

is equivalent to

$$\{0\} = U_{-\theta}(N(C_a + I) \cap N(C_0 + I)) = N(U_{-\theta}C_aU_\theta + I) \cap N(U_{-\theta}C_0U_\theta + I)$$

$$= N(C_r + I) \cap N(C_0 + I).$$

Then we may assume that $C_rf = f$ and $C_0f = f$ (i.e., f is odd). Decompose $f = f_+ + f_-$, with $f_+ \in \mathcal{H}^+$ and $f_- \in \mathcal{H}^-$. Clearly f_+, f_- are also odd, and then $C_0f_+ = -f_+$ and $C_0f_- = -f_-$. By Proposition 5.2 we know that $C_rf_+ = -f_+ - k$ and $C_rf_- = -f_- + k$, for some constant k. Since $f_+ \in \mathcal{H}^+$ we may regard it as a function in $\mathbb{D}$, and evaluate this identity at $z = 0$ (since f_+ is odd, $f_+(0) = 0$):

$$f_+(r) = f_+(\varphi_r(0)) = C_rf_+(0) = -f_+(0) - k = -k \tag{16}$$

Note also that the identity $C_rf_+ = -f_+ - k$ implies that $f_+ - \frac{k}{2} \in N(C_r - I)$. Therefore, $h = C_{\omega_r}(f_+ - \frac{k}{2})$ is an odd function in $\mathbb{D}$. In particular, $h(\omega_r) = -h(-\omega_r)$. On one hand

$$h(\omega_r) = f_+(\varphi_{\omega_r}(\omega_r) - \frac{k}{2} = f_+(0) - \frac{k}{2} = -\frac{k}{2}.$$

On the other hand

$$-h(-\omega_r) = -f_+(\varphi_{\omega_r}(-\omega_r) + \frac{k}{2}.$$

Note that $\varphi_{\omega_r}(-\omega_r) = \frac{2\omega_r}{1+\omega_r^2} = \Omega_{\omega_r} = r$ (recall (6)). Then the above equals

$$-h(-\omega_r) = -f_+(r) + \frac{k}{2}.$$

Then $h(\omega_r) = -h(-\omega_r)$ means that $-\frac{k}{2} = -f_+(r) + \frac{k}{2}$. That is $f_+(r) = k$, which together with (16) imply that $k = 0$. That is $f_+ \in N(C_r - I)$. That is $C_rf_+ = -f_r$ and $C_rf_- = -f_-$. Since $f_+ \in \mathcal{H}^+$, by Theorem 5.6 in [4] (Remark 5.4 above), we have that $f_+ = 0$.

We may use the isometry V ($Vf(z) = f(\bar{z})$), which commutes with C_r (see (13), because r is real. Clearly $Vf_- \in \mathcal{H}^-$. Then

$$C_rVf_- = VC_rf_- = -Vf_-,$$

and thus $f_- = 0$. □

Let us now consider intersections involving the orthogonal complements of the eigenspaces of C_a and C_b. We would like to consider arbitrary $a \neq b$. We were not able to achieve this, save for some special cases. We examine first the case $b = 0$:

Theorem 5.6. *Let $a \neq 0$. Then*

1. $N(C_a - I)^{\perp} \cap \mathcal{E} = \{0\} = N(C_a - I) \cap \mathcal{O}$.

2. $N(C_a + I)^{\perp} \cap \mathcal{O} = \{0\} = N(C_a + I) \cap \mathcal{E}$.

Proof. We prove first the left hand intersection in the first assertion. Let f even such that $f \perp N(C_a - I) = C_{\omega_a}(\mathcal{E})$. Put $f = f_+ + f_-$ with $f_+ \in \mathcal{H}^+$ and $f_- \in \mathcal{H}^-$. Clearly both f_+ and f_- are even. Then

$$0 = \langle f_+ + f_-, C_{\omega_a} z^{2m} \rangle.$$

For $m \geq 0$, since $C_{\omega_a}(z^{2m}) = \mathbf{C}_{\omega_a} z^{2m} \in \mathcal{H}^+$, we have

$$0 = \langle f_+, \mathbf{C}_{\omega_a} z^{2m} \rangle,$$

and thus $f_+ \in N(\mathbf{C}_0 - I) \cap N(\mathbf{C}_{\omega_a} - I)^{\perp}$. In [4] (Theorem 5.7) it was proven that this intersection is trivial. Then $f_+ = 0$. Thus, for $m < 0$, since $\varphi_{\omega_a}(z)$ has modulus 1 for $z \in \mathbb{T}$,

$$0 = \langle f_-, C_{\omega_a} z^{2m} \rangle \implies 0 = \langle \overline{f}_-, \overline{C_{\omega_a} z^{2m}} \rangle = \langle \overline{f}_-, \left((\varphi_{\omega_a}(z))^{2m} \right)^- \rangle = \langle \overline{f}_-, \varphi_{\omega_a}^{-2m} \rangle,$$

i.e., the even analytic element $\overline{f}_-$ is orthonogonal to $C_{\omega_a} z^k$ for $k \neq 0$. Clearly also $\overline{f}_- \perp 1$, so that by the same result cited above. $f_- = 0$.

The right hand intersection in the first assertion is simpler. Suppose $f = f_+ + f_-$ is odd and satisifies $C_a f = f$. Then by Proposition 5.2, we have that f_+ and f_- are odd and belong to $N(C_a - I)$. Then f_+ and $\overline{f}_-$ are odd (analytic) and belong to $N(\mathbf{C}_a - I)$. Therefore, again using Theorem 5.7 in [4]. we have that $f_+, f_- = 0$.

The proof of the second assertion is similar. □

There is an analogous result, which can be proved similarly:

Theorem 5.7. *Let $a \neq 0$. Then*

1. $N(C_a + I)^{\perp} \cap \mathcal{E} = \{0\}$.

2. $N(C_a - I)^{\perp} \cap \mathcal{O} = \{0\}$.

