One dimensional perturbation problem for linear relations

Itziar Baragaña[1,1], Francisco Martínez Pería[1,1,1,1], Alicia Roca[1,1,1,], Carsten Trunk[1,1]

[a] *Departamento de Ciencia de la Computación e I.A., Universidad del País Vasco UPV/EHU, Apartado 649, 20080 Donostia-San Sebastián, Spain*
[b] *Instituto Argentino de Matemática "Alberto P. Calderón" Saavedra 15, Piso 3 (1083) Buenos Aires, Argentina*
[c] *Centro de Matemática de La Plata – Facultad de Ciencias Exactas Universidad Nacional de La Plata, Calles 50 y 115 (1900) La Plata, Argentina*
[d] *Departamento de Matemática Aplicada, IMM, Universitat Politècnica de València, 46022 Valencia, Spain*
[e] *Department of Mathematics, Technische Universität Ilmenau, Postfach 100565, 98648 Ilmenau, Germany*

Abstract

Given two linear relations S and T in $\mathbb{C}^n$, we characterize when there exist linear relations $\tilde{S}$ and $\tilde{T}$ in $\mathbb{C}^n$, strictly equivalent to S and T, respectively, such that $\tilde{S}$ and $\tilde{T}$ are one dimensional perturbations of each other; i.e., $\tilde{S} \cap \tilde{T}$ is a linear subspace of codimension at most one both in $\tilde{S}$ and in $\tilde{T}$, but it cannot be equal to both of them simultaneously.

The result is achieved relating one dimensional perturbations of linear relations with rank-one perturbations of matrix pencils.

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1. Introduction

Linear relations are a natural generalization of linear operators and they can be traced back to [1], see also [7]. Linear relations in $\mathbb{C}^n$ are linear subspaces of $\mathbb{C}^n \times \mathbb{C}^n$. There is a well developed spectral theory for linear relations in $\mathbb{C}^n$, which is mainly expressed in terms of (proper) eigenvalues, Jordan and singular chains, and multishifts, see [1, 4, 5, 6, 7, 8, 9, 1].

*Corresponding author
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Linear relations are closely related to matrix pencils. In fact, associated to a matrix pencil

$$P(s) = sE - F \in \mathbb{C}[s]^{n\times m},$$

there are two linear relations: the kernel and the range representations. The kernel representation of $P(s)$ is

$$E^{-1}F = N(\begin{bmatrix} F & -E \end{bmatrix}) \subseteq \mathbb{C}^m \times \mathbb{C}^m,$$

where $N(X)$ stands for the kernel of a matrix X, and its range representation is given by

$$FE^{-1} = R\left(\begin{bmatrix} E \\ F \end{bmatrix}\right) \subseteq \mathbb{C}^n \times \mathbb{C}^n.$$

where $R(X)$ stands for the range of a matrix X.

Recently, the notion of the Weyr characteristic for a linear relation in $\mathbb{C}^n$ was introduced, both as a tool for developing a canonical form for linear relations in finite dimensional vector spaces [6] and also to relate the Kronecker invariants of a matrix pencil with the invariants of its kernel and range representations [1]. The Weyr characteristics of the kernel and range representations of a pencil $P(s)$ can be recovered from the Weyr characteristic of $P(s)$ [1]. Conversely, given a linear relation S, it is possible to find matrix pencils of different sizes whose range or kernel representations are S. This correspondence was used in [1, 1] to obtain new perturbation results for the Kronecker form of matrix pencils under rank-one perturbations.

The main goal of this paper is to use this correspondence in the opposite direction. Let S and T be linear relations in $\mathbb{C}^n$. Our aim is to find in terms of the Weyr characteristic of S and T a full characterization for the fact that there exist linear relations $\tilde{S}$ and $\tilde{T}$ in $\mathbb{C}^n$ which are strictly equivalent to S and T,

$$\bar{S} \overset{s.e.}{\sim} S \quad \text{and} \quad \bar{T} \overset{s.e.}{\sim} T$$

such that $\tilde{S}$ and $\tilde{T}$ are one dimensional perturbations of each other, i.e.

$$\max\left\{ \dim(\tilde{S}/\tilde{S}\cap\tilde{T}), \dim(\tilde{T}/\tilde{S}\cap\tilde{T})\right\} = 1.$$

In order to solve this problem we translate it into a matrix pencil completion problem, which in turn is related to the rank-one perturbation problem of matrix pencils. The solution to the problem we provide here, uses results from [2, 3, 1, 1].

The paper is organized as follows. Section 2 is devoted to preliminaries. In Section 3 we present some properties of linear relations, mainly of the kernel and range representations of matrix pencils. In Section 4 we state the problem to be studied, and we relate it to a matrix pencil completion problem. Section 5 contains some known results about matrix pencil completion problems which are used later, and we show that they can be stated in terms of the Weyr characteristics of the pencils involved. In Section 6 we obtain necessary conditions to solve the problem, and in Section 7 we solve the problem completely. Finally, in the Appendix we include some technical results.

2. Preliminaries

Let $\mathbb{C}$ be the field of complex numbers, and $\bar{\mathbb{C}}$ the extended field $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. $\mathbb{C}[s]$ denotes the ring of polynomials in the indeterminate s with coefficients in $\mathbb{C}$, $\mathbb{C}^{n\times m}$ is the vector space of $n \times m$ matrices over $\mathbb{C}$, and $\mathbb{C}[s]^{n\times m}$ is the bimodule of $n \times m$ matrices over $\mathbb{C}[s]$. $\mathrm{Gl}_n(\mathbb{C})$ is the general linear group of invertible matrices in $\mathbb{C}^{n\times n}$.

Given a matrix $X \in \mathbb{C}^{n\times m}$, $R(X) \subseteq \mathbb{C}^n$ is the subspace spanned by the columns of X and $N(X) \subseteq \mathbb{C}^m$ is the kernel of X.

In the main results below we compare finite sequences of different length. In order to simplify statements, we use two different types of sequences of integers: partitions and sequences.

A partition is a finite or infinite sequence of nonnegative integers $a = (a_1, a_2, \ldots)$, almost all being zero, such that $a_1 \geq a_2 \geq \ldots$. The number of nonzero components of a is the length of a (denoted $\ell(a)$) and $|a|$ is the sum of the components of a, i.e., $|a| = \sum_{i=1}^{\ell(a)} a_i$. Given a finite partition $a = (a_1, a_2, \ldots, a_n)$, if necessary, we take $a_i = 0$ if $i > n$. We identify two partitions that differ only in the number of zero components. The conjugate of a partition $a = (a_1, a_2, \ldots)$ is the partition $\bar{a} = (\bar{a}_1, \bar{a}_2, \ldots)$, where $\bar{a}_k := \#\{i : a_i \geq k\},\ k \geq 1$. Whenever we have a collection of partitions $w = (w^1, \ldots, w^s)$, we denote $|w| = \sum_{i=1}^{s} |w^i|$.

We will also work with finite sequences of integers $\mathbf{c} = (c_1, c_2, \ldots, c_m)$ such that $c_1 \geq c_2 \geq \ldots \geq c_m$. Observe that finite sequences have a fixed number of components, and they are written in bold to distinguish them from partitions. When necessary, we take $c_i = +\infty$ if $i < 1$ and $c_i = -\infty$ if $i > m$. As before, $|\mathbf{c}| = \sum_{i=1}^{m} c_i$. All along this paper, the sequences of integers involved have nonnegative components. The conjugate of a finite sequence of nonnegative integers $\mathbf{c} = (c_1, \ldots, c_m)$ is the conjugate partition of the partition $c = (c_1, \ldots, c_m, 0, \ldots)$. When necessary, we define the term $\bar{c}_0 = \#\{i : c_i \geq 0\} = m$.

A polynomial matrix of the form $P(s) = sE - F,\ E, F \in \mathbb{C}^{n\times m}$, is a matrix pencil. For basic notions on matrix pencils we refer to [1, Chapter XII]. Two matrix pencils $P_1(s) = sE_1 - F_1$ and $P_2(s) = sE_2 - F_2$ in $\mathbb{C}[s]^{n\times m}$ are strictly equivalent, denoted $P_1(s) \overset{s.e.}{\sim} P_2(s)$, if there exist invertible matrices $U \in \mathrm{Gl}_n(\mathbb{C})$, $V \in \mathrm{Gl}_m(\mathbb{C})$, such that $P_2(s) = UP_1(s)V$.

Given a matrix pencil $P(s) \in \mathbb{C}[s]^{n\times m}$, the normal rank of $P(s)$, denoted $\mathrm{rank}(P(s))$, is the order of the largest nonidentically zero minor of $P(s)$, i.e., it is the rank of $P(s)$ considered as a matrix on the field of fractions of $\mathbb{C}[s]$. The spectrum of the pencil $P(s) = sE - F$, denoted $\Lambda(P(s))$, is defined as $\Lambda(P(s)) = \{\lambda \in \bar{\mathbb{C}} : \mathrm{rank}(P(\lambda)) < \mathrm{rank}(P(s))\}$, where we agree that $P(\infty) = E$. The elements $\lambda \in \Lambda(P(s))$ are the eigenvalues of $P(s)$. If $\mathrm{rank}(P(s)) = r$ and $\Lambda(P(s)) = \{\lambda_1, \ldots, \lambda_\ell\}$, then the Kronecker invariants of $P(s)$ are ℓ partitions $n(\lambda_i) = (n_1(\lambda_i), n_2(\lambda_i), \ldots)$, where $n(\lambda_i)$ is the Segre characteristic at λ_i of $P(s)$ and $\ell(n(\lambda_i)) \leq r$, and two sequences of nonnegative integers, $\boldsymbol{\epsilon} = (\epsilon_1, \ldots, \epsilon_{m-r})$, $\boldsymbol{\eta} = (\eta_1, \ldots, \eta_{n-r})$, called column minimal indices and row minimal indices of

$P(s)$, respectively. The integers $\epsilon_{m-r} \leq \cdots \leq \epsilon_1$ ($\eta_{n-r} \leq \cdots \leq \eta_1$) are the increasing ordered list of degrees of the vector polynomials in any minimal basis of the right (left) null-space of $P(s)$ ([1]). They satisfy:

$$\sum_{i=1}^{\ell} |n(\lambda_i)| + |\boldsymbol{\epsilon}| + |\boldsymbol{\eta}| = r. \tag{1}$$

For $\lambda \in \bar{\mathbb{C}} \setminus \Lambda(P(s))$ we define $n(\lambda) = (0, 0, \dots)$.

The homogeneous invariant factors of $P(s)$ are homogeneous polynomials in the indeterminate s and t, $\phi_1(s,t) \mid \cdots \mid \phi_r(s,t)$, that collect the information of the finite and infinite eigenvalues. They are defined as

$$\phi_j(s,t) = t^{n_{r-j+1}(\infty)} \prod_{\lambda \in \Lambda(P(s)) \setminus \{\infty\}} (s - \lambda t)^{n_{r-j+1}(\lambda)}, \quad 1 \leq j \leq r.$$

As usual, we take $\phi_j(s,t) = 1$ for $j < 1$, and $\phi_j(s,t) = 0$ for $j > r$.

Following the notation of [1], we define a partition α and two finite sequences of nonnegative integers $\boldsymbol{\beta}, \boldsymbol{\gamma}$, as

$$\alpha = n(\infty), \quad \boldsymbol{\beta} = (\epsilon_1 + 1, \dots, \epsilon_{m-r} + 1), \quad \boldsymbol{\gamma} = (\eta_1 + 1, \dots, \eta_{n-r} + 1).$$

The Weyr characteristic of $P(s)$ is (w, b, c), where

$$w = (w(\lambda_1), \dots, w(\lambda_\ell)), \quad w(\lambda_i) = \overline{n(\lambda_i)}, \quad 1 \leq i \leq \ell,$$

$$b = \overline{\boldsymbol{\beta}}, \quad c = \overline{\boldsymbol{\gamma}},$$

(see [1, Definition 4.1]). Notice that (1) is equivalent to

$$|w| + (|b| - b_1) + (|c| - c_1) = r. \tag{2}$$

For $\lambda \in \bar{\mathbb{C}} \setminus \Lambda(P(s))$ we define $w(\lambda) = (0, 0, \dots)$.

