ON MEASURES DERIVED FROM ORBITAL INTEGRALS

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ABSTRACT. In this article we propose a novel framework to derive piecewise polynomial measures which result from invariant measures on adjoint orbits in the general context of compact and semi-simple Lie groups. The measures are computed from orbital integrals by a series of transformations on spaces of polynomials endowed with the apolar inner product. In the case of the unitary group we give a formula for the moments of the projection of an orbital measure.

Keywords. orbital integral, orbital measures, piecewise polynomial measure, moments of measure, apolar inner product, Fisher Bombieri inner product, reproducing kernel.

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1. Introduction

In this article we develop a novel approach to the derivation of some piecewise polynomial measures or Duistermat-Heckman type measures from Harish-Chandras (HC) orbital integral formula in the context of compact semi-simple Lie groups. These measures are derived from orbital measures, that is, invariant probability measures on adjoint orbits of compact semi-simple Lie groups. Specifically, we derive the pushforward of an orbital measure by the projection to the Cartan algebra, and we derive the radial part of the convolution of two orbital measures. In the case of type A root system we give a formula for the moments of the projection of an orbital measure. The characterization of these measures were previously addressed in [Z16, O13, F15] and in [F19, CMZ19, Z18]. They are probabilistic versions of the Schur-Horn theorem [H54] and Horn's problem [H62]. Given hermitian matrices X and Y with fixed eigenvalues the Schur-Horn theorem characterizes the possible diagonal entries of X as the convex closure of the permutation of eigenvalues, while Horn's problem asks for a characterization of the possible eigenvalues of the sum $X+Y$. The possible eigenvalues are given by a set of linear inequalities and the problem was solved by Klyatchko in [K98].

The approach applies directly to compact semi-simple Lie groups and is based on operations which can be done in finite dimensional spaces of polynomials endowed with the apolar inner product, which is the Segal-Bargmann-Fock space inner product restricted to polynomials. This operation if performed to both sides of an equation where the Fourier-Laplace transform of the unknown measure is written as an exponential polynomial involving the discriminant. The framework in the article was motivated in part by [M25] where the Harish-Chandra-Itzykson-Zuber (HCIZ) integral formula was put in the context of Segal-Bargmann spaces, and also [PS09] where divided difference operators and their adjoints were used to solve algebraic problems. In this article we took finite degree parts of power series and performed operations on these polynomials. Proofs in which finite degree terms are considered separately and an exponential function is constructed at the end are not uncommon in related literature, see for example the character expansion in the HCIZ integral in [MS17] and the proof of the Duistermaat-Heckman localization formula [MS17].

The article is organized as follows. In Section 2 we present the necessary results to state and prove the main theorems of the article. In Section 3 we give a characterization of the pushforward of an orbital measure, and we find the moments of the measure for type A root systems. In Section 4 we give a characterization of the radial part of the convolution of two orbital measures.

2. Preliminaries

We review the results which are necessary to prove the main results of the article, such as the Harish-Chandra integral formula and operators between spaces of polynomials endowed with the apolar inner product. Most results are known, to the best of our knowledge Propositions 2.5, Proposition 2.9 and Proposition 2.14 are new.

2.1. **Orbital measures and the Harish-Chandra integral formula.** Let G be a compact, semisimple and connected Lie group with Lie algebra $\mathfrak{g}$. The Killing form is the Ad-invariant inner product in $\mathfrak{g}$ and is denoted with $\langle\cdot,\cdot\rangle$, $\mathcal{W}$ is the Weyl group, $\epsilon(w)$ is the sign of $w\in\mathcal{W}$, that is, $\epsilon(w)=(-1)^{|w|}$ where $|w|$ is the number of reflections necessary to generate w. A compact Lie group admits a unique invariant probability measure called the Haar measure and the adjoint action induces a unique invariant probability measure

ν_a on the orbits $\mathcal{O}_a = \mathrm{Ad}_G(a)$ for $a \in \mathfrak{g}$. In [HC57] Harish-Chandra proved a formula for orbital integrals in Lie algebras in the case of compact, connected and semi-simple Lie groups, see the expository article [M21] and the references therein. The orbital integral admits an expression as an exponential polynomial. It is usually written

$$\Delta(x)\Delta(y) \int_G e^{\langle \mathrm{Ad}_g\, x, y\rangle} dg = \frac{[\Delta, \Delta]}{|\mathcal{W}|} \sum_{w \in \mathcal{W}} \epsilon(w) e^{\langle w(x), y\rangle},$$

where x and y are in a Cartan algebra $\mathfrak{t}_{\mathbb{C}}$ of $\mathfrak{g}_{\mathbb{C}}$. The discriminant

$$\Delta(x) = \prod_{\alpha \in \Phi^+} \langle \alpha, x\rangle$$

is the product of the positive roots Φ^+ considered as linear functionals. The term $[\Delta, \Delta]$ is computed with the apolar inner product which we a are going to introduce in the next section. We usually consider the element $a \in \mathfrak{t}$ of the orbit $\mathcal{O}_a$ as fixed and we write when $\Delta(a) \neq 0$

$$\mathcal{F}_a(x) = \int_G e^{\langle \mathrm{Ad}_g\, a, x\rangle} dg = \frac{[\Delta, \Delta]}{\Delta(a)\Delta(x)} \frac{1}{|\mathcal{W}|} \sum_{w \in \mathcal{W}} \epsilon(w) e^{\langle w(a), x\rangle} \tag{1}$$

as the Fourier-Laplace transform of the probability measure ν_a on the orbit $\mathcal{O}_a$.