We could not solve the case when $b \neq 0$. Let us state the following partial observations:

Remark 5.8. Let us briefly elaborate on the intersection $\mathcal{S} := N(C_a - I) \cap N(C_b - I)^{\perp}$, for arbitrary $a \neq b$ in $\mathbb{D}$. We compute $R_b \mathcal{S}$. First since R_b is a symmetry

$$R_b \left(N(C_b - I)^{\perp} \right) = \left(R_b N(C_b - I) \right)^{\perp} = \left(N(R_b C_b R_b - I) \right)^{\perp}.$$

Using (3) this equals

$$N(C_b^* - I)^\perp = R(C_b - I) = N(C_b + I).$$

On the other hand, $R_b(N(C_a - I)) = N(R_b C_a R_b - I)$, and using both expressions of R_b in (2), and (14), we get

$$R_b C_a R_b = M_{\frac{\sqrt{1-|b|^2}}{|1-\bar{b}z|}} C_b C_a C_b M_{\frac{|1-\bar{b}z|}{\sqrt{1-|b|^2}}} = M_{\frac{1}{|1-\bar{b}z|}} C_{b\bullet a} M_{|1-\bar{b}z|}.$$

Then

$$R_b \mathcal{S} = N(C_b + I) \cap M_{\frac{1}{|1-\bar{b}z|}} N(C_{b\bullet a} - I).$$

Or equivalently, applying $M_{\frac{|1-\bar{b}z|}{(1-|b|^2)^{1/2}}}$,

$$C_b \mathcal{S} = M_{|1-\bar{b}z|} R_b \mathcal{S} = N(C_{b\bullet a} - I) \cap M_{|1-\bar{b}z|} N(C_b + I).$$

6 Position of $N(C_a - I)$ and $N(C_a + I)$.

In this subsection we treat the special case of (the position of) $N(C_a - I)$ and $N(C_a + I)$. First note that $N(C_a - I) \cap N(C_a + I) = \{0\}$ and

$$N(C_a - I)^\perp \cap N(C_a + I)^\perp = \langle N(C_a - I) \vee N(C_a + I)\rangle^\perp = \mathcal{H}^\perp = \{0\}.$$

Suppose $a \neq 0$. As in the proof of Theorem 5.5, using the unitary U_θ (for $a = re^{i\theta}$, see (15)) we can reduce to the case $a = r \in (0, 1)$.

In order to further consider these subspaces we shall use their associated orthogonal projections $P_{N(C_a-I)}$ and $P_{N(C_a+I)}$. For an arbitrary idempotent Q, recall the formulas (see for instance T. Ando [1])

$$P_{R(Q)} = Q(Q + Q^* - I)^{-1} \quad \text{and} \quad P_{N(Q)} = (I - Q)(I - Q - Q^*)^{-1} \tag{17}$$

In our case $Q = \frac{1}{2}(C_a + I)$ and thus

$$P_{N(C_a-I)} = (C_a + I)\,(C_a + C_a^*)^{-1} = (C_a + I)\left(C_a + M_{\frac{1-|a|^2}{|1-\bar{a}z|^2}} C_a\right)^{-1}$$

$$= (C_a + I)\left(M_{1+\frac{1-|a|^2}{|1-\bar{a}z|^2}} C_a\right)^{-1} = (C_a + I) C_a M_{\psi_a} = (I + C_a) M_{\psi_a}$$

where $\psi_a(z) := \left(1 + \frac{1-|a|^2}{|1-\bar{a}z|^2}\right)^{-1}$, and similarly

$$P_{N(C_a+I)} = (I - C_a)(C_a + C_a^*)^{-1} = (I - C_a) M_{\psi_a}.$$

Proposition 6.1. *If $a \in \mathbb{D}$, $a \neq 0$, then*

$$\|P_{N(C_a-I)} - P_{N(C_a+I)}\| = 1$$

Proof. If $a = re^{i\theta}$, Then $C_a = U_\theta C_r U_{-\theta}$, $C_a^* = U_\theta C_r^* U_{-\theta}$, and thus

$$\|P_{N(C_a-I)} - P_{N(C_a+I)}\| = \|U_{-\theta} P_{N(C_r-I)} U_\theta - U_{-\theta} P_{N(C_r+I)} U_\theta\|$$

$$= \|U_{-\theta}(P_{N(C_r-I)} - P_{N(C_r+I)})U_\theta\| = \|P_{N(C_r-I)} - P_{N(C_r+I)}\|,$$

i.e., we reduce to the case $a = r \in (0,1)$. By the above formulas,

$$P_{N(C_r-I)} - P_{N(C_r+I)} = 2C_r M_{\psi_r}.$$

Then

$$(P_{N(C_r-I)} - P_{N(C_r+I)})^2 = (P_{N(C_r-I)} - P_{N(C_r+I)})^*(P_{N(C_r-I)} - P_{N(C_r+I)}) = 4M_{\psi_r} C_r^* C_r M_{\psi_r}$$

$$= 4M_{\psi_r} M_{\frac{1-r^2}{|1-rz|^2}} C_r C_r M_{\psi_r} = M_{4\psi_r^2 \frac{1-r^2}{|1-rz|^2}}.$$

If we identify $L^2(\mathbb{T}) \sim L^2([-\pi,\pi])$, $f(e^{it}) \sim f(t)$, then $|1-rz|^2 \sim 1 + r^2 - 2r\cos(t)$ and then

$$4\psi_r^2 \frac{1-r^2}{|1-rz|^2} \sim \frac{(1+r^2-2r\cos(t))}{(1-r\cos(t))^2}(1-r^2).$$

Thus, we have to compute

$$\sup_{t\in[-\pi,\pi]} \frac{(1+r^2-2r\cos(t))}{(1-r\cos(t))^2} = \sup_{s\in[-r,r]} \frac{1+r^2-s}{(1-s)^2} = \frac{1}{1-r^2}.$$

Then

$$\|P_{N(C_r-I)} - P_{N(C_r+I)}\|^2 = 1.$$

□

Remark 6.2. In [11] Chandler Davis studied operators which are differences of projections (see also [2]). An operator $A = P - Q$, for P, Q orthogonal projections, is a selfadjoint contraction, with the additional following spectral property:

1. For $|\lambda| < 1$, one has that $\lambda \in \sigma(A)$ if and only if $-\lambda \in \sigma(A)$. Moreover, the spectral multiplicity function of A is symmetric with respect to the origin. In particular, λ (with $|\lambda| < 1$) is an eigenvalue of A if and only if $-\lambda$ is also an eigenvalue of A, and in that case they have the same multiplicity.