Two matrix pencils $P_1(s)$ and $P_2(s)$ are strictly equivalent if and only if their Weyr characteristics (equivalently, their Kronecker invariants) coincide ([1, Chapter XII, Theorem 5]). A canonical form for the strict equivalence of matrix pencils is the Kronecker canonical form. It is a matrix pencil of the form

$$P_c(s) = sE_c - F_c = \begin{bmatrix} sI_{n_0} - J_0 & O & O & O \\ O & sN_\alpha - I_{|\alpha|} & O & O \\ O & O & sK_{\boldsymbol{\beta}} - L_{\boldsymbol{\beta}} & O \\ O & O & O & sK_{\boldsymbol{\gamma}}^T - L_{\boldsymbol{\gamma}}^T \end{bmatrix}, \tag{3}$$

where $n_0 = \sum_{\lambda \in \Lambda(P(s)) \cap \mathbb{C}} |n(\lambda)|$ and J_0 is a diagonal of Jordan blocks, $N_\alpha = \operatorname{diag}(N_{\alpha_1}, \dots, N_{\alpha_{\ell(\alpha)}})$,

$$N_k = \begin{bmatrix} 0 & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix} \in \mathbb{C}^{k \times k},$$

$$K_{\boldsymbol{\beta}} = \left[\operatorname{diag}(K_{\beta_1}, \dots, K_{\beta_{m_\beta}}) \quad O_{(|\boldsymbol{\beta}|-m+r)\times(m-r-m_\beta)}\right],$$
$$L_{\boldsymbol{\beta}} = \left[\operatorname{diag}(L_{\beta_1}, \dots, L_{\beta_{m_\beta}}) \quad O_{(|\boldsymbol{\beta}|-m+r)\times(m-r-m_\beta)}\right],$$

$m_\beta = \#\{i : \beta_i \geq 2\}$,

$$K_{\boldsymbol{\gamma}}^T = \begin{bmatrix} \operatorname{diag}(K_{\gamma_1}^T, \dots, K_{\gamma_{m_\gamma}}^T) \\ O_{(n-r-m_\gamma)\times(|\boldsymbol{\gamma}|-n+r)} \end{bmatrix},$$

$$L_{\boldsymbol{\gamma}}^T = \begin{bmatrix} \operatorname{diag}(L_{\gamma_1}^T, \dots, L_{\gamma_{m_\gamma}}^T) \\ O_{(n-r-m_\gamma)\times(|\boldsymbol{\gamma}|-n+r)} \end{bmatrix},$$

$m_\gamma = \#\{i : \gamma_i \geq 2\}$, and for $k > 1$,

$$K_k = \begin{bmatrix} 1 & 0 & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & 1 & 0 \end{bmatrix}, \quad L_k = \begin{bmatrix} 0 & 1 & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \end{bmatrix} \in \mathbb{C}^{(k-1)\times k}.$$

Observe that $m_\beta = b_2$, and $m - r - m_\beta$ is the number of column minimal indices equal to 0, i.e., $m - r - m_\beta = \#\{i : \beta_i = 1\} = b_1 - b_2$. Analogously, $m_\gamma = c_2$, and $n - r - m_\gamma$ is the number of row minimal indices equal to 0, i.e., $n - r - m_\gamma = \#\{i : \gamma_i = 1\} = c_1 - c_2$.

Along this work, a linear relation S is a linear subspace of $\mathbb{C}^m \times \mathbb{C}^n$. A matrix $X \in \mathbb{C}^{n\times m}$ can be identified with a linear relation in $\mathbb{C}^m \times \mathbb{C}^n$ via its graph $\Gamma(X) := \{(x, Xx) : x \in \mathbb{C}^m\}$. For basic notions and properties of linear relations we refer to [1, 1].

Here, we denote the *domain* and the *range* of a linear relation S in $\mathbb{C}^m \times \mathbb{C}^n$ by $\operatorname{dom} S$ and $R(S)$, respectively,

$$\operatorname{dom} S = \{x \in \mathbb{C}^m \; : \; (x, y) \in S \text{ for some } y \in \mathbb{C}^n\} \quad \text{and}$$
$$R(S) = \{y \in \mathbb{C}^n \; : \; (x, y) \in S \text{ for some } x \in \mathbb{C}^m\}.$$

Furthermore, $N(S)$ and $\operatorname{mul}(S)$ denote the *kernel* and the *multivalued* part of S,

$$N(S) = \{x \in \mathbb{C}^m \; : \; (x, 0) \in S\} \quad \text{and} \quad \operatorname{mul}(S) = \{y \in \mathbb{C}^n \; : \; (0, y) \in S\}.$$

Given a linear relation S and $\lambda \in \mathbb{C}$, define $\lambda S := \{(x, \lambda y) \; : \; (x, y) \in S\}$. For relations S and T in $\mathbb{C}^m \times \mathbb{C}^n$ the operator-like sum $S + T$ is the relation defined by

$$S + T = \{(x, y + z) \; : \; (x, y) \in S, \; (x, z) \in T\}.$$

The product of a linear relation S in $\mathbb{C}^m \times \mathbb{C}^n$ and a linear relation T in $\mathbb{C}^k \times \mathbb{C}^m$ is the linear relation in $\mathbb{C}^k \times \mathbb{C}^n$ defined by

$$ST = \{(x, z) \; : \; (y, z) \in S \text{ and } (x, y) \in T \text{ for some } y \in \mathbb{C}^m\}.$$

The inverse S^{-1} of a linear relation S in $\mathbb{C}^m \times \mathbb{C}^n$ always exists, and it is the linear relation in $\mathbb{C}^n \times \mathbb{C}^m$ given by

$$S^{-1} = \{(y,x) \ : \ (x,y) \in S\}.$$

In the following, we consider the case $m = n$. As mentioned in the Introduction, we call a linear relation in $\mathbb{C}^n \times \mathbb{C}^n$ briefly a linear relation in $\mathbb{C}^n$.

Let S be a linear relation in $\mathbb{C}^n$ and let $M \in \mathbb{C}^{n\times n}$. We denote by $\left[\begin{smallmatrix} M & 0 \\ 0 & M \end{smallmatrix}\right] \cdot S$ the action of M onto S:

$$\begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix} \cdot S = \{(Mx, My) \in \mathbb{C}^n \times \mathbb{C}^n : (x,y) \in S\}.$$

Two linear relations S and T are strictly equivalent, denoted $S \overset{s.e.}{\sim} T$, if there exists $P \in \mathrm{Gl}_n(\mathbb{C})$ such that

$$T = \begin{bmatrix} P & 0 \\ 0 & P \end{bmatrix} \cdot S.$$

The notions of eigenvalue, root manifolds and point spectrum also apply to linear relations. To this end, we introduce the identity relation which is given by $I = \{(x,x) \mid x \in \mathbb{C}^n\}$ (i.e. the graph of the identity matrix in $\mathbb{C}^n$). Given $\lambda \in \mathbb{C}$, $S - \lambda$ stands for the linear relation $S - \lambda I$:

$$S - \lambda = \{(x, y - \lambda x) \ : \ (x,y) \in S\}.$$

Then, $\lambda \in \mathbb{C}$ is an eigenvalue of S if $N(S-\lambda) \neq \{0\}$. On the other hand, we say that S has an eigenvalue at ∞ if $\mathrm{mul}(S) \neq \{0\}$. The point spectrum of S is the set $\sigma_p(S)$ consisting of the eigenvalues $\lambda \in \mathbb{C} \cup \{\infty\}$ of S.

We denote $S^0 = I$, where I denotes the identity relation. For $k = 1, 2, \ldots$ the k-th power of S is defined recursively by

$$S^k = S \cdot S^{k-1}.$$

The root manifolds $\mathcal{R}_\lambda(S)$ and $\mathcal{R}_\infty(S)$ are given by

$$\mathcal{R}_\lambda(S) = \bigcup_{k\in\mathbb{N}} N(S-\lambda)^k \quad \text{and} \quad \mathcal{R}_\infty(S) = \bigcup_{k\in\mathbb{N}} \mathrm{mul}(S^k),$$

respectively. Note that $\mathcal{R}_\lambda(S) = \mathcal{R}_0(S-\lambda)$ and $\mathcal{R}_\infty(S) = \mathcal{R}_\infty(S-\lambda)$ for all $\lambda \in \mathbb{C}$, see [5, Lemma 3.1].

The singular chains subspace associated to S is defined by

$$\mathcal{R}_c(S) = \mathcal{R}_0(S) \cap \mathcal{R}_\infty(S).$$

For $\lambda, \mu \in \mathbb{C} \cup \{\infty\}$ with $\lambda \neq \mu$, in [5] it is shown that

$$\mathcal{R}_c(S) = \mathcal{R}_\lambda(S) \cap \mathcal{R}_\mu(S).$$

If $\mathcal{R}_c(S) \neq \{0\}$ then $\sigma_p(S) = \mathbb{C} \cup \{\infty\}$ [1, Proposition 3.2]. Otherwise, if $\mathcal{R}_c(S) = \{0\}$, then the number of different eigenvalues in $\sigma_p(S)$ is bounded by the space dimension n. [1, Proposition 3.2, Lemma 4.3 and Theorem 4.6]. Based on this, we call an eigenvalue $\lambda \in \sigma_p(S)$ degenerate if $\mathcal{R}_\lambda(S) \subset \mathcal{R}_c(S)$ and proper otherwise. The set of proper eigenvalues of a linear relation S (see [1, Section 2] for the definition) will be denoted by $\Lambda(S)$.

Definition 2.1. Let S be a linear relation in $\mathbb{C}^n$ and let $\lambda \in \mathbb{C}$.

1. The Weyr characteristic corresponding to λ is given by the sequence $W(\lambda) = (W_k(\lambda))_{k\geq 1}$, where
$$W_k(\lambda) = \dim \frac{N((S-\lambda)^k) + \mathcal{R}_c(S)}{N((S-\lambda)^{k-1}) + \mathcal{R}_c(S)}.$$
2. The Weyr characteristic corresponding to ∞ is given by the sequence $A = (A_k)_{k\geq 1}$, where
$$A_k = \dim \frac{\operatorname{mul}(S^k) + \mathcal{R}_c(S)}{\operatorname{mul}(S^{k-1}) + \mathcal{R}_c(S)}.$$
3. The Weyr characteristic corresponding to the singular chains subspace is defined by the sequence $B = (B_k)_{k\geq 1}$, where
$$B_k = \dim \frac{N(S^k) \cap \mathcal{R}_c(S)}{N(S^{k-1}) \cap \mathcal{R}_c(S)}.$$
4. The Weyr characteristic corresponding to the multi-shifts is the sequence $C = (C_k)_{k\geq 1}$, where
$$C_k = \dim \frac{R(S^k) + \mathcal{R}_0(S)}{R(S^{k+1}) + \mathcal{R}_0(S)}.$$

Observe that $W(\lambda)$ is the zero sequence if λ is not a proper eigenvalue of S. Similarly, the sequence (A_k) is the zero sequence if ∞ is not a proper eigenvalue of S.

Each of the sequences $W(\lambda_j)$, A, B, and C is non-increasing [5]. Since the underlying space is finite-dimensional each of the Weyr characteristic sequences contains only a finite number of non-zero terms. In this work we set $W(\infty) = A$, and we collect the Weyr characteristics corresponding to the different proper eigenvalues in a single sequence: if these are $\lambda_1, \dots, \lambda_\ell \in \mathbb{C} \cup \{\infty\}$, we set
$$W = (W(\lambda_1), W(\lambda_2), \dots, W(\lambda_\ell)).$$

Definition 2.2. *The collection of partitions*
$$\big(W, B, C\big)$$
is called the Weyr characteristic of the linear relation S.

Two linear relations S and T are strictly equivalent if and only if their Weyr characteristics coincide ([1, Theorem 5.4]).

3. Linear relations and their dimension

In this section we show that there exists a close relationship between linear relations and matrix pencils, and we analyze how the corresponding Weyr characteristics are related.

