Algebras of reflections in $L^2(\mathbb{T})$

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Abstract

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. For $a \in \mathbb{D}$, consider $\varphi_a(z) = \dfrac{a - z}{1 - \bar{a}z}$ and C_a the composition operator in $L^2(\mathbb{T})$ induced by φ_a:

$$C_a f = f \circ \varphi_a.$$

Clearly C_a satisties $C_a^2 = I$, i.e., is a non-selfadjoint reflection. In this paper we study the operator algebras related to C_a: the C*-algebra generated by C_a, its commutant and its double commutant.

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1 Introduction

Let $\mathcal{H} = L^2(\mathbb{T})$ with normalized Lebesgue measure. For each $a \in \mathbb{D}$ let

$$\varphi_a(z) := \frac{a - z}{1 - \bar{a}z}.$$

Note that $\varphi_a(\varphi_a(z)) = z$ (for $z \neq \frac{a}{|a|^2}$) and $|\varphi_a(z)| = 1$ for $z \in \mathbb{T}$, and therefore the composition operator

$$C_a : \mathcal{H} \to \mathcal{H},\ C_a f = f \circ \varphi_a$$

is well defined, and is a *reflection* in $\mathcal{H}$: $C_a^2 = I$. A reflection C is a *symmetry* if additionally $C^* = C$. Clearly C_a is a symmetry only if $a = 0$. The adjoint of C_a is easily computed:

$$C_a^* = M_{\frac{1-|a|^2}{|1-\bar{a}z|^2}} C_a = C_a M_{\frac{|1-\bar{a}z|^2}{1-|a|^2}},$$

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where M_g denotes the multiplication operator (by g).

The purpose of this note is to study and characterize the C*-algebra $\mathcal{C}^*(C_a)$ generated by C_a, its commutant algebra $\{C_a, C_a^*\}'$ and the von Neumann algebra generated by C_a (all inside $\mathcal{B}(\mathcal{H})$).

We give an elementary description of $\mathcal{C}^*(C_a)$ in terms of C_a and multiplication operators: for instance, if $a = r \in (0,1)$, then

$$\mathcal{C}^*(C_r) = \{M_f + C_r M_g : f, g \in C(\mathbb{T}), f(\bar{z}) = f(z), g(\bar{z}) = g(z)\}.$$

For arbitrary $a = re^{i\theta}$ $\theta \in [-\pi, \pi)$, the description involves the rotation $U_\theta f(z) = f(e^{-\theta} z)$ (Theorem 3.1). This description facilitates the computation of $\mathcal{C}^*(C_a)''$: again for $a = r \in (0,1)$,

$$\mathcal{C}^*(C_r)'' = \{M_f + C_a M_g : f, g \in L^\infty(\mathbb{T}), f(\bar{z}) = f(z), g(\bar{z}) = g(z) \ (pp)\}.$$

This viewpoint allows also the explicit computation of:

- the faithful conditional expectation

$$E_a : \mathcal{C}^*(C_a) \to \{M_f : f \in C(\mathbb{T})\} \cap \mathcal{C}^*(C_a), \ E_a(M_f + CaM_g) = M_f;$$

- the center of $\mathcal{C}^*(C_a)$ and the center valued (tracial) conditional expectation F_a;
- the commutant $\mathcal{C}^*(C_a)' = \{C_a, C_a^*\}'$.

The algebra $\mathcal{C}^*(C_a)$ is also the C*-algebra generated by two projections, $P_{N(C_a - I)}$ and $P_{N(C_a + I)}$, associated to the eigenspaces of C_a. Therefore $\mathcal{C}^*(C_a)$ falls into the well studied class of C*-algebras generated by two projections (see for instance [14], or the survey [4] and the references therein). According to a result by Pedersen [14], one can characterize $\mathcal{C}^*(C_a)$ by means of the positive operator $P_{N(C_a - I)} P_{N(C_a + I)} P_{N(C_a - I)}$. We compute this operator and its spectra, equal to the interval $[0, |a|^2]$. To do these explicit computations, we need several elementary results on the relations between the spectra of $P - Q$, $P + Q$ and PQ, for P, Q orthogonal projections, as well as J. Dixmier's results [10] (see also [9] or [12]) characterizing the position of two subspaces.

The contents of the paper are the following. In Section 2 we transcibe basic facts and fix some notation. In Section 3 we give the elementary description of $\mathcal{C}^*(C_a)$ (Theorem 3.1). In Section 4 we state basic properties of pairs of projections P, Q. In Section 5 we present $\mathcal{C}^*(C_a)$ as the C*-algebra generated by two projections, and use Pedersen's Theorem [14], as presented by Böttcher and Spitkovsky (Theorem 4.1 in [4]), to describe $\mathcal{C}^*(C_a)$ (Corollary 5.3). In Section 6 we present the conditional expectation $E_a : \mathcal{C}^*(C_a) \to \{M_f : f \in C(\mathbb{T})\} \cap \mathcal{C}^*(C_a)$, compute the center $\mathcal{Z}_a$ of $\mathcal{C}^*(C_a)$, and a tracial center-valued trace F_a. In Section 7 we describe the von Neumann algebra $\mathcal{C}^*(C_a)''$ generated by C_a (Corollary 7.2). In Section 8 we study the commutant of $\mathcal{C}^*(C_a)$ inside $\mathcal{B}(\mathcal{H})$.

2 Preliminaries and notation

If $\mathcal{S}$ is a closed subspace of $\mathcal{H}$, $P_{\mathcal{S}}$ will denote the orthogonal projection onto $\mathcal{S}$. For a bounded operator T, $|T|$ will denote the modulus of T, i.e., $|T| = (T^*T)^{1/2}$; $N(T)$ and $R(T)$ will denote the nullspace and range of T, respectively.

Note that

$$C_a^*C_a = (1-|a|^2)M_{\frac{1}{|1-\bar{a}z|^2}} \text{ and } |C_a| = (C_a^*C_a)^{1/2} = (1-|a|^2)^{1/2}M_{\frac{1}{|1-\bar{a}z|}}. \tag{1}$$

In [7], Corach, Porta and Recht, made the following remarkable observation:

Remark 2.1. Let C be a reflection in $\mathcal{H}$, and let $C = R|C|$ be its polar decomposition. Then the unitary operator R is in fact a symmetry.

Let us denote by R_a the symmetry obtained in the polar decomposition of C_a. After routine computations one obtains that

$$R_a = C_a|C_a|^{-1} = \frac{1}{(1-|a|^2)^{1/2}}C_aM_{|1-\bar{a}z|} = (1-|a|^2)^{1/2}M_{\frac{1}{|1-\bar{a}z|}}C_a. \tag{2}$$

Note that if $C^2 = I$ and $C = R|C|$ is the polar decomposition, then $C^* = |C|R$, so that $RCR = C^*$. In particular, for $C = C_a$ we have

$$R_aC_aR_a = C_a^*. \tag{3}$$

There is another symmetry in $\mathcal{H}$ associated with φ_a, known in the literature (at least when acting in the (invariant) Hardy subspace $H^2(\mathbb{T}) \subset \mathcal{H}$, [8]):

$$W_a = (1-|a|^2)^{1/2}M_{\frac{1}{1-\bar{a}z}}C_a. \tag{4}$$

Note the resemblance between R_a in (2) and W_a in (4).

However, R_a belongs to $\mathcal{C}^*(C_a)$, but W_a does not (if $a \neq 0$): this would imply that $(1-|a|^2)^{1/2}W_aC_a = M_{\frac{1}{1-\bar{a}z}} \in \mathcal{C}^*(C_a)$, and we shall see below (Theorem 3.1) that for a continuous function f in $\mathbb{T}$, $M_f \in \mathcal{C}^*(C_a)$ only if $f(e^{i\theta}z) = f(e^{i\theta}\bar{z})$, which does not happen for $f(z) = \frac{1}{1-\bar{a}z}$. Neither does W_a belong to $\mathcal{C}^*(C_a)''$, by the same argument (see Corollary 7.2).

We shall say that an element $f \in \mathcal{H}$ is *even* if $f(z) = f(-z)$ *p.p.* in $\mathbb{T}$. Equivalently, if the Fourier coefficients $\hat{f}(n)$ of f are 0 for n odd. We shall denote by $\mathcal{E}$ the closed subspace of even elements. Similarly, $\mathcal{O}$ will denote the subspace of *odd* elements of $\mathcal{H}$ ($g \in \mathcal{H}$ such that $g(-z) = -g(z)$ *p.p.* in $\mathbb{T}$ or $\hat{g}(n) = 0$ for n even). Note that for $a = 0$, $C_0 = R_0$, and $N(C_0 - I) = \mathcal{E}$ and $N(C_0 + I) = \mathcal{O}$.