2. $N(A - I) = R(P) \cap N(Q)$ and $N(A + I) = N(P) \cap R(Q)$ may have different dimensions.

Theorem 6.3. *Let $a \in \mathbb{D}$, $a \neq 0$.*

$$\sigma(P_{N(C_a-I)} - P_{N(C_a+I)}) = [-1, -(1-|a|^2)^{1/2}] \cup [(1-|a|^2)^{1/2}, 1].$$

There are no eigenvalues. In particular

$$N(C_a - I) \cap N(C_a + I)^\perp = \{0\} = N(C_a - I)^\perp \cap N(C_a + I).$$

Proof. Again, if $a = re^{i\theta}$, it suffices to consider the case $a = r \in (0,1)$. Recall from the proof of Proposition 6.1, that $(P_{N(C_r-I)} - P_{N(C_r+I)})^2 = M_{\psi_r^2 \frac{1-r^2}{|1-rz|^2}}$. The minimum of

$$\psi_r^2 \frac{1-r^2}{|1-rz|^2} \sim \frac{(1+r^2-2r\cos(t))}{(1-r\cos(t))^2}(1-r^2)$$

for $t \in [-\pi, \pi]$ occurs at $t = \pm\pi$, and equals $1 - r^2$. It follows that

$$\sigma((P_{N(C_r-I)} - P_{N(C_r+I)})^2) = [1-r^2, 1]$$

with no eigenvalues. The result follows by the spectral symmetry of $P_{N(C_r-I)} - P_{N(C_r+I)}$. Note that ± 1 cannot be eigenvalues of $P_{N(C_r-I)} - P_{N(C_r+I)}$, because in either case 1 would be an eigenvalue of $(P_{N(C_r-I)} - P_{N(C_r+I)})^2$. □

Summarizing: $N(C_a - I)$ and $N(C_a + I)$ are in generic position.

When one studies the geometry of a pair of subspaces $\mathcal{S}$, $\mathcal{T}$, an important feature is the spectral picture of the product $P_{\mathcal{S}}P_{\mathcal{T}}P_{\mathcal{S}}$.

Lemma 6.4. *Suppose that $\mathcal{S}$ and $\mathcal{T}$ are closed subspaces of $\mathcal{H}$. If $P_{\mathcal{S}}P_{\mathcal{T}}P_{\mathcal{S}}$ has a an eigenvalue $\lambda \neq 0, 1$, then $\pm(1-\lambda^2)^{1/2}$ are eigenvalues of $P_{\mathcal{S}} - P_{\mathcal{T}}$.*

Proof. Suppose $P_{\mathcal{S}}P_{\mathcal{T}}P_{\mathcal{S}}f = \lambda f$ for $\lambda \neq 0, 1$ and $\|f\| = 1$. Then $f \in \mathcal{S}$ and therefore the subspace $\mathcal{V}$ generated by f and $P_{\mathcal{T}}f$ is invariant for $P_{\mathcal{S}}$ and $P_{\mathcal{T}}$:

$$P_{\mathcal{S}}f = f \ ; \ P_{\mathcal{S}}P_{\mathcal{T}}f = P_{\mathcal{S}}P_{\mathcal{T}}P_{\mathcal{S}}f = \lambda f \ ; \ P_{\mathcal{T}}f \ \text{ and } \ P_{\mathcal{T}}P_{\mathcal{T}}f = P_{\mathcal{T}}f$$

belong to $\mathcal{V}$. Then $P_{\mathcal{S}} - P_{\mathcal{T}}$ is a selfadjoint operator acting in $\mathcal{V}$, which is two dimensional (if $P_{\mathcal{T}}f$ where a multiple of f, then either $P_{\mathcal{T}}f = 0$ and then $P_{\mathcal{S}}P_{\mathcal{T}}P_{\mathcal{S}}f = P_{\mathcal{S}}P_{\mathcal{T}}f = 0$; or $P_{\mathcal{T}}f = \alpha f$ and thus $f \in \mathcal{S} \cap \mathcal{T}$ and therefore $P_{\mathcal{S}}P_{\mathcal{T}}P_{\mathcal{S}}f = f$). Then $\{f, P_{\mathcal{T}}f\}$ is a basis for $\mathcal{V}$ and the matrices of $P_{\mathcal{S}}$, $P_{\mathcal{T}}$ and $P_{\mathcal{S}} - P_{\mathcal{T}}$ as operators in $\mathcal{V}$ for this basis are, respectively:

$$\begin{pmatrix} 1 & \lambda \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & \lambda \\ -1 & -1 \end{pmatrix}.$$

Note that $\lambda \in (0,1)$: $P_{\mathcal{S}}P_{\mathcal{T}}P_{\mathcal{S}} \geq 0$. The eigenvalues of the third matrix are $-(1-\lambda^2)^{1/2}$ and $(1-\lambda^2)^{1/2}$. □

Corollary 6.5. *Let $a \in \mathbb{D}$, $a \neq 0$. Then $P_{N(C_a-I)}P_{N(C_a+I)}P_{N(C_a-I)}$ has no non zero eigenvalues.*

Proof. The fact that $N(C_a - I) \cap N(C_a + I) = \{0\}$, implies that 1 is not an eigenvalue of $P_{N(C_a-I)}P_{N(C_a+I)}P_{N(C_a-I)}$. Thus Lemma 6.4 applies: $P_{N(C_a-I)} - P_{N(C_a+I)}$ has no eigenvalues. □

There are several papers dealing with norm, invertibility and geometry of pairs of projections (including sums, differences and products of projections). See for instance the papers [12], [7] (or the survey paper [5] and references therein). Among these facts, let us state the following proposition.