We have to introduce some additional notions about linear relations which depend on a (fixed) underlying inner product in $\mathbb{C}^n$. The adjoint of S is the linear relation S^* in $\mathbb{C}^n$ given by

$$S^* = \{(u,v) \in \mathbb{C}^n \times \mathbb{C}^n : \ \langle x , v \rangle = \langle y , u \rangle \text{ for every } (x,y) \in S\},$$

where $\langle \cdot , \cdot \rangle$ stands for the inner product in $\mathbb{C}^n$. Geometrically, if $S^\perp$ denotes the orthogonal complement of S in $\mathbb{C}^n \times \mathbb{C}^n$, the adjoint of S can be described as $S^* = (-S^\perp)^{-1}$, where $-S = \{(x,-y) : \ (x,y) \in S\}$ (see [1]).

Given a matrix pencil $P(s) = sE - F \in \mathbb{C}[s]^{n\times m}$, the range and the kernel representations of $P(s)$ are the linear relations

$$FE^{-1} = \{(Ex, Fx) : x \in \mathbb{C}^m\} = R\left(\begin{bmatrix} E \\ F \end{bmatrix}\right) \subseteq \mathbb{C}^n \times \mathbb{C}^n,$$

and

$$E^{-1}F = \{(x,y) \in \mathbb{C}^m \times \mathbb{F}^m : Fx = Ey\} = N\left(\begin{bmatrix} F & -E \end{bmatrix}\right) \subseteq \mathbb{C}^m \times \mathbb{C}^m,$$

respectively, see [1]. Then,

$$\dim FE^{-1} = \operatorname{rank}\left(\begin{bmatrix} E \\ F \end{bmatrix}\right) \quad \text{and} \quad \dim E^{-1}F = 2m - \operatorname{rank}\left(\begin{bmatrix} F & -E \end{bmatrix}\right).$$

Remark 3.1. *Given $P(s) = sE - F \in \mathbb{C}[s]^{n\times m}$, define $P^*(s) := sE^* - F^* \in \mathbb{C}[s]^{m\times n}$. Then,*

$$(FE^{-1})^* = (F^*)^{-1}E^* \quad \textit{and} \quad (E^{-1}F)^* = F^*(E^*)^{-1},$$

i.e. the adjoint of the range representation of $P(s)$ is the kernel representation of $P^*(s)$ and the adjoint of the kernel representation of $P(s)$ is the range representation of $P^*(s)$.

In the next remark we analyze when two pencils have the same range or the same kernel representation. Also, we show that given a pencil, there exists another one having the same range (kernel) representation and minimal number of columns (rows).

Remark 3.2. *Given two pencils $P(s) = sE-F, \bar{P}(s) = s\bar{E}-\bar{F} \in \mathbb{C}[s]^{n\times m}$, then $FE^{-1} = \bar{F}\bar{E}^{-1}$ if and only if there exists an invertible matrix $V \in \mathrm{Gl}_m(\mathbb{C})$ such that $\begin{bmatrix} E \\ F \end{bmatrix} V = \begin{bmatrix} \bar{E} \\ \bar{F} \end{bmatrix}$, equivalently, $\bar{P}(s) = P(s)V$. Moreover, if $\dim FE^{-1} = d$, $V \in \mathrm{Gl}_m(\mathbb{C})$ can be chosen such that $\begin{bmatrix} E \\ F \end{bmatrix} V = \begin{bmatrix} E_1 & O \\ F_1 & O \end{bmatrix}$, where $\begin{bmatrix} E_1 \\ F_1 \end{bmatrix} \in$*

$\mathbb{C}^{(n+n)\times d}$ *has full (column) rank. Hence,* $P_1(s) = sE_1 - F_1 \in \mathbb{C}[s]^{n\times d}$ *has the same range representation as* $P(s)$*, i.e.* $F_1E_1^{-1} = FE^{-1}$*, and minimal number of columns.*

Analogously, $E^{-1}F = \bar{E}^{-1}\bar{F}$ *if and only if there exists* $U \in \mathrm{Gl}_n(\mathbb{C})$ *such that* $U\begin{bmatrix} F & -E \end{bmatrix} = \begin{bmatrix} \bar{F} & -\bar{E} \end{bmatrix}$*, equivalently,* $\bar{P}(s) = UP(s)$*. If* $\dim FE^{-1} = 2m - r$*, then* $U \in \mathrm{Gl}_n(\mathbb{C})$ *can be chosen such that* $U\begin{bmatrix} F & -E \end{bmatrix} = \begin{bmatrix} F_1 & -E_1 \\ O & O \end{bmatrix}$*, where* $\begin{bmatrix} F_1 & -E_1 \end{bmatrix} \in \mathbb{C}^{r\times(m+m)}$ *has full (row) rank. Hence,* $P_1(s) = sE_1 - F_1 \in \mathbb{C}[s]^{r\times m}$ *has the same kernel representation as* $P(s)$*, i.e.* $E_1^{-1}F_1 = E^{-1}F$*.*

Lemma 3.3 ([4, Theorem 3.3])**.** *Let* S *be a linear relation in* $\mathbb{C}^n$ *with* $\dim S = d$*. Then there exists a pencil* $P(s) = sE - F \in \mathbb{C}[s]^{n\times d}$ *with* $\operatorname{rank}\left(\begin{bmatrix} E \\ F \end{bmatrix}\right) = d$ *such that* $S = FE^{-1}$*.*

Moreover, for $r = 2n - d$ *there exists a pencil* $Q(s) = sG - H \in \mathbb{C}[s]^{r\times n}$ *with* $\operatorname{rank}(\begin{bmatrix} H & -G \end{bmatrix}) = r$ *such that* $S = G^{-1}H$*.*

The following lemma can be derived from [4, 1]. For simplicity we present here a short proof.

Lemma 3.4. *Given matrix pencils* $P(s) = sE - F$ *and* $\bar{P}(s) = s\bar{E} - \bar{F}$ *in* $\mathbb{C}[s]^{n\times m}$*, let* $S = E^{-1}F$ *and* $\bar{S} = \bar{E}^{-1}\bar{F}$ *be their kernel representations (resp. let* $S = FE^{-1}$ *and* $\bar{S} = \bar{F}\bar{E}^{-1}$ *be their range representations). Then,*

$$P(s) \overset{s.e.}{\sim} \bar{P}(s) \Leftrightarrow S \overset{s.e.}{\sim} \bar{S}.$$

Proof. If the matrix pencils are strictly equivalent, the strict equivalence of their kernel representations follows from [1, Proposition 4.3].

Conversely, if $S = E^{-1}F \overset{s.e.}{\sim} \bar{S} = \bar{E}^{-1}\bar{F}$ then there exists $T \in \mathrm{Gl}_m(\mathbb{C})$ such that $\bar{S} = \begin{bmatrix} T & O \\ O & T \end{bmatrix} \cdot S$. Therefore,

$$\begin{aligned}
\bar{S} &= \{(Tx_1, Ty_1) \ : \ (x_1, y_1) \in S\} = \{(Tx_1, Ty_1) \ : \ Fx_1 = Ey_1\} \\
&= \{(x_2, y_2) \ : \ FT^{-1}x_2 = ET^{-1}y_2\} = N\left(\begin{bmatrix} FT^{-1} & -ET^{-1} \end{bmatrix}\right) \\
&= (ET^{-1})^{-1}(FT^{-1}).
\end{aligned}$$

Now, let $P'(s) = P(s)T^{-1} = sET^{-1} - FT^{-1}$. Since $\bar{E}^{-1}\bar{F} = (ET^{-1})^{-1}(FT^{-1})$, by Remark 3.2 there exists $U \in \mathrm{Gl}_n(\mathbb{C})$ such that $P'(s) = U\bar{P}(s)$. Hence, $P(s) = U\bar{P}(s)T$.

The result for range representations can be proved similarly. □

In the next lemma we calculate the dimensions of the range and kernel representations of a given pencil.

Lemma 3.5. *Let* $P(s) = sE - F \in \mathbb{C}[s]^{n\times m}$ *be a matrix pencil with Weyr characteristic* (w, b, c)*. Then,*

$$\dim FE^{-1} = m - b_1 + b_2, \quad \dim E^{-1}F = 2m - n + c_1 - c_2.$$

Proof. Notice that $P(s) \overset{s.e.}{\sim} P_c(s)$, where $P_c(s) = sE_c - F_c$ is its Kronecker canonical form (3). By Lemma 3.4, we have that $E^{-1}F \overset{s.e.}{\sim} E_c^{-1}F_c$ and $FE^{-1} \overset{s.e.}{\sim} F_cE_c^{-1}$. Then, $\dim E^{-1}F = \dim E_c^{-1}F_c$ and $\dim FE^{-1} = \dim F_cE_c^{-1}$.

It is easy to see that rank $\begin{bmatrix} K_{\boldsymbol{\beta}} \\ L_{\boldsymbol{\beta}} \end{bmatrix} = |\boldsymbol{\beta}| - b_1 + b_2$ and rank $\begin{bmatrix} K_{\boldsymbol{\gamma}}^T \\ L_{\boldsymbol{\gamma}}^T \end{bmatrix} = |\boldsymbol{\gamma}| - c_1$. Hence,

$$\begin{aligned}\dim FE^{-1} &= \operatorname{rank}\begin{bmatrix} E_c \\ F_c \end{bmatrix} = \operatorname{rank}\begin{bmatrix} I_{n_0} \\ J_0 \end{bmatrix} + \operatorname{rank}\begin{bmatrix} N_\alpha \\ I_{|\alpha|} \end{bmatrix} + \operatorname{rank}\begin{bmatrix} K_{\boldsymbol{\beta}} \\ L_{\boldsymbol{\beta}} \end{bmatrix} + \operatorname{rank}\begin{bmatrix} K_{\boldsymbol{\gamma}}^T \\ L_{\boldsymbol{\gamma}}^T \end{bmatrix} \\ &= n_0 + |\alpha| + |\boldsymbol{\beta}| + |\boldsymbol{\gamma}| - c_1 - b_1 + b_2 = m - b_1 + b_2.\end{aligned}$$

Analogously, rank $\begin{bmatrix} L_{\boldsymbol{\beta}} & -K_{\boldsymbol{\beta}} \end{bmatrix} = |\boldsymbol{\beta}| - b_1$ and rank $\begin{bmatrix} L_{\boldsymbol{\gamma}}^T & -K_{\boldsymbol{\gamma}}^T \end{bmatrix} = |\boldsymbol{\gamma}| - c_1 + c_2$, meanwhile $\begin{bmatrix} J_0 & -I_{n_0} \end{bmatrix}$ and $\begin{bmatrix} I_{|\alpha|} & -N_\alpha \end{bmatrix}$ have full rank. Therefore,

$$\operatorname{rank}\begin{bmatrix} F_c & -E_c \end{bmatrix} = n_0 + |\alpha| + |\boldsymbol{\beta}| - b_1 + |\boldsymbol{\gamma}| - c_1 + c_2,$$

and

$$\begin{aligned}\dim E^{-1}F &= 2m - \operatorname{rank}\begin{bmatrix} F_c & -E_c \end{bmatrix} = 2m - (n_0 + |\alpha| + |\boldsymbol{\beta}| - b_1 + |\boldsymbol{\gamma}| - c_1 + c_2) \\ &= 2m - n + c_1 - c_2.\end{aligned}$$

□

As an immediate consequence of Lemma 3.5, we obtain

$$\dim FE^{-1} = m \quad \Leftrightarrow \quad b_1 = b_2,$$

and

$$\dim E^{-1}F = 2m - n \quad \Leftrightarrow \quad c_1 = c_2.$$

In [1] the relationship between the eigenvalues and the Weyr characteristic of a matrix pencil and those of its kernel and range representations was obtained. We state those results adapting the notation to the one used in this work.