Remark 2.2. The map φ_a has a unique fixed point ω_a inside $\mathbb{D}$, namely

$$\omega_a = \frac{1}{\bar{a}}(1-\sqrt{1-|a|^2}). \tag{5}$$

The following identity is easy to verify:

$$\varphi_{\omega_a} \circ \varphi_a = -\varphi_{\omega_a}. \tag{6}$$

It is useful in the description of the two eigenspaces of C_a. In [3] (Theorem 3.2) it was shown that

$$N(C_a - I) = C_{\omega_a}\mathcal{E} \ \text{ and } \ N(C_a + I) = C_{\omega_a}\mathcal{O}. \tag{7}$$

we shall use the symmetry V in $\mathcal{H}$, given by

$$Vf(z) = f(\bar{z}), \ f \in \mathcal{H}. \tag{8}$$

It is clearly well defined, isometric and satisfies $V^2 = I$. Also it is easy to see that

$$VC_a = C_{\bar{a}}V. \tag{9}$$

3 The C*-algebra generated by C_a

In this section we present an elementary description of the C*-algebra $\mathcal{C}^*(C_a)$ generated by C_a.

First note that if $a = 0$, $\mathcal{C}^*(C_0)$ is the algebra generated by (the identity I and) the orthogonal projection $P_\mathcal{E} = \frac{1}{2}(C_0 + I)$ onto $\mathcal{E}$. It consists of the elements of the form $\alpha P_\mathcal{E} + \beta P_\mathcal{O}$. Its commutant consists of the operators which are diagonal wih respect to the decomposition $\mathcal{H} = \mathcal{E} \oplus \mathcal{O}$, i.e., the operators whose matrices in terms of this decomposition are of the form $\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}$.

For $\theta \in [-\pi, \pi)$, denote by U_θ the unitary operator $U_\theta f(z) = f(e^{-i\theta}z)$.

Note that if $a = re^{i\theta}$, an elementary computation shows that that $U_{-\theta}C_aU_\theta = C_r$.

Note also that if $\varphi \in L^\infty(\mathbb{T})$, then $U_{-\theta}M_\varphi U_\theta = M_{U_{-\theta}\varphi}$. Denote by $C(\mathbb{T})$ the algebra of continuous functions in $\mathbb{T}$.

Theorem 3.1. *Let $a \in \mathbb{D}$, $a = re^{i\theta}$ ($r \neq 0$). Then*

$$\mathcal{C}^*(C_a) = \{M_f + C_aM_g : f, g \in C(\mathbb{T}), f(e^{i\theta}z) = f(e^{i\theta}\bar{z}) \text{ and } g(e^{i\theta}z) = g(e^{i\theta}\bar{z})\}$$

$$= \{M_f + C_aM_g : f, g \in C(\mathbb{T}), U_{-\theta}f, U_{-\theta}g \in N(V - I)\}.$$

Moreover, this presentation $M_f + C_aM_g$ is unique.

Proof. Suppose first that $a = r \in (0, 1)$. Recall that $C_r^* = M_{\frac{1-r^2}{|1-rz|^2}}C_r = C_rM_{\frac{|1-rz|^2}{1-r^2}}$. Then

$$C_r^*C_r = M_{\frac{1-r^2}{|1-rz|^2}} \ \text{ and } \ C_rC_r^* = M_{\frac{|1-rz|^2}{1-r^2}}.$$

Since $C_r^2=(C_r^*)^2=I$, it follows that any polynomial expression in C_r, C_r^* can be rewritten as

$$M_p+C_rM_q, \quad \text{for } p,q \text{ polynomials in } |1-rz|^2 \text{ and } |1-rz|^{-2}.$$

Conversely, any such expression belongs to $\mathcal{C}^*(C_r)$. Therefore $\mathcal{C}^*(C_r)$ consists of elements of the form $M_f+M_gC_r$, for $f,g\in\mathcal{C}^*(|1-rz|^2)\subset C(\mathbb{T})$, where $\mathcal{C}^*(|1-rz|^2)$ denotes the C*-algebra generated by the continuous function $|1-rz|^2$. If one puts $z=e^{it}$, and identifies $\mathcal{H}=L^2(\mathbb{T})$ with $L^2(-\pi,\pi)$, $f(e^{it})\sim f(t)$, one has

$$|1-rz|^2\sim 1+r^2-2r\cos(t).$$

Then $\mathcal{C}^*(|1-rz|^2)$ identifies with the sub-C*-algebra of $C([-\pi,\pi])$ generated by the constant functions and $\cos(t)$. This is the subalgebra of continuous even functions in $[-\pi,\pi]$ (even in the parameter t, not to be confused with $\mathcal{E}$). Short proof: clearly $\mathcal{C}^*(\cos(t))$ is contained in the algebra of continuous even functions in $[-\pi,\pi]$; conversely, it is elementary that $\cos(kt)$, $k\in\mathbb{N}$, can be expressed as a polynomial in $\cos(t)$, therefore any trigonometric polynomial $\frac{a_0}{2}+\sum_{k=1}^n a_k\cos(kt)$ is a polynomial in $\cos(t)$, and these finite Fourier sums are uniformly dense in the set of continuous even functions. Thus $\mathcal{C}^*(C_r)$ consists of operators of the form $M_f+M_gC_r$ for f,g even and continuous.

Now, that $f(t)$ is even in the variable t, for the variable $z=e^{it}$, means that $f(z)$ satisfies that $f(\bar{z})=f(z)$.

Clearly, if $a=re^{i\theta}$,

$$\mathcal{C}^*(C_a)=U_\theta\mathcal{C}^*(C_r)U_{-\theta},$$

and consists of operators of the form

$$U_\theta\left(M_f+M_gC_r\right)U_{-\theta}=M_{U_\theta f}+M_{U_\theta g}C_a,$$

where $U_\theta f, U_\theta g$ are again continuous, with $f(z)=f(\bar{z})$, $g(z)=g(\bar{z})$. Denote $f_\theta=U_\theta f$. Then

$$f_\theta(w)=f(e^{-i\theta}w)=f(\overline{e^{-i\theta}w})=f(e^{i\theta}\bar{w})=f(e^{-i\theta}e^{2i\theta}\bar{w})=f_\theta(e^{2i\theta}\bar{w}),$$

or equivalently, putting $w=e^{i\theta}z$: $f_\theta(e^{i\theta}z)=f_\theta(e^{i\theta}\bar{z})$; and similarly for g.

Let us check now that the presentation of the elements of $\mathcal{C}^*(C_a)$ given above is unique. It suffices to consider the case $a=r\in(0,1)$. Also, it reduces to consider $M_f=C_rM_g$. Evaluating at $1\in\mathcal{H}$ we get $f(z)=g(\varphi_r(z))$, and evaluating at $z\in\mathcal{H}$ we get $zf(z)=\varphi_r(z)g(\varphi_r(z))$. Then $zg(\varphi_r(z))=\varphi_r(z)g(\varphi_r(z))$, which implies that $g=0$, and thus $f=0$. □

Remark 3.2. The above symmetry condition with respect to the angle θ, in the variable t reads:

$$f(\theta+t)=f(\theta-t), \text{ modulo } 2\pi.$$

Indeed, for $z = e^{it}$,

$$f(t+\theta) \sim f(e^{i\theta}e^{it}) = f(e^{i\theta}e^{-it}) \sim f(\theta - t).$$

Also, in terms of the symmetry V ($Vg(z) = g(\bar{z})$), this means

$$U_\theta f = VU_\theta f,$$

i.e., $f \in N(U_{-\theta}VU_\theta - I)$.

Remark 3.3. In general, the C*-algebra generated by a non selfadjoint idempotent (i.e., by a non unitary reflection) turns out to be the algebra generated by two othogonal projections (see [4]). Indeed, for Q an idempotent, note the formulas (see for instance T. Ando [1]),

$$P_{R(Q)} = Q(Q+Q^*-I)^{-1} \quad \text{and} \quad P_{N(Q)} = (I-Q)(I-Q-Q^*)^{-1}$$

and for E, F orthogonal projections such that $R(E)\dot{+}R(F) = \mathcal{H}$, one has that $E-F$ is invertible and

$$P_{R(E)\|R(F)} = E(E-F)^{-1},$$

(see [5] and [6]). Thus $\mathcal{C}^*(Q) = \mathcal{C}^*(I-Q) = \mathcal{C}^*(P_{R(Q)}, P_{N(Q)})$. In our case $Q = \frac{1}{2}(C_a + I)$ (indeed, note that $R(Q) = \{f \in \mathcal{H} : C_a f = f\} = N(C_a - I)$), we have

$$P_{N(C_a-I)} = \frac{1}{2}(C_a+I)\{\frac{1}{2}(C_a+I) + \frac{1}{2}(C_a^*+I) - I\}^{-1} = \frac{1}{4}(C_a+I)(C_a+C_a^*)^{-1}$$

$$= (C_a+I)\{C_a+M_{\frac{1-|a|^2}{|1-\bar{a}z|^2}}C_a\}^{-1} = (C_a+I)\{(I+M_{\frac{1-|a|^2}{|1-\bar{a}z|^2}})C_a\}^{-1} = (C_a+I)C_a\{I+M_{\frac{1-|a|^2}{|1-\bar{a}z|^2}}\}^{-1}.$$

Thus, using that $C_a^2 = I$,

$$P_{N(C_a-I)} = (I+C_a)M_{\psi_a}, \tag{10}$$

where $\psi_a(z) = (1 + \frac{1-|a|^2}{|1-\bar{a}z|^2})^{-1}$. Similarly

$$P_{N(C_a+I)} = (I-C_a)M_{\psi_a}. \tag{11}$$

There is a vast bibliography concerning this subject. See for instance the survey article [4] and the references therein, in particular, the paper [14] by G.K. Pedersen will be useful. Pedersen remarked that the key data to characterize the C*-algebra generated by two projections P and Q, is the spectrum $\sigma(PQP)$ of PQP. In order to compute the spectrum of $P_{N(C_a-1)}P_{N(C_a+1)}P_{N(C_a-1)}$, in the next section we shall present several basic facts on the theory of two subspaces, according to [10].