Proposition 6.6. ([12], [8])

Let P,Q be orthogonal projections. Then $\|PQ\| < 1$ if and only if $P^\perp + Q^\perp$ is invertible.

In our case, we obtain the preliminary estimation:

Corollary 6.7. *Let $a \in \mathbb{D}$, $a \neq 0$. Then*

$$\|P_{N(C_a-I)}P_{N(C_a+I)}\| < 1.$$

Proof. Let us check that $P^\perp_{N(C_a-I)} + P^\perp_{N(C_a+I)}$ is invertible. Note that

$$P^\perp_{N(C_a-I)} + P^\perp_{N(C_a+I)} = 2 - P_{N(C_a-I)} + P_{N(C_a+I)} = 2 - (C_a+I)M_{\psi_r} - (I - C_a)M_{\psi_a} = 2 - 2M_{\psi_a},$$

which is invertible if and only if the continuous function $\psi_a - 1$ does not vanish in $\mathbb{T}$. Note that

$$\psi_a(z) - 1 = \psi_a(z)\frac{1 - |a|^2}{|1 - \bar{a}z|},$$

does not vanish in $\mathbb{T}$. □

We may refine this result, by yet another consequence of Lemma 6.4.

Corollary 6.8. *Let $a \in \mathbb{D}$, $a \neq 0$. Then*

$$\sigma(P_{N(C_a-I)}P_{N(C_a+I)}P_{N(C_a-I)}) \subset [0, |a|].$$

with $|a|$ belonging to this spectrum. In particular, $\|P_{N(C_a-I)}P_{N(C_a+I)}P_{N(C_a-I)}\| = |a|$.

Proof. Write $A := P_{N(C_a-I)}P_{N(C_a+I)}P_{N(C_a-I)}$. Note that $A \geq 0$. Consider the universal representation $\pi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{J})$, obtained as the sum of all GNS representations of $\mathcal{B}(\mathcal{H})$. Note that $\|A\|$ is an eigenvalue for $\pi(A)$ (if φ is a state in $\mathcal{B}(\mathcal{H})$ such that $\varphi(A) = \|A\|$, then, in the Hilbert space $\mathcal{J}_\varphi$ associated to GNS representation corresponding to the state φ, $\|A\|$ is an eigenvalue $\pi(A)$, with associated eigenvector in $\mathcal{J}_\varphi$, given by the class of 1 in the space $\mathcal{J}_\varphi \subset \mathcal{J}$). Note from Corollary 6.7, that $\|A\| < 1$. Then since $\|A\| \neq 0, 1$ is an eignevalue of $\pi(A) = \pi(P_{N(C_a-I)}P_{N(C_a+I)}P_{N(C_a-I)})$, by Lemma 6.4, $(1 - \|A\|^2)^{1/2}$ is an eigenvalue of $\pi(P_{N(C_a-I)} - P_{N(C_a+I)})$. Since π is faithful, it is an injective $*$-homomorphism, and thus

$$\sigma(\pi(P_{N(C_a-I)} - P_{N(C_a+I)})) = \sigma(P_{N(C_a-I)} - P_{N(C_a+I)}).$$

i.e., (using Theorem 6.3)

$$(1 - \|A\|^2)^{1/2} = (1 - |a|^2)^{1/2},$$

i.e., $\|A\| = |a|$. □

Note that if $a = 0$, the product $P_{N(C_0-I)}P_{N(C_0+I)}P_{N(C_0-I)} = P_{\mathcal{E}}P_{\mathcal{O}}P_{\mathcal{E}} = 0$ and this result is trivially valid.

7 Pushing with the R and W symmetries

Recall from (3) that

$$R_aC_aR_a = C_a^* = M_{\frac{1-|a|^2}{|1-\bar{a}z|^2}} C_a.$$

If we conjugate C_a with the symmetry W_a of Remark 3.8.2, we get:

Lemma 7.1. *Let $a \in \mathbb{D}$. Then*

$$W_aC_aW_a = M_{\frac{1-|a|^2}{(1-\bar{a}z)^2}} C_a = M_{\frac{k_a}{\|k_a\|_2}} W_a.$$

Proof. Direct computation:

$$W_aC_aW_af(z) = W_aC_a\left(\frac{(1-|a|^2)^{1/2}}{1-\bar{a}z}f(\varphi_a(z))\right) = W_a\left(\frac{1-\bar{a}z}{(1-|a|^2)^{1/2}}f(z)\right)$$

$$= \frac{1-|a|^2}{(1-\bar{a}z)^2}f(\varphi_a(z)).$$

□

Recall the definition of Ω_a in (6). We have the following:

Lemma 7.2. *Let $a \in \mathbb{D}$. Then*

1.

$$R_aC_0R_a = R_{\Omega_a}.$$

2.

$$W_aC_0W_a = W_{\Omega_a}$$

Proof. We prove the first assertion, the second is analogous:

$$R_aC_0R_af(z) = R_aC_0\left(\frac{(1-|a|^2)^{1/2}}{|1-\bar{a}z|}f(\frac{a-z}{1-\bar{a}z})\right) = R_a\left(\frac{(1-|a|^2)^{1/2}}{|1+\bar{a}z|}f(\frac{a+z}{1+\bar{a}z})\right)$$

$$= \frac{(1-|a|^2)}{|1-\bar{a}z|}\frac{1}{|1+\bar{a}\frac{a-z}{1-\bar{a}z}|}f\left(\frac{a+\frac{a-z}{1-\bar{a}z}}{1+\bar{a}\frac{a-z}{1-\bar{a}z}})\right) = \frac{1-|a|^2}{|1+|a|^2-2\bar{a}z|}f\left(\frac{a-|a|^2z+a-z}{1-\bar{a}z+|a|^2-\bar{a}z}\right)$$

$$= \frac{1-|a|^2}{1+|a|^2}\frac{1}{|1-\bar{\Omega}_az|}f\left(\frac{\frac{2a}{1+|a|^2}-z}{1-\frac{2\bar{a}}{1+|a|^2}z}\right) = \frac{1-|a|^2}{1+|a|^2}\frac{1}{|1-\bar{\Omega}_az|}f(\varphi_{\Omega_a}(z)).$$

