Optimal frames of translates with prescribed norms in shift invariant spaces

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Abstract

For a given finitely generated shift invariant (FSI) subspace $\mathcal{W} \subset L^2(\mathbb{R}^k)$ we obtain a simple criterion for the existence of shift generated (SG) Bessel sequences $E(\mathcal{F})$ induced by finite sequences of vectors $\mathcal{F} = \{f_i\}_{1\leq i\leq n} \in \mathcal{W}^n$ such that $\|f_i\|^2 = \alpha_i$ for $1 \leq i \leq n$, where $0 < \alpha_1 \leq \ldots \leq \alpha_n$ are positive numbers, and a prescribed fine spectral structure i.e., such that the spectra of $S_{E(\mathcal{F})}$ is prescribed in each fiber of $\mathrm{Spec}(\mathcal{W}) \subset \mathbb{T}^k$. Then, we characterize the finite sequences $\mathcal{F} \in \mathcal{W}^n$ as above and such that the fine spectral structure of the shift generated Bessel sequences $E(\mathcal{F})$ has minimal spread (i.e. we show the existence of optimal SG Bessel sequences with prescribed norms); in this context, the spread of the spectra is measured in terms of the convex potential $P^{\mathcal{W}}_\varphi$ induced by $\mathcal{W}$ and an arbitrary convex function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$. Indeed, we show that there exist universal optimal SG Bessel sequences $E(\mathcal{F})$ in the sense that they are minimizers of $P^{\mathcal{W}}_\varphi$ for every convex function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$.

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1 Introduction

Let $\mathcal{W}$ be a closed subspace of a separable complex Hilbert space $\mathcal{H}$ and let $\mathbb{I}$ be a finite or countable infinite set. A sequence $\mathcal{F} = \{f_i\}_{i\in\mathbb{I}}$ in $\mathcal{W}$ is a frame for $\mathcal{W}$ if vectors $h \in \mathcal{W}$ can be represented as infinite series $h = \sum_{i\in\mathbb{I}} \alpha_i\, f_i$ for some (not necessarily unique) sequence $(\alpha_i)_{i\in\mathbb{I}} \in \ell^2(\mathbb{I})$, in such a way that the representation is continuous. Thus, a frame $\mathcal{F}$ for $\mathcal{W}$ allows for linear (typically redundant) and stable encoding-decoding schemes of vectors (signals) in $\mathcal{W}$. Indeed, if $\mathcal{V}$ is a closed subspace of $\mathcal{H}$ such that $\mathcal{V} \oplus \mathcal{W}^\perp = \mathcal{H}$ (e.g. $\mathcal{V} = \mathcal{W}$) then it is possible to find frames $\mathcal{G} = \{g_i\}_{i\in\mathbb{I}}$ for $\mathcal{V}$ such that

$$h = \sum_{i\in\mathbb{I}} \langle h, g_i\rangle\ f_i\ , \quad \text{for } h \in \mathcal{W}\,. \tag{1}$$

The representation above lies within the theory of oblique duality (see [21, 23, 24, 25]). In applied situations, it is usually desired to develop encoding-decoding schemes as above, with some additional features related to the stability of the scheme. In some cases, we search for schemes such that the sequence of norms $\{\|f_i\|^2\}_{i\in\mathbb{I}}$ as well as the spectral properties of the family $\mathcal{F}$ are given in advance, leading to what is known in the literature as the frame design problem (see [3, 9, 17, 19, 31, 36] and the papers [26, 33, 34, 35] for the more general frame completions problem with prescribed norms). It is well known that both the spread of the sequences of norms as well as the spread of the spectra of the frame $\mathcal{F}$ are linked with numerical properties of $\mathcal{F}$. Once we have constructed

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a frame $\mathcal{F}$ for $\mathcal{W}$ with the desired properties, we turn our attention to the construction of frames $\mathcal{G}$ for $\mathcal{V}$ satisfying Eq.(1) and having some prescribed features related to their numerical stability (see [4, 6, 23, 33, 35]).

It is well known that the frame design problem has an equivalent formulation in terms of the relation between the main diagonal of a positive semi-definite operator and its spectra; in the finite-dimensional setting, this relation is characterized in the Schur-Horn theorem from matrix analysis. There have been recent important advances in both the frame design problems as well as the Schur-Horn theorems in infinite dimensions, mainly due to the interactions of these problems (see [3, 12, 13, 14, 15, 28, 29]). There are also complete parametrizations of all finite frames with prescribed norms and eigenvalues (of their frame operators) in terms of the so-called eigen-steps sequences [17]. On the other hand, the spectral structure of oblique duals (that include classical duals) of a fixed frame can be described in terms of the relations between the spectra of a positive semi-definite operator and the spectra of its compressions to subspaces. In the finite-dimensional context (see [4, 33]), these relations are known as the Fan-Pall inequalities (that include the so-called interlacing inequalities as a particular case). Yet, in general, the corresponding results in frame theory do not take into consideration any additional structure of the frame. For example, regarding the frame design problem, it seems natural to wonder whether we can construct a structured frame (e.g., wavelet, Gabor or a shift-generated frame) with prescribed structure; similarly, in case we fix a structured frame $\mathcal{F}$ for $\mathcal{W}$ it seems natural to wonder whether we can construct structured oblique dual frames with further prescribed properties.

In [6], as a first step towards a detailed study of the spectral properties of structured oblique duals of shift generated systems induced by finite families of vectors in $L^2(\mathbb{R}^k)$, we extended the Fan-Pall theory to the context of measurable fields of positive semi-definite matrices and their compressions by measurable selections of subspaces; this allowed us to give an explicit description of what we called *fine spectral structure* of the shift generated duals of a fixed shift generated (SG) frame for a finitely generated shift invariant (FSI) subspace $\mathcal{W}$ of $L^2(\mathbb{R}^k)$. Given a convex function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ we also introduced the convex potential associated to the pair $(\varphi, \mathcal{W})$, that is a functional on SG Bessel sequences that measures the spread of the fine spectral structure of the sequence; there we showed that these convex potentials detect tight frames as their minimizers (under some normalization conditions).

Our initial motivation for the study of duality theory for shift-generated sequences was [21]. At that time, we learned about the general theory of shift-generated sequences from the seminal papers [11, 22, 37]. Nevertheless, it was not until we learned about [1, 2] that we understood much of how the general theory worked; furthermore, we understood that (measurable versions of) matrix analysis methods could be used to tackle some relevant problems in the theory of finitely generated shift-invariant systems. That was the *leit motiv* for the works [6, 5].

Indeed, in [5] building on an extension of the Schur-Horn theorem for measurable fields of positive semi-definite matrices, we characterized the possible *fine structures* of SG Bessel sequences in FSI subspaces (see Section 2.2 for preliminaries on SG Bessel sequences, Remark 3.2 and Theorem 3.5); in turn, we solved a frame design problem, where the prescribed features of the SG Bessel sequences are described in terms of some local internal (or fine) structure, relative to a finitely generated shift invariant subspace $\mathcal{W}$. At the end of that work, we posed a natural extension of a *optimal* frame design problem involving some global norm restrictions (as opposed to local restrictions in terms of the fine structure) and arbitrary convex potentials for finitely generated shift-invariant systems; nevertheless, we only obtained partial results toward the solution of this problem. These partial results showed that such optimal shift-generated systems as above had a discrete fine structure. This feature of the optimal shift-generated systems was our motivation for introducing the multi-designs in [7] (see also [8]); we also posed a discrete version of the optimal multi-design problem with prescribed weights that was a natural analogue in that setting of the frame design problem with prescribed norms for shift generated sequences. In the study of optimal multi-designs, we introduced some new results and tools and obtained the complete description of such optimal constructions.

In this paper, we revisit the construction of shift-generated systems from a finite number of initial vectors that satisfy some norm restrictions and that minimize convex potentials in this context, since we are trying to find the "tightest" shift-generated Bessel sequences. We make use of natural extensions of the ideas that lead to the construction of optimal multi-designs with prescribed weights in [7] and obtain a complete description of such optimal shift-generated systems. Furthermore, we show that there exist minimizers that are *universal*, in the sense that they minimize every convex potential. These results recover some well-known facts from finite frame theory (see [18, 26, 31, 32], originating from the landmark paper [9]).

The paper is organized as follows. In Section 2 we recall some general facts about frames for subspaces in separable complex Hilbert spaces and several aspects of the theory of frames of integer translates in finitely generated shift invariant subspaces (FSI subspaces). We conclude this section with a description of the convex potentials associated with frames of translates in FSI subspaces. In Section 3 we introduce the main problem under consideration and our main contributions. Indeed, in Section 3.1 we obtain a simple criterion for the existence of families $\mathcal{F} = \{f_i\}_{i\in\mathbb{I}_n} \in \mathcal{W}^n$, where $\mathcal{W}$ is a FSI subpace, such that the squared norms $(\|f_i\|^2)_{i\in\mathbb{I}_n}$ are prescribed by some fixed positive weights $(\alpha_i)_{i\in\mathbb{I}_n}$ and such that the fine spectral structure of the shift generated sequence $E(\mathcal{F}) = \{T_k f_i\}_{(k,i)\in\mathbb{Z}\times\mathbb{I}_n}$ is also prescribed. In Section 3.2 we construct families $\mathcal{F}^{\rm op}$ with a specified spectral structure and show that these families are minimizers of every convex potential on the set of families $\mathcal{F} = \{f_i\}_{i\in\mathbb{I}_n} \in \mathcal{W}^n$ with prescribed norms, which is our main result. The paper also contains an appendix section in which we recall some basic facts about majorization theory in finite measure spaces.

2 Preliminaries

In this section, we recall some basic facts related to frames for subspaces and shift-generated frames for shift-invariant (SI) subspaces of $L^2(\mathbb{R}^k)$. We also recall the notion of convex potentials of finitely generated shift-invariant systems.

General Notations

Throughout this work, we shall use the following notation: the space of complex $d\times d$ matrices is denoted by $\mathcal{M}_d(\mathbb{C})$, the real subspace of self-adjoint matrices is denoted $\mathcal{H}(d)$ and $\mathcal{M}_d(\mathbb{C})^+$ denotes the set of positive semi-definite matrices; $\mathcal{G}l(d)$ is the group of invertible elements of $\mathcal{M}_d(\mathbb{C})$, $\mathcal{U}(d)$ is the subgroup of unitary matrices and $\mathcal{G}l(d)^+ = \mathcal{M}_d(\mathbb{C})^+ \cap \mathcal{G}l(d)$. If $T\in\mathcal{M}_d(\mathbb{C})$, we denote by $\|T\|$ its spectral norm, by $\operatorname{rk} T = \dim R(T)$ the rank of T, and by $\operatorname{tr} T$ the trace of T. If $W\subseteq\mathbb{C}^d$ is a subspace we denote by $P_W\in\mathcal{M}_d(\mathbb{C})^+$ the orthogonal projection onto W.