2.2. **Spaces of polynomials and operators between them.** Let V be a vector space of dimension $n \in \mathbb{N}$ with the inner product $\langle \cdot, \cdot \rangle$. This space has coordinates $(x_1, \dots, x_n)$ given by an orthonormal basis. We define the space of polynomials on V with degree not greater than $k \in \mathbb{N}_0$ as

$$\mathcal{P}^k(V) = \{f \in \mathcal{P}(V) : \deg(f) \leq k\}.$$

This space is endowed with the apolar inner product

$$[f, g] = f(\partial) g(x)|_{x=0}$$

for $f, g \in \mathcal{P}^k(V)$. Here $f(\partial)$ is the polynomial $f(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n})$. This inner product is sometimes called Fisher or Bombieri inner product. If $(x_1, \dots, x_n)$ are coordinates given by an orthonormal basis of V then the monomials given by $x^\beta / \sqrt{\beta!}$ are orthonormal, where $\beta \in \mathbb{N}^n$ is a multi-index and $\beta! := \beta_1! \beta_2! \dots \beta_n!$. Note that

$$\left[x^\beta, x^\gamma\right] = \begin{cases} \beta! & \text{if } \beta = \gamma \\ 0 & \text{if } \beta \neq \gamma \end{cases}$$

Let

$$f(x) = \sum_\beta f_\beta x^\beta \qquad \text{and} \qquad g(x) = \sum_\beta g_\beta x^\beta$$

be two polynomials on V where β runs on multi-indices, then

$$[f, g] = f(\partial) g(x)|_{x=0} = \sum_\beta f_\beta g_\beta \beta!.$$

From this formula it follows that the apolar inner product is symmetric.

Remark 2.1. *The apolar inner product is the Segal-Bargmann-Fock inner product* [Ha00, N11] *restricted to real polynomials. If f, g are polynomials defined over the complex numbers then*

$$g^*(\partial)f(z)|_{z=0} = \frac{1}{\pi^n}\int_{\mathbb{C}^n} f(z)\overline{g(z)}e^{-|z|^2}dz$$

where $g^*(z) = \overline{g(\overline{z})}$.

A multivariate version of Taylor's theorem applied to polynomials asserts that for a polynomial f and $x, a \in V$

$$f(x+a) = e^{\langle a,\partial\rangle}f(x) = \sum_\beta a^\beta \partial^\beta f(x).$$

If for $a \in V$ we denote by $T_a f(x) = f(x+a)$ the translation operator on polynomials then $T_a = e^{\langle a,\partial\rangle}$ as operators on polynomials. In the space $\mathcal{P}^k(V)$ we define for $a \in V$ the functions

$$q_a^k(x) = \sum_{l=0}^{k} \langle x, a\rangle^l$$

which are truncations of exponentials $e^{\langle x,a\rangle}$. We denote with $\mathrm{ev}_a(f) = f(a)$ the evaluation functionals at $a \in V$.

Proposition 2.2. *For $a \in V$ the functions q_a^k has the reproducing property*

$$\left[f, q_a^k\right] = \mathrm{ev}_a(f) = f(a)$$

for $f \in \mathcal{P}^k(V)$.

Proof. For $a \in V$ and $f \in \mathcal{P}^k(V)$

$$\begin{aligned}\left[f, q_a^k\right] &= q_a^k(\partial)f(x)|_{x=0} = e^{\langle \partial, a\rangle}f(x)|_{x=0}\\ &= f(x+a)|_{x=0} = f(a)\end{aligned}$$

where we used the definition of the the apolar inner product and Taylor's theorem for polynomials. This property also follows from the reproducing property of Segal-Bargmann spaces restricted to polynomials, see Section 5 in [Ha00]. □

Consider the case when the inner product space is endowed with a finite refection group $\mathcal{W}$ which has an alternating character $\epsilon : \mathcal{W} \to \{1, -1\}$. We can define the space of alternating polynomials

$$\mathcal{P}_{\mathrm{alt}}^k(V) = \{f \in \mathcal{P}^k(V) : f(w(x)) = \epsilon(w)f(x) \text{ for all } w \in \mathcal{W}\}$$

and the space of symmetric polynomials

$$\mathcal{P}_{\mathrm{sym}}^k(V) = \{f \in \mathcal{P}^k(V) : f(w(x)) = f(x) \text{ for all } w \in \mathcal{W}\}.$$

In $\mathcal{P}^k(V)$ we define the orthogonal projection onto the alternating polynomials P_{alt} and the orthogonal projection onto the symmetric P_{sym}, they are given by

$$P_{\mathrm{alt}}f(x) = \frac{1}{|\mathcal{W}|}\sum_{w\in\mathcal{W}} \epsilon(w)f(w(x)) \qquad \text{and} \qquad P_{\mathrm{sym}}f(x) = \frac{1}{|\mathcal{W}|}\sum_{w\in\mathcal{W}} f(w(x)).$$

Lemma 2.3. *For $f \in \mathcal{P}^k(V)$ and $g \in \mathcal{P}_{\mathrm{alt}}^k(V)$ we have*

$$[P_{\mathrm{alt}}f, g] = [f, g]$$

Proof. This follows from $[P_{\text{alt}}f, g] = [f, P_{\text{alt}}g] = [f, g]$, since $P_{\text{alt}}^* = P_{\text{alt}}$ and projections fix the vectors in their range. ☐

We define for $a \in V$ the function

$$r_a^k = P_{\text{alt}} q_a^k.$$

Proposition 2.4. *For $a \in V$ the function r_a^k has the reproducing property*

$$\left[r_a^k, g\right] = g(a)$$

in $\mathcal{P}_{\text{alt}}^k(V)$.

Proof. Note that

$$\begin{aligned}\left[r_a^k, g\right] &= \left[P_{\text{alt}} q_a^k, g\right] = \left[q_a^k, P_{\text{alt}} g\right] \\ &= \left[q_a^k, g\right] = g(a)\end{aligned}$$

where we used the definition of r_a^k, $P_{\text{alt}}^* = P_{\text{alt}}$ and the reproducing property of q_a^k sated in Proposition 2.2. ☐

Analogous properties hold in the space of symmetric polynomials $\mathcal{P}_{\text{sym}}^k(V)$ but we are not going to use them.