Finally, note that

$$1-|\Omega_a|^2 = 1-\frac{4|a|^2}{(1+|a|^2)^2} = \frac{(1-|a|^2)^2}{(1+|a|^2)^2},$$

so that the computation above equals

$$(1-|\Omega_a|^2)^{1/2}\frac{1}{|1-\bar{\Omega}_az|}f(\varphi_{\Omega_a}(z)) = R_{\Omega_a}f(z).$$

□

We can use the formulas in Remark 3 and Lemma 7.2.1 applied to the intersections computed in Theorems 5.5, 5.6 and 5.7 to obtain the following:

Theorem 7.3. *Let $a \in \mathbb{D}$, Then we have:*

1. $N(R_{\Omega_a} - I) \cap N(C_a + I)^{\perp} = \langle \frac{1}{|1-\bar{a}z|} \rangle$.
2. $N(R_{\Omega_a} + I) \cap N(C_a - I)^{\perp} = \{0\}$.
3. $N(R_{\Omega_a} - I) \cap N(C_a + I) = \{0\}$.
4. $N(R_{\Omega_a} + I) \cap N(C_a + I)^{\perp} = \{0\}$.
5. $N(R_{\Omega_a} + I) \cap N(C_a - I) = \{0\}$.
6. $N(R_{\Omega_a} + I) \cap N(C_a - I)^{\perp} = \{0\}$.
7. $N(R_{\Omega_a} - I) \cap N(C_a - I) = \{0\}$.
8. $N(R_{\Omega_a} + I) \cap N(C_a + I) = \{0\}$.

Proof. We write the proofs of the first two, and indicate how to do the rest.

1. From Theorem 5.5 we have that $N(C_0 - I) \cap N(C_a - I) = \langle 1 \rangle$. Apply the symmetry R_a to this formula, noting that $R_a(N(C_b \pm I)) = N(R_a C_b R_a \pm I)$:

$$\langle R_a 1 \rangle = R_a(N(C_0 - I)) \cap R_a(N(C_a - I)) = N(R_a C_0 R_a - I) \cap N(R_a C_a R_a - I)$$

$$= N(R_{\Omega_a} - I) \cap N(C_a^* - I).$$

Note that $R_a 1 = \frac{(1-|a|^2)^{1/2}}{|1-\bar{a}z|} C_a 1 = \frac{(1-|a|^2)^{1/2}}{|1-\bar{a}z|}$, and that $R(C_a \pm I) = N(C_a \mp I)$, so that

$$N(C_a^* - I) = R(C_a - I)^{\perp} = N(C_a + I)^{\perp}.$$

2. From Theorem 5.5 we have that $N(C_0 + I) \cap N(C_a + I) = \{0\}$. Applying R_a as above we get

$$\{0\} = R_a(N(C_0+I)) \cap R_a(N(C_a+I)) = N(R_{\Omega_a}+I) \cap N(C_a^*+I) = N(R_{\Omega_a}+I) \cap R(C_a+I)^{\perp}$$

$$= N(R_{\Omega_a} + I) \cap N(C_a - I)^{\perp}.$$

3. Apply R_a to $N(C_0 - I) \cap N(C_a - I)^{\perp} = \{0\}$ from Theorem 5.6.
4. Apply R_a to $N(C_0 + I) \cap N(C_a - I) = \{0\}$ from Theorem 5.6.
5. Apply R_a to $N(C_0 + I) \cap N(C_a + I)^{\perp} = \{0\}$ from Theorem 5.6.
6. Apply R_a to $N(C_0 + I) \cap N(C_a - I) = \{0\}$ from Theorem 5.6.
7. Apply R_a to $N(C_0 - I) \cap N(C_a + I)^{\perp} = \{0\}$ from Theorem 5.7.
8. Apply R_a to $N(C_0 + I) \cap N(C_a - I)^{\perp} = \{0\}$ from Theorem 5.7. □

Lemma 7.4. *Let $a \in \mathbb{D}$, then*

1.

$$R_{\omega_a} C_a R_{\omega_a} = M_{\frac{|1+\bar{\omega}_a z|}{|1-\bar{\omega}_a z|}} C_0.$$

2.

$$W_{\omega_a} C_a W_{\omega_a} = M_{\frac{1+\bar{\omega}_a z}{1-\bar{\omega}_a z}} C_0.$$

Proof. We prove only assertion 1., the proof of assertion 2. is similar:

$$R_{\omega_a} C_a R_{\omega_a} f(z) = R_{\omega_a} C_a \left(\frac{(1-|\omega_a|^2)^{1/2}}{|1-\bar{\omega}_a z|} f(\varphi_{\omega_a}(z)) \right) = R_{\omega_a} \left(\frac{(1-|\omega_a|^2)^{1/2}}{|1-\bar{\omega}_a \frac{a-z}{1-\bar{a}z}|} f(\varphi_{\omega_a}(\varphi_a(z))) \right).$$

Recall from (7) that $\varphi_{\omega_a} \circ \varphi_a = -\varphi_{\omega_a}$. Then the above expression equals

$$R_{\omega_a} \left(\frac{(1-|\omega_a|^2)^{1/2}|1-\bar{a}z|}{|1-\bar{\omega}_a - z(\bar{a}-\bar{\omega}_a)|} f(-\varphi_a(z)) \right) = \frac{(1-|\omega_a|^2)|1-\bar{a}\frac{\omega_a - z}{1-\bar{\omega}_a z}|}{|1-\bar{\omega}_a z||1-\bar{\omega}_a a - (\bar{\omega}_a - \bar{a})\frac{\omega_a - z}{1-\bar{\omega}_a z}|} f(-z)$$

$$= \frac{(1-|\omega_a|^2)|1-\bar{a}\omega_a - z(\bar{\omega}_a - \bar{a})|}{|1-\bar{\omega}_a z||1-\bar{\omega}_a a - \omega_a \bar{a} + |\omega_a|^2 - z(2\bar{\omega}_a - \bar{a} - \bar{\omega}_a^2 a)|} f(-z).$$