Given $d\in\mathbb{N}$, we denote by $\mathbb{I}_d=\{1,\dots,d\}\subseteq\mathbb{N}$ and we set $\mathbb{I}_0=\emptyset$. For a vector $v\in\mathbb{R}^d$, we denote by $v^\downarrow\in\mathbb{R}^d$ the rearrangement of v in non-increasing order. We denote by $(\mathbb{R}^d)^\downarrow=\{v\in\mathbb{R}^d : v=v^\downarrow\}$ the set of downwards ordered vectors. Given $S\in\mathcal{H}(d)$, we write $\lambda(S)=\lambda^\downarrow(S)=(\lambda_1(S)\,,\,\dots\,,\,\lambda_d(S)\,)\in(\mathbb{R}^d)^\downarrow$ for the vector of eigenvalues of S - counting multiplicities - arranged in decreasing order. Finally, given $n\in\mathbb{N}$, we let $\mathbb{1}_n=(1,\dots,1)\in\mathbb{R}^n$.

2.1 Frames for subspaces

In what follows $\mathcal{H}$ denotes a separable complex Hilbert space and $\mathbb{I}$ denotes a finite or countably infinite set. Let $\mathcal{W}$ be a closed subspace of $\mathcal{H}$: recall that a sequence $\mathcal{F}=\{f_i\}_{i\in\mathbb{I}}$ in $\mathcal{W}$ is a *frame* for $\mathcal{W}$ if there exist constants $0 < a \le b$ such that

$$a\,\|f\|^2 \le \sum_{i\in\mathbb{I}} |\langle f, f_i\rangle|^2 \le b\,\|f\|^2 \quad \text{for every} \quad f \in \mathcal{W}\,. \tag{2}$$

In general, if $\mathcal{F}$ satisfies the inequality to the right in Eq. (2) we say that $\mathcal{F}$ is a b-Bessel sequence for $\mathcal{W}$. Moreover, we shall say that a sequence $\mathcal{G} = \{g_i\}_{i\in\mathbb{I}}$ in $\mathcal{H}$ is a Bessel sequence - without explicit reference to a closed subspace - whenever $\mathcal{G}$ is a Bessel sequence for its closed linear span; notice that this is equivalent to the fact that $\mathcal{G}$ is a Bessel sequence for $\mathcal{H}$.

Given a Bessel sequence $\mathcal{F} = \{f_i\}_{i\in\mathbb{I}}$ we consider its *synthesis operator* $T_\mathcal{F} \in L(\ell^2(\mathbb{I}), \mathcal{H})$ given by $T_\mathcal{F}((a_i)_{i\in\mathbb{I}}) = \sum_{i\in\mathbb{I}} a_i\ f_i$ which, by hypothesis on $\mathcal{F}$, is a bounded linear transformation. We also consider $T_\mathcal{F}^* \in L(\mathcal{H}, \ell^2(\mathbb{I}))$ called the *analysis operator* of $\mathcal{F}$, given by $T_\mathcal{F}^*(f) = (\langle f, f_i\rangle)_{i\in\mathbb{I}}$ and the *frame operator* of $\mathcal{F}$ defined by $S_\mathcal{F} = T_\mathcal{F}\,T_\mathcal{F}^*$. It is straightforward to check that

$$\langle S_\mathcal{F} f, f\rangle = \sum_{i\in\mathbb{I}} |\langle f, f_i\rangle|^2 \quad \text{for every} \quad f \in \mathcal{H}\ .$$

Hence, $S_\mathcal{F}$ is a positive semi-definite bounded operator; moreover, a Bessel sequence $\mathcal{F}$ in $\mathcal{W}$ is a frame for $\mathcal{W}$ if and only if $S_\mathcal{F}$ is an invertible operator when restricted to $\mathcal{W}$ or equivalently, if the range of $T_\mathcal{F}$ coincides with $\mathcal{W}$.

If $\mathcal{V}$ is a closed subspace of $\mathcal{H}$ such that $\mathcal{V} \oplus \mathcal{W}^\perp = \mathcal{H}$ (e.g. $\mathcal{V} = \mathcal{W}$), then it is possible to find frames $\mathcal{G} = \{g_i\}_{i\in\mathbb{I}}$ for $\mathcal{V}$ such that

$$f = \textstyle\sum_{i\in\mathbb{I}} \langle f, g_i\rangle\ f_i\ , \qquad \text{for } f \in \mathcal{W}\,.$$

The above representation lies within the theory of oblique duality (see [21, 23, 24, 25]). In this note, we shall not be concerned with oblique duals; nevertheless, notice that the numerical stability of the encoding-decoding scheme above depends both on the numerical stability corresponding to $\mathcal{F}$ and $\mathcal{G}$ as above. One way to measure the stability of the encoding or decoding algorithms is to measure the spread of the spectra of the frame operators corresponding to $\mathcal{F}$ and $\mathcal{G}$. Therefore, both the task of constructing optimally stable $\mathcal{F}$ together with obtaining optimally stable duals $\mathcal{G}$ of $\mathcal{F}$ are of fundamental interest in frame theory.

2.2 SI subspaces, frames of translates and their convex potentials

In what follows, we consider $L^2(\mathbb{R}^k)$ (with respect to Lebesgue measure) as a separable and complex Hilbert space. Recall that a closed subspace $\mathcal{V} \subseteq L^2(\mathbb{R}^k)$ is *shift-invariant* (SI) if $f \in \mathcal{V}$ implies $T_\ell f \in \mathcal{V}$ for any $\ell \in \mathbb{Z}^k$, where $T_y f(x) = f(x-y)$ is the translation by $y \in \mathbb{R}^k$. For example, take a subset $\mathcal{A} \subset L^2(\mathbb{R}^k)$ and set

$$\mathcal{S}(\mathcal{A}) = \overline{\text{span}}\,\{T_\ell f:\ f \in \mathcal{A}\,,\ \ell \in \mathbb{Z}^k\}\,.$$

Then, $\mathcal{S}(\mathcal{A})$ is a shift-invariant subspace called the *SI subspace generated by* $\mathcal{A}$; indeed, $\mathcal{S}(\mathcal{A})$ is the smallest closed and SI subspace that contains $\mathcal{A}$. We say that a SI subspace $\mathcal{V}$ is *finitely generated* (FSI) if there exists a finite set $\mathcal{A} \subset L^2(\mathbb{R}^k)$ such that $\mathcal{V} = \mathcal{S}(\mathcal{A})$. We further say that $\mathcal{W}$ is a *principal* SI subspace if there exists $f \in L^2(\mathbb{R}^k)$ such that $\mathcal{W} = \mathcal{S}(f)$.

To describe the fine structure of a SI subspace, we consider the following representation of $L^2(\mathbb{R}^k)$ (see [22, 11, 37] and [16] for extensions of these notions to the more general context of actions of locally compact abelian groups). Let $\mathbb{T} = [-1/2, 1/2)$ endowed with the Lebesgue measure and let $L^2(\mathbb{T}^k, \ell^2(\mathbb{Z}^k))$ be the Hilbert space of square integrable $\ell^2(\mathbb{Z}^k)$-valued functions that consists of all vector valued measurable functions $\phi: \mathbb{T}^k \to \ell^2(\mathbb{Z}^k)$ with the norm

$$\|\phi\|^2 = \textstyle\int_{\mathbb{T}^k} \|\phi(x)\|^2_{\ell^2(\mathbb{Z}^k)}\ dx < \infty.$$

Then, $\Gamma : L^2(\mathbb{R}^k) \to L^2(\mathbb{T}^k, \ell^2(\mathbb{Z}^k))$ defined for $f \in L^1(\mathbb{R}^k) \cap L^2(\mathbb{R}^k)$ by

$$\Gamma f : \mathbb{T}^k \to \ell^2(\mathbb{Z}^k) \ , \quad \Gamma f(x) = (\hat{f}(x+\ell))_{\ell\in\mathbb{Z}^k}, \tag{3}$$

extends uniquely to an isometric isomorphism between $L^2(\mathbb{R}^k)$ and $L^2(\mathbb{T}^k, \ell^2(\mathbb{Z}^k))$, where

$$\hat{f}(x) = \int_{\mathbb{R}^k} f(y)\ e^{-2\pi\, i\, \langle y,\, x\rangle}\ dy \quad \text{ for } \quad x \in \mathbb{R}^k ,$$

denotes the Fourier transform of $f \in L^1(\mathbb{R}^k) \cap L^2(\mathbb{R}^k)$.

Let $\mathcal{V} \subset L^2(\mathbb{R}^k)$ be a SI subspace. Then, there exists a function $J_\mathcal{V} : \mathbb{T}^k \to \{$ closed subspaces of $\ell^2(\mathbb{Z}^k)\}$ such that: if $P_{J_\mathcal{V}(x)}$ denotes the orthogonal projection onto $J_\mathcal{V}(x)$ for $x \in \mathbb{T}^k$, then for every $\xi,\, \eta \in \ell^2(\mathbb{Z}^k)$ the function $x \mapsto \langle P_{J_\mathcal{V}(x)}\, \xi\, ,\, \eta\rangle$ is measurable and

$$\mathcal{V} = \{f \in L^2(\mathbb{R}^k) : \Gamma f(x) \in J_\mathcal{V}(x) \text{ for a.e. } x \in \mathbb{T}^k\}. \tag{4}$$

The function $J_\mathcal{V}$ is the so-called *measurable range function* associated with $\mathcal{V}$. By [11, Prop.1.5], Eq. (4) establishes a bijection between SI subspaces of $L^2(\mathbb{R}^k)$ and measurable range functions. In this context, we define $\operatorname{Spec}(\mathcal{V})$ as the essential support of the measurable function $d(x) = \dim J_\mathcal{V}(x) \in [0, \infty]$, for $x \in \mathbb{T}^k$. In case $\mathcal{V} = S(\mathcal{A}) \subseteq L^2(\mathbb{R}^k)$ is the SI subspace generated by $\mathcal{A} = \{h_i : i \in \mathbb{I}\} \subset L^2(\mathbb{R}^k)$, where $\mathbb{I}$ is a finite or countable infinite set, then for a.e. $x \in \mathbb{T}^k$ we have that

$$J_\mathcal{V}(x) = \overline{\operatorname{span}}\, \{\Gamma h_i(x) :\ i \in \mathbb{I}\}\, . \tag{5}$$

Recall that a bounded linear operator $S \in L(L^2(\mathbb{R}^k))$ is *shift preserving* (SP) if $T_\ell\, S = S\, T_\ell$ for every $\ell \in \mathbb{Z}^k$. In this case (see [11, Thm 4.5]), there exists a (weakly) measurable field of operators $[S]_{(\cdot)} : \mathbb{T}^k \to L(\ell^2(\mathbb{Z}^k)\,)$ (i.e. such that for every $\xi,\, \eta \in \ell^2(\mathbb{Z}^k)$ the function $\mathbb{T}^k \ni x \mapsto \langle [S]_x\, \xi\, ,\, \eta\rangle$ is measurable) and essentially bounded (i.e. the function $\mathbb{T}^k \ni x \mapsto \|\, [S]_x\, \|$ is essentially bounded) such that

$$[S]_x\big(\Gamma f(x)\,\big) = \Gamma(Sf)(x) \quad \text{ for a.e. } x \in \mathbb{T}^k \ , \quad f \in L^2(\mathbb{R}^k)\, . \tag{6}$$

Moreover, $\|S\| = \operatorname{ess\,sup}_{x\in\mathbb{T}^k} \|\, [S]_x\, \|$. Conversely, if $s : \mathbb{T}^k \to L(\ell^2(\mathbb{Z}^k))$ is a weakly measurable and essentially bounded field of operators, then there exists a unique bounded operator $S \in L(L^2(\mathbb{R}^k))$ that is SP and such that $[S] = s$. For example, let $\mathcal{V}$ be a SI subspace and consider $P_\mathcal{V} \in L(L^2(\mathbb{R}^k))$, the orthogonal projection onto $\mathcal{V}$; then, $P_\mathcal{V}$ is SP so that $[P_\mathcal{V}] : \mathbb{T}^k \to L(\ell^2(\mathbb{Z}^k))$ is given by $[P_\mathcal{V}]_x = P_{J_\mathcal{V}(x)}$ i.e., the orthogonal projection onto $J_\mathcal{V}(x)$, for a.e. $x \in \mathbb{T}^k$.