For $a \in V$ we have the translation operator

$$T_a : \mathcal{P}^k(V) \to \mathcal{P}^k(V) \qquad \text{given by} \qquad T_a f(x) = f(x+a).$$

This operator has an adjoint $T_a^* : \mathcal{P}^k(V) \to \mathcal{P}^k(V)$. To characterize the adjoint we use the notation F^k to truncate polynomials, if we have a polynomial $f(x) = \sum_\beta x^\beta$ then

$$F^k(f(x)) = \sum_{\beta : |\beta| \leq k} x^\beta.$$

We also denote with $M_f g = f.g$ the multiplication of polynomials. Define the operator

$$F^k M_{q_a^k} : \mathcal{P}^k(V) \to \mathcal{P}^k(V)$$

which consists of multiplying by q_a^k and discarding in the result the terms of degree greater than k.

Proposition 2.5. *For $a \in V$ and $k \in \mathbb{N}_0$ we have*

$$T_a^* = F^k M_{q_a^k}.$$

Proof. For $f, g \in \mathcal{P}^k(V)$

$$\begin{aligned}\left[F^k M_{q_a^k} f, g\right] &= e^{\langle \partial, a \rangle} f(\partial) g(x)|_{x=0} \\ &= f(\partial) e^{\langle \partial, a \rangle} g(x)|_{x=0} \\ &= f(\partial) g(x+a)|_{x=0} \\ &= f(\partial) T_a g(x)|_{x=0} \\ &= \left[f, T_a g\right].\end{aligned}$$

An alternative proof can be given by checking the equation in monomials $f(x) = x^\beta$ and $g(x) = x^\beta$ using the binomial formula for several variables. We have for multi-indices $\beta \leq \gamma$

$$\begin{aligned}\left[x^\beta, T_a x^\gamma\right] = \left[x^\beta, (x+a)^\gamma\right] &= \left[x^\beta, \sum_{\delta \leq \gamma} \binom{\gamma}{\delta} x^\delta a^{\beta-\delta}\right] \\ &= \left[x^\beta, \binom{\gamma}{\beta} x^\beta a^{\gamma-\beta}\right] = a^{\gamma-\beta}\beta! \frac{\gamma!}{\beta!(\gamma-\beta)!}.\end{aligned}$$

One the other hand, with a slight abuse of notation when using the not truncated exponential, we have

$$\left[e^{\langle a,x\rangle} x^\beta, x^\gamma\right] = \left[\sum_\delta \frac{1}{\delta!} a^\delta x^{\delta+\beta}, x^\gamma\right] = \left[\frac{1}{(\gamma-\beta)!} a^{\gamma-\beta} x^\gamma, x^\gamma\right] = a^{\gamma-\beta}\frac{\gamma!}{(\gamma-\beta)!}.$$

□

Until the end of this section we consider vector spaces which are the Cartan algebras $\mathfrak{t}$ of Lie algebras of compact semi-simple Lie groups. One of the essential properties of the discriminant $\Delta : \mathfrak{t} \to \mathbb{R}$ is that it skew with respect to the action of $\mathcal{W}$, $\Delta(w(x)) = \epsilon(w)\Delta(x)$. This follows from the fact that if α is a simple root the reflection through the plane $\langle \alpha, x\rangle = 0$ sends $\alpha \mapsto -\alpha$ and permutes the other positive roots. The next proposition is well known.

Proposition 2.6. *Every polynomial $f \in \mathcal{P}^k_{\text{alt}}(\mathfrak{t})$ is divisible by Δ.*

Proof. For each reflection $w_\alpha \in \mathcal{W}$ we have $f(w_\alpha(x)) = -f(x)$ so f vanishes on the hyperplane $\langle \alpha, x\rangle = 0$ and the polynomial $\langle \alpha, x\rangle$ divides f. Since the Lie algebra is semi-simple its root system is reduced, that is, the only scalar multiple of a root α is $-\alpha$. Therefore, the positive roots are relatively prime, since they only divide each other if they are scalar multiples. Since for each positive root $\langle \alpha, x\rangle = 0$ divides f the discriminant Δ divides the polynomial f □

This proposition implies that

$$\mathcal{P}^{k+|\Phi^+|}_{\text{alt}}(\mathfrak{t}) = \Delta . \mathcal{P}^k_{\text{sym}}(\mathfrak{t}).$$

Definition 2.7. *The division by the discriminant is defined as*

$$D_\Delta : \mathcal{P}^{k+|\Phi^+|}_{\text{alt}}(\mathfrak{t}) \to \mathcal{P}^k_{\text{sym}}(\mathfrak{t}) \qquad \textit{given by} \qquad D_\Delta f(x) = \frac{f(x)}{\Delta(x)}$$

and its adjoint $I_\Delta = D^*_\Delta$ is

$$I_\Delta = D^*_\Delta : \mathcal{P}^k_{\text{sym}}(\mathfrak{t}) \to \mathcal{P}^{k+|\Phi^+|}_{\text{alt}}(\mathfrak{t}).$$

For the case $\mathfrak{t} \simeq \mathbb{R}$ the polynomials are of one variable and we get the characterization of D^*_Δ as an anti-derivative $I_\Delta f(x) = \int_0^x f(t)dt$.

Proposition 2.8. *In the case of the one dimensional Cartan algebra $\mathfrak{t} \simeq \mathbb{R}$ the adjoint $D^*_\Delta = I_\Delta$ of the division operator is the anti-derivative.*

Proof. It is enough to verify the identity in the monomials x^k and x^l. We have

$$D_\Delta(x^k) = \frac{1}{x}x^k = x^{k-1}$$

and

$$I_\Delta(x^l) = \int_0^x t^l dt = \frac{1}{l+1} x^{l+1}.$$

It is easy to very that $\left[x^{k-1}, x^l\right] = \left[x^k, \frac{1}{l+1}x^{l+1}\right]$. If $k \neq l+1$ then both side are zero and if $k = l+1$ then both sides are equal to $l!$. □

The adjoint D_Δ^* of D_Δ can be characterized as follows in the general case

Proposition 2.9. *The operator* D_Δ^* *is the inverse of the discriminant with differential variables* $\Delta(\partial)$*, that is,*

$$D_\Delta^* = \Delta(\partial)^{-1}.$$

Proof. For symmetric polynomials f and g

$$\begin{aligned} [f, g] &= [D_\Delta(\Delta.f), g] \\ &= [\Delta.f, D_\Delta^* g] \\ &= [f, \Delta(\partial) D_\Delta^* g], \end{aligned}$$

therefore $\Delta(\partial) D_\Delta^* = Id$. Also, for alternating polynomials r and s

$$\begin{aligned} [r, s] &= [\Delta.D_\Delta(r), s] \\ &= [D_\Delta(r), \Delta(\partial)s] \\ &= [r, D_\Delta^* \Delta(\partial)s], \end{aligned}$$

so that $D_\Delta^* \Delta(\partial) = Id$. □

We have also the following characterization of I_Δ on the functions q_a^k.