Note that $2\bar{\omega}_a - \bar{a} - \bar{\omega}_a^2 a = 0$. Also note that

$$1-\bar{a}\omega_a - z(\bar{\omega}_a - \bar{a}) = (1-\bar{a}\omega_a)(1 - z\frac{\bar{\omega}_a - \bar{a}}{1-\bar{a}\omega_a})$$

and that $\frac{\bar{\omega}_a - \bar{a}}{1-\bar{a}\omega_a} = \overline{\frac{\omega_a - a}{1-\bar{\omega}_a a}} = \overline{\varphi_{\omega_a}(a)} = -\bar{\omega}_a$, because (7) implies that

$$-\omega_a = -\varphi_{\omega_a}(0) = \varphi_{\omega_a}(\varphi_a(0)) = \varphi_{\omega_a}(a).$$

Thus we get

$$R_{\omega_a} C_a R_{\omega_a} f(z) = \frac{(1-|\omega_a|^2)|1-\bar{a}\omega|}{|1+|\omega_a|^2 - \bar{\omega}_a a - \omega_a \bar{a}|} \frac{|1+\bar{\omega}_a z|}{|1-\bar{\omega}_a z|} f(-z).$$

The proof finishes by showing that

$$\frac{(1-|\omega_a|^2)|1-\bar{a}\omega|}{|1+|\omega_a|^2 - \bar{\omega}_a a - \omega_a \bar{a}|} = 1,$$

which is a straightforwad computation. □

Remark 7.5. Note that $M_{\frac{|1+\bar{\omega}_a z|}{|1-\bar{\omega}_a z|}}$ is positive, and that

$$R_{\omega_a} C_a^* R_{\omega_a} = C_0 M_{\frac{|1+\bar{\omega}_a z|}{|1-\bar{\omega}_a z|}}.$$

So that this expression is the (unique) polar decomposition of $R_{\omega_a} C_a^* R_{\omega_a}$. In particular

$$|R_{\omega_a} C_a^* R_{\omega_a}| = R_{\omega_a} |C_a^*| R_{\omega_a} = M_{\frac{|1+\bar{\omega}_a z|}{|1-\bar{\omega}_a z|}}.$$

8 Geometry of the Grassmann manifold of $\mathcal{H}$

Let $Gr(\mathcal{H}) := \{\mathcal{S} \subset \mathcal{H} : \mathcal{S}$ is a closed subspace of $\mathcal{H}\}$. In [13] and [9], H. Porta, L. Recht and G. Corach (in the latter paper) studied the differential geometric structure of $Gr(\mathcal{H})$, by identifying a subspace $\mathcal{S}$ with the orthogonal projection $P_{\mathcal{S}}$ onto $\mathcal{S}$, or alernatively, with the symmetry $\epsilon_{\mathcal{S}} := P_{\mathcal{S}} - I$ which is the identity at $\mathcal{S}$ (see also [3] for an abridged survey of these results).

Let us briefly recall the facts of the geometry of $Gr(\mathcal{H})$ needed here:

Remark 8.1. [13], [9]

1. $Gr(\mathcal{H})$, regarded as the space of orthogonal projections, is a complemented submanifold of $\mathcal{B}(\mathcal{H})$. Its tangent spaces are complemented subspaces of $\mathcal{B}(\mathcal{H})$. Thus $Gr(\mathcal{H})$ has the Finsler metric which consists in the norm of $\mathcal{B}(\mathcal{H})$ at every tangent space.

2. Additionally $Gr(\mathcal{H})$ is a homogeneous space of the unitary group $\mathcal{U}(\mathcal{H})$ of $\mathcal{H}$ (as in the classical finite dimensional setting): the natural left action of (unitary) operators on closed subspaces, $U \cdot \mathcal{S} = U(\mathcal{S})$, for $U \in \mathcal{U}(\mathcal{H})$ and $\mathcal{S} \in Gr(\mathcal{H})$ (which at the operator level is $U \cdot P_{\mathcal{S}} = P_{U(\mathcal{S})} = UP_{\mathcal{S}}U^*$, or $U \cdot \epsilon_{\mathcal{S}} = U\epsilon_{\mathcal{S}}U^*$).

3. This action admits a natural *reductive* structure, based on the notion of diagonaltiy / co-diagonality at every point $\mathcal{S} \in Gr(\mathcal{H})$: an operator $A \in \mathcal{B}(\mathcal{H})$ is diagonal with respect to $\mathcal{S}$ if $A\mathcal{S} \subset \mathcal{S}$ and $A\mathcal{S}^\perp \subset \mathcal{S}^\perp$ (i.e., A commutes with $\epsilon_{\mathcal{S}}$); $B \in \mathcal{B}(\mathcal{H})$ si co-diagonal with respect to $\mathcal{S}$ is $B\mathcal{S} \subset \mathcal{S}^\perp$ and $B\mathcal{S}^\perp \subset \mathcal{S}$ (this is equivalent to saying that B anti-commutes with $\epsilon_{\mathcal{S}}$). Clearly any operator in $\mathcal{B}(\mathcal{H})$ decomposes uniquely as an $\mathcal{S}$-diagonal operator plus an $\mathcal{S}$-co-diagonal operator. The tangent space of $Gr(\mathcal{H})$ at $\mathcal{S}$ identifies with the space of selfadjoint operators in $\mathcal{B}(\mathcal{H})$ which are $\mathcal{S}$-codiagonal. The reductive structure induces a linear connection: if $X(t)$ is a vector field which is tangent along the smooth curve $\mathcal{S}_t$ in $Gr(\mathcal{H})$ for $t \in [a, b]$ (i.e., $X(t)$ is a smooth path of selfadjoint operators, pointwise $\mathcal{S}_t$-co-diagonal at every $t \in [a, b]$), then the covariant derivative of $X(t)$ is
$$\frac{DX(t)}{dt} = \mathcal{S}_t - \text{co-diagonal part of } \frac{d}{dt}X(t),$$
where $\frac{d}{dt}X(t)$ is the usual derivative of $X(t)$.