4 Basics on two projections

Definition 4.1. *Let P, Q be orthogonal projections in $\mathcal{H}$, they are said to be in generic position if*

$$R(P) \cap R(Q) = N(P) \cap N(Q) = R(P) \cap N(Q) = N(P) \cap R(Q) = \{0\}.$$

The main result in [10] states the following

Theorem 4.2. [10] *Let P, Q be orhogonal projections in $\mathcal{H}$, in generic position. Then there exists an isometric isomemorphism between $\mathcal{H}$ and a product Hilert space $\mathcal{L} \times \mathcal{L}$ and a positive operator X in $\mathcal{B}(\mathcal{L})$ with $\|X\| \leq \pi/2$ and $N(X) = \{0\}$, such that the projections P and Q are carried to*

$$P \simeq \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } Q \simeq \begin{pmatrix} C^2 & CS \\ CS & S^2 \end{pmatrix},$$

where $C = \cos(X)$ and $S = \sin(X)$.

The following elementary results will also be useful.

Lemma 4.3. *Let P and Q be projections in generic position. Then for $\lambda \neq 0$ we have that*

$$\lambda \in \sigma(PQP) \iff 1 \pm \lambda^{1/2} \in \sigma(P+Q).$$

Proof. We can reason with the operators in $\mathcal{L} \times \mathcal{L}$. Then

$$PQP \simeq \begin{pmatrix} C^2 & 0 \\ 0 & 0 \end{pmatrix},$$

and thus $\sigma(PQP) = \{\cos^2(t) : t \in \sigma(X)\} \cup \{0\}$. Also

$$P + Q \simeq \begin{pmatrix} 1 + C^2 & CS \\ CS & S^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} C^2 & CS \\ CS & -C^2 \end{pmatrix}.$$

Note that

$$\begin{pmatrix} C^2 & CS \\ CS & -C^2 \end{pmatrix} = \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix} \begin{pmatrix} C & S \\ S & -C \end{pmatrix}.$$

The right hand matrices commute, and the matrix $\Sigma = \begin{pmatrix} C & S \\ S & -C \end{pmatrix}$ is a symmetry: $\Sigma^* = \Sigma$ and $\Sigma^2 = 1$. It is well known that if a, b are elements of a C*-algebra such that $ab = ba$, then $\sigma(ab) \subset \{\lambda\mu : \lambda \in \sigma(a), \mu \in \sigma(b)\}$. Therefore, since $\sigma(\Sigma) = \{-1; 1\}$, we have that

$$\sigma\left(\begin{pmatrix} C^2 & CS \\ CS & -C^2 \end{pmatrix}\right) \subset \{\pm\cos(t) : t \in \sigma(X)\}.$$

Thus,

$$\sigma(P+Q) \subset \{1 \pm \cos(t) : t \in \sigma(X)\}.$$

Conversely, note that

$$\begin{pmatrix} C^2 & CS \\ CS & -C^2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} C & S \\ S & -C \end{pmatrix} \{ \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix} - \lambda \begin{pmatrix} C & S \\ S & -C \end{pmatrix} \}.$$

This product is invertible if and only if the right hand factor is invertible, which implies that its square is positive and invertible, and therefore the diagonal entries of this square are invertible, i.e. $(C \pm \lambda 1)^2$ are invertible. Therefore

$$\sigma(P+Q) = \sigma\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} C^2 & CS \\ CS & -C^2 \end{pmatrix} \right) = \{1 \pm \cos(t) : t \in \sigma(X)\}.$$

□

For eigenvalues:

Lemma 4.4. *Let P, Q be in generic position. If $\lambda \neq 0$ (i.e., $\lambda > 0$) is an eigenvalue of PQP, then if $1 \pm \lambda^{1/2}$ are eigenvalues of $P+Q$*

Proof. Suppose that $\lambda > 0$ and $f \in \mathcal{H}$ with $\|f\| = 1$ satisfy $PQPf = \lambda f$. Clearly $f \in R(P)$, and the subspace $\mathbb{V}$ generated by f and Qf is two dimensional and invariant both for P and Q. Indeed, the first assertion: if Qf were a multiple of f, then $Qf \in R(P) \cap R(Q) = \{0\}$; the second: $Pf = f, PQf = PQPf = \lambda f, Qf \in \mathbb{V}$. The matrices of P, Q and $P + Q\big|_{\mathbb{V}}$ in the basis $\{f, Qf\}$ of $\mathbb{V}$ are

$$\begin{pmatrix} 1 & \lambda \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & \lambda \\ 1 & 1 \end{pmatrix}.$$

The eigenvalues of $P + Q\big|_{\mathbb{V}}$ are $1 \pm \lambda^{1/2}$. □

In order to use these results, let us check that $P_{N(C_a-1)}$ and $P_{N(C_a+1)}$ are in generic position:

Proposition 4.5. *If a in $\mathbb{D}$, then $P_{N(C_a-1)}$ and $P_{N(C_a+1)}$ are in generic position.*

Proof. First note that $N(C_a - 1) \cap N(C_a + 1) = \{0\}$ and

$$N(C_a-1)^\perp \cap N(C_a+1)^\perp = \langle N(C_a-1); N(C_a+1) \rangle^\perp = \mathcal{H}^\perp = \{0\}.$$

That $N(C_a-1) \cap N(C_a+1)^\perp = \{0\} = N(C_a-1)^\perp \cap N(C_a+1)$ was proved in [3] (Theorem 6.3). □

Proposition 4.6. *Le $a \in \mathbb{D}$, $a \neq 0$. Then*

$$\sigma(P_{N(C_a-1)}P_{N(C_a+1)}P_{N(C_a-1)}) = [0, |a|^2],$$

with 0 *the only eigenvalue.*

Proof. That 0 is an eigenvalue is clear. Let us see that it is the only eigenvalue, and compute the rest of the spectrum. We consider the spectrum of $P_{N(C_a-1)} + P_{N(C_a+1)}$ instead. Recall from (3.3) that

$$P_{N(C_a-I)} = (I + C_a)M_{\psi_a} \ , \text{ and } \ P_{N(C_a+I)} = (I - C_a)M_{\psi_a}$$

where $\psi_a(z) = \left(1 + \frac{1-|a|^2}{|1-\bar{a}z|^2}\right)^{-1}$. Then

$$P_{N(C_a-I)} + P_{N(C_a+I)} = 2M_{\psi_a}.$$

This operator has no eigenvalues, which implies that $P_{N(C_a-I)}P_{N(C_a+I)}P_{N(C_a-I)}$ has no non nil eigenvalues (Lemma 4.4).

Morever, the spectrum of this operator is the image of the continuous map $2\psi_a(z)$ ($z \in \mathbb{T}$). It is an elementary computation that this set equal $[1-|a|, 1+|a|]$. Using now that $\lambda > 0$ lies in $\sigma(P_{N(C_a-I)}P_{N(C_a+I)}P_{N(C_a-I)})$ if and only if $1 \pm \lambda^{1/2}$ lies in $\sigma(P_{N(C_a-I)} + P_{N(C_a+I)})$ (Lemma 4.3), we obtain that

$$\sigma(P_{N(C_a-I)}P_{N(C_a+I)}P_{N(C_a-I)}) = [0, |a|^2].$$

□

Remark 4.7. In particular,

$$\|P_{N(C_a-I)}P_{N(C_a+I)}\| = \|P_{N(C_a-I)}P_{N(C_a+I)}P_{N(C_a-I)}\|^{1/2} = |a|.$$

5 Pedersen's presentation

Recall that using Dixmier's theory [10] (transcribed here as Theorem 4.2) for the projections $P_{N(C_a-I)}$ and $P_{N(C_a+I)}$, we have that $\mathcal{H} \simeq \mathcal{L} \times \mathcal{L}$ and there exists a positive operator $X \in \mathcal{B}(\mathcal{L})$, $X \leq \pi/2$, $N(X) = \{0\}$, such that the isomorphism that carries $\mathcal{H}$ onto $\mathcal{L} \times \mathcal{L}$ maps the projections $P_{N(C_a-I)}$ and $P_{N(C_a+I)}$ onto

$$P_{N(C_a-I)} \simeq \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad P_{N(C_a+I)} \simeq \begin{pmatrix} C^2 & CS \\ CS & S^2 \end{pmatrix},$$

where $C = \cos(X)$ and $S = \sin(X)$. The algebra $\mathcal{C}^*(C_a)$ can be described in terms of $H = \sin(X)^2$. Since $\sigma(\cos^2(X)) = \sigma(C^2) = \sigma(P_{N(C_a-I)}P_{N(C_a+I)}P_{N(C_a-I)}) = [0, |a|^2]$, we have that $\sigma(X) = [\arccos(|a|), \pi/2]$. It follows that $\sigma(H) = [1 - |a|^2, 1]$. Then rephrasing Theorem 4.1 in [4], we have:

Theorem 5.1. *Let $a \in \mathbb{D}$, $a \neq 0$. Then $\mathcal{C}^*(C_a)$ is $*$-isomorphic to*

$$\{\begin{pmatrix} f_{00}(H) & f_{01}(H) \\ f_{10}(H) & f_{11}(H) \end{pmatrix} : f_{ij} \in C(1-|a|^2, 1), f_{01}(1) = f_{10}(1) = 0\}.$$

□

This result shows the need to focus on the operator $H = \sin^2(X) = S^2$. Recall that

$$P_{N(C_a-I)} + P_{N(C_a+I)} - I = M_{2\psi_a - 1} \simeq \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix} \begin{pmatrix} C & S \\ S & -C \end{pmatrix}.$$

Theses matrices commute, so that, after elementary computations

$$M^2_{2\psi_a-1} = M_{(2\psi_a-1)^2} \simeq \begin{pmatrix} C^2 & 0 \\ 0 & C^2 \end{pmatrix} = I - \begin{pmatrix} S^2 & 0 \\ 0 & S^2 \end{pmatrix}. \tag{12}$$

Therefore we have

Proposition 5.2. *The operator H, acting in $L^2(\mathbb{T})$, equals the restriction of the multiplication operator wih symbol $\nu_a = 1 - (2\psi_a - 1)^2$, restricted to the inavriant subspace $N(C_a - I) = C_{\omega_a}(\mathcal{E})$:*

$$H \simeq M_{\nu_a}\Big|_{N(C_a-I)} \in \mathcal{B}(N(C_a - I)).$$

Proof. As seen above, $H = \sin(X)^2$ is unitarily equivalent to the restriction of $M_{1-(2\psi_a-1)^2}$ to the invariant subspace which corresponds to the first of the subspaces in the decomposition which induces the matrix representation above, i.e., $N(C_a - I)$. □

Therefore, we may specify Theorem 5.1:

Corollary 5.3. *Let $a \in \mathbb{D}$, $a \neq 0$. Then $\mathcal{C}^*(C_a)$ is $*$-isomorphic to*

$$\left\{ \begin{pmatrix} M_{f_{00}(\nu_a)}\Big|_{N(C_a-I)} & M_{f_{01}(\nu_a)}\Big|_{N(C_a-I)} \\ & \\ M_{f_{10}(\nu_a)}\Big|_{N(C_a-I)} & M_{f_{11}(\nu_a)}\Big|_{N(C_a-I)} \end{pmatrix} : f_{ij} \in C(1-|a|^2, 1), f_{01}(1) = f_{10}(1) = 0 \right\}.$$

Proof. Clearly, since $H \simeq M_{\nu_a}\Big|_{N(C_a-I)}$, it follows that

$$f(H) \simeq f(M_{\nu_a}\Big|_{N(C_a-I)}) = f(M_{\nu_a})\Big|_{N(C_a-I)},$$

for any continuous function in the spectrum of H (equal to $[1-|a|^2, 1]$). □

For $a = r \in (0,1)$, the function $\nu_r = 1 - (2\psi_r - 1)^2$ equals

$$\nu_r = -\frac{4}{r}(1-r^2)\frac{z(z^2 - z(r+\frac{1}{r}) + 1)}{(z-\omega_r)^2(z-\alpha_r)^2} = \frac{4}{r^2}(1-r^2)\frac{z(z-r)(1-rz)}{(z-\omega_r)^2(z-\alpha_r)^2}, \tag{13}$$

where $\alpha_r = \frac{1}{r}(1+\sqrt{1-r^2})$ is the (other than ω_r) fixed point of φ_r, the one outside $\mathbb{D}$.

If $a = re^{i\theta}$, then $\nu_a = U_\theta \nu_r$, i.e.,

$$\nu_a = \frac{4}{|a|^2}(1-|a|^2)\frac{e^{-\theta}z(e^{-\theta}z - r)(1 - re^{-\theta}z)}{(e^{-\theta}z-\omega_r)^2(e^{-\theta}z-\alpha_r)^2} = \frac{4}{|a|^2}(1-|a|^2)e^{2\theta}z\frac{(z-a)(1-\bar{a}z)}{(z-e^{i\theta}\omega_r)^2(z-e^{i\theta}\alpha_r)^2}$$

$$= \frac{4}{(\bar{a})^2}(1-|a|^2)z\frac{(z-a)(1-\bar{a}z)}{(z-\omega_a)^2(z-\alpha_a)^2},$$

where, analogously, α_a is the fixed point of φ_a with $|\alpha_a| > 1$.

Remark 5.4. At a first glance, it is not obvious (though it is a consequence of the facts above) that M_{ν_a} should leave $N(C_a - 1)$ invariant. Suppose for simplicity that $a = r \in (0,1)$. Then a straightforward computation shows that

$$C_r(\nu_r) = \nu_r.$$

Since $\{z^{2k} : k \in \mathbb{Z}\}$ is an orthonormal basis for $\mathcal{E}$, then $\{\varphi_{\omega_r}(z^{2k}) = \left(\frac{\omega_r - z}{1-\omega_r z}\right)^{2k} : k \in \mathbb{Z}\}$ is a (Schauder) basis for $C_{\omega_r}(\mathcal{E}) = N(C_r - I)$ (recall (7)). Then, using the identity (6): $\varphi_{\omega_r} \circ \varphi_r = -\varphi_{\omega_r}$, we have

$$C_r C_{\omega_r} = C_{\varphi_{\omega_r} \circ \varphi_r} = C_{-\varphi_{\omega_r}},$$

and then

$$C_r(\nu_r \varphi_{\omega_r}(z^{2k})) = C_r(\nu_r)C_r(C_{\omega_r}(z^{2k})) = \nu_r C_{-\varphi_{\omega_r}}(z^{2k})) = \nu_r(-\varphi_{\omega_r}(z))^{2k} = \nu_r \varphi_{\omega_r}(z^{2k}),$$

i.e., $C_r(N(C_r - I)) = N(C_r - I)$.

The same computation using odd powers, which generate $N(C_r + 1)$), shows that

$$M_{\nu_r} N(C_r + I) = N(C_r + I),$$

a fact which does not follow from the previous results. In other words, M_{ν_r} is diagonal with respect to the (non orthogonal) decomposition $\mathcal{H} = N(C_r - I) \dot{+} N(C_r + I)$. For arbitrary $a \in \mathbb{D}$, using the unitary U_θ for $a = re^{i\theta}$, the fact remains, that M_{ν_a} is diagonal for the decomposition $\mathcal{H} = N(C_a - I) \dot{+} N(C_a + I)$.

Using Theorem 5.1 we have that

Proposition 5.5. *Let $a, b \in \mathbb{D} \setminus \{0\}$. Then $\mathcal{C}^*(C_a) \simeq \mathcal{C}^*(C_b)$.*

Proof. Theorem 5.1 states that $\mathcal{C}^*(C_a)$ consists of 2×2 matrices with entries in $\mathcal{C}^*(H_a)$, where H_a is the operator corresponding to the Dixmier representation of $P_{N(C_a+I)}$; the off-diagonal entries must belong to the ideal $\{g(H_a) : g \in C([1-|a|,1), g(1) = 0\}$. Denote by H_b the operator corresponding to the representation of $\mathcal{C}^*(C_b)$. It is clear that the C*-algebras $\mathcal{C}^*(H_a)$ and $\mathcal{C}^*(H_b)$ are $*$-isomorphic: by the Gelfand-Naimark theorem, the former is isomorphic to $C([1-|a|,1])$ and the latter to $C([1-|b|,1]$. Moreover, the Gelfand isomorphisms carry $f(H_a)$ and $g(H_b)$ onto the functions $f \in C([1-|a|,1])$ and $g \in C([1-|b|,1])$, respectively. Pick a homeomorphism $h : [1-|b|,1] \to [1-|a|,1]$ such that $h(1) = 1$. Then the transpose map $\tau_h : C([1-|a|,1]) \to C([1-|b|,1])$, $\tau_h f = f \circ h$ is a $*$-isomorphism which carries the ideal $\{f \in C([1-|a|,1) : f(1) = 0\}$ onto the ideal $\{g \in C([1-|b|,1) : g(1) = 0\}$. Therefore, composing with the respective Gelfand maps, we obtain a $*$-isomorphism Φ,

$$\Phi : \mathcal{C}^*(H_a) \to \mathcal{C}^*(H_b)$$

which carries the ideal $\{f(H_a) : f \in C([1-|a|,1]), f(1) = 0\}$ onto $\{g(H_a) : g \in C([1-|b|,1]), g(1) = 0\}$. This isomorphism Φ clearly induces a $*$-isomorphism between the matrix algebras, and therefore between $\mathcal{C}^*(C_a)$ and $\mathcal{C}^*(C_b)$. □

Note also that:

Proposition 5.6. *Let $a, b \in \mathbb{D}$. Then $\mathcal{C}^*(C_a)$ is $*$-isomorphic to $\mathcal{C}^*(C_b)$ with an isomorphism that carries C_a to C_b if and only if $|a| = |b|$.*

Proof. Suppose that $|a| = |b|$. Then $b = ae^{i\theta}$ for some $\theta \in [-\pi, \pi)$. Then, it is clear that $C_b = U_\theta C_a U_\theta$, and therefore also $C_b^* = U_\theta C_a^* U_\theta$. Thus, the inner automorphism implemented by U_θ provides a $*$-isomorphism between $\mathcal{C}^*(C_a)$ and $\mathcal{C}^*(C_b)$.