Lemma 3.6 ([1, Proposition 4.2]). *Let $P(s) = sE - F \in \mathbb{C}[s]^{n\times m}$ be a matrix pencil, then*

$$\Lambda(P(s)) = \Lambda(E^{-1}F) = \Lambda(FE^{-1}).$$

Lemma 3.7 ([1, Theorem 5.1]). *Let $P(s) = sE - F \in \mathbb{C}[s]^{n\times m}$ be a matrix pencil with Weyr characteristic (w, b, c). If (W, B, C) is the Weyr characteristic of the kernel representation $E^{-1}F$, then $W = w$, $B = b$ and $C = (c_3, c_4, \dots)$.*

Lemma 3.8 ([1, Proposition 5.2]). *Let S be a linear relation in $\mathbb{C}^m$ with Weyr characteristic (W, B, C). If S is the kernel representation of a pencil $P(s) = sE - F \in \mathbb{C}[s]^{n\times m}$, then the Weyr characteristic (w, b, c) of $P(s)$ is given by $w = W$, $b = B$, and $c = (n - m + B_1, m - |W| - |B| - |C|, C_1, C_2, \dots)$.*

Lemma 3.9 ([1, Theorem 6.1]). *Let $P(s) = sE - F \in \mathbb{C}[s]^{n\times m}$ be a matrix pencil with Weyr characteristic (w, b, c). If (W, B, C) is the Weyr characteristic of the range representation FE^{-1}, then $W = w$, $B = (b_2, b_3, \dots)$ and $C = (c_2, c_3, \dots)$.*

Lemma 3.10 ([1, Proposition 6.2]). *Let S be a linear relation in $\mathbb{C}^m$ with Weyr characteristic (W,B,C). If S is the range representation of a pencil $P(s)=sE-F\in\mathbb{C}[s]^{n\times m}$, then the Weyr characteristic (w,b,c) of $P(s)$ is given by $w=W$, $b=(m-|W|-|B|-|C|,B_1,B_2,\dots)$ and $c=(n-|W|-|B|-|C|,C_1,C_2,\dots)$.*

Notice that given a matrix pencil $P(s)=sE-F\in\mathbb{C}[s]^{n\times m}$ with Weyr characteristic (w,b,c), and (W_k,B_k,C_k) and (W_r,B_r,C_r) as Weyr characteristics of its kernel $E^{-1}F\subseteq\mathbb{C}^m\times\mathbb{C}^m$ and range representations $FE^{-1}\subseteq\mathbb{C}^n\times\mathbb{C}^n$, respectively, by (2) and Lemmas 3.5, 3.8 and 3.10 we obtain

$$\begin{aligned}\operatorname{rank}(P(s))&=|W_k|+|B_k|-B_{k,1}+|C_k|+c_2=|W_r|+|B_r|+|C_r|,\qquad(4)\\ \dim E^{-1}F&=2m-n+c_1-c_2=|W_k|+|B_k|+|C_k|+B_{k,1}\quad\text{and}\\ \dim FE^{-1}&=m-b_1+b_2=|W_r|+|B_r|+|C_r|+B_{r,1}.\end{aligned}$$

Lemma 3.11. *Given an integer $n\ge 0$, a finite subset $\{\lambda_1,\dots,\lambda_\ell\}\subset\bar{\mathbb{C}}$, two partitions B, C, and a collection of partitions $W=(W(\lambda_1),\dots,W(\lambda_\ell))$, there exists a linear relation S in $\mathbb{C}^n$ with Weyr characteristic (W,B,C) if and only if*

$$n-(|W|+|B|+|C|)\ge C_1.$$

Proof. Assume that (W,B,C) is the Weyr characteristic of a linear relation S in $\mathbb{C}^n$ and denote $d=\dim S$. By Lemma 3.3, there exists a pencil $P(s)=sE-F\in\mathbb{F}[s]^{n\times d}$ such that $S=FE^{-1}$. Let (w,b,c) the Weyr characteristic of $P(s)$. Then by Lemma 3.9, $w=W$, $b=(b_1,B_1,B_2,\dots)$, $c=(c_1,C_1,C_2,\dots)$, and by (4) we have $\operatorname{rank}(P(s))=|W|+|B|+|C|$ and

$$C_1=c_2\le c_1=n-\operatorname{rank}(P(s))=n-(|W|+|B|+|C|).$$

Conversely, assume that $n-(|W|+|B|+|C|)\ge C_1$. Define $w=W$, $b=(B_1,B_1,B_2,\dots)$, $c=(n-(|W|+|B|+|C|),C_1,C_2,\dots)$ and let $P(s)=sE-F$ be a pencil in Kronecker canonical form with Weyr characteristic (w,b,c)). Note that

$$\operatorname{rank}(P(s))=|w|+(|b|-b_1)+(|c|-c_1)=|W|+|B|+|C|.$$

The number of rows of $P(s)$ is given by $c_1+\operatorname{rank}(P(s))=n-(|W|+|B|+|C|)+\operatorname{rank}(P(s))=n$. Hence, $S:=FE^{-1}$ is a linear relation in $\mathbb{C}^n$ and, by Lemma 3.9, its Weyr characteristic is (W,B,C). □

4. Rank one perturbation of linear relations

Given two linear relations S and T in $\mathbb{C}^n$, consider their intersection $S\cap T$ and the quotient (sub)spaces $\frac{S}{S\cap T}$ and $\frac{T}{S\cap T}$. We define

$$r(S,T)=\max\left\{\dim\frac{S}{S\cap T},\dim\frac{T}{S\cap T}\right\}.$$

Notice that $r(S,T) \geq 0$, and $r(S,T) = 0$ if and only if $S = T$. More precisely, $r(S,T)$ measures the maximum between the codimensions of $S \cap T$ in S and T, respectively. Also, $r(S,T)$ can be alternatively calculated as

$$r(S,T) = \max\left\{\dim \frac{S\hat{+}T}{T}, \dim \frac{S\hat{+}T}{S}\right\},$$

where $S\hat{+}T$ is the sum of S and T as subspaces of $\mathbb{C}^n \times \mathbb{C}^n$:

$$S\hat{+}T = \{(x+u, y+v) : \ (x,y) \in S, \ (u,v) \in T\}.$$

We can represent linear relations in $\mathbb{C}^n$ as range or kernel representations of suitable matrix pencils with n rows or n columns, respectively. In the next lemma we analyze the case when a linear relation contains another one and the difference of their dimensions is r.

Lemma 4.1. *Given two linear relations S and U in $\mathbb{C}^n$ such that $U \subseteq S$, denote $d = \dim S$, $g = \dim U$ and $m = 2n - g = \dim U^{\perp}$. Let $r \geq 1$ be an integer. The following statements are equivalent:*

(i) $\dim \frac{S}{U} = r$.

(ii) There exist pencils $P(s) = sE - F \in \mathbb{C}[s]^{n\times d}$, $P_1(s) = sE_1 - F_1 \in \mathbb{C}[s]^{n\times(d-r)}$ and $P_2(s) = sE_2 - F_2 \in \mathbb{C}[s]^{n\times r}$ such that

$$P(s) = \begin{bmatrix} P_1(s) & P_2(s) \end{bmatrix}, \quad FE^{-1} = S \ \text{ and } \ F_1E_1^{-1} = U.$$

(iii) There exist pencils $Q(s) = sG - H \in \mathbb{C}[s]^{m\times n}$, $Q_1(s) = sG_1 - H_1 \in \mathbb{C}[s]^{(m-r)\times n}$ and $Q_2(s) = sG_2 - H_2 \in \mathbb{C}[s]^{r\times n}$ such that

$$Q(s) = \begin{bmatrix} Q_1(s) \\ Q_2(s) \end{bmatrix}, \quad G_1^{-1}H_1 = S \ \text{ and } \ G^{-1}H = U.$$

Proof. Assume that (i) holds. Then $g = \dim U = d - r$. By Lemma 3.3 there exists a pencil $P_1(s) = sE_1 - F_1 \in \mathbb{C}[s]^{n\times(d-r)}$ such that $\operatorname{rank}\begin{bmatrix} E_1 \\ F_1 \end{bmatrix} = d - r$ and $F_1E_1^{-1} = U$. Note that the columns of $\begin{bmatrix} E_1 \\ F_1 \end{bmatrix}$ form a basis of $U \subset S$. Let $\{(e_1, f_1), \ldots, (e_r, f_r)\}$ be a basis for a subspace $V \subset S$ such that $S = U \dot{+} V$, and let $E_2, F_2 \in \mathbb{C}^{n\times r}$ be the matrices whose columns are $\{e_1, \ldots, e_r\}$ and $\{f_1, \ldots, f_r\}$, respectively. Then, defining $P_2(s) := sE_2 - F_2$ and $P(s) := \begin{bmatrix} P_1(s) & P_2(s) \end{bmatrix} = sE - F$, it is immediate that $FE^{-1} = R\left(\begin{bmatrix} E_1 & E_2 \\ F_1 & F_2 \end{bmatrix}\right) = U \dot{+} V = S$, where $U \dot{+} V$ denotes the direct sum of the subspaces U and V. This proves (ii).

Conversely, assume that (ii) holds. Then, $d = \dim S = \dim R\left(\begin{bmatrix} E \\ F \end{bmatrix}\right) = \dim R\left(\begin{bmatrix} E_1 & E_2 \\ F_1 & F_2 \end{bmatrix}\right)$. Hence, both $\begin{bmatrix} E \\ F \end{bmatrix}$ and $\begin{bmatrix} E_1 \\ F_1 \end{bmatrix}$ have full (column) rank.

Therefore, $g = \dim U = \dim R\left(\begin{bmatrix} E_1 \\ F_1 \end{bmatrix}\right) = d - r$, or equivalently, $\dim \frac{S}{U} = r$. This completes the proof of the equivalence $(i) \Leftrightarrow (ii)$.

The equivalence $(i) \Leftrightarrow (iii)$ is obtained applying the above case to the inclusion $S^\perp \subset U^\perp$, i.e., $\dim \frac{U^\perp}{S^\perp} = r$ if and only if there exist pencils $Q^*(s) = sG^* - H^* \in \mathbb{C}[s]^{n\times m}$, $Q_1^*(s) = sG_1^* - H_1^* \in \mathbb{C}[s]^{n\times(m-r)}$ and $Q_2^*(s) = sG_2^* - H_2^* \in \mathbb{C}[s]^{n\times r}$ such that $Q^*(s) = \begin{bmatrix} Q_1^*(s) & Q_2^*(s) \end{bmatrix}$, $H^*(G^*)^{-1} = U^\perp$ and $H_1^*(G_1^*)^{-1} = S^\perp$. Moreover, $H^*(G^*)^{-1} = U^\perp$ if and only if $G^{-1}H = U$, and $H_1^*(G_1^*)^{-1} = S^\perp$ if and only if $G_1^{-1}H_1 = S$. □

Now we state the main problem studied in this paper.

Problem 4.2. *(rank-one perturbation for linear relations). Given two linear relations S and T in $\mathbb{C}^n$, find necessary and sufficient conditions for the existence of linear relations $\bar{S}$ and $\bar{T}$ in $\mathbb{C}^n$, such that*

$$\bar{S} \overset{s.e.}{\sim} S, \quad \bar{T} \overset{s.e.}{\sim} T \quad \text{and} \quad r(\bar{S}, \bar{T}) = 1.$$

Remark 4.3. *Notice that the case $r(\bar{S}, \bar{T}) = 0$ is trivial, for there exist $\bar{S}, \bar{T}$ in $\mathbb{C}^n$ such that $\bar{S} \overset{s.e.}{\sim} S$, $\bar{T} \overset{s.e.}{\sim} T$ and $r(\bar{S}, \bar{T}) = 0$ if and only if $S \overset{s.e.}{\sim} T$. In fact, if $r(\bar{S}, \bar{T}) = 0$ then $\bar{S} = \bar{T} = \bar{S} \cap \bar{T}$, hence $S \overset{s.e.}{\sim} \bar{S} \overset{s.e.}{\sim} \bar{T} \overset{s.e.}{\sim} T$. Conversely, if $S \overset{s.e.}{\sim} T$, then taking $\bar{S} = \bar{T} = T$ we have $\bar{S} \overset{s.e.}{\sim} S$, $\bar{T} \overset{s.e.}{\sim} T$ and $r(\bar{S}, \bar{T}) = r(T, T) = 0$.*

As mentioned, we deal with linear relations as kernel or range representations of appropriate pencils. Note that every rank one matrix pencil in $\mathbb{C}[s]^{n\times m}$ can be written in one of the following ways:

$$(su - v)w^*, \quad (0, 0) \neq (u, v) \in \mathbb{C}^n \times \mathbb{C}^n,\ 0 \neq w \in \mathbb{C}^m, \tag{5}$$

or

$$w(su^* - v^*), \quad (0, 0) \neq (u, v) \in \mathbb{C}^m \times \mathbb{C}^m,\ 0 \neq w \in \mathbb{C}^n. \tag{6}$$

In [1] we have the following result.