4. The geodesics of this connection which start at $\mathcal{S}$ are of the form
$$\gamma(t) = e^{tZ}\mathcal{S},$$
where $Z^* = -Z$ is $\mathcal{S}$-codiagonal. We shall say that a geodesic is normalized if additionally $\|Z\| \leq \pi/2$. In terms of projections, the geodesic is $\gamma(t) = e^{tZ}P_{\mathcal{S}}e^{-tZ}$, in terms of symmetries it is $\gamma(t) = e^{tZ}\epsilon_{\mathcal{S}}e^{-tZ} = e^{2tZ}\epsilon_{\mathcal{S}}$ (because Z anti-commutes with $\epsilon_{\mathcal{S}}$)

5. This item was shown in [2] (see also [3]). There exists a geodesic which joins $\mathcal{S}$ and $\mathcal{T}$ if and only if
$$\dim \mathcal{S} \cap \mathcal{T}^{\perp} = \dim \mathcal{S}^{\perp} \cap \mathcal{T}.$$
The geodesic can be chosen normalized. There existis a unique normalized geodesic joining $\mathcal{S}$ and $\mathcal{T}$ at $t = 0$ and $t = 1$ if and only if $\mathcal{S} \cap \mathcal{T}^{\perp} = \{0\} = \mathcal{S}^{\perp} \cap \mathcal{T}$.

6. Normalized geodesics have minimal length for $|t| \leq 1$.

First, we use the facts contained in Section 6 about the (generic) position of $N(C_a - I)$ and $N(C_a + I)$. Note that for $a = 0$, $N(C_0 - I) = \mathcal{E}$ and $N(C_0 + I) = \mathcal{O} = \mathcal{E}^{\perp}$, we have that $\mathcal{E} \cap \mathcal{O}^{\perp} = \mathcal{E}$ and $\mathcal{E}^{\perp} \cap \mathcal{O} = \mathcal{O}$ are both infinite dimensional, and therefore there exist infinitely many geodesics joining $\mathcal{E}$ and $\mathcal{O}$.

Corollary 8.2. *Let $a \in \mathbb{D}$, $a \neq 0$. Then there exists a unique normalized geodesic joining $N(C_a - I)$ and $N(C_a + I)$ a $t = 0$ and $t = 1$.*

Proof. By Theorem 6.3, $N(C_a - I) \cap N(C_a + I)^{\perp} = \{0\} = N(C_a - I)^{\perp} \cap N(C_a + I)$. □

Next, using Theorems 5.6 and 5.7 we have the following consequences:

Corollary 8.3. *Let $a \neq 0$ in $\mathbb{D}$. The following pairs subspaces can be joined by a unique normalized geodesic of the Grassmann manifold of $\mathcal{H} = L^2(\mathbb{T})$:*

1. *$N(C_a - I)$ and $\mathcal{E}$.*

2. *$N(C_a + I)$ and $\mathcal{O}$.*

3. *$N(C_a + I)$ and $\mathcal{E}$.*

We can deduce also the following negative result:

Corollary 8.4. *Let $a \neq 0$. Then the $N(C_a - I)$ and $\mathcal{O}$ cannot be joined by a geodesic of the Grassmann manifold of $\mathcal{H}$.*

Proof. Note from Theorem 5.5 (with $b = 0$) that $N(C_a - I) \cap \mathcal{O}^{\perp} = N(C_a - I) \cap \mathcal{E}$ is one dimensional. Whereas from Proposition 5.7 we have that
$$N(C_a - I)^{\perp} \cap \mathcal{O} = \{0\}.$$
□

Remark 8.5. If γ is a geodesic in $Gr(\mathcal{H})$ and $U \in \mathcal{U}(\mathcal{H})$, then $U \cdot \gamma$ is also a geodesic in $Gr(\mathcal{H})$. Therefore $\mathcal{S}$ and $\mathcal{T}$ can be joined by a geodesic (respectively, unique normalized geodesic) if and only if $U \cdot \mathcal{S}$ and $U \cdot \mathcal{T}$ can be joined by a geodesic (respectively, unique normalized geodesic).

The analogous statement is valid for the orthocomplements $\mathcal{S}^{\perp}$, $\mathcal{T}^{\perp}$ (the same uniparameter unitary group that induces a geodesic between $\mathcal{S}$ and $\mathcal{T}$, induces a geodesic between $\mathcal{S}^{\perp}$ and $\mathcal{T}^{\perp}$).

We may use these fact to obtain the following consequences:

Corollary 8.6. *Let $a \in \mathbb{D}$, $a \neq 0$. Then the following pairs of subspaces can be joined by a unique normalized geodesic of $Gr(\mathcal{H})$ at $t = 0$ and $t = 1$:*

1. $N(C_a + I)$ *and* $N(R_{\Omega_a} - I)^{\perp} = N(R_{\Omega_a} + I)$.
2. $N(C_a - I)$ *and* $N(R_{\Omega_a} + I)^{\perp} = N(R_{\Omega_a} - I)$.
3. $N(C_a - I)$ *and* $N(R_{\Omega_a} - I)^{\perp} = N(R_{\Omega_a} + I)$

Proof. By Corollary 8.3.1, we know that $N(C_a - I)$ and $N(C_0 - I)$ can be joined by a unique normalized geodesic at $t = 0$ and $t = 1$. Therefore the same holds for

$$R_a(N(C_a - I)) = N(C_a + I)^{\perp} \quad \text{and} \quad R_a(N(C_0 - I)) = N(R_{\Omega_a} - I),$$

and therefore also for

$$N(C_a + I) \quad \text{and} \quad N(R_{\Omega_a} - I)^{\perp}.$$

Assertions 2. and 3. follow similarly using Corollaries 8.3.2 and 8.3.3. □

Similarly, using Corollary 8.4

Corollary 8.7. *Let $a \in \mathbb{D}$, $a \neq 0$. Then the subspaces*

$$N(C_a + I) \quad \textit{and} \quad N(R_{\Omega_a} + I)^{\perp} = N(R_{\Omega_a} - I)$$

cannot be joined by a geodesic of $Gr(\mathcal{H})$.