The previous notions associated with SI subspaces and SP operators allow for developing a detailed study of frames of translates. Indeed, let $\mathcal{F} = \{f_i\}_{i\in\mathbb{I}}$ be a (possibly finite) sequence in $L^2(\mathbb{R}^k)$. In what follows, we consider the sequence of integer translates of $\mathcal{F}$, denoted $E(\mathcal{F})$ and given by

$$E(\mathcal{F}) = \{T_\ell\, f_i\}_{(\ell,\, i)\, \in\, \mathbb{Z}^k\times\mathbb{I}}\, .$$

For $x \in \mathbb{T}^k$, let $\Gamma\mathcal{F}(x) = \{\Gamma f_i(x)\}_{i\in\mathbb{I}}$, which is a (possibly finite) sequence in $\ell^2(\mathbb{Z}^k)$. Then $E(\mathcal{F})$ is a b-Bessel sequence if and only if $\Gamma\mathcal{F}(x)$ is a b-Bessel sequence for a.e. $x \in \mathbb{T}^k$ (see [11, 37]). In this case, we consider the synthesis operator $T_{\Gamma\mathcal{F}(x)} : \ell^2(\mathbb{I}) \to \ell^2(\mathbb{Z}^k)$ and frame operator $S_{\Gamma\mathcal{F}(x)} : \ell^2(\mathbb{Z}^k) \to \ell^2(\mathbb{Z}^k)$ of $\Gamma\mathcal{F}(x)$, for $x \in \mathbb{T}^k$. It is straightforward to check that $S_{E(\mathcal{F})}$ is a SP operator.

If $\mathcal{F} = \{f_i\}_{i\in\mathbb{I}}$ and $\mathcal{G} = \{g_i\}_{i\in\mathbb{I}}$ are such that $E(\mathcal{F})$ and $E(\mathcal{G})$ are Bessel sequences then (see [27, 37]) the following fundamental relation holds:

$$[T_{E(\mathcal{G})}\, T^*_{E(\mathcal{F})}]_x = T_{\Gamma\mathcal{G}(x)}\, T^*_{\Gamma\mathcal{F}(x)} \ , \quad \text{ for a.e } x \in \mathbb{T}^k\, . \tag{7}$$

These equalities have several consequences. For example, if $\mathcal{W}$ is a SI subspace of $L^2(\mathbb{R}^k)$ and we assume that $\mathcal{F}, \in \mathcal{W}^n$ then, for every $f,\, g \in L^2(\mathbb{R}^k)$,

$$\langle S_{E(\mathcal{F})}\, f,\, g\rangle = \int_{\mathbb{T}^k} \langle S_{\Gamma\mathcal{F}(x)}\ \Gamma f(x),\, \Gamma g(x)\rangle_{\ell^2(\mathbb{Z}^k)}\ dx\ .$$

This last fact implies that $[S_{E(\mathcal{F})}]_x = S_{\Gamma\mathcal{F}(x)}$ for a.e. $x \in \mathbb{T}^k$. Moreover, $E(\mathcal{F})$ is a frame for $\mathcal{W}$ with frame bounds $0 < a \le b$ if and only if $\Gamma\mathcal{F}(x)$ is a frame for $J_{\mathcal{W}}(x)$ with frame bounds $0 < a \le b$ for a.e. $x \in \mathbb{T}^k$ (see [11]).

We end this section with the notion of convex potentials in FSI introduced in [6] ; to describe these potentials, we consider the sets

$$\operatorname{Conv}(\mathbb{R}_+) = \{\varphi : \mathbb{R}_+ \to \mathbb{R}_+ \ , \ \varphi \ \text{ is a convex function } \} \tag{8}$$

and $\operatorname{Conv_s}(\mathbb{R}_+) = \{\varphi \in \operatorname{Conv}(\mathbb{R}_+) \ , \ \varphi \text{ is strictly convex } \}$.

Definition 2.1. Let $\mathcal{W}$ be a FSI subspace in $L^2(\mathbb{R}^k)$, let $\mathcal{F} = \{f_i\}_{i\in\mathbb{I}_n} \in \mathcal{W}^n$ be such that $E(\mathcal{F})$ is a Bessel sequence and consider $\varphi \in \operatorname{Conv}(\mathbb{R}_+)$. The convex potential associated with $(\varphi, \mathcal{W})$ on $E(\mathcal{F})$, denoted $P_\varphi^{\mathcal{W}}(E(\mathcal{F}))$, is given by

$$P_\varphi^{\mathcal{W}}(E(\mathcal{F})) = \int_{\operatorname{Spec}(\mathcal{W})} \operatorname{tr}(\varphi(S_{\Gamma\mathcal{F}(x)})\,[P_{\mathcal{W}}]_x)\ dx \ , \tag{9}$$

where $\varphi(S_{\Gamma\mathcal{F}(x)})$ denotes the functional calculus of the positive and finite rank operator $S_{\Gamma\mathcal{F}(x)} \in L(\ell^2(\mathbb{Z}^k))^+$ and $\operatorname{tr}(\cdot)$ the usual semi-finite trace in $L(\ell^2(\mathbb{Z}^k))$. △

Example 2.2. Let $\mathcal{W}$ be a FSI subspace of $L^2(\mathbb{R}^k)$ and let $\mathcal{F} = \{f_i\}_{i\in\mathbb{I}_n} \in \mathcal{W}^n$. If we set $\varphi(x) = x^2$ for $x \in \mathbb{R}_+$ then, the corresponding potential on $E(\mathcal{F})$, that we shall denote $\operatorname{FP}(E(\mathcal{F}))$, is given by

$$\operatorname{FP}(E(\mathcal{F})) = \int_{\mathbb{T}^k} \operatorname{tr}(S^2_{\Gamma\mathcal{F}(x)})\ dx = \int_{\mathbb{T}^k}\ \sum_{i,\,j\in\mathbb{I}_n} |\langle \Gamma f_i(x), \Gamma f_j(x)\rangle|^2\ dx\,,$$

where we have used the fact that $\varphi(0) = 0$ in this case. Hence, $\operatorname{FP}(E(\mathcal{F}))$ is a natural extension of the Benedetto-Fickus frame potential (see [9]). △

With the notation of Definition 2.1, it is shown in [6] that $P_\varphi^{\mathcal{W}}(E(\mathcal{F}))$ is a well-defined functional on the class of Bessel sequences $E(\mathcal{F})$ induced by a finite sequence $\mathcal{F} = \{f_i\}_{i\in\mathbb{I}_n} \in \mathcal{W}^n$ as above. The main motivation for considering convex potentials is that, under some natural normalization hypothesis, they detect tight frames as their minimizers (see [6, Theorem 3.9.]); that is, convex potentials provide simple scalar measures of stability that can be used to compare shift-generated frames. Therefore, the convex potentials for FSI are natural extensions of the convex potentials in finite dimensions introduced in [32].

3 Main results

To describe the main problem and main contribution of this manuscript, we consider the following:

Definition 3.1. Let $\mathcal{W}$ be a FSI subspace of $L^2(\mathbb{R}^k)$ and let $\alpha = (\alpha_i)_{i\in\mathbb{I}_n} \in (\mathbb{R}^n_{>0})^\downarrow$. We let

$$\mathfrak{B}_\alpha(\mathcal{W}) = \{\mathcal{F} = \{f_i\}_{i\in\mathbb{I}_n} \in \mathcal{W}^n :\ E(\mathcal{F}) \text{ is Bessel},\ \|f_i\|^2 = \alpha_i\,,\ i \in \mathbb{I}_n\}\ , \tag{10}$$

the set of SG Bessel sequences in $\mathcal{W}$ with norms prescribed by α. △

The restrictions on the families $\mathcal{F} = \{f_i\}_{i\in\mathbb{I}_n} \in \mathfrak{B}_\alpha(\mathcal{W})$ (namely $\|f_i\|^2 = \alpha_i$ for $i \in \mathbb{I}_n$) are of a *global* nature, as opposed to the local nature of restrictions coming from the decomposition map $\Gamma : L^2(\mathbb{R}^k) \to L^2(\mathbb{T}^k, \ell^2(\mathbb{Z}^k))$ (see Eq. (3)). Our problem is to describe those $\mathcal{F} \in \mathfrak{B}_\alpha(\mathcal{W})$ such that the encoding schemes associated to their corresponding Bessel sequences $E(\mathcal{F})$ are as stable as possible. Ideally, we would search for sequences $\mathcal{F}$ such that $E(\mathcal{F})$ are tight frames for $\mathcal{W}$; yet, there are obstructions for the existence of such sequences (see Remark 3.9 below).

By [6, Theorem 3.9.] (and a simple re-scaling argument), we know that if there exists $\mathcal{F}_0 \in \mathfrak{B}_\alpha(\mathcal{W})$ such that $E(\mathcal{F}_0)$ is a tight frame for $\mathcal{W}$, then $E(\mathcal{F}_0)$ is a minimizer in $\mathfrak{B}_\alpha(\mathcal{W})$ of every convex potential $P_\varphi^{\mathcal{W}}$ for any convex function $\varphi \in \mathrm{Conv}(\mathbb{R}_+)$ and moreover, in case $\varphi \in \mathrm{Conv_s}(\mathbb{R}_+)$ is a strictly convex function, then every such $\mathcal{F} \in \mathfrak{B}_\alpha(\mathcal{W})$ for which $P_\varphi^{\mathcal{W}}(E(\mathcal{F})) = P_\varphi^{\mathcal{W}}(E(\mathcal{F}_0))$ is a tight frame for $\mathcal{W}$. This suggests that in the general case, to search for $\mathcal{F} \in \mathfrak{B}_\alpha(\mathcal{W})$ such that the encoding schemes associated with their corresponding Bessel sequences $E(\mathcal{F})$ are as stable as possible, we could study the minimizers in $\mathfrak{B}_\alpha(\mathcal{W})$ of the convex potential $P_\varphi^{\mathcal{W}}$ corresponding to a strictly convex function $\varphi \in \mathrm{Conv_s}(\mathbb{R}_+)$. Therefore, in what follows we show the existence of finite sequences $\mathcal{F}^{\mathrm{op}} \in \mathfrak{B}_\alpha(\mathcal{W})$ such that

$$P_\varphi^{\mathcal{W}}(E(\mathcal{F}^{\mathrm{op}})) = \min\{P_\varphi^{\mathcal{W}}(E(\mathcal{F})) : \ \mathcal{F} \in \mathfrak{B}_\alpha(\mathcal{W})\} \ \text{ for every } \ \varphi \in \mathrm{Conv}(\mathbb{R}_+) \ .$$

We also describe the fine spectral structure of the frame operator of $E(\mathcal{F}^{\mathrm{op}})$ (Theorem 3.22). If $\varphi(x) = x^2$, our results extend some results from [9, 18, 32] for the frame potential to the context of SG Bessel sequences lying in a FSI subspace $\mathcal{W}$.