Lemma 4.4 ([1, Lemma 7.3])**.** *Let $P(s) = sE - F, \bar{P}(s) = s\bar{E} - \bar{F} \in \mathbb{C}[s]^{n\times m}$,*

(a) If $\bar{P}(s) - P(s)$ is a rank one matrix pencil as in (5), then $r(FE^{-1}, \bar{F}\bar{E}^{-1}) \leq 1$.

(b) If $\bar{P}(s) - P(s)$ is a rank one matrix pencil as in (6), then $r(E^{-1}F, \bar{E}^{-1}\bar{F}) \leq 1$.

The next two corollaries follow straightforward from Lemma 4.1.

Corollary 4.5. *Given two linear relations S and T in $\mathbb{C}^n$, denote $d = \dim S$, $g = \dim(S \cap T)$ and $m = 2n - g = (S \cap T)^\perp$. Then, the following statements are equivalent:*

(i) $\dim \frac{S}{S\cap T} = 1$ *and* $\dim \frac{T}{S\cap T} = 0$.

(ii) *There exist pencils* $P(s) = sE - F \in \mathbb{C}[s]^{n\times d}$, $P_1(s) = sE_1 - F_1 \in \mathbb{C}[s]^{n\times(d-1)}$ *and* $u(s) = se - f \in \mathbb{C}[s]^{n\times 1}$ *such that* $P(s) = \begin{bmatrix} P_1(s) & u(s) \end{bmatrix}$, $FE^{-1} = S$ *and* $F_1E_1^{-1} = T$.

(iii) *There exist pencils* $Q_1(s) = sG_1 - H_1 \in \mathbb{C}[s]^{(m-1)\times n}$, $Q(s) = sG - H \in \mathbb{C}[s]^{m\times n}$ *and* $v(s) = sg - h \in \mathbb{C}[s]^{n\times 1}$ *such that* $Q(s) = \begin{bmatrix} Q_1(s) \\ v(s)^* \end{bmatrix}$, $G_1^{-1}H_1 = S$ *and* $G^{-1}H = T$.

Corollary 4.6. *Given two linear relations* S *and* T *in* $\mathbb{C}^n$ *denote* $d = \dim S$, $g = \dim(S\cap T)$ *and* $m = 2n - g = \dim(S\cap T)^\perp$. *Then, the following statements are equivalent:*

(i) $\dim \frac{S}{S\cap T} = \dim \frac{T}{S\cap T} = 1$.

(ii) $\dim S = \dim T$ *and there exist pencils* $P(s) = sE - F, \bar{P}(s) = s\bar{E} - \bar{F} \in \mathbb{C}[s]^{n\times d}$, $P_1(s) = sE_1 - F_1 \in \mathbb{C}[s]^{n\times(d-1)}$, *and* $u(s) = se - f, \bar{u}(s) = s\bar{e} - \bar{f} \in \mathbb{C}[s]^{n\times 1}$ *such that* $P(s) = \begin{bmatrix} P_1(s) & u(s) \end{bmatrix}$, $\bar{P}(s) = \begin{bmatrix} P_1(s) & \bar{u}(s) \end{bmatrix}$, $FE^{-1} = S$, $\bar{F}\bar{E}^{-1} = T$ *and* $F_1E_1^{-1} = S\cap T$.

(iii) *There exist pencils* $Q_1(s) = sG_1 - H_1, \bar{Q}_1(s) = s\bar{G}_1 - \bar{H}_1 \in \mathbb{C}[s]^{(m-1)\times n}$, $Q(s) = sG - H, \bar{Q}(s) = s\bar{G} - \bar{H} \in \mathbb{C}[s]^{m\times n}$, *and* $v(s) = sg - h, \bar{v}(s) = s\bar{g} - \bar{h}, \in \mathbb{C}[s]^{n\times 1}$ *such that* $Q(s) = \begin{bmatrix} Q_1(s) \\ v(s)^* \end{bmatrix}$, $\bar{Q}(s) = \begin{bmatrix} \bar{Q}_1(s) \\ \bar{v}(s)^* \end{bmatrix}$, $G_1^{-1}H_1 = S$, $\bar{G}_1^{-1}\bar{H}_1 = T$ *and* $G^{-1}H = \bar{G}^{-1}\bar{H} = S\cap T$.

Now, we can prove the converse of Lemma 4.4.

Theorem 4.7. *Given two linear relations* S *and* T *in* $\mathbb{C}^n$, *denote* $d = \dim S$, $g = \dim T$, *and* $m = \dim T^\perp = 2n - g$. *Then, the following statements are equivalent:*

(i) $r(S,T) \leq 1$.

(ii) *There exist pencils* $P(s) = sE - F, \bar{P}(s) = s\bar{E} - \bar{F} \in \mathbb{C}[s]^{n\times d}$ *such that* $S = FE^{-1}$, $T = \bar{F}\bar{E}^{-1}$ *and*

$$\bar{P}(s) - P(s) = (su - v)w^*, \quad (u,v) \in \mathbb{C}^n \times \mathbb{C}^n, \quad w \in \mathbb{C}^d.$$

(iii) *There exist pencils* $Q(s) = sG - H, \bar{Q}(s) = s\bar{G} - \bar{H} \in \mathbb{C}[s]^{m\times n}$ *such that* $S = G^{-1}H$, $T = \bar{G}^{-1}\bar{H}$ *and*

$$\bar{Q}(s) - Q(s) = w(su^* - v^*), \quad (u,v) \in \mathbb{C}^n \times \mathbb{C}^n, \quad w \in \mathbb{C}^m.$$

Proof. The implications $(ii) \Rightarrow (i)$ and $(iii) \Rightarrow (i)$ are immediate consequences of Lemma 4.4.

Conversely, assume that (i) holds.

If $r(S,T) = 0$, then $S = T$ and $g = d$. By Lemma 3.3 there exists a pencil $P(s) = sE - F \in \mathbb{C}[s]^{n\times d}$ such that $FE^{-1} = S = T$. Taking $\bar{P}(s) = P(s)$, (ii) follows.

If $\dim \frac{S}{S\cap T} = 1$ and $\dim \frac{T}{S\cap T} = 0$, then $g = \dim(S\cap T) = d-1$. By Corollary 4.5 there exist pencils $P(s) = sE - F \in \mathbb{C}[s]^{n\times d}$, $P_1(s) = sE_1 - F_1 \in \mathbb{C}[s]^{n\times (d-1)}$ and $u(s) = se - f \in \mathbb{C}[s]^{n\times 1}$ such that $P(s) = \begin{bmatrix} P_1(s) & u(s)\end{bmatrix}$, $FE^{-1} = S$ and $F_1E_1^{-1} = T$. Let $\bar{P}(s) = s\bar{E} - \bar{F} = \begin{bmatrix} sE_1 - F_1 & O\end{bmatrix} \in \mathbb{C}[s]^{n\times d}$. Then $\bar{F}\bar{E}^{-1} = F_1E_1^{-1} = T$ and $\bar{P}(s) - P(s) = \begin{bmatrix} O & -u(s)\end{bmatrix} = u(s)w^*$, where $w^* = \begin{bmatrix} O & -1\end{bmatrix} \in \mathbb{C}[s]^{1\times((d-1)+1)}$.

If $\dim \frac{S}{S\cap T} = 0$ and $\dim \frac{T}{S\cap T} = 1$, the proof is analogous.

If $\dim \frac{S}{S\cap T} = \dim \frac{T}{S\cap T} = 1$, then $g = d$. By Corollary 4.6 there exist pencils $P(s) = sE - F, \bar{P}(s) = s\bar{E} - \bar{F} \in \mathbb{C}[s]^{n\times d}$, $P_1(s) = sE_1 - F_1 \in \mathbb{C}[s]^{n\times(d-1)}$, $u(s) = se - f, \bar{u}(s) = s\bar{e} - \bar{f} \in \mathbb{C}[s]^{n\times 1}$ such that $P(s) = \begin{bmatrix} P_1(s) & u(s)\end{bmatrix}$, $\bar{P}(s) = \begin{bmatrix} P_1(s) & \bar{u}(s)\end{bmatrix}$, $FE^{-1} = S$, $\bar{F}\bar{E}^{-1} = T$ and $F_1E_1^{-1} = S\cap T$. Hence,

$$\bar{P}(s) - P(s) = \begin{bmatrix} O & \bar{u}(s) - u(s)\end{bmatrix} = (\bar{u}(s) - u(s))w^*,$$

where $w^* = \begin{bmatrix} O & 1\end{bmatrix} \in \mathbb{C}[s]^{1\times((d-1)+1)}$.

We can prove $(i) \Rightarrow (iii)$ in a similar way, or it can be derived from case (ii) applied to the adjoint pencils (see Remark 3.1).

□

5. Matrix pencil completion theorems

As we have seen in Section 4, the rank perturbation problem of linear relations is related to a matrix pencil completion problem, which consists in describing the possible Kronecker invariants of a pencil with prescribed columns or rows (see [1] for a more general setting). In this section we introduce, in Lemmas 5.2 and 5.4, some known results about the latter problem. Although they are valid over arbitrary fields, here we state them over $\mathbb{C}$. To state the results we need to define the 1step-majorization, which is a particular case of generalized majorization [1, Definition 2].

Definition 5.1. *Given two finite sequences of integers* $\mathbf{g} = (g_1, \dots, g_m)$ *and* $\mathbf{d} = (d_1, \dots, d_{m+1})$*, we say that* $\mathbf{d}$ *is 1step-majorized by* $\mathbf{g}$ *(denoted by* $\mathbf{d} \prec' \mathbf{g}$*) if*

$$g_i = d_{i+1}, \quad h \le i \le m,$$

where $h = \min\{i = 1, \dots, m : \ g_i < d_i\}$.

Lemmas 5.2 and 5.4 are particular cases of [1, Theorem 4.3] and they can also be seen in [2, Lemmas 4.3 and 4.4].

Lemma 5.2. *Given two matrix pencils* $H_1(s) \in \mathbb{C}[s]^{(n+p)\times(n+m)}$, $H(s) \in \mathbb{C}[s]^{(n+p+1)\times(n+m)}$ *of* $\operatorname{rank}(H_1(s)) = \operatorname{rank}(H(s)) = n$*, let* $\pi_1^1(s,t) \mid \cdots \mid \pi_n^1(s,t)$, $g_1 \ge \cdots \ge g_m \ge 0$ *and* $t_1 \ge \cdots \ge t_p \ge 0$ *be the homogeneous invariant factors, the column and the row minimal indices of* $H_1(s)$*, respectively, and let* $\pi_1(s,t) \mid \cdots \mid \pi_n(s,t)$, $k_1 \ge \cdots \ge k_m \ge 0$ *and* $u_1 \ge \cdots \ge u_{p+1} \ge 0$ *be the homogeneous invariant factors, the column and the row minimal indices indices of* $H(s)$*, respectively.*

Let $\mathbf{g} = (g_1, \dots, g_m)$, $\mathbf{t} = (t_1, \dots, t_p)$, $\mathbf{k} = (k_1, \dots, k_m)$, $\mathbf{u} = (u_1, \dots, u_{p+1})$. *There exists a pencil* $h(s) \in \mathbb{C}[s]^{1\times(n+m)}$ *such that* $H(s) \overset{s.e.}{\sim} \begin{bmatrix} h(s) \\ H_1(s) \end{bmatrix}$ *if and only if*

$$\pi_i(s,t) \mid \pi_i^1(s,t) \mid \pi_{i+1}(s,t), \quad 1 \le i \le n, \tag{7}$$

$$\mathbf{u} \prec' \mathbf{t}, \tag{8}$$

$$\mathbf{g} = \mathbf{k}. \tag{9}$$

Remark 5.3. *Let* $\theta = \#\{i \;:\; t_i > 0\}$ *and* $\bar{\theta} = \#\{i \;:\; u_i > 0\}$. *Lemma 4.3 in [2] also contains the condition*

$$\bar{\theta} \ge \theta. \tag{10}$$

But we show that (7)-(9) implies (10): we have $\operatorname{rank}(H(s)) = n = \sum_{i=1}^n \deg(\pi_i) + \sum_{i=1}^m k_i + \sum_{i=1}^{p+1} u_i$ *and* $\operatorname{rank}(H_1(s)) = n = \sum_{i=1}^n \deg(\pi_i^1) + \sum_{i=1}^m g_i + \sum_{i=1}^p t_i$. *Therefore* $\sum_{i=1}^p t_i = \sum_{i=1}^{p+1} u_i + \sum_{i=1}^n (\deg(\pi_i) - \deg(\pi_i^1)) + \sum_{i=1}^m (k_i - g_i)$. *From (7) and (9) we obtain* $\sum_{i=1}^p t_i \le \sum_{i=1}^{p+1} u_i$. *Then, by (8) and Lemma 5.10 of [2] we derive (10).*