Analogously as with Corollaries 8.6 and 8.7, but using $U = W_a$ we obtain:

Corollary 8.8. *Let $a \in \mathbb{D}$, $a \neq 0$. Then the following subspaces can be joined by a unique normalized geodesic of $Gr(\mathcal{H})$ at $t = 0$ and $t = 1$:*

1. $M_{\frac{1}{1-\bar{a}z}} N(C_a - I)$ *and* $N(W_{\Omega_a} - I)$.
2. $M_{\frac{1}{1-\bar{a}z}} N(C_a + I)$ *and* $N(W_{\Omega_a} + I)$.
3. $M_{\frac{1}{1-\bar{a}z}} N(C_a - I)$ *and* $N(W_{\Omega_a} + I)$.

Proof. Recall from Corollary 8.3 that there exists a unique geodesic between $N(C_a - I)$ and $\mathcal{E} = N(C_0 - I)$, and from Lemmas 7.1 and 7.2.2 the formulas

$$W_a C_a W_a = M_{\frac{1-|a|^2}{(1-\bar{a}z)^2}} C_a \quad \text{and} \quad W_a C_0 W_a = W_{\Omega_a}.$$

Note also (after a straightforward computation) that $M_{\frac{\sqrt{1-|a|^2}}{1-\bar{a}z}} C_a = C_a M_{\frac{1-\bar{a}z}{\sqrt{1-|a|^2}}}$, so that

$$W_a C_a W_a = M_{\frac{\sqrt{1-|a|^2}}{1-\bar{a}z}} C_a M_{\frac{1-\bar{a}z}{\sqrt{1-|a|^2 2}}} = M_{\frac{\sqrt{1-|a|^2}}{1-\bar{a}z}} C_a M^{-1}_{\frac{\sqrt{1-|a|^2}}{1-\bar{a}z}}.$$

Then

$$W_a(N(C_a - I)) = N(W_aC_aW_a - I) = N(M_{\frac{\sqrt{1-|a|^2}}{1-\bar{a}z}}C_aM^{-1}_{\frac{\sqrt{1-|a|^2}}{1-\bar{a}z}} - 1) = M_{\frac{\sqrt{1-|a|^2}}{1-\bar{a}z}}N(C_a - I),$$

and

$$W_aN(C_0 - I) = N(W_aC_0W_a - I) = N(W_{\Omega_a} - I),$$

and the proof of the first assertion follows from Remark 8.5. The other two assertions follow similarly. □

Likewise, from Corollary 8.4 and similar reasoning we have that

Corollary 8.9. *The subspaces*

$$M_{\frac{1}{1-\bar{a}z}}N(C_a - I) \quad \text{and} \quad N(W_{\Omega_a} + I)$$

cannot be joined by a geodesic of $Gr(\mathcal{H})$.

Remark 8.10. The result of existence of unique normalized geodesics between subspaces, could also be stated in terms of the corresponding projections, or symmetries. For instance, the fact that there exists unique such geodesics joining

1. the subspaces $N(C_a - I)$ and $N(C_a + I)$ (from Corollary 8.2), or
2. the subspaces $N(C_a + I)$ and $N(R_{\Omega_a} - I)$ (from Corollary 8.6.3),

can be rephrased: there exist unique geodesics between (respectively)

1. the projections $(I + C_a)M_{\psi_a}$ and $(I - C_a)M_{\psi_a}$ (where $\psi_a(z) = \left(1 + \frac{1-|a|^2}{|1-\bar{a}z|^2}\right)^{-1}$),
2. the symmetries $\epsilon_{N(C_a+I)}$ and R_{Ω_a}.

Note that

$$\epsilon_{N(C_a+I)} = 2P_{N(C_a+I)} - I = M_{2\psi_a - 1} - C_aM_{\psi_a}.$$

References

[1] T. Ando, Unbounded or bounded idempotent operators in Hilbert space, Linear Algebra Appl. 438 (2013), 3769 - 3775.

[2] E. Andruchow, Operators which are the difference of two projections. J. Math. Anal. Appl. 420 (2014), no. 2, 1634–1653.

[3] E. Andruchow. The Grassmann manifold of a Hilbert space. Proceedings of the XIIth "Dr. Antonio A. R. Monteiro" Congress, 41–55, Actas Congr. "Dr. Antonio A. R. Monteiro", Univ. Nac. del Sur, Bahía Blanca, 2014.

[4] E. Andruchow; G. Corach; L. Recht. Symmetries and reflections from composition operators in the disk. Integral Equations Operator Theory 97 (2025), no. 3, Paper No. 16

[5] A. Böttcher; I.M. Spitkovsky. A gentle guide to the basics of two projections theory. Linear Algebra Appl. 432 (2010), 1412–1459.

[6] E. Berkson, Composition operators isolated in the uniform operator topology. Proc. Amer. Math. Soc. 81 (1981), no. 2, 230–232.

[7] D. Buckholtz, Hilbert space idempotents and involutions. Proc. Amer. Math. Soc. 128 (2000), no. 5, 1415–1418.

[8] Y.-N. Chen; H–K. Du; Y.-F. Pang, A simplification of the Kovarik formula. J. Math. Anal. Appl. 331 (2007), no. 1, 13–20.

[9] G. Corach; H. Porta; L. Recht. The geometry of spaces of projections in C^*-algebras. Adv. Math. 101 (1993), no. 1, 59–77.

[10] C.C. Cowen; B.D. MacCluer. Composition operators on spaces of analytic functions. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1995.

[11] C. Davis, Separation of two linear subspaces, Aca Sci. Math. (Szeged) 19 (1958), 172–187.

[12] F. Deutsch. The angle between subspaces of a Hilbert space. Approximation theory, wavelets and applications (Maratea, 1994), 107–130, NATO Adv. Sci. Inst. Ser. C: Math. Phys. Sci., 454, Kluwer Acad. Publ., Dordrecht, 1995.

[13] H. Porta; L. Recht. Minimality of geodesics in Grassmann manifolds. Proc. Amer. Math. Soc. 100 (1987), no. 3, 464–466.