3.1 Existence of shift generated frames with prescribed struture

In this section, we characterize the existence of finite families $\mathcal{F} = \{f_i\}_{i\in\mathbb{I}_n} \in \mathcal{B}_\alpha(\mathcal{W})$, for $\mathcal{W}$ is a FSI subspace and a positive sequence $\alpha = (\alpha_i)_{i\in\mathbb{I}_n}$, such that the shift generated sequence $E(\mathcal{F})$ has a prescribed fine spectral structure (see Theorem 3.7 below). As we shall see, the possible fine structure of $E(\mathcal{F})$ can be characterized in terms of majorization relations. Our approach is based on previous results from [5]. Next, we introduce several notions that we need to state and prove the results.

Remark 3.2. Let $\mathcal{W}$ be a FSI subspace with $d(x) = \dim J_{\mathcal{W}}(x)$ for $x \in \mathbb{T}^k$, and let $\mathcal{F} = \{f_i\}_{i\in\mathbb{I}_n} \in \mathcal{W}^n$ be a finite sequence in $L^2(\mathbb{R}^k)$ such that $E(\mathcal{F})$ is a Bessel sequence. In what follows, we consider:

1. the *fine spectral structure of* $E(\mathcal{F})$, i.e. the weakly measurable function

$$\mathbb{T}^k \ni x \mapsto (\lambda_j([S_{E(\mathcal{F})}]_x)\,)_{j\in\mathbb{N}} \in \ell^1_+(\mathbb{Z}^k) \ ,$$

   with $\lambda_j([S_{E(\mathcal{F})}]_x) = \lambda_j(x)$, for $j \in \mathbb{I}_{d(x)}$, and $\lambda_j([S_{E(\mathcal{F})}]_x) = 0$ for $j \geq d(x)+1$ and $x \in \mathbb{T}^k$. Thus, the fine spectral structure of $\mathcal{F}$ describes the eigenvalues of the positive finite rank operator $[S_{E(\mathcal{F})}]_x = S_{\Gamma\mathcal{F}(x)} \in L(\ell^2(\mathbb{Z}^k))$, counting multiplicities and arranged in non-increasing order.

2. The *fine structure of* $E(\mathcal{F})$ given by the fine spectral structure together with the measurable vector valued function $\mathbb{T}^k \ni x \mapsto (\|\Gamma f_i(x)\|^2)_{i\in\mathbb{I}_n} \in \mathbb{R}^n_+$. △

Remark 3.3. To state the next result, we shall need the notion of vector majorization from matrix analysis. First some notation: Given $x = (x_i)_{i\in\mathbb{I}_d} \in \mathbb{R}^d$ we denote by $x^\downarrow = (x_i^\downarrow)_{i\in\mathbb{I}_d}$ the vector obtained by rearranging the entries of x in non-increasing order. We denote by $(\mathbb{R}^d)^\downarrow = \{x^\downarrow : \ x \in \mathbb{R}^d\}$, $(\mathbb{R}^d_{\geq 0})^\downarrow = \{x^\downarrow : \ x \in \mathbb{R}^d_{\geq 0}\}$.

In what follows, we use the notion of majorization between vectors of non-negative entries, and possibly different sizes: given $a = (a_i)_{i\in\mathbb{I}_n} \in \mathbb{R}^n_{\geq 0}$ and $b = (b_i)_{i\in\mathbb{I}_m} \in \mathbb{R}^m_{\geq 0}$ we say that a is *majorized* by b, denoted $a \prec b$, if

$$\sum_{i\in\mathbb{I}_k} a_i^\downarrow \leq \sum_{i\in\mathbb{I}_k} b_i^\downarrow \ , \quad 1 \leq k \leq \min\{n,m\} \quad \text{and} \quad \sum_{i\in\mathbb{I}_n} a_i = \sum_{i\in\mathbb{I}_m} b_i \, . \qquad \triangle$$

Notation 3.4. Let us consider the following objects:

1. A FSI subspace $\mathcal{W}$ in $L^2(\mathbb{R}^k)$ with $d(x) = \dim J_{\mathcal{W}}(x)$, for $x \in \mathbb{T}^k$.
2. Measurable functions $\alpha_j : \mathbb{T}^k \to \mathbb{R}_+$, for $j \in \mathbb{I}_n$, and $\lambda_j : \mathbb{T}^k \to \mathbb{R}_+$, for $j \in \mathbb{N}$. △

The next result solves the problem of the existence of shift-generated sequences with prescribed fine structure.

Theorem 3.5 ([5]). *Consider Notation 3.4. Then the following conditions are equivalent:*

1. *There exists $\mathcal{F} = \{f_j\}_{j\in\mathbb{I}_n} \in \mathcal{W}^n$ such that $E(\mathcal{F})$ is a Bessel sequence and:*
   (a) $\|\Gamma f_j(x)\|^2 = \alpha_j(x)$ *for* $j \in \mathbb{I}_n$ *and a.e.* $x \in \mathbb{T}^k$;
   (b) $\lambda_j([S_{E(\mathcal{F})}]_x) = \lambda_j(x)$ *for* $j \in \mathbb{N}$ *and a.e.* $x \in \mathbb{T}^k$.
2. *The following admissibility conditions hold:*
   (a) $\lambda_j(x) = 0$ *for a.e.* $x \in \{y \in \mathbb{T}^k : j \geq \min\{d(y), n\} + 1\}$, $j \in \mathbb{N}$;
   (b) $(\alpha_j(x))_{j\in\mathbb{I}_n} \prec (\lambda_j(x))_{j\in\mathbb{I}_{d(x)}}$ *for a.e.* $x \in \mathbb{T}^k$. □

It is worth emphasizing that the preceding result does not fully meet possible prescribed structural requirements for the sequence $E(\mathcal{F})$. In particular, the conditions imposed on the norms and on the spectrum are formulated in a local manner, which does not allow for deriving directly global characterizations involving the norms of the vectors in $\mathcal{F}$. To obtain such global characterizations, we introduce Theorem 3.7 below, which is based on ideas developed in [7].

Remark 3.6. We will need the following well-known properties of the majorization preorder (see [10] for a detailed exposition of majorization theory).

1. If $a(i) \in \mathbb{R}^n_{\geq 0}$ and $b(i) \in \mathbb{R}^m_{\geq 0}$ are such that $a(i) \prec b(i)$, for $i \in \mathbb{I}_q$ then $\sum_{i\in\mathbb{I}_q} a(i) \prec \sum_{i\in\mathbb{I}_q} b(i)^{\downarrow}$. From this fact, we conclude that: if $a : \mathbb{T}^k \to \mathbb{R}^n_{\geq 0}$ and $b : \mathbb{T}^k \to \mathbb{R}^m_{\geq 0}$ are measurable functions, such that $a(x) \prec b(x)$, for a.e. $x \in \mathbb{T}^k$ then
$$\int_{\mathbb{T}^k} a(x)\ dx \prec \int_{\mathbb{T}^k} b(x)^{\downarrow}\ dx \in (\mathbb{R}^m_{\geq 0})^{\downarrow}\,.$$
2. If $a(i) \in \mathbb{R}^{n_i}_{\geq 0}$ and $b(i) \in \mathbb{R}^{m_i}_{\geq 0}$ are such that $a(i) \prec b(i)$, for $i = 1, 2$, then $(a(1), a(2)) \prec (b(1), b(2)) \in \mathbb{R}^{m_1+m_2}_{\geq 0}$.
3. Let $a \in \mathbb{R}^n_{\geq 0}$ and $b \in \mathbb{R}^m_{\geq 0}$, with $m \leq n$. Then, $a \prec b$ if and only if there exists a doubly stochastic matrix D of size n such that $D(b, 0_{n-m}) = a$ (recall that D has non-negative entries and is such that $D\mathbb{1}_n = D^t\mathbb{1}_n = \mathbb{1}_n$). △

Theorem 3.7. *Consider Notation 3.4. Let $\alpha = (\alpha_j)_{j\in\mathbb{I}_n} \in (\mathbb{R}^n_{>0})^{\downarrow}$ be a fixed vector of weights. Assume that $n \geq d(x)$ a.e. Then the following conditions are equivalent:*

1. *There exists $\mathcal{F} = \{f_j\}_{j\in\mathbb{I}_n} \in \mathcal{B}_\alpha(\mathcal{W})$ such that*
$$\lambda_j([S_{E(\mathcal{F})}]_x) = \lambda_j(x)\ , \quad \textit{for} \quad j \in \mathbb{N} \quad \textit{and a.e.} \quad x \in \mathbb{T}^k\,.$$
2. *The following admissibility conditions hold:*
   (a) $\lambda_j(x) = 0$ *for a.e.* $x \in \{y \in \mathbb{T}^k : j \geq d(y) + 1\}$, $j \in \mathbb{N}$;
   (b) *The following majorization relation holds:*
$$(\alpha_j)_{j\in\mathbb{I}_n} \prec \int_{\mathbb{T}^k} (\lambda_1(x), \ldots, \lambda_n(x)\,)\ dx \in (\mathbb{R}^n_{\geq 0})^{\downarrow}\,.$$