Lemma 5.4. *Given matrix pencils* $H_1(s) \in \mathbb{C}[s]^{(n+p)\times(n+m)}$ *with* $\operatorname{rank}(H_1(s)) = n$, *and* $H(s) \in \mathbb{C}[s]^{(n+p+1)\times(n+m)}$ *with* $\operatorname{rank}(H(s)) = n+1$, *let* $\pi_1^1(s,t) \mid \cdots \mid \pi_n^1(s,t)$, $g_1 \ge \cdots \ge g_m \ge 0$ *and* $t_1 \ge \cdots \ge t_p \ge 0$ *be the homogeneous invariant factors, the column and the row minimal indices of* $H_1(s)$, *respectively, and let* $\pi_1(s,t) \mid \cdots \mid \pi_{n+1}(s,t)$, $k_1 \ge \cdots \ge k_{m-1} \ge 0$ *and* $u_1 \ge \cdots \ge u_p \ge 0$ *be the homogeneous invariant factors, the column and the row minimal indices of* $H(s)$, *respectively.*

Let $\mathbf{g} = (g_1, \dots, g_m)$, $\mathbf{t} = (t_1, \dots, t_p)$, $\mathbf{k} = (k_1, \dots, k_{m-1})$, *and* $\mathbf{u} = (u_1, \dots, u_p)$. *There exists a pencil* $h(s) \in \mathbb{C}[s]^{1\times(n+m)}$ *such that* $H(s) \overset{s.e.}{\sim} \begin{bmatrix} h(s) \\ H_1(s) \end{bmatrix}$ *if and only if (7),*

$$\mathbf{g} \prec' \mathbf{k}, \tag{11}$$

$$\mathbf{t} = \mathbf{u}. \tag{12}$$

To solve Problem 4.2, we express these results in terms of the Weyr characteristics of the pencils involved.

Lemma 5.5 ([1, Lemma 3.2], see also [3, Lemma 4.3])**.** *Let* a *and* b *be partitions. Let* $p = \overline{a}$ *and* $q = \overline{b}$ *be the conjugate partitions. Let* $k \ge 0$ *be an integer. Then,*

$$a_j \ge b_{j+k}, \quad j \ge 1,$$

if and only if

$$p_j \ge q_j - k, \quad j \ge 1.$$

Lemma 5.6. *For* $i = 1, 2$ *let* $P^i(s) \in \mathbb{C}[s]^{n^i \times m^i}$ *be matrix pencils such that* $\operatorname{rank}(P^i(s)) = \rho_i$. *Let* $\phi_1^i(s,t) \mid \cdots \mid \phi_{\rho_i}^i(s,t)$ *be the homogeneous invariant factors of* $P^i(s)$.

For $\lambda \in \bar{\mathbb{C}}$, let $n^i(\lambda)$ be the Segre characteristic at λ of $P^i(s)$, and let $w^i(\lambda) = \overline{n^i(\lambda)}$ be the conjugate partition of $n^i(\lambda)$.

Let $x \geq \rho_2 - \rho_1$ be an integer. Then, the following statements are equivalent

(i) $\phi^1_j(s,t) \mid \phi^2_{j+x}(s,t), \quad j \geq 1,$

(ii) $n^1_{j+\rho_1-\rho_2+x}(\lambda) \leq n^2_j(\lambda), \quad \lambda \in \bar{\mathbb{C}}, \quad j \geq 1,$

(iii) $w^1_j(\lambda) + \rho_2 - \rho_1 - x \leq w^2_j(\lambda), \quad \lambda \in \bar{\mathbb{C}}, \quad j \geq 1.$

Proof. The equivalence between (i) and (ii) is immediate, it is enough to take into account that

$$\phi^i_j(s,t) = t^{n^i_{\rho_i-j+1}(\infty)} \prod_{\lambda \in \Lambda(P^i(s))\setminus\{\infty\}} (s-\lambda t)^{n^i_{\rho_i-j+1}(\lambda)}, \quad 1 \leq j \leq \rho_i.$$

The equivalence between (ii) and (iii) follows from Lemma 5.5. □

Definition 5.7 ([3, Definition 4.1]). *Given two partitions a and b, we say that a is* <u>*conjugate majorized*</u> *by b if $b_1 = a_1 + 1$ and*

$$b_i = a_i + 1, \quad 1 \leq i \leq g,$$

where $g = \max\{i : b_i > a_i\}$. If a is <u>*conjugate majorized*</u> *by b we write*

$$a \angle b.$$

Notation. Given two partitions a and b, if $a_j \leq b_j$ for $j \geq 1$ we write

$$a \leq b.$$

Remark 5.8. *Notice that, if $(a_1, a_2, \dots) \angle (b_1, b_2, \dots)$ and $k \geq 2$ is an integer, then $(a_k, a_{k+1}, \dots) \leq (b_k, b_{k+1}, \dots)$ or $(b_k, b_{k+1}, \dots) \angle (a_k, a_{k+1}, \dots)$.*

Lemma 5.9 ([3, Proposition 4.5]). *Given two finite sequences of nonnegative integers $\mathbf{k} = (k_1, \dots, k_{m+1})$ and $\mathbf{d} = (d_1, \dots, d_m)$, let $(r_1, \dots) = \overline{(k_1, \dots, k_{m+1})}$, $(s_1, \dots) = \overline{(d_1, \dots, d_m)}$ be the conjugate partitions, $r_0 = m+1 = s_0 + 1$, and $r = (r_0, r_1, \dots)$, $s = (s_0, s_1, \dots)$. Then $\mathbf{k} \prec' \mathbf{d}$ if and only if $s \angle r$.*

Applying Lemmas 5.6 and 5.9, Lemmas 5.2 and 5.4 can be expressed in terms of the Weyr characteristics of the pencils $H(s)$ and $H_1(s)$. By transposition the results also apply for column completion instead of row completion. For convenience, we present next the second option.

Lemma 5.10 (restatement of Lemma 5.2). *Given two matrix pencils $H_1(s) \in \mathbb{C}[s]^{(n+p)\times(n+m)}$, $H(s) \in \mathbb{C}[s]^{(n+p)\times(n+m+1)}$ of $\operatorname{rank}(H_1(s)) = \operatorname{rank}(H(s)) = n$, let (w, b, c) and (w^1, b^1, c^1) be the Weyr characteristics of $H(s)$ and $H_1(s)$, respectively.*

There exists a pencil $h(s) \in \mathbb{C}[s]^{(n+p)\times 1}$ such that $H(s) \overset{s.e.}{\sim} \begin{bmatrix} h(s) & H_1(s) \end{bmatrix}$ if and only

$$w_j(\lambda) \leq w^1_j(\lambda) \leq w_j(\lambda) + 1, \quad \lambda \in \bar{\mathbb{C}}, \quad j \geq 1, \tag{13}$$

$$b^1 \angle b. \tag{14}$$

$$c = c^1, \tag{15}$$

Lemma 5.11 (restatement of Lemma 5.4)**.** *Given two matrix pencils* $H_1(s) \in \mathbb{C}[s]^{(n+p)\times(n+m)}$*,* $H(s) \in \mathbb{C}[s]^{(n+p)\times(n+m+1)}$ *of* $\operatorname{rank}(H_1(s)) = n$ *and* $\operatorname{rank}(H(s)) = n+1$*, let* (w, b, c) *and* (w^1, b^1, c^1) *be the Weyr characteristics of* $H(s)$ *and* $H_1(s)$*, respectively.*

There exists a pencil $h(s) \in \mathbb{C}[s]^{(n+p)\times 1}$ *such that* $H(s) \overset{s.e.}{\sim} \begin{bmatrix} h(s) & H_1(s) \end{bmatrix}$ *if and only if*

$$w_j(\lambda) - 1 \leq w_j^1(\lambda) \leq w_j(\lambda), \quad \lambda \in \bar{\mathbb{C}}, \quad j \geq 1, \tag{16}$$

$$b = b^1, \tag{17}$$

$$c \angle c^1. \tag{18}$$

6. Necessary conditions for Problem 4.2

The aim of this section is to find necessary conditions for solving problem 4.2. Let S and T be two linear relations such that $r(S,T) = 1$. Assume that $\dim S \geq \dim T$. Then, $\dim S = \dim T = \dim(S \cap T) + 1$, or $\dim S = \dim(S \cap T) + 1 = \dim T + 1$. If $\dim T = \dim(S \cap T)$, then $S \cap T = T \subset S$. We start analyzing the case when one of the linear relations is included in the other one.

Theorem 6.1. *Let* S, U *be two linear relations in* $\mathbb{C}^n$ *such that* $U \subseteq S$*,* $\dim S = d$ *and* $\dim \frac{S}{U} = 1$*.*

Let (W, B, C) *and* (W^1, B^1, C^1) *be the Weyr characteristics of* S *and* U*, respectively. Then one of the two following conditions holds:*

(a)

$$W_j(\lambda) \leq W_j^1(\lambda) \leq W_j(\lambda) + 1, \quad j \geq 1, \quad \lambda \in \bar{\mathbb{C}}, \tag{19}$$

$$B^1 \angle B, \tag{20}$$

$$C = C^1. \tag{21}$$

(b)

$$W_j(\lambda) - 1 \leq W_j^1(\lambda) \leq W_j(\lambda), \quad j \geq 1, \quad \lambda \in \bar{\mathbb{C}}, \tag{22}$$

$$B = B^1, \tag{23}$$

$$c \angle c^1, \tag{24}$$

where $c = (n - d + B_1, C_1, C_2, \dots)$ *and* $c^1 = (n - d + B_1 + 1, C_1^1, C_2^1 \dots)$*.*

Remark 6.2. *Condition (24) is equivalent to*

$$C \angle C^1 \text{ or } C^1 \leq C.$$

Proof. By Lemma 4.1, there exist pencils $P(s) = sE - F \in \mathbb{C}[s]^{n\times d}$, $P_1(s) = sE_1 - F_1 \in \mathbb{C}[s]^{n\times(d-1)}$ and $u(s) = se - f \in \mathbb{C}[s]^{n\times 1}$ such that $P(s) = \begin{bmatrix} P_1(s) & u(s) \end{bmatrix}$, $FE^{-1} = S$ and $F_1E_1^{-1} = U$.

Let (w, b, c) and (w^1, b^1, c^1) be the Weyr characteristics of $P(s)$ and $P_1(s)$, respectively. Then by Lemma 3.9 $w = W$, $w^1 = W^1$,

$$b_j = B_{j-1}, \quad b^1_j = B^1_{j-1}, \quad c_j = C_{j-1}, \quad c^1_j = C^1_{j-1}, \quad j \geq 2.$$

As $\dim S = \dim FE^{-1} = d$ and $\dim U = \dim F_1E_1^{-1} = d-1$, we have $b_1 = b_2 = B_1$ and $b^1_1 = b^1_2 = B^1_1$. Then $\operatorname{rank}(P(s)) = d - b_1 = d - B_1$ and $\operatorname{rank}(P_1(s)) = d - 1 - b^1_1 = d - 1 - B^1_1$; hence $c_1 = n - d + B_1$ and $c^1_1 = n - d + 1 + B^1_1$.

We have $\operatorname{rank}(P_1(s)) \leq \operatorname{rank}(P(s)) \leq \operatorname{rank}(P_1(s)) + 1$.

(a) If $\operatorname{rank}(P(s)) = \operatorname{rank}(P_1(s))$, then, by Lemma 5.10, conditions (13), (14) and (15) hold. From (13) and (15) we obtain inmediatly (19) and (21).