Suppose that there exists a $*$-isomorphism $\Phi : \mathcal{C}^*(C_a) \to \mathcal{C}^*(C_b)$ such that $\Phi(C_a) = C_b$. Since

$$P_{N(C_a-I)} = (C_a + I)(C_a + C_a^*)^{-1} \quad \text{and} \quad P_{N(C_a+I)} = (I - C_a)(C_a + C_a^*)^{-1},$$

it follows that $\Phi(P_{N(C_a-I)}) = P_{N(C_b-I)}$ and $\Phi(P_{N(C_a+I)}) = P_{N(C_b+I)}$. Then, using Remark 4.7, it follows that $|a| = |b|$. □

6 A center valued conditional expectation

The uniqueness of the presentation of the elements of $\mathcal{C}^*(C_a)$ in Theorem 3.1, enables several computations.

First, the following conditional expectation

$$E_a : \mathcal{C}^*(C_a) \to \{M_f : f \in C(\mathbb{T}), U_{-\theta} f \in N(V - I)\} \subset \mathcal{C}^*(C_a),$$

$$E_a(M_f + C_a M_g) := M_f. \tag{14}$$

Proposition 6.1. *The map E_a is a faithful conditional expectation.*

Proof. Since E_a is well defined, it is linear. Also it is clear that it is the identity when restricted to its range. Consider first the case $a = r \in (0,1)$. Recall that $M_h C_r = C_r M_{C_r h}$. Put $A = M_f + C_r M_g \in \mathcal{C}^*(C_r)$. Then

$$M_h A M_k = M_h(M_f + C_r M_g)M_k = M_{hfk} + M_h C_r M_{gk} = M_{hfk} + C_r M_{gkC_r h},$$

and thus

$$E_r(M_h A M_k) = M_{hfk} = M_h E_r(A) M_k.$$

Similarly, after elementary computations

$$E_r(A^*A) = E\left((M_{\bar f} + M_{\bar g}C_r^*)(M_f + C_r M_g)\right) = M_{|f|^2} + M_{\bar g}C_r^* C_r M_g = M_{|f|^2+|g|^2\frac{1-r^2}{|1-rz1^2}} \geq 0.$$

This computation shows that E_r is positive and faithful. For arbitrary $a = re^{i\theta}$, we have that $E_a(A) = U_\theta E_r(U_{-\theta} A U_\theta)U_{-\theta}$: if $A = M_f + C_a M_g$, denoting, for an invertible operator $G \in \mathcal{B}(\mathcal{H})$, $Ad_G : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$, $Ad_G(T) = GTG^{-1}$,

$$Ad_{U_\theta} E_r Ad_{U_{-\theta}}(A) = U_\theta E_r(U_{-\theta}(M_f + C_a M_g)U_\theta)U_{-\theta}$$

$$= U_\theta E_r(U_{-\theta} M_f U_\theta + U_{-\theta} C_r U_\theta U_{-\theta} M_g U_\theta)U_{-\theta} = U_\theta E_r(M_{U_\theta f} + C_r M_{U_\theta g})U_{-\theta} = U_\theta(M_{U_\theta f})U_{-\theta}$$

$$= M_f = E_a(A).$$

Therefore E_a is also a faithful conditional expectation. □

Next, we can compute the *center* $\mathcal{Z}_a$ of $\mathcal{C}^*(C_a)$.

Proposition 6.2. *The center $\mathcal{Z}_a$ of $\mathcal{C}^*(C_a)$ is*

$$\mathcal{Z}_a = \{M_f \in \mathcal{B}(\mathcal{H}) : f \in N(C_a - I) \cap C(\mathbb{T}), U_{-\theta} f \in N(V - I)\}.$$

Proof. Consider first the case $a = r \in (0,1)$. Suppose that $A = M_f + C_r M_g$ commutes with C_r and with M_h for any $h(z) = h(\bar z)$ in $C(\mathbb{T})$. Then

$$A M_h = M_{hf} + C_r M_{gh} = M_h A = M_{fh} + C_r M_{gC_r h},$$

for all h as above. Then $g(z)h(z) = g(z)h(\varphi_r(z))$ *pp*. This implies that $g = 0$. Therefore M_f commutes with C_r:

$$C_r M_f = M_f C_r = C_r M_{C_r f}, \text{ i.e., } f = C_r f.$$

That is, $f \in N(C_r - I)$. Clearly $f \in N(V - I)$.

For arbitrary $a = re^{i\theta} \neq 0$, the center $\mathcal{Z}_a$ of $\mathcal{C}^*(C_a) = U_\theta \mathcal{C}^*(C_r) U_{-\theta}$ consists of operators of the form $U_\theta M_k U_{-\theta} = M_{U_\theta k}$ with k in $\mathcal{Z}_r$, i.e., $k(\bar z) = k(z)$ and $C_r k = k$. Thus

$$U_{-\theta} U_\theta k = k \in N(V - I) \text{ and } C_a U_{-\theta} f = U_{-\theta} C_r U_\theta U_{-\theta} k = U_{-\theta} C_r k = U_{-\theta} k.$$

Therefore

$$\mathcal{Z}_a = \{M_f \in \mathcal{B}(\mathcal{H}) : f \in N(C_a - I) \cap C(\mathbb{T}), U_\theta f \in N(V - I)\}.$$

$\square$

Consider $r \in (0,1)$. One can obtain explicitly a conditional expectation

$$\Omega_r : \{f \in C(\mathbb{T}) : f(z) = f(\bar{z})\} \to \{f \in C(\mathbb{T}) : f(z) = f(\bar{z})\} \cap N(C_r - I).$$

Namely, $\Omega_r(f) = \frac{1}{2}(f + C_r f)$ (i.e., the restriction of the idempotent $\frac{1}{2}(C_r + I)$ to the algebra $\{f \in C(\mathbb{T}) : f(z) = f(\bar{z})\}$). Clearly it is well defined: it preserves continuous functions, as well as the condition $f(\bar{z}) = f(z)$ due to $C_r V = V C_r$, and its range lies inside $N(C_r - I)$. Also it is a projection onto its range. Note that if $k \in \{f \in C(\mathbb{T}) : f(z) = f(\bar{z})\} \cap N(C_r - I)$, then

$$\Omega_r(kf) = \frac{1}{2}(C_r(kf) + kf) = \frac{1}{2}(Cr(k)Cr(f) + kf) = \frac{1}{2}(kC_r f + kf) = k\Omega_r(f),$$

because $C_r(fg) = C_r(f)C_r(g)$ whenever f, g and $fg \in \mathcal{H}$, as is the case here. Finally, though C_r is not selfadjoint, it preserves the positive structure of $C(\mathbb{T})$: if $f(z) \geq 0$ for all $z \in \mathbb{T}$, then $C_r f(z) = f(\varphi_r(z)) \geq 0$ as well. it follows that if $f \geq 0$, then $\Omega_r(f) \geq 0$. Also, it is clear that if $f \geq 0$ and $\Omega_r(f) = 0$, then $f = 0$, i.e., Ω_r is faithful.

We can construct a center-valued faithful conditional expectation

$$F_a : \mathcal{C}^*(C_a) \to Z_a \subset \mathcal{C}^*(C_a).$$

Consider first $a = r \in (0,1)$. Then the map $\mathbf{\Omega}_r : M_f \mapsto M_{\Omega_r f}$ provides a faithful conditional expectation from $\{M_f : f \in C(\mathbb{T}), f(\bar{z}) = f(z)\}$ onto $\mathcal{Z}_r$, which can be composed with E_r above,

$$F_r := \mathbf{\Omega}_r E_r : \mathcal{C}^*(C_r) \to \mathcal{Z}_r. \tag{15}$$

Lemma 6.3. *F_r is tracial, i.e., $F_r(AB) = F_r(BA)$, for $A, B \in \mathcal{C}^*(C_r)$.*

Proof. Put $A = M_f + C_r M_g$ and $B = M_h + C_r M_k$ for $f, g, k, h \in C(\mathbb{T}) \cap N(V - I)$. Then (using the multiplication rules above)

$$AB = M_{fh + C_r(g)k} + C_r M_{gh + C_r(f)h} \text{ and } BA = M_{fh + C_r(k)g} + C_r M_{fk + C_r(h)g}.$$