Proposition 2.10. *For* $k \in \mathbb{N}_0$ *and* $a \in \mathfrak{t}$ *such that* $\Delta(a) \neq 0$ *the identity*

$$I_\Delta(q_a^k) = \frac{1}{\Delta(a)} r_a^{k+|\Phi^+|}$$

holds.

Proof. Using the fact that $\langle \partial, \alpha \rangle e^{\langle x, a \rangle} = \langle \alpha, a \rangle e^{\langle x, a \rangle}$ we obtain the following well known identity for power series

$$\Delta(\partial) \left(\frac{1}{|\mathcal{W}|} \sum_{w \in \mathcal{W}} \epsilon(w) e^{\langle x, w(a) \rangle} \right) = \Delta(a) \sum_{w \in \mathcal{W}} e^{\langle x, w(a) \rangle}.$$

If we take polynomial parts by applying the truncation operator F^k we get

$$\Delta(\partial) \left(r_a^{k+|\Phi^+|} \right) = \Delta(a) q_a^k.$$

If we apply $\Delta(\partial)^{-1}$ in the case $\Delta(a) \neq 0$ we obtain the formula stated in the theorem. □

Next we give a computation in the case of polynomials of lowest degree. We denote with $\mathbf{1}$ the constant polynomial $\mathbf{1}(x) = 1$.

Proposition 2.11. *The operator* I_Δ *satisfies*

$$I_\Delta \mathbf{1} = \frac{\Delta}{[\Delta, \Delta]}.$$

Proof. We have $D_\Delta(\mathbb{R}\,\Delta) = \mathbb{R}\,\mathbf{1}$ and $D_\Delta(\{\mathbb{R}\,\Delta\}^\perp) = \{\mathbb{R}\,\mathbf{1}\}^\perp$. Therefore $I_\Delta \mathbf{1} = D_\Delta^* \mathbf{1} = d\Delta$ for $d \in \mathbb{R}$. We have

$$1 = [D_\Delta \Delta, \mathbf{1}] = [\Delta, I_\Delta \mathbf{1}] = [\Delta, d\Delta] = d\,[\Delta, \Delta]\,.$$

Hence $d = [\Delta, \Delta]^{-1}$ and the conclusion of the proposition follows. $\square$

In the case of type A root systems one can give the following characterization of the operator $I_\Delta = D_\Delta^*$ by computing its matrix coefficients and taking the transpose of a matrix. We assume that the group is the unitary group $\mathrm{U}(n)$, its center is not trivial but this makes the link to the theory of symmetric polynomials more direct. The unitary orbital integral in this case is known as the Harish-Chandra-Itzykson-Zuber (HCIZ) integral [IZ80]. It is usually written

$$\begin{aligned}\int_{\mathrm{U}(n)} e^{\mathrm{Tr}(uAu^{-1}B)} du &= \left(\prod_{p=1}^{n-1} p!\right) \frac{\det\left[e^{a_i b_j}\right]_{i,j=1}^n}{\Delta(A)\Delta(B)} \\ &= \left(\prod_{p=1}^{n} p!\right) \frac{\frac{1}{n!}\sum_{w\in S_n} e^{\langle A, w(B)\rangle}}{\Delta(A)\Delta(B)}\end{aligned}$$

where det is the determinant of a matrix, $\mathrm{U}(n)$ is the group of n-by-n unitary matrices, A and B are fixed n-by-n diagonal matrices with eigenvalues $a_1 < \cdots < a_n$ and $b_1 < \cdots < b_n$ respectively, and

$$\Delta(A) = \prod_{i<j}(a_j - a_i)$$

is the Vandermonde determinant.

Remark 2.12. *The HCIZ has become an important identity in quantum field theory, random matrix theory, and algebraic combinatorics.*

Let $\mathbb{N}_0$ stand for the non negative integers. For $\mu \in \mathbb{N}_0^n$ we denote the monomials as usual with $x^\mu = x_1^{\mu_1} \dots x_n^{\mu_n}$ and we use the notations $\mu! = \mu_1! \dots \mu_n!$ and $|\mu| = \mu_1 + \cdots + \mu_n$. The set Π of partitions is defined as

$$\Pi = \{\lambda \in \mathbb{N}_0^n : \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n\}.$$

For $k \in \mathbb{N}_0$ we denote with

$$\Pi^k = \{\lambda \in \Pi : |\lambda| \leq k\}$$

the partitions of total size not greater than k. We set $\delta = (n-1, n-2, \dots, 0)$. For $\lambda \in \Pi$, if $(x_1, \dots, x_n)$ are the eigenvalues of a hermitian matrix x we define

$$\chi_\lambda(x) = s_\lambda(x_1, \dots, x_n),$$

where s_λ is a Schur polynomial. These polynomials are defined by

$$s_\lambda(x_1, \dots, x_n) = \frac{a_{\lambda+\delta}(x_1, \dots, x_n)}{a_\delta(x_1, \dots, x_n)},$$

where

$$a_\mu(x_1, \dots, x_n) = \det\left[x_i^{\mu_j}\right]_{i,j=1}^n.$$

Note that $a_\delta(x_1, \dots, x_n) = \Delta(x_1, \dots, x_n)$ is the Vandermonde determinant.