Proof. In both cases, we have that the maps $\lambda_j(x) \equiv 0$ for $j > n$. We shall denote by

$$\lambda(x) \stackrel{\text{def}}{=} (\lambda_1(x), \dots, \lambda_n(x)\,) = (\lambda_1(x), \dots, \lambda_{d(x)}(x), \underbrace{0, \dots, 0}_{n-d(x)}) \in (\mathbb{R}^n_{\geq 0})^{\downarrow}$$

for a.e. $x \in \mathbb{T}^k$. Assume that condition 1. holds. We now set $\alpha_j(x) = \|\Gamma f_j(x)\|^2$, for $j \in \mathbb{I}_n$ and $x \in \mathbb{T}^k$; set $\alpha(x) = (\alpha_j(x))_{j\in\mathbb{I}_n} \in \mathbb{R}^n_{\geq 0}$, and notice that in general $\alpha(x) \neq \alpha(x)^{\downarrow}$, for $x \in \mathbb{T}^k$. By Theorem 3.5 we see that $(\alpha_j(x))_{j\in\mathbb{I}_n} \prec (\lambda_j(x))_{j\in\mathbb{I}_{d(x)}}$, for a.e. $x \in \mathbb{T}^k$. Since $\lambda(x) \in (\mathbb{R}^n_{\geq 0})^{\downarrow}$ for a.e. $x \in \mathbb{T}^k$, then (see item 1. in Remark 3.6)

$$\alpha = \textstyle\int_{\mathbb{T}^k} \alpha(x)\ dx \prec \int_{\mathbb{T}^k} \lambda(x)\ dx \in (\mathbb{R}^n_{\geq 0})^{\downarrow}\,,$$

and condition 2. holds. Conversely, if condition 2. holds there exists a doubly stochastic matrix D of size n, such that

$$\alpha = D\left(\textstyle\int_{\mathbb{T}^k} \lambda(x)\ dx\right) = \int_{\mathbb{T}^k} D\,\lambda(x)\ \ dx\,.$$

We now set $\alpha(x) = (\alpha_j(x))_{j\in\mathbb{I}_n} = D\,\lambda(x)$, for $x \in \mathbb{T}^k$. Then, by construction we get that $\alpha(x) \prec \lambda(x)$ for a.e. $x \in \mathbb{T}^k$ (see item 3. in Remark 3.6). Thus, by Theorem 3.5, there exists $\mathcal{F} = (f_j)_{j\in\mathbb{I}_n} \in \mathcal{W}^n$ such that $E(\mathcal{F})$ is a Bessel sequence, $\|\Gamma f_j(x)\|^2 = \alpha_j(x)$, for $j \in \mathbb{I}_n$, and $\lambda_j([S_{E(\mathcal{F})}]_x) = \lambda_j(x)$, for $j \in \mathbb{I}_{d(x)}$, for a.e. $x \in \mathbb{T}^k$. In particular, we get that

$$\|f_j\|^2 = \textstyle\int_{\mathbb{T}^k} \|\Gamma_j(x)\|^2\, dx = \int_{\mathbb{T}^k} (D\,\lambda(x))_j\, dx = \left(\int_{\mathbb{T}^k} D\,\lambda(x)\ \ dx\right)_j = \alpha_j\ ,$$

and condition 1. holds in this case. □

Corollary 3.8. *With the notation of Theorem 3.7, there exists $\mathcal{F} \in \mathcal{B}_\alpha(\mathcal{W})$ such that $E(\mathcal{F})$ is a c-tight frame for $\mathcal{W}$ if and only if*

$$\alpha \prec c \cdot h_{\mathcal{W}} \quad \text{where} \quad h_{\mathcal{W}} = \textstyle\int_{\mathbb{T}^k} \mathbb{1}_{d(x)} \oplus 0_{n-d(x)}\ dx \in \mathbb{R}^n_{\geq 0}\,.$$

In this case, $c = \frac{\operatorname{tr}(\alpha)}{\operatorname{tr}(h_{\mathcal{W}})}$ is uniquely determined (by α and $\mathcal{W}$).

Proof. Notice that $\mathcal{F} \in \mathcal{B}_\alpha(\mathcal{W})$ is such that $E(\mathcal{F})$ is a c-tight frame for $\mathcal{W}$ if and only if $\lambda([S_{E(\mathcal{F})}]_x) = c\,\mathbb{1}_{d(x)}$, for $x \in \mathbb{T}^k$. Hence, the result is a direct consequence of Theorem 3.7. □

Remark 3.9. Corollary 3.8 shows that there are restrictions for the existence of $\mathcal{F} \in \mathcal{B}_\alpha(\mathcal{W})$ such that $E(\mathcal{F})$ is a tight frame for $\mathcal{W}$. Indeed, consider the particular case where $\mathcal{W}$ is such that $d(x) = d_1 = n \geq 2$, for $x \in \mathbb{T}^k$. Hence, in this case $h_{\mathcal{W}} = \mathbb{1}_n$ and then, $\alpha \prec c\,\mathbb{1}_{d_1}$ which implies that $c = \frac{\operatorname{tr}(\alpha)}{n}$ and that $\alpha = \frac{\operatorname{tr}(\alpha)}{n}\,\mathbb{1}_n$. △

3.2 Optimal frames with prescribed norms for FSI subspaces

In this section we state and prove our main result on the existence of universal optimal sequences $\mathcal{F}^{\rm op} \in \mathcal{B}_\alpha(\mathcal{W})$ (see Theorem 3.22). We first recall some previous results from [5] related to the discrete structure of minimizers of convex potentials in $\mathcal{B}_\alpha(\mathcal{W})$.

Let us fix some general notions and notation for future reference:

Notation 3.10. In what follows, we consider:

1. A FSI subspace $\mathcal{W}$ in $L^2(\mathbb{R}^k)$;
2. $d(x) = \dim J_{\mathcal{W}}(x) = \operatorname{tr}([P_{\mathcal{W}}]_x)$, for $x \in \mathbb{T}^k$;
3. The Lebesgue measure on $\mathbb{T}^k$; denoted $|\cdot|$;

4. $d_1 := \|d\|_\infty$ the essential supremum of the measurable function $d(\cdot)$;

5. $Z_i = d^{-1}(i) \subseteq \mathbb{T}^k$ and $p_i = |Z_i|$, $i \in \mathbb{I}_{d_1}$;

6. The spectrum of $\mathcal{W}$, denoted $\operatorname{Spec}(\mathcal{W}) = \bigcup_{i\in\mathbb{I}_{d_1}} Z_i = \{x \in \mathbb{T}^k : d(x) \geq 1\}$;

7. $\alpha = (\alpha_i)_{i\in\mathbb{I}_n} \in (\mathbb{R}^n_{>0})^\downarrow$, such that $d_1 \leq n$; △

Theorem 3.11 ([5]). *Consider the Notations 3.10 and fix $\varphi \in \operatorname{Conv}(\mathbb{R}_+)$. Then, there exists $\mathcal{F}^{\rm op} \in \mathfrak{B}_\alpha(\mathcal{W})$ such that:*

1. *$\lambda_j([S_{E(\mathcal{F}^{\rm op})}]_x) =: \psi^{\rm op}_{i,j} \in \mathbb{R}_+$ is a.e. constant for $x \in Z_i$, $j \in \mathbb{I}_i$ and $i \in \mathbb{I}_{d_1}$;*

2. *For every $\mathcal{F} \in \mathfrak{B}_\alpha(\mathcal{W})$ we have that*

$$\textstyle\sum_{i\in\mathbb{I}_{d_1}} p_i \left(\sum_{j\in\mathbb{I}_i} \varphi(\psi^{\rm op}_{i,j})\right) = P^{\mathcal{W}}_\varphi(E(\mathcal{F}^{\rm op})) \leq P^{\mathcal{W}}_\varphi(E(\mathcal{F}))\ .$$

If we further assume that $\varphi \in \operatorname{Conv_s}(\mathbb{R}_+)$ then:

a) If $\mathcal{F} \in \mathfrak{B}_\alpha(\mathcal{W})$ is such that $P^{\mathcal{W}}_\varphi(E(\mathcal{F})) = P^{\mathcal{W}}_\varphi(E(\mathcal{F}^{\rm op}))$ then $S_{E(\mathcal{F})}$ has the same fine spectral structure as $S_{E(\mathcal{F}^{\rm op})}$.

b) If φ is differentiable in $\mathbb{R}_+$ then $E(\mathcal{F})$ is a frame for $\mathcal{W}$. □

We point out that in the result above, we first fix a convex function φ and then consider minimizers of the induced convex potential on $\mathcal{B}_\alpha(\mathcal{W})$. We will not make use of the discrete structure that was derived from the previous result. Nevertheless, we will make use of the uniqueness result in item *a)*.

Next, we will consider several results that will allow us to construct the optimal families $\mathcal{F}^{\rm op} \in \mathcal{B}_\alpha(\mathcal{W})$ (see Remark 3.14). Building on Theorem 3.7, we will first construct an admissible fine spectral structure for these optimal families $\mathcal{F}^{\rm op}$.

Remark 3.12. Consider Notation 3.10 and set

$$H_j := \{x \in \mathbb{T}^k \ : \ d(x) \geq j\} \quad \text{and} \quad h_j = |H_j|\ , \quad \text{for} \quad i \in \mathbb{I}_{d_1}\ .$$

We also introduce $h := (h_j)_{j\in\mathbb{I}_{d_1}} \in (\mathbb{R}^{d_1}_{>0})^\downarrow$; notice that in this case,

$$h = \textstyle\int_{\mathbb{T}^k} \mathbb{1}_{d(x)} \oplus 0_{d_1-d(x)}\ dx\,. \tag{11}$$

Consider further $\alpha = (\alpha_j)_{j\in\mathbb{I}_n} \in (\mathbb{R}^n_{>0})^\downarrow$, for some $n \geq d_1$. For $1 \leq s \leq t \leq d_1$, denote by $P_{s,t}$ and Q_t the ratios

$$P_{s\,,\,t} = \frac{\sum\limits_{i=s}^{t} \alpha_i}{\sum\limits_{i=s}^{t} h_i} \quad \text{and} \quad Q_t = \frac{\sum\limits_{i=t}^{n} \alpha_i}{\sum\limits_{i=t}^{d_1} h_i}\ . \tag{12}$$

Proposition 3.13. *Consider Notation 3.10 and the vector $h = (h_i)_{i\in\mathbb{I}_{d_1}}$ as in Eq. (11). Then, there exists*

$$p \in \mathbb{I}_{d_1} \quad \textit{and} \quad g_1, \ldots, g_p \in \mathbb{N} \quad \textit{with} \quad 0 = g_0 < g_1 < \cdots < g_p = d_1$$

such that, if we define $\gamma_i = P_{g_{i-1}+1\,,\,g_i}$, for $i \in \mathbb{I}_{p-1}$ and $\gamma_p = Q_{g_{p-1}+1}$ according to Eq. (12), then

1. *$\gamma_1 > \ldots > \gamma_p > 0$;*

2. They satisfy the following "block" majorizations:

$$\begin{aligned} &(\gamma_i\, h_k)_{k=g_{i-1}+1}^{g_i} \succ (\alpha_i)_{k=g_{i-1}+1}^{g_i} \quad \textit{for} \quad i \in \mathbb{I}_{p-1} \quad \textit{and} \\ &(\gamma_p\, h_k)_{k=g_{p-1}+1}^{d_1} \succ (\alpha_i)_{k=g_{p-1}+1}^{n} \,. \end{aligned} \tag{13}$$

In particular, by Remark 3.6,

$$(\gamma_1\, \mathbb{1}_{g_1-g_0}\,,\ \gamma_2\, \mathbb{1}_{g_2-g_1}\,,\ \ldots\,,\ \gamma_p\, \mathbb{1}_{g_p-g_{p-1}}) \circ h \succ \alpha\ , \tag{14}$$

where $\circ$ *denotes the entry-wise (or Hadamard) product.*

Proof. First note that d_1 clearly satisfies $Q_{d_1} \geq P_{d_1\,,\,d_1}$. Then we can define the index

$$s^* = \min\{j \in \mathbb{I}_{d_1} : Q_j \geq P_{j\,,\,k} \quad \text{for every} \quad j \leq k \leq d_1\} \tag{15}$$

We denote $c = Q_{s^*}$. Therefore, by Eq. (12), $c \sum_{i=s^*}^{d_1} h_i = \sum_{i=s^*}^{n} \alpha_i$ and

$$c \sum_{i=s^*}^{k} h_i \geq \sum_{i=s^*}^{k} \alpha_i \quad \text{for every} \quad s^* \leq k \leq d_1\ .$$

Since $\alpha = \alpha^\downarrow$ and $h = h^\downarrow$ we get that $(\alpha_k)_{k=s^*}^{n} \prec (h_k\, c)_{k=s^*}^{d_1}$. If $s^* = 1$ then we set $p = 1$, $g_0 = 0$, $g_1 = d_1$ and $\gamma_1 = c > 0$. Then items 1. and 2. of the statement are satisfied in this case.