Then

$$E_r(AB) = M_{fh + C_r(g)k} \text{ and } E_r(BA) = M_{fh + C_r(k)g}.$$

Note that (using that C_r is multiplicative for these functions)

$$\Omega_r(fh + C_r(g)k) = \frac{1}{2}(C_r(fh + C_r(g)k) + fh + C_r(g)k)$$

$$= \frac{1}{2}(C_r(f)C_r(h) + fh + gC_r(k) + C_r(g)k),$$

and

$$\Omega_r(fh + C_r(k)g) = \frac{1}{2}(C_r(fh + C_r(k)h) + fh + C_r(k)h)$$
$$= \frac{1}{2}(C_r(f)C_r(h) + fh + kC_r(h) + C_r(k)h),$$

i.e,

$$F_r(AB) = \mathbf{\Omega}_r E_r(AB) = \mathbf{\Omega}_r E_r(BA) = F_r(BA).$$

□

For arbitrary $a = re^{i\theta} \neq 0$, we put

$$F_a(A) = U_\theta F_r(U_{-\theta} A U_\theta) U_{-\theta}, \tag{16}$$

which provides a center valued faithful and tracial conditional expectation for $\mathcal{C}^*(C_a)$ (note that it is a composition of positive faithful maps). Extending the previous maps for arbitrary $a \in \mathbb{D}$, put

$$\Omega_a : C(\mathbb{T}) \to C(\mathbb{T}),\ \Omega_a f = \frac{1}{2}(f + C_a f),$$

and

$$\mathbf{\Omega}_a : \{M_f : f \in C(\mathbb{T})\} \to \{M_f : f \in C(\mathbb{T})\},\ \mathbf{\Omega}_a M_f = M_{\Omega_a f}.$$

Then we have

Corollary 6.4. *The center valued faithful and tracial conditional expectation* $F_a : \mathcal{C}^*(C_a) \to \mathcal{Z}_a \subset \mathcal{C}^*(C_a)$*, is given by*

$$F_a = \mathbf{\Omega}_a E_a.$$

Proof. If $a = re^{i\theta}$, we have that $F_a = Ad_{U_\theta} F_r Ad_{U_{-\theta}}$. Then, using that $E_a = Ad_{U_\theta} E_r Ad_{U_{-\theta}}$,

$$F_a = Ad_{U_\theta} \mathbf{\Omega_r} E_r Ad_{U_{-\theta}} = Ad_{U_\theta} \mathbf{\Omega_r} Ad_{U_{-\theta}} E_a.$$

Note that $U_\theta M_f U_{-\theta} = M_{U_\theta f}$. Then

$$Ad_{U_\theta} \mathbf{\Omega}_r Ad_{U_{-\theta}}(M_f) = Ad_{U_\theta} \mathbf{\Omega}_r (M_{U_{-\theta} f}) = Ad_{U_\theta}(\frac{1}{2} M_{U_{-\theta} f + C_r U_{-\theta} f})$$

$$= \frac{1}{2} M_{U_\theta (U_{-\theta} f + C_r U_{-\theta} f)} = \frac{1}{2} M_{f + U_\theta C_r U_{-\theta} f} = \frac{1}{2} M_{f + C_a f} = \mathbf{\Omega}_a(M_f),$$

and the proof follows. □

7 The von Neumann algebra generated by C_a

Let us describe now the weak closure $\mathcal{C}^*(C_a)''$ of $\mathcal{C}^*(C_a)$ in $\mathcal{B}(\mathcal{H})$. To this effect the following fact will be useful:

Lemma 7.1. *Let $r \in (0,1)$ and $f_n, g_n \in C(\mathbb{T})$ such that $\|M_{f_n} + C_r M_{g_n}\| \le 1$ and $M_{f_n} + C_r M_{g_n} \to A$ in the strong operator topology. Then there exist $f_0, g_0 \in L^\infty(\mathbb{T})$ such that $A = M_{f_0} + C_r M_{g_0}$.*

Proof. We know that $f_n h + C_r(g_n h) = f_n h + g_n(\varphi_r) h(\varphi_r) \to Ah$ in $\mathcal{H}$ for all $h \in \mathcal{H}$. Put $h = 1$ and $h = z$, and we get

$$f_n + g_n(\varphi_r) \to A1 \quad \text{and} \quad f_n z + g_n(\varphi_r)\varphi_r \to Az.$$

Applying M_z to the left hand term we get $f_n z + g_n(\varphi_r) z \to zA1$, which combined with the right hand term yields that $g_n(\varphi_r)(\varphi_r - z)$ is convergent in $\mathcal{H}$. Note that the continuous function $\varphi_r - z$ does not vanish in $\mathbb{T}$, and therefore the operator $M_{\varphi_r - z}$ is invertible in $\mathcal{H}$. It follows that $g_n(\varphi_r) = C_r g_n$ and g_n are convergent in $\mathcal{H}$, say $g_n \to g_0$. Then also f_n is convergent, $f_n \to f_0$. These elements f_0, g_0 belong to $\mathcal{H} = L^2(\mathbb{T})$, let us show that they belong to $L^\infty(\mathbb{T})$. Note that

$$E_r\left((M_{f_n} + C_r M_{g_n})^*(M_{f_n} + C_r M_{g_n})\right) = M_{|f_n|^2 + |g_n|^2 \frac{1-r^2}{|1-rz|^2}},$$

and thus

$$\sup_{z\in\mathbb{T}} |f_n|^2 + |g_n|^2 \frac{1-r^2}{|1-rz|^2} = \|E_r\left((M_{f_n} + C_r M_{g_n})^*(M_{f_n} + C_r M_{g_n})\right)\| \le \|M_{f_n} + C_r M_{g_n}\|^2 \le 1.$$

Thus there exists a constant C such that $|f_n|, |g_n| \le C$ for all $n \ge 1$. Since $f_n \to f_0$ in $L^2(\mathbb{T})$, there exists a subsequence which converges almost everywhere, and then $|f_0| \le C$ *pp*, and similarly for g_0. It follows that $M_{f_0} + C_r M_{g_0} \in \mathcal{B}(\mathcal{H})$ and $M_{f_n} + C_r M_{g_n} \to M_{f_0} + C_r M_{g_0}$ in the strong operator topology, i.e., $A = M_{f_0} + C_r M_{g_0}$. ☐

Corollary 7.2. *Let $a \neq 0$ in $\mathbb{D}$. Then*

$$\mathcal{C}^*(C_a)'' = \{M_f + C_a M_g : f, g \in L^\infty(\mathbb{T}) : U_{-\theta} f, U_{-\theta} g \in N(V - I)\}.$$

The center of this algebra is

$$\mathcal{Z}(\mathcal{C}^*(C_a)'') = \{M_f \in \mathcal{B}(\mathcal{H}) : f \in N(C_a - I) \cap L^\infty(\mathbb{T}), U_{-\theta} f \in N(V - I)\}.$$

The conditional expectation F_a of Corollary 6.4 extends to the (unique) normal faithful center valued conditional, which is also tracial. It follows that $\mathcal{C}^*(C_a)''$ is a finite von Neumann algebra.

Proof. The first two assertions are clear consequences of the results in the previous section and Lemma 7.1. Only the third assertion needs proof, and it suffices to reason in the case $a = r \in (0, 1)$. From Lemma 7.1 we know that if $M_{f_n} + C_r M_{g_n} \to M_{f_0} + C_r M_{g_0}$ in the strong operator topology, for a bounded sequence $M_{f_n} + C_r M_{g_n}$ with $f_n, g_n \in C(\mathbb{T})$, then $f_n \to f_0$ in $\mathcal{H}$, and these functions are uniformly (essentially) bounded. It follows that $M_{f_n} \to M_{f_0}$ in the strong operator topology, i.e., the expectation E_r extends to a normal expectation in the weak operator closure of $\mathcal{C}^*(C_a)$. The map Ω_r is also weak operator continuous: if $f_n \to f_0$ s.o.t and $|f_n| \leq C$ for all n, then $C_r f_n \to C_r f_0$ in $\mathcal{H}$, and also $|C_r f_n(z)| = |f_n(\varphi_r(z))| \leq C$ *pp.* $\square$

8 The commutant of $\{C_a, C_a^*\}$.

We focus now on the commutants of these algebras.

Theorem 8.1. *Let $a \in \mathbb{D}$, $a = re^{i\theta}$, $r \neq 0$. Denote by $T_a := C_{\omega_a} T C_{\omega_a}$. Then*

$$T \in \{C_a, C_a^*\}' \iff \begin{cases} \bullet \ T_a \text{ is diagonal with respect to } \mathcal{H} = \mathcal{E} \oplus \mathcal{O} \\ \bullet \ T_a \text{ commutes with } U_\theta (S + S^*) U_{-\theta}, \end{cases}$$

where $S = M_z$ is the bilateral shift.