Remark 2.13. *The irreducible polynomial representations of the general linear group are labelled by Young diagrams, which we think of as vectors $\lambda \in \Pi$. The character of the λ-representation is given by χ_λ, the Schur polynomial evaluated at the eigenvalues of an invertible matrix.*

For $\mu \in \Pi$ we denote with m_μ the monomial symmetric polynomial which we defined as the sum of x^λ where λ ranges over distinct permutations of μ, therefore m_μ is the sum of $|\mathcal{W} \cdot \mu| = |S_n \cdot \mu|$ distinct monomials. Here $|S_n \cdot \mu|$ is the cardinality of the Weyl orbit of μ.

Schur polynomials can be expressed as linear combinations of monomial symmetric functions with non-negative integer coefficients $K_{\lambda\mu}$ called Kostka numbers

$$s_\lambda = \sum_\mu K_{\lambda\mu} m_\mu.$$

The Kostka numbers $K_{\lambda\mu}$ are given by the number of semi-standard Young tableaux of shape $\lambda \in \Pi$ and weight $\mu \in \Pi$. References for Kostka numbers are Section VI.1 in [M05] and Theorem 2.11.2 in [S01] for the representation theoretic aspect.

It is known that $(m_\mu)_{\mu \in \Pi^k}$ and $(a_{\lambda+\delta})_{\delta \in \Pi^k}$ form algebraic bases of $\mathcal{P}^k_{\mathrm{sym}}(\mathbb{R}^n)$ and $\mathcal{P}^{k+l}_{\mathrm{alt}}(\mathbb{R}^n)$ respectively, where $l = |\Phi^+| = \frac{n(n-1)}{2}$. We have

$$[a_{\lambda+\delta}, a_{\lambda+\delta}] = n!(\lambda+\delta)! \qquad \text{and} \qquad [m_\mu, m_\mu] = |S_n \cdot \mu| \mu!.$$

We scale these polynomials to be of unit norm and define for $\lambda \in \Pi^k$

$$a'_{\lambda+\delta} = \frac{1}{\sqrt{n!(\lambda+\delta)}} a_{\lambda+\delta} \qquad \text{and} \qquad m'_\mu = \frac{1}{|S_n \cdot \mu| \mu!} m_\mu.$$

Therefore, for $k \in \mathbb{N}$ the set $(a'_{\lambda+\delta})_{\lambda \in \Pi^k}$ is an orthonormal basis of $\mathcal{P}^{k+l}_{\mathrm{alt}}(\mathbb{R}^n)$ and the set $(m'_\mu)_{\mu \in \Pi^k}$ is an orthonormal basis of $\mathcal{P}^k_{\mathrm{sym}}(\mathbb{R}^n)$.

We have the following characterization of the operator I_Δ given by its matrix coefficients.

Proposition 2.14. *For $k \in \mathbb{N}$ the operator $D_\Delta : \mathcal{P}^{k+l}_{\mathrm{alt}}(\mathbb{R}^n) \to \mathcal{P}^k_{\mathrm{sym}}(\mathbb{R}^n)$ and its adjoint $I_\Delta = D^*_\Delta$ are given by*

$$D_\Delta a_{\lambda+\delta} = \sum_{\lambda \in \Pi} K_{\lambda\mu} m_\mu$$

and

$$I_\Delta m_\mu = D^*_\Delta m_\mu = \sum_{\lambda \in \Pi} \frac{|S_n \cdot \mu| \mu!}{n!(\lambda+\delta)!} K_{\lambda\mu} a_{\lambda+\delta}.$$

Proof. Note that for $\lambda \in \Pi^k$

$$D_\Delta a_{\lambda+\delta} = \frac{a_{\lambda+\delta}}{a_\delta} = s_\lambda = \sum_\lambda K_{\lambda\mu} m_\mu,$$

Therefore

$$D_\Delta a'_{\lambda+\delta} = s_\lambda = \sum_\lambda K'_{\lambda\mu} m'_\mu,$$

with

$$K'_{\lambda\mu} = \frac{\sqrt{|S_n \cdot \mu| \mu!}}{\sqrt{n!(\lambda+\delta)!}} K_{\lambda\mu}.$$

Hence for $\mu \in \Pi^k$

$$D^*_\Delta m'_\mu = \sum_\lambda K'_{\lambda\mu} a'_{\lambda+\mu},$$

and this implies that

$$D_\Delta^* m_\mu = \sum_\lambda \frac{|S_n \cdot \mu| \mu!}{n!(\lambda+\delta)!} K_{\lambda\mu} a_{\lambda+\delta}.$$

□

3. The projection of an orbital measure to the Cartan algebra

For $a \in \mathfrak{t}$ let ν_a be the orbital measure on $\mathcal{O}_a$ and define the measure μ_a on the Cartan algebra as $\mu_a = \mathrm{pr}_*(\nu_a)$ where $\mathrm{pr} : \mathfrak{g} \to \mathfrak{t}$ is the orthogonal projection. The measure μ_a was characterized in [Z16, O13, F15], it can also be characterized as a Duistermaat-Heckman measure in the context of Hamiltonian torus actions, see Section 5 in the book [GLS96] and the references therein. In this section we give an alternative characterization and provide a formula for the moments in the case of type A root systems. We start with an equation satisfied by the measure μ_a.