Otherwise, $s^* > 1$ and we proceed to find the step g_1. First, we define γ_1:

$$\gamma_1 = \max\{P_{1,k}\ :\ 1 \leq k \leq s^* - 1\}\,,$$

and then we define g_1 as:

$$g_1 = \max\{j \in \mathbb{I}_{s^*-1}\ : P_{1,j} = \gamma_1\}\,.$$

By construction, we obtain that

$$(\gamma_1 h_i)_{i=1}^{g_1} \succ (\alpha_i)_{i=1}^{g_1}\,.$$

Now, if $g_1 = s^* - 1$ then we set $p = 2$, $g_0 = 0$, $g_2 = d_1$ and $\gamma_2 = c > 0$.

Otherwise, $g_1 < s^* - 1$ (and having the index s^* fixed), we define g_2 in a similar way:

$$\gamma_2 = \max\{P_{g_1+1,k}\ :\ g_1 < k \leq s^* - 1\}$$

and then,

$$g_2 = \max\{g_1 + 1 \leq j \leq s^* - 1\ : P_{g_1+1,j} = \gamma_2\}\,.$$

Again, by construction, we have that

$$(\gamma_2 h_i)_{i=g_1+1}^{g_2} \succ (\alpha_i)_{i=g_1+1}^{g_2}\,.$$

We claim that $\gamma_1 > \gamma_2$. Indeed, suppose that $P_{g_1+1,g_2} = \gamma_2 \geq \gamma_1 = P_{1,g_1}$. Then,

$$\begin{aligned} P_{1,g_2} - P_{1,g_1} &= \frac{\sum_{i=1}^{g_1} \alpha_i + \sum_{i=g_1+1}^{g_2} \alpha_i}{\sum_{i=1}^{g_1} h_i + \sum_{i=g_1+1}^{g_2} h_i} - \frac{\sum_{i=1}^{g_1} \alpha_i}{\sum_{i=1}^{g_1} h_i} = \\ &= \frac{\left(\sum_{i=1}^{g_1} h_i\right)\left(\sum_{i=g_1+1}^{g_2} \alpha_i\right) - \left(\sum_{i=g_1+1}^{g_2} h_i\right)\left(\sum_{i=1}^{g_1} \alpha_i\right)}{\sum_{i=1}^{g_1} h_i \left(\sum_{i=1}^{g_1} h_i + \sum_{i=g_1+1}^{g_2} h_i\right)} \\ &= \frac{\left(\sum_{i=1}^{g_1} h_i\right)\left(\sum_{i=g_1+1}^{g_2} h_i\right)(\gamma_2 - \gamma_1)}{\sum_{i=1}^{g_1} h_i \left(\sum_{i=1}^{g_1} h_i + \sum_{i=g_1+1}^{g_2} h_i\right)} \geq 0\,. \end{aligned}$$

Hence, $P_{1,g_2} = P_{1,g_1} = \gamma_1$ which contradicts the definition of g_1, so the claim is proved.

We can continue inductively with this process, that is, once we find $g_{k-1} < s^* - 1$ we compute first γ_k as the maximum among $P_{g_{k-1}+1,l}$, with $g_{k-1} + 1 \le l \le s^* - 1$ and then define $g_k \le s^* - 1$ as the maximum index $g_{k-1} + 1 \le l \le s^* - 1$ such that $P_{g_{k-1}+1,l} = \gamma_k$. As before, this construction guarantees the corresponding block majorization.

Notice that in the last step, corresponding to the $p-1$ iteration of the process, we necessarily have $g_{p-1} = s^* - 1$. Define $\gamma_p = Q_{g_{p-1}+1} = c > 0$ and $g_p = d_1$.

By construction, and the previous remarks we have that $\gamma_1 > \gamma_2 > \ldots > \gamma_{p-1}$ and Eq. 13 is satisfied. It remains to prove that $\gamma_{p-1} > \gamma_p$.

Suppose, on the contrary, that $\gamma_p \ge \gamma_{p-1}$. Consider $\bar{c} = Q_{g_{p-2}+1}$. Then $\bar{c}$ is a convex combination of γ_p and γ_{p-1}. Indeed, let $A = \sum_{i=g_{p-2}+1}^{g_{p-1}} h_i$, $B = \sum_{i=g_{p-1}+1}^{l} h_i$ and $C = \sum_{i=l+1}^{d_1} h_i$; if we let $t = \frac{A}{A+B+C}$ then $\bar{c} = t\gamma_{p-1} + (1-t)\gamma_p$. In particular, $\gamma_{p-1} \le \bar{c} \le \gamma_p$. Therefore, we have that

$$P_{g_{p-2}+1,l} \le \gamma_{p-1} \le \bar{c}\,, \ \text{ for } g_{p-2}+1 \le l \le g_{p-1} \tag{16}$$

Let $g_{p-1} + 1 \le l < d_1$. Since $\gamma_{p-1} \le \gamma_p$,

$$\frac{A}{A+B}\,\gamma_{p-1} + \frac{B}{A+B}\,\gamma_p \le \frac{A}{A+B+C}\,\gamma_{p-1} + \frac{B+C}{A+B+C}\,\gamma_p\,.$$

Hence, since by definition of γ_p we have $P_{g_{p-1}+1\,,\,l} \le \gamma_p$, we obtain

$$\frac{A}{A+B}\,\gamma_{p-1} + \frac{B}{A+B} P_{g_{p-1}+1\,,\,l} \le \frac{A}{A+B+C}\,\gamma_{p-1} + \frac{B+C}{A+B+C}\,\gamma_p\,. \tag{17}$$

Using that $\frac{A}{A+B}\,\gamma_{p-1} + \frac{B}{A+B}\,P_{g_{p-1}+1\,,\,l} = P_{g_{p-2}+1\,,\,l}$ and $\frac{A}{A+B+C}\,\gamma_{p-1} + \frac{B+C}{A+B+C}\,\gamma_p = \bar{c}$, we deduce

$$P_{g_{p-2}+1\,,\,l} \le \bar{c} \ , \qquad \text{for} \qquad g_{p-1} + 1 \le l \le d_1 \tag{18}$$

Therefore, Eqs. (16) and (18) imply that, for $j = g_{p-2} + 1 < s^* = g_{p-1} + 1$,

$$P_{j\,,\,l} \le Q_j \ , \qquad \text{for} \qquad l = j, \ldots, d_1$$

which contradicts the construction of s^*. So we can conclude that $\gamma_{p-1} > \gamma_p$ and the theorem is proved. □

Remark 3.14 (Construction of the optimal families $\mathcal{F}^{\rm op} \in \mathcal{B}_\alpha(\mathcal{W})$)**.** Consider Notation 3.10. Let $0 = g_0 < \cdots < g_p = d_1$ and $\gamma_1 > \cdots > \gamma_p > 0$ be as in Proposition 3.13. Then, we set:

1. $\lambda^{\rm op} = (\lambda_j^{\rm op})_{j\in\mathbb{I}_{d_1}} := (\gamma_k \mathbb{1}_{g_k - g_{k-1}})_{k=1}^p \in (\mathbb{R}_{>0}^{d_1})^\downarrow$.
2. $(\lambda_j^{\rm op}(x))_{j\in\mathbb{I}_{d(x)}} = (\lambda_j^{\rm op})_{j\in\mathbb{I}_{d(x)}} \in (\mathbb{R}_{>0}^{d(x)})^\downarrow$, for $x \in \mathbb{T}^k$.

We further construct

$$\Lambda^{\rm op} = (\Lambda_j^{\rm op})_{j\in\mathbb{I}_{d_1}} := \int_{\mathbb{T}^k} \lambda^{\rm op}(x) \oplus 0_{n-d(x)}\; dx \in (\mathbb{R}_{\ge 0}^n)^\downarrow\,.$$

Given $1 \le j \le d_1$, then we have that

$$\Lambda_j^{\rm op} = \int_{\mathbb{T}^k} (\lambda^{\rm op}(x) \oplus 0_{n-d(x)})_j\; dx = \int_{H_j} \lambda_j^{\rm op}\; dx = h_j\, \lambda_j^{\rm op}\ .$$

Thus, $\Lambda^{\rm op} = (\gamma_k \mathbb{1}_{g_k-g_{k-1}})_{k=1}^p \circ h \succ \alpha$. By Theorem 3.7 there exist $\mathcal{F}^{\rm op} \in \mathcal{B}_\alpha(\mathcal{W})$ such that $\lambda([S_{E(\mathcal{F}^{\rm op})}]_x) = \lambda^{\rm op}(x)$, for $x \in \mathbb{T}^k$. △

In Theorem 3.22 (main result) we will show that the families $\mathcal{F}^{\rm op} \in \mathcal{B}_\alpha(\mathcal{W})$ constructed as in the previous remark are universal minimizers of convex potentials in $\mathcal{B}_\alpha(\mathcal{W})$. To show this, we need some further tools wich we will introduce below.