Proof. We treat first the case $a = r \in (0, 1)$. If T commutes with C_r, then for $f \in N(C_r - I)$ and $g \in N(C_r + I)$ it follows that $Tf \in N(C_r - I)$ and $Tg \in N(C_r + I)$. Since $N(C_r - I) = C_{\omega_r}\mathcal{E}$, it follows that $C_{\omega_r} T C_{\omega_r}$ maps $\mathcal{E}$ into $\mathcal{E}$: if $h \in \mathcal{E}$, $C_{\omega_r} h \in N(C_r - I)$ then

$$C_{\omega_r} T C_{\omega_r} h \in C_{\omega_r} T(N(C_r - I)) \subset C_{\omega_r}(N(C_r - I)) = \mathcal{E},$$

and similarly $C_{\omega_r} T C_{\omega_r} \mathcal{O} \subset \mathcal{O}$. The fact that T commutes with C_r and C_r^*, implies that T commutes with $M_{|1-rz|^2}$. Then $T_r = C_{\omega_r} T C_{\omega_r}$ commutes with $C_{\omega_r} M_{|1-rz|^2} C_{\omega_r} = \sqrt{1-r^2} M_{\frac{|1+\omega_r z|^2}{|1-\omega_r z|^2}}$. The function $\frac{|1+\omega_r z|^2}{|1-\omega_r z|^2}$ in $L^2(\mathbb{T})$ identifies with $c(t) = \frac{1+\omega_r^2+2\omega_r\cos(t)}{1+\omega_r^2-2\omega_r\cos(t)}$ in $L^2(-\pi, \pi)$. Then T_r commutes with any element in the continuous functional calculus of this function $c(t)$, for instance, with $f(c(t)) = \cos(t)$, for $f(s) = \frac{1+\omega_r^2}{2\omega_r}\frac{s-1}{s+1}$. The function $\cos(t)$ is $Re(z) = \frac{1}{2}(z + \bar{z}) = \frac{1}{2}(z + \frac{1}{z})$ for $z = e^{it}$. Thus T_r is diagonal with respect to the decomposition $\mathcal{H} = \mathcal{E} \oplus \mathcal{O}$ and commutes with $M_{Re(z)} = \frac{1}{2}(S + S^*)$.

Suppose now that $a = re^{i\theta}$. Then T commutes with C_a, C_a^* if and only if $U_{-\theta} T U_\theta$ commutes with C_r, C_r^*. This is in turn equivalent to the facts that $C_{\omega_r} U_{-\theta} T U_\theta C_{\omega_r}$ is diagonal in the decomposition $\mathcal{H} = \mathcal{E} \oplus \mathcal{O}$ and commutes with $M_{Re(z)}$. Note that

$$U_\theta C_{\omega_r} f(z) = f\left(\frac{\omega_r - e^{-i\theta} z}{1 - \omega_r e^{-i\theta} z}\right) = f\left(e^{-i\theta}\frac{e^{i\theta}\omega_r - z}{1 - \overline{\omega_r e^{i\theta}} z}\right),$$

and that (using that $r = |a|$)

$$e^{i\theta}\omega_r = \frac{e^{i\theta}}{r}(1 - \sqrt{1-r^2}) = \frac{1}{\bar{a}}(1 - \sqrt{1-|a|^2}) = \omega_a.$$

Thus, $U_\theta C_{\omega_r} = C_{\omega_a} U_\theta$ and

$$C_{\omega_r} U_{-\theta} = (U_\theta C_{\omega_r})^{-1} = (C_{\omega_a} U_\theta)^{-1} = U_{-\theta} C_{\omega_a}.$$

Therefore

$$C_{\omega_r} U_{-\theta} T U_\theta C_{\omega_r} = U_{-\theta} C_{\omega_a} T C_{\omega_a} U_\theta.$$

Then $T_a = C_{\omega_a} T C_{\omega_a}$ is diagonal with respect to the decomposition

$$\mathcal{H} = U_\theta \mathcal{E} \oplus U_\theta \mathcal{O} = \mathcal{E} \oplus \mathcal{O},$$

since clearly $U_\theta \mathcal{E} = \mathcal{E}$ and $U_\theta \mathcal{O} = \mathcal{O}$. Moreover $U_{-\theta} C_{\omega_a} T C_{\omega_a} U_\theta$ commutes with $M_{Re(z)} = S + S^*$, which means that $C_{\omega_a} T C_{\omega_a}$ commutes with

$$U_\theta (S + S^*) U_{-\theta}.$$

□

In the proof above, it was shown that

$$T \in \{C_a, C_a^*\}' \iff C_{\omega_a} T C_{\omega_a} \in \{C_0, U_\theta (S + S^*) U_{-\theta}\}', \tag{17}$$

where S is the bilateral shift M_z in $L^2(\mathbb{T})$. In other words, $Ad_{C_{\omega_a}}$, which is a global multiplicative linear isomorphism of $\mathcal{B}(\mathcal{H})$, maps these C^*-algebras onto one another. Clearly, if $a \neq 0$, $Ad_{C_{\omega_a}}$ is not a global $*$-morphism. However, one has the following:

Proposition 8.2. *Let $a \in \mathbb{D}$. Then the map $Ad_{C_{\omega_a}}$, given by $Ad_{C_{\omega_a}}(T) = C_{\omega_a} T C_{\omega_a}$ induces $*$-isomorphisms*

$$Ad_{C_{\omega_a}}\big|_{\{C_a, C_a^*\}'} : \{C_a, C_a^*\}' \to \{C_0, U_\theta (S + S^*) U_{-\theta}\}',$$

with (the same) inverse

$$Ad_{C_{\omega_a}}\big|_{\{C_0, U_\theta (S + S^*) U_{-\theta}\}'} : \{C_0, U_\theta (S + S^*) U_{-\theta}\}' \to \{C_a, C_a^*\}'.$$

Proof. Clearly, it suffices to show that one of the two maps is $*$-preserving, for instance $Ad_{C_{\omega_a}}\big|_{\{C_0, U/\theta (S + S^*) U_{-\theta}\}'}$. Let us reason first in the case $a = r \in (0,1)$ (i.e., $\theta = 0$). Pick $T \in \{C_0, S + S^*\}'$. Then

$$Ad_{C_{\omega_r}}(T)^* = C_{\omega_r}^* T^* C_{\omega_r}^* = C_{\omega_r} M_{\frac{|1-\omega_r z|^2}{\sqrt{1-\omega_r^2}}} T M_{\frac{\sqrt{1-\omega_r^2}}{|1-\omega_r z|^2}} C_{\omega_r}$$

As seen above, identifying $L^2(\mathbb{T})$ with $L^2(-\pi, \pi)$, T commutes with $S + S^*$ if and only if it commutes with $M_{\cos(t)}$, which in turn implies that T commutes with $M_{1+\omega_r^2 - 2\omega_r \cos(t)}$, i.e. with $M_{|1-\omega_2 z|^2}$ (going back to $L^2(\mathbb{T})$). Therefore

$$Ad_{C_{\omega_r}}(T)^* = C_{\omega_r} T^* C_{\omega_r} = Ad_{C_{\omega_r}}(T^*).$$

For $a = re^{i\theta}$, recall from the previous proof that $U_\theta C_{\omega_r} = C_{\omega_a} U_\theta$, or equivalently $C_{\omega_a} = U_\theta C_{\omega_r} U_{-\theta}$, and therefore

$$Ad_{C_{\omega_a}}(T) = U_\theta Ad_{C_{\omega_r}}(U_{-\theta} T U_\theta) U_{-\theta}, \text{ i.e., } Ad_{C_{\omega_a}} = Ad_{U_\theta} \circ Ad_{C_{\omega_r}} \circ Ad_{U_{-\theta}},$$

is a $*$-isomorphism between $\{C_0, U_\theta(S + S^*)U_{-\theta}\}'$ and $\{C_a, C_a^*\}'$. ☐

Since $U_\theta C_0 U_{-\theta} = C_0$, the inner automorphism Ad_{U_θ} maps $\{C_0, S+S^*\}'$ onto $\{C_0, U_\theta(S+S^*)U_{-\theta}\}'$. Then:

Corollary 8.3. *Let $a, b \in \mathbb{D} \setminus \{0\}$. The von Neumann algebras $\{C_a, C_a^*\}'$ and $\{C_b, C_b^*\}'$ are $*$-isomorphic, with a normal $*$-isomorphism.*

Proof. The isomorphisms $Ad_{C_{\omega_a}}$ and Ad_{U_θ} are weakly continuous. ☐

Remark 8.4. The above proof in fact shows that for $a = r$, the operator $|C_{\omega_r}|^2 = M_{\frac{1-\omega_r^2}{|1-\omega_r z|^2}}$, and thus also $|C_{\omega_r}|$, commute with $\{C_0, S + S^*\}'$. Recall that R_{ω_r} denotes the unitary part in the polar decomposition $C_{\omega_r} = R_{\omega_r}|C_{\omega_r}| = |C_{\omega_r}|^{-1} R_{\omega_r}$. Therefore, for any $T \in \{C_0, S + S^*\}$, we have that

$$C_{\omega_r} T C_{\omega_r} = R_{\omega_r}|C_{\omega_r}| \; T \; |C_{\omega_r}|^{-1} R_{\omega_r} = R_{\omega_r} T R_{\omega_r},$$

i.e. the automorphism $Ad_{C_{\omega_r}}$ acting in $\{C_0, S+S^*\}'$, coincides in this algebra with $Ad_{R_{\omega_r}}$. For arbitrary $a = re^{i\theta}$,

$$Ad_{C_{\omega_a}} = Ad_{U_\theta} \circ Ad_{R_{\omega_r}} \circ Ad_{U_{-\theta}} = Ad_{U_\theta R_{\omega_r} U_{-\theta}},$$

in $\{C_0, U_\theta(S + S^*)U_{-\theta}\}'$.