Proposition 3.1. *For $a \in \mathfrak{t}$ such that $\Delta(a) \neq 0$ the measure μ_a satisfies*

$$\frac{[\Delta,\Delta]}{\Delta(a)\Delta(x)} \frac{1}{|\mathcal{W}|} \sum_{w\in\mathcal{W}} \epsilon(w) e^{\langle w(a),x\rangle} = \int_{\mathfrak{t}} e^{\langle c,x\rangle} d\mu_a(x). \tag{2}$$

Proof. Since $\langle \mathrm{Ad}_g\, x, y\rangle = \langle \mathrm{pr}(\mathrm{Ad}_g\, x), y\rangle$ for $y \in \mathfrak{t}$, $x \in \mathfrak{g}$ and $g \in G$ it is easy to very that the measure $\mu_a = \mathrm{pr}_*(\nu_a)$ satisfies

$$\int_G e^{\langle \mathrm{Ad}_g\, a, x\rangle} dg = \int_G e^{\langle \mathrm{pr}(\mathrm{Ad}_g\, a), x\rangle} dg = \int_{\mathfrak{t}} e^{\langle c,x\rangle} d\mu_a(x).$$

Therefore, the orbital integral (1) implies that the measure μ_a satisfies

$$\mathcal{F}_a(x) = \frac{[\Delta,\Delta]}{\Delta(a)\Delta(x)} \frac{1}{|\mathcal{W}|} \sum_{w\in\mathcal{W}} \epsilon(w) e^{\langle w(a),x\rangle} = \int_{\mathfrak{t}} e^{\langle c,x\rangle} d\mu_a(x).$$

□

Theorem 3.2. *For $a \in \mathfrak{t}$ such that $\Delta(a) \neq 0$ the measure μ_a is characterized by*

$$\int_{\mathfrak{t}} f(x) d\mu_a(x) = \frac{[\Delta,\Delta]}{\Delta(a)} \mathrm{ev}_a \left(I_\Delta P_{\mathrm{sym}} f\right), \tag{3}$$

for every $f \in \mathcal{P}(\mathfrak{t})$.

Proof. We write $\eta = [\Delta,\Delta]/\Delta(a)$ and $l = |\Phi^+|$. If we take the part of degree not greater than $k \in \mathbb{N}_0$ in the equation of Proposition 3.1 then we have

$$\eta . \frac{r_a^{k+l}(x)}{\Delta(x)} = \int_{\mathfrak{t}} q_c^k(x) d\mu_a(x).$$

If for a polynomial $f \in \mathcal{P}^k(\mathfrak{t})$ we take the inner product of both sides with $f_{\mathrm{sym}} = P_{\mathrm{sym}} f$ then in the LHS we get

$$\begin{aligned}
\left[\eta D_\Delta r_a^{k+l}, f_{\mathrm{sym}}\right] &= \eta . \left[D_\Delta r_a^{k+l}, f_{\mathrm{sym}}\right] \\
&= \eta . \left[r_a^{k+l}, I_\Delta f_{\mathrm{sym}}\right] \\
&= \eta . \left[q_a^{k+l}, I_\Delta f_{\mathrm{sym}}\right] \\
&= \eta . \mathrm{ev}_a \left(I_\Delta f_{\mathrm{sym}}\right)
\end{aligned}$$

where we used Definition 2.7 in the second equality, Lemma 2.3 in the third and Proposition 2.4 in the final equality. On the other hand, in the LHS we get

$$\begin{aligned}\left[\int_{\mathfrak{t}} q_c^k d\mu_a(x), f_{\text{sym}}\right] &= \int_{\mathfrak{t}} \left[q_c^k, f_{\text{sym}}\right] d\mu_a(x)\\ &= \int_{\mathfrak{t}} \text{ev}_c(f_{\text{sym}}) d\mu_a(x)\\ &= \int_{\mathfrak{t}} f_{\text{sym}}(x) d\mu_a(x)\\ &= \int_{\mathfrak{t}} f(x) d\mu_a(x)\end{aligned}$$

where we used Proposition 2.2 in the second equality and the symmetry of the measure μ_a in the final equality. Hence, the conclusion of the theorem holds. ☐

Corollary 3.3. *We can rewrite (3) as*

$$(I_\Delta f)(a) = \frac{\Delta(a)}{[\Delta, \Delta]} \int_{\mathfrak{t}} f(x) d\mu_a(x)$$

for every $f \in \mathcal{P}_{\text{sym}}(\mathfrak{t})$. If we change the symbols a and x to xx and y we obtain

$$\left(\Delta(\partial)^{-1} f\right)(x) = (I_\Delta f)(x) = \frac{\Delta(x)}{[\Delta, \Delta]} \int_{\mathfrak{t}} f(y) d\mu_x(y).$$

This provides an alternative characterization of $I_\Delta = \Delta(\partial)^{-1}$.

Remark 3.4. *In Proposition 2.10 we showed that*

$$I_\Delta(q_b^k) = \frac{1}{\Delta(b)} r_b^{k+l}.$$

Therefore if take $f = q_b^k$ in Theorem 3.2 we get

$$\begin{aligned}\int_{\mathfrak{t}} q_a^k(x) d\mu_a(x) &= \frac{[\Delta, \Delta]}{\Delta(a)} \text{ev}_a \left(I_\Delta P_{\text{sym}} q_a^k\right)\\ &= \frac{[\Delta, \Delta]}{\Delta(a)\Delta(b)} r_b^{k+l}(a)\end{aligned}$$

which was the main assumption of the theorem.

Example 3.5. *Let us check that the measure μ_a characterized in this way is a probability measure. If we take as polynomial $f = \mathbf{1}$ then by Theorem 3.2 and Proposition 2.11*

$$\int_{\mathfrak{t}} 1 d\mu(x) = \frac{[\Delta, \Delta]}{\Delta(a)} \text{ev}_a(I_\Delta \mathbf{1}) = \frac{[\Delta, \Delta]}{\Delta(a)} \text{ev}_a \left(\frac{\Delta}{[\Delta, \Delta]}\right) = \frac{[\Delta, \Delta]}{\Delta(a)} \frac{\Delta(a)}{[\Delta, \Delta]} = 1.$$

Example 3.6. *Consider the case in which $G = \mathrm{SU}(2)$. Using Proposition 2.8 and the identification $\mathfrak{t} \simeq \mathbb{R}$ given by $(x, -x) \mapsto x$ we have*

$$\begin{aligned}\int_{\mathfrak{t}} f(x) d\mu_a(x) &= \eta. \text{ev}_a(I_\Delta P_{\text{sym}} f)\\ &= \frac{1}{a} \int_0^a \frac{1}{2} (f(a - t) + f(-a + t)) dt.\end{aligned}$$

It is easy to see that the density of the measure μ_a is given by the indicator function in the interval $[a, -a]$ of $\mathfrak{t}$. Since adjoint orbits are spheres, that is, $\mathcal{O}_a = \mathrm{Ad}_{\mathrm{SU}(2)}(a) \simeq S^2$ this is the fact known to Archimedes that the projection of the surface measure of a sphere to an axis is given by an indicator function.