As $b_1 = b_2$ and $b^1_1 = b^1_2$, from (14) we obtain $b_2 = b^1_2 + 1$. By Remark 5.8, from (14) we derive $(b^1_2, \dots) \angle (b_2, \dots)$, equivalently (20).

(b) If $\operatorname{rank}(P(s)) = \operatorname{rank}(P_1(s)) + 1$, then, by Lemma 5.11, we obtain (16), (17) and (18), which are equivalent to (22)-(24).

□

As an immediate consequence of Theorem 6.1 we obtain in the next theorem necessary conditions for Problem 4.2.

Theorem 6.3. *Let S, T be two linear relations in $\mathbb{C}^n$ such that $r(S, T) = 1$ and $\dim S = d \geq \dim T$.*

Let (W, B, C) and $(\bar{W}, \bar{B}, \bar{C})$ be the Weyr characteristics of S and T, respectively.

1. *If $\dim \frac{S}{S\cap T} = 1$ and $\dim \frac{T}{S\cap T} = 0$ then one of the two following conditions holds:*

   (a)
$$W_i(\lambda) \leq \bar{W}_i(\lambda) \leq W_i(\lambda) + 1, \quad i \geq 1, \quad \lambda \in \bar{\mathbb{C}}, \tag{25}$$
$$\bar{B} \angle B, \tag{26}$$
$$C = \bar{C}. \tag{27}$$

   (b)
$$W_i(\lambda) - 1 \leq \bar{W}_i(\lambda) \leq W_i(\lambda), \quad i \geq 1, \quad \lambda \in \bar{\mathbb{C}}, \tag{28}$$
$$B = \bar{B}, \tag{29}$$
$$c \angle \bar{c}, \tag{30}$$

   where $c = (n-d+B_1, C_1, C_2 \dots)$ and $\bar{c} = (n-d+B_1+1, \bar{C}_1, \bar{C}_2, \dots)$.

2. *If $\dim \frac{S}{S\cap T} = \dim \frac{T}{S\cap T} = 1$, let (W^1, B^1, C^1) be the Weyr characteristic of $S \cap T$. Then one of the four following conditions holds:*

(c)

$$\max\{W_i(\lambda),\bar{W}_i(\lambda)\} \leq W_i^1(\lambda) \leq \min\{W_i(\lambda),\bar{W}_i(\lambda)\}+1,\ i\geq 1,\ \lambda\in\bar{\mathbb{C}}, \quad (31)$$

$$B^1 \angle B, \quad B^1 \angle \bar{B} \quad (32)$$

$$C=\bar{C}=C^1, \quad (33)$$

(d)

$$\max\{W_i(\lambda),\bar{W}_i(\lambda)-1\} \leq W_i^1(\lambda) \leq \min\{W_i(\lambda)+1,\bar{W}_i(\lambda)\},\ i\geq 1,\ \lambda\in\bar{\mathbb{C}}, \quad (34)$$

$$\bar{B}=B^1 \angle B \quad (35)$$

$$C=C^1 \text{ and } \bar{c} \angle c, \quad (36)$$

where $c=(n-d+B_1,C_1,C_2,\dots)$ *and* $\bar{c}=(n-d+B_1-1,\bar{C}_1,\bar{C}_2,\dots,)$.

(e)

$$\max\{W_i(\lambda)-1,\bar{W}_i(\lambda)\} \leq W_i^1(\lambda) \leq \min\{W_i(\lambda),\bar{W}_i(\lambda)+1\},\ i\geq 1,\ \lambda\in\bar{\mathbb{C}}, \quad (37)$$

$$B=B^1 \angle \bar{B} \quad (38)$$

$$\bar{C}=C^1 \text{ and } c \angle \bar{c}, \quad (39)$$

where $c=(n-d+B_1,C_1,C_2,\dots)$ *and* $\bar{c}=(n-d+B_1+1,\bar{C}_1,\bar{C}_2,\dots)$.

(f)

$$\max\{W_i(\lambda),\bar{W}_i(\lambda)\}-1 \leq W_i^1(\lambda) \leq \min\{W_i(\lambda),\bar{W}_i(\lambda)\},\ i\geq 1,\ \lambda\in\bar{\mathbb{C}}, \quad (40)$$

$$B=\bar{B}=B^1, \quad (41)$$

$$c \angle c^1 \text{ and } \bar{c} \angle c^1, \quad (42)$$

where $c=(n-d+B_1,C_1,C_2,\dots)$, $\bar{c}=(n-d+B_1,\bar{C}_1,\bar{C}_2,\dots)$, *and* $c^1=(n-d+B_1+1,C_1^1,C_2^1,\dots)$.

Proof.

1. If $\dim\frac{S}{S\cap T}=1$ and $\dim\frac{T}{S\cap T}=0$ then $T=S\cap T\subset S$ and the result follows from Theorem 6.1.
2. If $\dim\frac{S}{S\cap T}=\dim\frac{T}{S\cap T}=1$, then $\dim S=\dim T=\dim(S\cap T)+1$. Applying Theorem 6.1 to S and $S\cap T$, either (19)-(21) or (22)-(24) are satisfied. Applying Theorem 6.1 to T and $S\cap T$, either $(\overline{19})$-$(\overline{21})$ or $(\overline{22})$-$(\overline{24})$ are satisfied, where conditions $(\overline{19})$ and $(\overline{22})$ are, respectively, conditions (19) and (22) substituting $W_j(\lambda)$ by $\bar{W}_j(\lambda)$, and similarly for the other conditions. If (19)-(21) and $(\overline{19})$-$(\overline{21})$ are satisfied, then (c) holds. Analogously we obtain (d), (e) and (f).

□

In the next section we prove that the necessary conditions obtained in Theorem 6.3 (item 1) are sufficient to solve Problem 4.2 if $\dim S = \dim T + 1$ (see Theorem 7.3). When $\dim S = \dim T$, the necessary conditions of Theorem 6.3 (item 2) involve partitions (W^1, B^1, C^1) which are the Weyr characteristic of $S \cap T$, hence, they satisfy the conditions (a) or (b) of Theorem 6.3 with respect to the Weyr characteristic of S and with respect to the Weyr characteristic of T. In Theorem 7.4 we state necessary and sufficient conditions for the existence of such partitions (W^1, B^1, C^1), therefore they are necessary and sufficient conditions to solve Problem 4.2 when $\dim S = \dim T$.

7. A solution to Problem 4.2

Lemma 7.1. *Let S, U be two linear relations in $\mathbb{C}^n$. Then, there exists a linear relation $\bar{S}$ in $\mathbb{C}^n$ such that $\bar{S} \overset{s.e.}{\sim} S$ and $U \subseteq \bar{S}$ if and only if there exists a linear relation $\bar{U}$ in $\mathbb{C}^n$ such that $\bar{U} \overset{s.e.}{\sim} U$ and $\bar{U} \subseteq S$.*

Proof. Let us assume that $\bar{S} \overset{s.e.}{\sim} S$ and $U \subseteq \bar{S}$. Then, there exists $T \in \mathrm{Gl}_n(\mathbb{C})$ such that $\bar{S} = \begin{bmatrix} T & O \\ O & T \end{bmatrix} S$. Let $\bar{U} = \begin{bmatrix} T^{-1} & O \\ O & T^{-1} \end{bmatrix} U$. Then $\bar{U} \overset{s.e.}{\sim} U$ and $\bar{U} \subseteq \begin{bmatrix} T^{-1} & O \\ O & T^{-1} \end{bmatrix} \bar{S} = S$.

Conversely, let us assume that $\bar{U} \overset{s.e.}{\sim} U$ and $\bar{U} \subseteq S$. Then, there exists $V \in Gl_n(\mathbb{C})$ such that $\bar{U} = \begin{bmatrix} V & O \\ O & V \end{bmatrix} U$. Let $\bar{S} = \begin{bmatrix} V^{-1} & O \\ O & V^{-1} \end{bmatrix} S$. Then $\bar{S} \overset{s.e.}{\sim} S$ and $U = \begin{bmatrix} V^{-1} & O \\ O & V^{-1} \end{bmatrix} \bar{U} \subseteq \begin{bmatrix} V^{-1} & O \\ O & V^{-1} \end{bmatrix} S = \bar{S}$. □

Theorem 7.2. *Let S, U be two linear relations in $\mathbb{C}^n$ such that $\dim S = d = \dim U + 1$ and let (W, B, C) and (W^1, B^1, C^1) be the Weyr characteristics of S and U, respectively. Then there exists a linear relation $\bar{S}$ in $\mathbb{C}^n$ such that $\bar{S} \overset{s.e.}{\sim} S$ and $U \subset \bar{S}$ (equivalently, there exists a linear relation $\bar{U}$ in $\mathbb{C}^n$ such that $\bar{U} \overset{s.e.}{\sim} U$ and $\bar{U} \subset S$) if and only if one of the conditions (a) or (b) of Theorem 6.1 holds.*

Proof. Assume that there exists $\bar{S}$ such that $\bar{S} \overset{s.e.}{\sim} S$, $U \subset \bar{S}$. Then $\dim \frac{\bar{S}}{U} = \dim S - \dim U = 1$. By Theorem 5.4 of [1], (W, B, C) is the Weyr characteristic of $\bar{S}$. Then, conditions (a) or (b) of Theorem 6.1 holds.

By Lemma 3.3 there exist pencils $P(s) = sE - F \in \mathbb{C}[s]^{n \times d}$, $P_1(s) = sE_1 - F_1 \in \mathbb{C}[s]^{n \times (d-1)}$ such that $\operatorname{rank} \begin{bmatrix} E \\ F \end{bmatrix} = d$, $\operatorname{rank} \begin{bmatrix} E_1 \\ F_1 \end{bmatrix} = d - 1$, $FE^{-1} = S$. and $F_1 E_1^{-1} = U$.

Let (w, b, c) and (w^1, b^1, c^1) be the Weyr characteristics of $P(s)$ and $P_1(s)$, respectively. As in the proof of Theorem 6.1, $\operatorname{rank}(P(s)) = d - B_1$, $\operatorname{rank}(P_1(s)) = d - 1 - B_1^1$,

$$w = W, \quad b = (B_1, B_1, B_2, \dots), \quad c = (n - d + B_1, C_1, C_2, \dots),$$

$$w^1 = W^1, \quad b^1 = (B_1^1, B_1^1, B_2^1, \dots), \quad c^1 = (n-d+1+B_1^1, C_1^1, C_2^1, \dots).$$

- Assume that (a) holds. Condition (19) is equivalent to (13). From (20) we derive $B_1 = B_1^1 + 1$; hence $b_1 = b_1^1 + 1$, $c_1 = c_1^1$ and $\operatorname{rank}(P(s)) = \operatorname{rank}(P_1(s))$. Thus, from (20) and (21) we obtain (14) and (15). By Lemma 5.10 here exists a pencil $u(s) = se - f \in \mathbb{F}[s]^{n\times 1}$ such that $P(s) \overset{s.e.}{\sim} \begin{bmatrix} P_1(s) & u(s) \end{bmatrix}$.
- Assume that (b) holds. Condition (22) is equivalent to (16). From (23) we derive $B_1 = B_1^1$; hence $b_1 = b_1^1$, $c_1 = c_1^1+1$ and $\operatorname{rank}(P(s)) = \operatorname{rank}(P_1(s))+1$. Thus, from (23) and (24) we obtain (17) and (18). By Lemma 5.11 here exists a pencil $u(s) = se - f \in \mathbb{F}[s]^{n\times 1}$ such that $P(s) \overset{s.e.}{\sim} \begin{bmatrix} P_1(s) & u(s) \end{bmatrix}$.

In both cases, let $\bar{P}(s) = \begin{bmatrix} P_1(s) & u(s) \end{bmatrix} = s\begin{bmatrix} E_1 & e \end{bmatrix} - \begin{bmatrix} F_1 & f \end{bmatrix}$ and $\bar{S} = \begin{bmatrix} F_1 & f \end{bmatrix}\begin{bmatrix} E_1 & e \end{bmatrix}^{-1} = R\left(\begin{bmatrix} E_1 & e \\ F_1 & f \end{bmatrix}\right)$. It is clear that $U \subset \bar{S}$. By Lemma 3.4, $\bar{S} \overset{s.e.}{\sim} S$. □

Given two linear relations S and T in $\mathbb{C}^n$ such that $\dim S \geq \dim T$ and $r(S,T) = 1$, then $\dim S = \dim(S\cap T)+1 \geq \dim T$. Therefore, $\dim S = \dim T+1$ or $\dim S = \dim T$. As an immediate consequence of Theorem 7.2 we obtain a solution to Problem 4.2 when $\dim S = \dim T + 1$.