Definition 3.15. *Let $(X, \mathcal{X}, \mu), (Y, \mathcal{Y}, \nu)$ be two measure spaces; we consider their direct sum, denoted $X \bigoplus Y$, given by the three-tuple $(X \oplus Y, \mathcal{X} \bigoplus \mathcal{Y}, \mu \oplus \nu)$, where*

1. *$X \oplus Y = X \overset{d}{\cup} Y$ (the disjoint union of the sets); we further consider the canonical inclusions $\eta_X : X \to X \oplus Y$ and $\eta_Y : Y \to X \oplus Y$ of X and Y into their disjoint union; hence η_X and η_Y are injective functions such that $\eta_X(X) \cap \eta_Y(Y) = \emptyset$ and $\eta_X(X) \cup \eta_Y(Y) = X \oplus Y$.*

2. *$\mathcal{X} \bigoplus \mathcal{Y} = \{A \oplus B = \eta_X(A) \cup \eta_Y(B) : A \in \mathcal{X},\ B \in \mathcal{Y}\}$;*

3. *$\mu \oplus \nu$ is the measure given by $\mu \oplus \nu(A \oplus B) = \mu(A) + \nu(B)$;*

4. *If $f \in L^1(X, \mathcal{X}, \mu)$ and $g \in L^1(Y, \mathcal{Y}, \nu)$ then we can set $f \oplus g : X \oplus Y \to \mathbb{C}$ given by $f \oplus g(w) = f(w)$ if $w \in X$ or $f \oplus g(w) = g(w)$ if $w \in Y$. In this case, $f \oplus g \in L^1(X \oplus Y, \mathcal{X} \oplus \mathcal{Y}, \mu \oplus \nu)$ and*
$$\textstyle\int_{X \oplus Y} f \oplus g \ d\mu \oplus \nu = \int_X f \ d\mu + \int_Y g \ d\nu \,.$$

Notice that using the maps η_X and η_Y we can consider (as we sometimes do) $X, Y \subset X \oplus Y$. $\triangle$

Remark 3.16. Let $(X, \mathcal{X}, \mu)$, $(Y, \mathcal{Y}, \nu)$ and $(Z, \mathcal{Z}, \omega)$ be measure spaces. Then, the measure spaces $(X \bigoplus Y) \bigoplus Z$ and $X \bigoplus (Y \bigoplus Z)$ are the same, in the sense that there exists a bijective function f between the underlying sets, such that f and f^{-1} are both measurable and measure-preserving. Hence, we will simply write $X \bigoplus Y \bigoplus Z$.

Definition 3.17. *Consider Notation 3.10. We define*

1. *$Z_\mathcal{W} := \bigoplus_{i \in \mathbb{I}_{d_1}} Z_i \times \mathbb{I}_i$ endowed with the measure $\mu_\mathcal{W}$ that corresponds to the direct sum (see Definition 3.15) of the product measure $\mu_i = |\cdot| \times \#(\cdot)$ of Lebesgue measure on Z_i and the counting measure $\#(\cdot)$ on $\mathbb{I}_i$, for $i \in \mathbb{I}_{d_1}$.*

2. *If $\mathcal{F} \in \mathcal{B}_\alpha(\mathcal{W})$ then we let $\mathcal{M}_\mathcal{F} \in L^\infty(Z_\mathcal{W})$ be the non-negative function given by*
$$\mathcal{M}_\mathcal{F}(x, j) = \lambda_j([S_{E(\mathcal{F})}]_x) \quad \text{for} \quad x \in Z_i \ , \quad j \in \mathbb{I}_i \ , \quad i \in \mathbb{I}_{d_1} \,. \qquad \triangle$$

Remark 3.18. In the next result, we will make use of the notion of majorization between functions in finite measure spaces, which we now describe. Indeed, given a finite measure space $(X, \mathcal{X}, \mu)$ and an essentially bounded non-negative function $f \in L^\infty(X, \mu)^+$ we consider its *decreasing rearrangement* (see [30]), denoted $f^* : [0, \mu(X)) \to \mathbb{R}_+$, that is given by

$$f^*(s) \overset{\rm def}{=} \sup\{t \in \mathbb{R}_+ : \ \mu\{x \in X : \ f(x) > t\} > s\} \quad \text{for} \quad s \in [0, \mu(X))\,. \tag{19}$$

If $g \in L^\infty(X, \mu)^+$ then we say that f *majorizes* g (in $(X, \mathcal{X}, \mu)$), denoted $g \prec f$, if

$$\textstyle\int_0^s g^*(t) \ dt \le \int_0^s f^*(t) \ dt \quad \text{for every} \quad 0 \le s \le \mu(X)$$

and $\int_0^{\mu(X)} g^*(t) \ dt = \int_0^{\mu(X)} f^*(t) \ dt$. $\triangle$

The next result shows the importance of majorization relations between the functions $\mathcal{M}_\mathcal{F} \in L^\infty(Z_\mathcal{W}, \mu_\mathcal{W})^+$ associated to $\mathcal{F} \in \mathcal{B}_\alpha(\mathcal{W})$.

Proposition 3.19. *Consider Notation 3.10, let $\mathcal{F} \in \mathcal{B}_\alpha(\mathcal{W})$ and let $\mathcal{M}_\mathcal{F} : \bigoplus_{i\in\mathbb{I}_{d_1}} Z_i \times \mathbb{I}_i \to \mathbb{R}_{\geq 0}$ be as in Definition 3.17. Then, for every $\varphi \in \operatorname{Conv}(\mathbb{R}_+)$ we have that*

$$P_\varphi^{\mathcal{W}}(E(\mathcal{F})) = \int_{Z_\mathcal{W}} \varphi \circ \mathcal{M}_\mathcal{F}(x)\ d\mu_\mathcal{W}\,. \tag{20}$$

Then, if $\mathcal{F}' \in \mathcal{B}_\alpha(\mathcal{W})$ is such that $\mathcal{M}_{\mathcal{F}'} \prec \mathcal{M}_\mathcal{F}$ then $P_\varphi^{\mathcal{W}}(E(\mathcal{F}')) \leq P_\varphi^{\mathcal{W}}(E(\mathcal{F}))$.

Proof. Let $\varphi \in \operatorname{Conv}(\mathbb{R}_+)$; by item 4. in Definition 3.15, we have that

$$\int_{Z_\mathcal{W}} \varphi \circ \mathcal{M}_\mathcal{F}(x)\ d\mu_\mathcal{W} = \sum_{i\in\mathbb{I}_{d_1}} \int_{Z_i\times\mathbb{I}_i} \varphi \circ \mathcal{M}_\mathcal{F}(x,j) d\mu_i\,.$$

Notice that $\varphi \circ \mathcal{M}_\mathcal{F}(x,j) = \varphi(\lambda_j([S_{E(\mathcal{F})}]_x))$, for $x \in Z_i$ and $j \in \mathbb{I}_i$. Using the previous identity and iterated integrals and the identity $\sum_{j\in\mathbb{I}_i} \varphi(\lambda_j([S_{E(\mathcal{F})}]_x)) = \operatorname{tr}(\varphi([S_{E(\mathcal{F})}]_x)[P_\mathcal{W}]_x))$ for $x \in Z_i$, we get that

$$\int_{Z_i\times\mathbb{I}_i} \varphi \circ \mathcal{M}_\mathcal{F}(x,j) d\mu_i = \int_{Z_i} \operatorname{tr}(\varphi([S_{E(\mathcal{F})}]_x)[P_\mathcal{W}]_x))\, dx\,.$$

The previous facts prove Eq. (20). The last claim of the statement is a straightforward consequence of the properties of majorization (see item 3. in Theorem 4.2). □

The previous proposition shows that the functions $\mathcal{M}_\mathcal{F}$ play an important role in the computation of $P_\varphi^{\mathcal{W}}(E(\mathcal{F}))$. Furthermore, if $\mathcal{M}_{\mathcal{F}'} \prec \mathcal{M}_\mathcal{F}$, then a comparison between the convex potentials of $E(\mathcal{F})$ and $E(\mathcal{F}')$ follows, valid for *all* convex functions $\varphi \in \operatorname{Conv}(\mathbb{R}_+)$. In turn, to get a majorization relation between these functions, we need to understand the decreasing re-arrangements $\mathcal{M}_\mathcal{F}^*$ (see Remark 3.18). Next, we focus on $\mathcal{M}_{\mathcal{F}^{\rm op}}^*$, where $\mathcal{F}^{\rm op}$ is constructed as in Remark 3.14.

Remark 3.20. Consider the Notation from Remark 3.14. Let $0 = g_0 < \cdots < g_p = d_1$ and $\gamma_1 > \cdots > \gamma_p > 0$ be as in Proposition 3.13. Let $\mathcal{F}^{\rm op}$ be constructed as in Remark 3.14 and let $\mathcal{M}_{\mathcal{F}^{\rm op}} \in L^\infty(Z_\mathcal{W})$ be constructed as in Definition 3.17. Then

$$\mathcal{M}_{\mathcal{F}^{\rm op}}(x,j) = \gamma_1 \quad \text{for} \quad x \in Z_i\ ,\ \ j \in \mathbb{I}_{\min\{i,\,g_1\}}\ ,\ \ i \in \mathbb{I}_{d_1}\,.$$

Similarly, if $2 \leq \ell \leq p$ we have that

$$\mathcal{M}_{\mathcal{F}^{\rm op}}(x,j) = \gamma_\ell \quad \text{for} \quad x \in Z_i\ ,\ \ g_{\ell-1}+1 \leq j \leq \min\{i,\, g_\ell\}\ ,\ \ g_{\ell-1}+1 \leq i \leq d_1\,.$$

The previous facts imply that if we set $t_0 = 0$ and inductively, for $1 \leq \ell \leq p$

$$t_\ell \ = t_{\ell-1} + \sum_{i=g_{\ell-1}+1}^{d_1} (\min\{i,\, g_\ell\} - g_{\ell-1})\ |Z_i| \tag{21}$$

then, we get that

$$\mathcal{M}_{\mathcal{F}^{\rm op}}^*(t) = \gamma_\ell \quad \text{for} \quad t \in [t_{\ell-1}, t_\ell] \quad \text{for} \quad \ell \in \mathbb{I}_p\,.$$

Finally, notice that $\mu_\mathcal{W}(Z_\mathcal{W}) = t_p = \sum_{\ell\in\mathbb{I}_{d_1}} i\,|Z_i|$. △

Remark 3.21. Consider the Notation from Remark 3.20 and let $\mathcal{F} \in \mathcal{B}_\alpha(\mathcal{W})$. In order to check that $\mathcal{M}_{\mathcal{F}^{\rm op}} \prec \mathcal{M}_\mathcal{F}$ we only need to show that

$$\int_0^{t_\ell} \mathcal{M}_{\mathcal{F}^{\rm op}}^*(t)\ dt \leq \int_0^{t_\ell} \mathcal{M}_\mathcal{F}^*(t)\ dt\ , \quad \text{for} \quad \ell \in \mathbb{I}_p\,. \tag{22}$$

Indeed, assume that Eq. (22) and let $s \in [0, t_p]$ be such that

$$\int_0^s \mathcal{M}_{\mathcal{F}^{\rm op}}^*(t)\ dt > \int_0^s \mathcal{M}_\mathcal{F}^*(t)\ dt\,. \tag{23}$$