We give a formula for the moments of the measure μ_a in the case of type A root systems.

Theorem 3.7. *In the case of the unitary group* $\mathrm{U}(n)$ *and for a partition* $\eta \in \Pi$ *the following equations for the moments of the monomial symmetric function* $m^\eta(x)$ *and the monomial* x^η *hold*

$$\int_{\mathbb{R}^n} m_\eta(x) d\mu_b(x) = [\Delta, \Delta] \sum_{\lambda \in \Pi} \frac{|S_n \cdot \eta| \eta!}{n!(\lambda+\delta)!} K_{\lambda\eta} s_\lambda(b),$$

$$\int_{\mathbb{R}^n} x^\eta d\mu_b(x) = [\Delta, \Delta] \sum_{\lambda \in \Pi} \frac{\eta!}{n!(\lambda+\delta)!} K_{\lambda\eta} s_\lambda(b).$$

Proof. For a monomial symmetric polynomial m_η Theorem 3.2 and Proposition 2.14 imply that

$$\begin{aligned}\int_{\mathbb{R}^n} m_\eta(x) d\mu_b(x) &= \frac{[\Delta,\Delta]}{\Delta(b)} \operatorname{ev}_b (I_\Delta m_\eta) \\ &= \frac{[\Delta,\Delta]}{\Delta(b)} \operatorname{ev}_b \left(\sum_\lambda \frac{|S_n \cdot \eta| \eta!}{n!(\lambda+\delta)!} K_{\lambda\eta} a_{\lambda+\delta} \right) \\ &= \frac{[\Delta,\Delta]}{a_\delta(b)} \left(\sum_\lambda \frac{|S_n \cdot \eta| \eta!}{n!(\lambda+\delta)!} K_{\lambda\eta} a_{\lambda+\delta}(b) \right) \\ &= [\Delta,\Delta] \sum_\lambda \frac{|S_n \cdot \eta| \eta!}{n!(\lambda+\delta)!} K_{\lambda\eta} s_\lambda(b),\end{aligned}$$

therefore the first equation of the theorem holds. Also, it is easy to check that for a monomial symmetric polynomial m^η

$$P_{\mathrm{sym}}(x^\eta) = \frac{1}{|S_n \cdot \eta|} m^\eta(x).$$

Hence, the second equation of the theorem follows. □

4. The radial part of the convolution of orbital measures

Let ν_a and ν_b be the orbital measures on $\mathcal{O}_a$ and $\mathcal{O}_b$ for $a, b \in \mathfrak{t}$. We characterize the radial part $\nu_{a,b}$ of the convolution of measures $\nu_a * \nu_b$. This was done in previous research in [F19, IZ80, CMZ19] using mainly Fourier analytic techniques. We start with an equation satisfied by the radial measure $\nu_{a,b}$.

Proposition 4.1. *For* $a, b \in \mathfrak{t}$ *such that* $\Delta(a) \neq 0$ *and* $\Delta(b) \neq 0$ *the radial measure* $\nu_{a,b}$ *satisfies*

$$\begin{aligned}(4) \qquad &\frac{[\Delta,\Delta]}{\Delta(a)\Delta(b)} \frac{1}{\Delta(x)} \left(\frac{1}{|\mathcal{W}|} \sum_{w \in \mathcal{W}} \epsilon(w) e^{\langle w(a), x\rangle} \right) \left(\frac{1}{|\mathcal{W}|} \sum_{w \in \mathcal{W}} \epsilon(w) e^{\langle w(b), x\rangle} \right) \\ &= \int_{\mathfrak{t}} \frac{1}{\Delta(x)} \left(\frac{1}{|\mathcal{W}|} \sum_{w \in \mathcal{W}} \epsilon(w) e^{\langle w(x), x\rangle} \right) d\nu_{a,b}(x).\end{aligned}$$

Proof. By Proposition 2.1 in [F19] the radial measure $\nu_{a,b}$ of the convolution $\nu_a * \nu_b$ is given by the Fourier-Laplace transform as

$$\mathcal{F}_a(z)\mathcal{F}_b(z) = \int_{\mathfrak{t}} \mathcal{F}_c(z) d\nu_{a,b}(x).$$

If we express the Fourier-Laplace transform terms $\mathcal{F}_a, \mathcal{F}_b$ and $\mathcal{F}_c$ using the orbital integral formula (1), and if we multiply both sides by $\Delta(x)$ and divide both sides by $[\Delta, \Delta]$ then we obtain the equation (4) in the statement of the proposition. □

Theorem 4.2. *For $a, b \in \mathfrak{t}$ such that $\Delta(a) \neq 0$ and $\Delta(b) \neq 0$ the radial part $\nu_{a,b}$ of the convolution $\nu_a * \nu_b$ of orbital measures is characterized by*

$$\int_{\mathfrak{t}} \frac{f_{\text{alt}}(x)}{\Delta(x)} d\nu_{a,b}(x) = \frac{[\Delta, \Delta]}{\Delta(a)\Delta(b)} \operatorname{ev}_b \left(I_\Delta \left(\frac{1}{|\mathcal{W}|} \sum_{w \in \mathcal{W}} \epsilon(w) T_{w(a)} f_{\text{alt}} \right) \right). \tag{5}$$

for every alternating polynomial $f_{\text{alt}} \in \mathcal{P}_{alt}(\mathfrak{t})$.