Remark 8.5. Let $a \in \mathbb{D}$, $a = re^{i\theta}$, $r \neq 0$ and $T \in \{C_a, C_a^*\}'$. The matrix of $C_{\omega_a} T C_{\omega_a}$ in terms of the decomposition $\mathcal{H} = \mathcal{E} \oplus \mathcal{O}$ is

$$C_{\omega_a} T C_{\omega_a} = \begin{pmatrix} A_e & 0 \\ 0 & A_o \end{pmatrix},$$

where $A_e : \mathcal{E} \to \mathcal{E}$ and $A_o : \mathcal{O} \to \mathcal{O}$. Note that for $z \in \mathbb{T}$, $Re(e^{-i\theta} z) = \frac{1}{2}(e^{-i\theta} z + \frac{e^{i\theta}}{z})$ is an odd element in $C(\mathbb{T})$, and therefore $M_{Re(e^{-i\theta} z)} = U_\theta(S + S^*)U_{-\theta}$ maps $\mathcal{E}$ into $\mathcal{O}$, and $\mathcal{O}$ into $\mathcal{E}$. Its matrix in terms of $\mathcal{H} = \mathcal{E} \oplus \mathcal{O}$ is therefore

$$M_{Re(e^{-i\theta} z)} = \begin{pmatrix} 0 & M_{Re(e^{-i\theta} z)}\big|_{\mathcal{O}} \\ M_{Re(e^{-i\theta} z)}\big|_{\mathcal{E}} & 0 \end{pmatrix}.$$

Thus, the condition that $C_{\omega_a} T C_{\omega_a}$ commutes with $M_{Re(e^{-i\theta} z)}$ means that

$$A_e M_{Re(e^{-i\theta} z)}\big|_{\mathcal{O}} = M_{Re(e^{-i\theta} z)}\big|_{\mathcal{O}} A_o \quad \text{and} \quad A_o M_{Re(e^{-i\theta} z)}\big|_{\mathcal{E}} = M_{Re(e^{-i\theta} z)}\big|_{\mathcal{E}} A_e.$$

Examples 8.6. Notice the following examples of operators in $\{C_a, C_a^*\}'$:

1. If $f \in \mathcal{E} \cap L^\infty(\mathbb{T})$, then clearly $M_f \mathcal{E} \subset \mathcal{E}$ and $M_f \mathcal{O} \subset \mathcal{O}$, and M_f commutes with $M_{Re(e^{-i\theta}z)}$. Therefore $C_{\omega_a} M_f C_{\omega_a} \in \{C_a, C_a^*\}'$. Note that $C_{\omega_a} M_f C_{\omega_a} = M_{C_{\omega_a} f}$. In other words, if $h \in N(C_a - I) \cap L^\infty(\mathbb{T})$, then $M_h \in \{C_a, C_a^*\}'$.

2. Recall from Remark 5.4 the function
$$\nu_a = 1 - (2\psi_a - 1)^2 = \frac{4}{(\bar{a})^2}(1 - |a|^2) z \frac{(z-a)(1-\bar{a}z)}{(z-\omega_a)^2 (z-\alpha_a)^2},$$
where $\psi_a(z) = (1 + \frac{1-|a|^2}{|1-\bar{a}z|^2})^{-1}$. In Remark 5.4, it was shown that M_{ν_a} is diagonal with respect to the (non orthogonal) decomposition $\mathcal{H} = N(C_a - I) \oplus N(C_a + I)$. The fact that $M_{\nu_a}(N(C_a - I)) \subset N(C_a - I)$, together with the fact that M_{ν_a} is selfadjoint (ν_a is real), implies that M_{ν_a} commutes with $P_{N(C_a - I)}$. Also $M_{\nu_a}(N(C_a + I)) \subset N(C_a + I)$, and therefore M_{ν_a} commutes with $P_{N(C_a + I)}$. Thus, M_{ν_a} commutes with C_a and C_a^*. Note that this does not fall in the previous case, since ν_a is not even.

3. If $a = r \in (0,1)$, recall that the operator V (given by $Vf(z) = f(\bar{z})$) commutes with C_r. Since it is unitary, it also commutes with C_r^*. Thus $V \in \{C_r, C_r^*\}'$. For arbitrary $a = re^{i\theta}$, we have that $U_\theta V U_{-\theta} = U_{2\theta} V = V U_{-2\theta}$ belongs to $\{C_a, C_a^*\}'$.

In view of Remark 8.5, one can address the problem of further characterizing $\{C_0, S + S^*\}'$ (and therefore all algebras $\{C_a, C_a^*\}'$) in the following fashion. Given that $\{C_0, S + S^*\}'$ is a C*-algebra, it suffices to exhibit which are its selfadjoint elements. To this effect, in view of the matrix representation in terms of the decomposition $\mathcal{H} = \mathcal{E} \oplus \mathcal{O}$, we ask if a selfadjoint operator $A_e : \mathcal{E} \to \mathcal{E}$ can be extended to an element in $\{C_0, S + S^*\}'$, i.e., if there exists a selfadjoint operator $X_o : \mathcal{O} \to \mathcal{O}$ such that
$$A_e \oplus X_o = \begin{pmatrix} A_e & 0 \\ 0 & X_o \end{pmatrix} \text{ commutes with } S + S^*.$$
Let us call $A_e \oplus 0$ the operator in $\mathcal{H}$, which coincides with the former A_e in $\mathcal{E}$ and is zero in $\mathcal{O}$. Consider also $0 \oplus X_o$, which is zero in $\mathcal{E}$ and coincides with X_o in $\mathcal{O}$. As we remarked above, $S + S^*$ is co-diagonal with respect to $\mathcal{H} = \mathcal{E} \oplus \mathcal{O}$: $S + S^* = P_{\mathcal{E}}(S + S^*)P_{\mathcal{O}} + P_{\mathcal{O}}(S + S^*)P_{\mathcal{E}}$. The conditions that $A_e \oplus X_o$ must satisfy to belong to $\{C_0, S + S^*\}'$ are the equations given in Remark 8.5:
$$A_e(S + S^*) = (S + S^*)X_0 \quad \text{and} \quad X_0(S + S^*) = (S + S^*)A_e.$$
Clearly, since the operators are sefadjoint, this is equivalent to
$$A_e(S + S^*) = (S + S^*)X_o, \tag{18}$$
This relates to the operator equation $\mathbf{A}X = \mathbf{B}$ (here $\mathbf{A} = S + S^*$, $\mathbf{B} = A_o(S + S^*)$) considered by R. Douglas in [11]. More specifically, we are looking for selfadjoint solutions, so the following extension of Douglas' Theorem by W. Liang and C. Deng [13] is useful:

Theorem 8.7. (*W. Liang, C. Deng* [13], *Thm* **3.1 ii**) *Given operators* $\mathbf{A}$, $\mathbf{B}$ *acting in* $\mathcal{H}$, *the equation* $\mathbf{AX} = \mathbf{B}$ *has a selfadjoint solution* $\mathbf{X}$ *if and only if* $R(\mathbf{B}) \subset R(\mathbf{A})$ *and* $\mathbf{AB}^*$ *is selfadjoint.*

Applied to our situation, we get the following (note that $(S+S^*)^2$ is diagonal in the decomposition $\mathcal{H} = \mathcal{E} \oplus \mathcal{O}$):

Corollary 8.8. *A selfadjoint operator* $A_e : \mathcal{E} \to \mathcal{E}$ *can be extended to a selfadjoint operator* $A : \mathcal{H} \to \mathcal{H}$ *in* $\{C_0, S+S^*\}'$ *if and only if*

$$A_e\left((S+S^*)(\mathcal{O})\right) \subset R(S+S^*) \quad \textit{and} \quad A_e \textit{ commutes with } (S+S^*)^2\big|_{\mathcal{E}}.$$

Proof. Denote $A = A_e \oplus 0$ as above. Then we have selfadjpoint solutions of $\mathbf{AX} = \mathbf{B}$ if and only if $R(\mathbf{B}) = R(A(S+S^*)) = A_e\left((S+S^*)\mathcal{O}\right) \subset R(\mathbf{A}) = R(S+S^*)$, and

$$\mathbf{AB}^* = (S+S^*)^2 A \ \text{ is selfadjoint.}$$

Since both $(S+S^*)^2$ and A are selfadjoint, this happens if and only if they commute:

$$\begin{pmatrix} (S+S^*)\big|_{\mathcal{E}} A_e & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} (S+S^*)\big|_{\mathcal{E}} & 0 \\ 0 & (S+S^*)^2\big|_{\mathcal{O}} \end{pmatrix} \begin{pmatrix} A_e & 0 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} A_e & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} (S+S^*)\big|_{\mathcal{E}} & 0 \\ 0 & (S+S^*)^2\big|_{\mathcal{O}} \end{pmatrix} = \begin{pmatrix} A_e (S+S^*)\big|_{\mathcal{E}} & 0 \\ 0 & 0 \end{pmatrix},$$

i.e., A_e and $(S+S^*)^2\big|_{\mathcal{E}}$ commute. □

Note that A commutes with $(S+S^*)^2 = 2 + S^2 + (S^*)^2$ if and only if it commutes with $S^2 + (S^*)^2$. Also note that an elementary computation shows that $(S+S^*)\mathcal{O} = M_{1+z^2}\mathcal{E}$.

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