Then, there exists $j \in \mathbb{I}_p$ such that $t_{j-1} < s < t_j$; in this case, using Eqs. (22) for $\ell = j-1$ and (23), we have that

$$(s - t_{j-1})\,\gamma_j = \int_{t_{j-1}}^s \mathcal{M}_{\mathcal{F}^{\rm op}}^*(t)\ dt > \int_{t_{j-1}}^s \mathcal{M}_\mathcal{F}^*(t)\ dt$$

which implies that $\mathcal{M}^*_{\mathcal{F}}(s) < \gamma_j$; again, since $\mathcal{M}^*_{\mathcal{F}}(t)$ is non-increasing we get that

$$(t_j - s)\,\gamma_j = \int_s^{t_j} \mathcal{M}^*_{\mathcal{F}^{\rm op}}(t)\; dt > \int_s^{t_j} \mathcal{M}^*_{\mathcal{F}}(t)\; dt$$

$$\implies \int_0^{t_j} \mathcal{M}^*_{\mathcal{F}^{\rm op}}(t)\; dt > \int_0^{t_j} \mathcal{M}^*_{\mathcal{F}}(t)\; dt\;,$$

using Eq. (23). Notice that the last inequality above contradicts Eq. (22) for $\ell = j$. We finally notice that for $\ell = p$ we always get an equality in Eq. (22), since

$$\int_0^{t_p} \mathcal{M}^*_{\mathcal{F}}(t)\; dt = \int_{Z_{\mathcal{W}}} \mathcal{M}_{\mathcal{F}}(w)\; d\mu_{\mathcal{W}} = \int_{\mathbb{T}^k} \operatorname{tr}([S_{E(\mathcal{F})}]_x)\; dx = \sum_{i\in\mathbb{I}_n} \alpha_i\,. \qquad \triangle$$

Theorem 3.22. *With the notation from Remark 3.14 and Definition 3.17, if*

$$\mathcal{F} \in \mathcal{B}_\alpha(\mathcal{W}) \implies \mathcal{M}_{\mathcal{F}^{\rm op}} \prec \mathcal{M}_{\mathcal{F}}$$

in $(Z_{\mathcal{W}}, \mathcal{Z}_{\mathcal{W}}, \mu_{\mathcal{W}})$. In particular, for every $\varphi \in \operatorname{Conv}(\mathbb{R}_+)$ then

$$P^{\mathcal{W}}_\varphi(E(\mathcal{F}^{\rm op})) \le P^{\mathcal{W}}_\varphi(E(\mathcal{F}))\,.$$

Furthermore, if there exists $\psi \in \operatorname{Conv}_{\rm s}(\mathbb{R}_+)$ such that $P^{\mathcal{W}}_\psi(E(\mathcal{F}^{\rm op})) = P^{\mathcal{W}}_\psi(E(\mathcal{F}))$ then $E(\mathcal{F})$ has the same fine spectral structure as $E(\mathcal{F}^{\rm op})$.

Proof. By Theorem 3.5, if we let $\alpha_j(x) = \|\Gamma f_j(x)\|^2$, for $x \in \mathbb{T}^k$ and $j \in \mathbb{I}_n$, and $\lambda_j(x) = \lambda_j([S_{E(\mathcal{F})}]_x)$ for $x \in \mathbb{T}^k$ and $j \in \mathbb{I}_{d(x)}$, then we have that $(\alpha_j(x))_{j\in\mathbb{I}_n} \prec (\lambda_j(x))_{j\in\mathbb{I}_{d(x)}}$. Hence, for $x \in \mathbb{T}^k$ we conclude that

$$\sum_{i\in\mathbb{I}_s} \alpha_j(x) \;\le \sum_{i=1}^{\min\{s,d(x)\}} \lambda_i(x) \quad \text{for} \quad s \in \mathbb{I}_n\,. \tag{24}$$

By Remark 3.21 we just need to check Eq. (22) for every $1 \le \ell \le p-1$. Hence, let $\ell \in \mathbb{I}_{p-1}$ and let

$$S_\ell = \{(x,j)\ : x \in Z_i\ ,\ 1 \le j \le \min\{g_\ell,\, i\}\ ,\ i \in \mathbb{I}_{d_1}\}\,.$$

It follows that

$$\mu_{\mathcal{W}}(S_\ell) \;= \sum_{i\in\mathbb{I}_{d_1}} \min\{g_\ell,\, i\}\,|Z_i| = t_\ell\,.$$

Hence, using that $d(x) = i$ for $x \in Z_i$ we get that

$$\begin{aligned}\int_{S_\ell} \mathcal{M}_{\mathcal{F}}(w)\; d\mu_{\mathcal{W}}(w) &= \sum_{i\in\mathbb{I}_{d_1}} \int_{Z_i} \sum_{j=1}^{\min\{g_\ell,i\}} \lambda_j(x)\; dx \ge \sum_{i\in\mathbb{I}_{d_1}} \int_{Z_i} \sum_{j=1}^{g_\ell} \alpha_j(x)\; dx \\ &= \int_{\mathbb{T}^k} \sum_{j=1}^{g_\ell} \alpha_j(x)\; dx = \sum_{j=1}^{g_\ell} \alpha_j\end{aligned}$$

By Proposition 3.13, for $m \in \mathbb{I}_p$ we see that

$$\begin{aligned}\sum_{j=g_{m-1}+1}^{g_m} \alpha_j &= \gamma_m \sum_{j=g_{m-1}+1}^{g_m} h_j = \gamma_m \sum_{j=g_{m-1}+1}^{g_m} \sum_{i=j}^{d_1} |Z_i| \\ &= \gamma_m \sum_{i=g_{m-1}+1}^{d_1} (\min\{i,\, g_m\} - g_{m-1})\; |Z_i|\,.\end{aligned}$$

Hence, using the previous fact and Eq. (21) for the definition of t_m for $m \in \mathbb{I}_p$,

$$\sum_{j=g_{m-1}+1}^{g_m} \alpha_j \;= \gamma_m\,(t_m - t_{m-1}) \quad \text{for} \quad m \in \mathbb{I}_p\,.$$

Putting everything together, we get that (see item 2. in Remark 4.1)

$$\begin{aligned}\int_0^{t_\ell} \mathcal{M}^*_{\mathcal{F}}(x)\; dx &\ge \int_{S_\ell} \mathcal{M}_{\mathcal{F}}(w)\; d\mu_{\mathcal{W}}(w) \ge \sum_{j=1}^{g_\ell} \alpha_j = \sum_{m=1}^{\ell} \sum_{j=g_{m-1}+1}^{g_m} \alpha_j \\ &= \sum_{m=1}^{\ell} \gamma_m\,(t_m - t_{m-1}) = \int_0^{t_\ell} \mathcal{M}^*_{\mathcal{F}^{\rm op}}(x)\; dx\,.\end{aligned} \tag{25}$$

The last claim of the statement is a consequence of item $a)$ in Theorem 3.11. $\square$

4 Appendix: Majorization in finite measure spaces

Majorization between vectors (see [10, 30]) has played a key role in frame theory. On the one hand, majorization allows us to characterize the existence of frames with prescribed properties (see [3, 17, 19]). On the other hand, majorization is a preorder relation that implies a family of tracial inequalities; this last fact can be used to explain the structure of minimizers of general convex potentials, which include the Benedetto-Fickus' frame potential (see [9, 18, 31, 32, 33, 34, 35]). We will be dealing with convex potentials in the context of Bessel families of integer translates of finite sequences; accordingly, we will need the following general notion of majorization between functions in probability spaces.

Throughout this section the triple $(X, \mathcal{X}, \mu)$ denotes a finite measure space i.e. $\mathcal{X}$ is a σ-algebra of sets in X and μ is a finite measure defined on $\mathcal{X}$. We shall denote by $L^\infty(X,\mu)^+ = \{f \in L^\infty(X,\mu) : f \geq 0\}$. For $f \in L^\infty(X,\mu)^+$, recall that the *decreasing rearrangement* of f, denoted $f^* : [0, \mu(X)) \to \mathbb{R}_+$, is given by

$$f^*(s) \stackrel{\text{def}}{=} \sup\{t \in \mathbb{R}_+ : \ \mu\{x \in X : \ f(x) > t\} > s\} \quad \text{for} \quad s \in [0, \mu(X))\,.$$

Remark 4.1. We mention some facts related to the decreasing rearrangement of functions. Let $f \in L^\infty(X,\mu)^+$, then:

1. f^* is a right-continuous and non-increasing function.
2. f and f^* are equimeasurable i.e. for every Borel set $A \subset \mathbb{R}$ then $\mu(f^{-1}(A)) = |(f^*)^{-1}(A)|$, where $|B|$ denotes the Lebesgue measure of the Borel set $B \subset \mathbb{R}$. In turn, this implies that
$$\int_S f \ d\mu \leq \int_0^s f^*(t) \ dt \quad \text{for every} \quad S \in \mathcal{X}\ ,\ \mu(S) \leq s\,. \tag{26}$$
3. Using that f and f^* are equimeasurable we can deduce that for every continuous $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ then: $\varphi \circ f \in L^\infty(X,\mu)$ iff $\varphi \circ f^* \in L^\infty([0,1])$ and in this case
$$\int_X \varphi \circ f \ d\mu = \int_0^{\mu(X)} \varphi \circ f^* \ dx\ . \tag{27}$$
4. If $g \in L^\infty(X,\mu)$ is such that $f \leq g$ then $0 \leq f^* \leq g^*$; moreover, in case $f^* = g^*$ then $f = g$. $\triangle$

Recall that given $f, g \in L^\infty(X,\mu)^+$ we say that f *majorizes* g (in $(X, \mathcal{X}, \mu)$), denoted $g \prec f$, if

$$\int_0^s g^*(t) \ dt \leq \int_0^s f^*(t) \ dt \quad \text{for every} \quad 0 \leq s \leq \mu(X)$$

and $\int_0^{\mu(X)} g^*(t) \ dt = \int_0^{\mu(X)} f^*(t) \ dt$. To check that majorization holds between functions in probability spaces, we can consider the so-called *doubly stochastic maps*. Recall that a linear operator D acting on $L^\infty(X,\mu)$ is a doubly-stochastic map if D is unital, positive and trace preserving i.e. $D(1_X) = 1_X$, $D\big(L^\infty(X,\mu)^+\big) \subseteq L^\infty(X,\mu)^+$ and

$$\int_X D(f)(x) \ d\mu(x) = \int_X f(x) \ d\mu(x) \quad \text{for every } f \in L^\infty(X,\mu)\ .$$

It is worth pointing out that D is necessarily a contractive map in the $\|\cdot\|_\infty$-norm.

Our interest in majorization lies in its relation to integral inequalities involving convex functions. The following result summarizes this relation as well as the role of the doubly stochastic maps (see [20, 38]). Recall that $\text{Conv}(\mathbb{R}_+)$ and $\text{Conv}_s(\mathbb{R}_+)$ (see Eq. (8)) denote the sets of convex and strictly convex functions $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, respectively.

Theorem 4.2. Let $f,\, g \in L^\infty(X,\mu)^+$. Then the following conditions are equivalent:

1. $g \prec f$;
2. There is a doubly stochastic map D acting on $L^\infty(X, \mu)$ such that $D(f) = g$;
3. For every $\varphi \in \text{Conv}(\mathbb{R}_+)$ we have that

$$\int_X \varphi(g(x))\ d\mu(x) \leq \int_X \varphi(f(x))\ d\mu(x)\ . \tag{28}$$

Similarly, $g \prec_w f \Leftrightarrow$ Eq. (28) holds for every non-decreasing convex function φ. □

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