Proof. We set $\varphi = [\Delta, \Delta]/\Delta(a)\Delta(b)$ and $l = |\Phi^+|$. If we take the part of degree not greater than $k \in \mathbb{N}_0$ of equation (4) we get

$$\varphi . F^k M_{r_a^k} \frac{r_b^{k+l}(x)}{\Delta(x)} = \int_{\mathfrak{t}} \frac{1}{\Delta(x)} r_c^k(x) d\nu_{a,b}(x).$$

For an alternating polynomial $f_{\text{alt}} \in \mathcal{P}^k_{\text{alt}}(\mathfrak{t})$ we take the inner product of both sides with f_{alt}. On the LHS we have

$$\begin{aligned}
\left[\varphi . F^k M_{r_a^k} D_\Delta r_b^{k+l}, f_{\text{alt}}\right] &= \varphi . \left[\frac{1}{|\mathcal{W}|} \sum_{w \in \mathcal{W}} \epsilon(w) F^k M_{q^k_{w(a)}} D_\Delta r_b^{k+l}, f_{\text{alt}} \right] \\
&= \frac{1}{|\mathcal{W}|} \sum_{w \in \mathcal{W}} \varphi . \left[\epsilon F^k M_{q^k_{w(a)}} D_\Delta r_b^{k+l}, f_{\text{alt}} \right] \\
&= \frac{1}{|\mathcal{W}|} \sum_{w \in \mathcal{W}} \varphi . \left[D_\Delta r_b^{k+l}, \epsilon(w) T_{w(a)} f_{\text{alt}} \right] \\
&= \varphi . \left[D_\Delta r_b^{k+l}, \frac{1}{|\mathcal{W}|} \sum_{w \in \mathcal{W}} \epsilon(w) T_{w(a)} f_{\text{alt}} \right] \\
&= \varphi . \left[r_b^{k+l}, I_\Delta \left(\frac{1}{|\mathcal{W}|} \sum_{w \in \mathcal{W}} \epsilon(w) T_{w(a)} f_{\text{alt}} \right) \right] \\
&= \varphi . \left[q_b^{k+l}, I_\Delta \left(\frac{1}{|\mathcal{W}|} \sum_{w \in \mathcal{W}} \epsilon(w) T_{w(a)} f_{\text{alt}} \right) \right] \\
&= \varphi . \operatorname{ev}_b \left(I_\Delta \left(\frac{1}{|\mathcal{W}|} \sum_{w \in \mathcal{W}} \epsilon(w) T_{w(a)} f_{\text{alt}} \right) \right)
\end{aligned}$$

where we used the definition $r_a^k = P_{\text{alt}} q_a^k$ in the first equality, Proposition 2.5 in the third, Definition 2.7 in the fifth, Lemma 2.3 in the sixth and Proposition 2.2 in the final equality. On the other hand, on the RHS we get

$$\begin{aligned}
\left[\int_{\mathfrak{t}} \frac{1}{\Delta(x)} r_c^k(x) d\nu_{a,b}(x), f_{\text{alt}} \right] &= \int_{\mathfrak{t}} \frac{1}{\Delta(x)} \left[r_c^k(x), f_{\text{alt}} \right] d\nu_{a,b}(x) \\
&= \int_{\mathfrak{t}} \frac{1}{\Delta(x)} \operatorname{ev}_c(f_{\text{alt}}) d\nu_{a,b}(x) \\
&= \int_{\mathfrak{t}} \frac{1}{\Delta(x)} f_{\text{alt}}(x) d\nu_{a,b}(x),
\end{aligned}$$

where we used Proposition 2.4 in the second equality. Combining the result of taking the inner product on both sides we obtain the equation in the statement of the theorem. Since all symmetric polynomials are of the form f_{alt}/Δ for an alternating polynomial f_{alt} and since the measure $\nu_{a,b}$ is symmetric we see that (5) completely characterizes the measure $\nu_{a,b}$. ☐

We can get a slight variation of the characterization in Theorem 4.2 of the radial measure $\nu_{a,b}$.

Corollary 4.3. *The radial part $\nu_{a,b}$ of the convolution $\nu_a * \nu_b$ is characterized by*

$$\int_{\mathfrak{t}} g(x) d\nu_{a,b}(x) = \frac{[\Delta,\Delta]}{\Delta(a)\Delta(b)} \operatorname{ev}_b \left(I_\Delta \left(\frac{1}{|\mathcal{W}|} \sum_{w\in\mathcal{W}} \epsilon(w) T_{w(a)} (\Delta . P_{\text{sym}} g) f \right)\right)$$

for every polynomial $g \in \mathcal{P}(\mathfrak{t})$.

Proof. An alternating polynomial $f_{\text{alt}} \in \mathcal{P}_{\text{alt}}(\mathfrak{t})$ can be written as $f_{\text{alt}} = \Delta . P_{\text{sym}} g$ for a polynomial $g \in \mathcal{P}(\mathfrak{t})$. Making this substitution in formula (5) and noting that

$$\int_{\mathfrak{t}} (P_{\text{sym}} h)(x) d\nu_{a,b}(x) = \int_{\mathfrak{t}} h(x) d\nu_{a,b}(x)$$

for any polynomial $h \in \mathcal{P}(\mathfrak{t})$ the corollary follows. ☐

Example 4.4. *Consider the case in which $G = \mathrm{SU}(2)$. For $f_{\text{alt}} \in \mathcal{P}_{\text{alt}}(\mathfrak{t})$, using the identification $\mathfrak{t} \simeq \mathbb{R}$ given by $(x,-x) \to x$ and Proposition 2.8 we have*

$$\begin{aligned}\int_{\mathfrak{t}} f_{\text{alt}}(x) \frac{1}{\Delta(x)} d\nu_{a,b}(x) &= \varphi . \operatorname{ev}_b \left(I_\Delta \left(\frac{1}{2} \left(T_a f_{\text{alt}} - T_{-a} f_{\text{alt}} \right)\right)\right) \\ &= \frac{1}{2ab} \int_0^b \frac{1}{2} \left(f_{\text{alt}}(b+a-t) - f_{\text{alt}}(b-a-t)\right) dt\end{aligned}$$

One can verify that the density function ϕ of the measure $\nu_{a,b}$ is given when $0 < b < a$ by

$$\phi(x) = \begin{cases} \frac{x}{4ab} & \text{if } a-b \le x \le a+b \\ \frac{-x}{4ab} & \text{if } -a-b \le x \le -a+b \\ 0 & \text{otherwise.} \end{cases}$$

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