Theorem 7.3. *Let S, T be two linear relations in $\mathbb{C}^n$ such that $\dim S = d = \dim T + 1$. Let (W, B, C) and $(\bar{W}, \bar{B}, \bar{C})$ be the Weyr characteristics of S and T, respectively. Then there exists a linear relation $\bar{S}$ in $\mathbb{C}^n$ such that $\bar{S} \overset{s.e.}{\sim} S$ and $r(\bar{S}, T) = 1$ (equivalently, there exists a linear relation $\bar{T}$ in $\mathbb{C}^n$ such that $\bar{T} \overset{s.e.}{\sim} T$ and $r(S, \bar{T}) = 1$) if and only if one of the conditions (a) or (b) of Theorem 6.3 holds.*

Proof. There exists $\bar{S}$ such that $\bar{S} \overset{s.e.}{\sim} S$ and $r(\bar{S}, T) = 1$ (there exists $\bar{T}$ such that $\bar{T} \overset{s.e.}{\sim} T$ and $r(S, \bar{T}) = 1$) if and only if there exists $\bar{S}$ such that $\bar{S} \overset{s.e.}{\sim} S$, $\dim \frac{\bar{S}}{\bar{S}\cap T} = 1$ and $\dim \frac{T}{\bar{S}\cap T} = 0$ (there exists $\bar{T}$ such that $\bar{T} \overset{s.e.}{\sim} T$, $\dim \frac{S}{S\cap\bar{T}} = 1$ and $\dim \frac{\bar{T}}{S\cap\bar{T}} = 0$) if and only if there exists $\bar{S}$ such that $\bar{S} \overset{s.e.}{\sim} S$, and $T \subset \bar{S}$ (there exists $\bar{T}$ such that $\bar{T} \overset{s.e.}{\sim} S$, and $\bar{T} \subset S$). By Theorem 7.2 this occurs if and only if one of the conditions (a) or (b) of Theorem 6.3 holds. □

The solution to the case $\dim S = \dim T$ is given in the next theorem. The proof follows the ideas of [2, Theorem 5.1]. We need some technical lemmas from [2] and [3], which, for the reader's convenience, we include in Appendix A.

Theorem 7.4. *Let S, T be two linear relations in $\mathbb{C}^n$ such that $\dim S = \dim T = d$ and $S \overset{s.e.}{\not\sim} T$. Let (W, B, C) and $(\bar{W}, \bar{B}, \bar{C})$ be the Weyr characteristics of S and T, respectively, and let $\Lambda(S) \cup \Lambda(T) = \{\lambda_1, \dots, \lambda_\ell\}$. There exist linear relations $\bar{S}, \bar{T}$ in $\mathbb{C}^n$ such that $\bar{S} \overset{s.e.}{\sim} S$, $\bar{T} \overset{s.e.}{\sim} T$ and $r(\bar{S}, \bar{T}) = 1$ if and only if*

1. *If $B = \bar{B}$ and $C = \bar{C}$:*

$$W_i(\lambda) - 1 \leq \bar{W}_i(\lambda) \leq W_i(\lambda) + 1, \quad i \geq 1, \quad \lambda \in \bar{\mathbb{C}}. \tag{43}$$

2. *If $B = \bar{B}$ and $C \neq \bar{C}$: (43) and*

$$X \leq \sum_{i \geq 1} \min\{C_i, \bar{C}_i\} + \max\{e, e'\}, \tag{44}$$

where $x = \min\{i \ : \ C_i \neq \bar{C}_i\}$, $e = \min\{i \geq x-1 : \ \bar{C}_{i+1} \geq C_{i+1}\}$, $e' = \min\{i \in \{i \geq x-1 : \ C_{i+1} \geq \bar{C}_{i+1}\}$, *and*

$$X = \mid W \mid + \mid C \mid - \sum_{i=1}^{\ell} \sum_{j \geq 1} \min\{W_j(\lambda_i), \bar{W}_j(\lambda_i)\} - 1.$$

3. *If $B \neq \bar{B}$ and $B_1 = \bar{B}_1$: (43),*

$$C = \bar{C}, \tag{45}$$

and

$$Y \geq \sum_{i \geq 1} \max\{B_i, \bar{B}_i\} - \max\{\bar{e}, \bar{e}'\}, \tag{46}$$

where $\bar{x} = \min\{i \ : \ B_i \neq \bar{B}_i\}$, $\bar{e} = \min\{i \geq \bar{x}-1 : \ \bar{B}_{i+1} \geq B_{i+1}\}$, $\bar{e}' = \min\{i \in \{i \geq \bar{x}-1 : \ B_{i+1} \geq \bar{B}_{i+1}\}$,*, and*

$$Y = \mid W \mid + \mid B \mid - \sum_{i=1}^{\ell} \sum_{j \geq 1} \max\{W_j(\lambda_i), \bar{W}_j(\lambda_i)\}.$$

4. *If $B_1 \neq \bar{B}_1$: one of the two following conditions hold:*
   (a)
   $$\bar{W}_j(\lambda) - 2 \leq W_j(\lambda) \leq \bar{W}_j(\lambda), \quad j \geq 1, \quad \lambda \in \bar{\mathbb{C}}, \tag{47}$$
   $$\bar{B} \angle B, \quad \bar{c} \angle c, \tag{48}$$
   where $c = (n-d+B_1, C_1, C_2, \dots)$, $\bar{c} = (n-d+\bar{B}_1, \bar{C}_1, \bar{C}_2, \dots)$,
   $$\sum_{i=1}^{\ell} \sum_{j \geq 1} \max\{W_j(\lambda_i), \bar{W}_j(\lambda_i) - 1\} \leq x \leq \sum_{i=1}^{\ell} \sum_{j \geq 1} \min\{W_j(\lambda_i) + 1, \bar{W}_j(\lambda_i)\}, \tag{49}$$
   where $x = \mid W \mid + \mid B \mid - \mid \bar{B} \mid$.

   (b)
   $$W_j(\lambda) - 2 \leq \bar{W}_j(\lambda) \leq W_j(\lambda), \quad j \geq 1, \quad \lambda \in \bar{\mathbb{C}}, \tag{50}$$
   $$B \angle \bar{B}, \quad c \angle \bar{c}, \tag{51}$$
   where $c = (n-d+B_1, C_1, C_2, \dots)$, $\bar{c} = (n-d+\bar{B}_1, \bar{C}_1, \bar{C}_2, \dots)$,
   $$\sum_{i=1}^{\ell} \sum_{j \geq 1} \max\{W_j(\lambda_i) - 1, \bar{W}_j(\lambda_i)\} \leq y \leq \sum_{i=1}^{\ell} \sum_{j \geq 1} \min\{W_j(\lambda_i), \bar{W}_j(\lambda_i) + 1\}, \tag{52}$$
   where $y = \mid \bar{W} \mid + \mid \bar{B} \mid - \mid B \mid$.

Remark 7.5. *If* $\lambda \notin \{\lambda_1, \ldots, \lambda_\ell\}$*, then* $\min\{W_j(\lambda), \bar{W}_j(\lambda)\} = 0$*. Therefore, in item 2. we can define*

$$X = \mid W \mid + \mid C \mid - \sum_{\lambda \in \bar{\mathbb{C}}} \sum_{j \geq 1} \min\{W_j(\lambda), \bar{W}_j(\lambda)\} - 1.$$

Analogously, in item 3.,

$$Y = \mid W \mid + \mid B \mid - \sum_{\lambda \in \bar{\mathbb{C}}} \sum_{j \geq 1} \max\{W_j(\lambda), \bar{W}_j(\lambda)\}.$$

and, in item 4. conditions (49) and (52) can be written, respectively, as

$$\sum_{\lambda \in \bar{\mathbb{C}}} \sum_{j \geq 1} \max\{W_j(\lambda), \bar{W}_j(\lambda) - 1\} \leq x \leq \sum_{\lambda \in \bar{\mathbb{C}}} \sum_{j \geq 1} \min\{W_j(\lambda) + 1, \bar{W}_j(\lambda)\},$$

and

$$\sum_{\lambda \in \bar{\mathbb{C}}} \sum_{j \geq 1} \max\{W_j(\lambda) - 1, \bar{W}_j(\lambda)\} \leq y \leq \sum_{\lambda \in \bar{\mathbb{C}}} \sum_{j \geq 1} \min\{W_j(\lambda), \bar{W}_j(\lambda) + 1\}.$$

Proof. Necessity. Let us assume that there exist linear relations $\bar{S}, \bar{T}$ in $\mathbb{C}^n$ such that $\bar{S} \overset{s.e.}{\sim} S$, $\bar{T} \overset{s.e.}{\sim} T$ and $r(\bar{S}, \bar{T}) = 1$. As $\dim \bar{S} = \dim \bar{T}$ and $\bar{S} \neq \bar{T}$, we have $\dim \frac{\bar{S}}{\bar{S} \cap \bar{T}} = \dim \frac{\bar{T}}{\bar{S} \cap \bar{T}} = 1$. Let (W^1, B^1, C^1) be the Weyr characteristic of $\bar{S} \cap \bar{T}$. Then one of the four conditions (c), (d), (e) or (f) of Theorem 6.3 hold.

1. If $B = \bar{B}$ and $C = \bar{C}$, then (c) or (f) holds. Condition (43) is derived from (31) if (c) holds, and from (40) if (f) holds.
2. If $B = \bar{B}$ and $C \neq \bar{C}$, then (f) holds. From (40) we derive (43). By Lemma A.8, from (42) we have that

$$\mid C^1 \mid \leq \sum_{i \geq 1} \min\{C_i, \bar{C}_i\} + \max\{e, e'\}. \tag{53}$$

   We have $\mid W^1 \mid + \mid B^1 \mid + \mid C^1 \mid + B_1^1 = \dim(\bar{S} \cap \bar{T}) = d - 1 = \mid W \mid + \mid B \mid + \mid C \mid + B_1 - 1$. From (41) we obtain $\mid C^1 \mid = \mid W \mid + \mid C \mid - \mid W^1 \mid - 1$. From (40), $X \leq \mid C^1 \mid$. Therefore, from (53) we obtain (44).
3. If $B \neq \bar{B}$ and $B_1 = \bar{B}_1$, then (c) holds. From (31) we derive (43) and from (33), condition (45) is immediate. From (32), for any integer $Z \geq B_1 = \bar{B}_1$, $(Z-1, B_1^1, B_2^1 \ldots) \angle (Z, B_1, B_2, \ldots,)$ and $(Z-1, B_1^1, B_2^1 \ldots) \angle (Z, \bar{B}_1, \bar{B}_2, \ldots,)$. By Lemma A.8,

$$\mid B^1 \mid \geq \sum_{i \geq 1} \max\{B_i, \bar{B}_i\} - \max\{\bar{e}, \bar{e}'\}. \tag{54}$$

   We have $\mid W^1 \mid + \mid B^1 \mid + \mid C^1 \mid + B_1^1 = \dim(\bar{S} \cap \bar{T}) = d - 1 = \mid W \mid + \mid B \mid + \mid C \mid + B_1 - 1$. From (32) and (33) we obtain $\mid B^1 \mid = \mid W \mid + \mid B \mid - \mid W^1 \mid$. From (31), $Y \geq \mid B^1 \mid$. Therefore, from (54) we obtain (46).

4. If $B_1 \neq \bar{B}_1$, then (d) or (e) holds.
Assume that (d) holds. From (34) we derive (47) and from (35) and (36), condition (48) is immediate. We have $\mid W^1 \mid + \mid B^1 \mid + \mid C^1 \mid + B_1^1 = \dim(\bar{S} \cap \bar{T}) = d - 1 = \mid W \mid + \mid B \mid + \mid C \mid + B_1 - 1$. From (35) and (36) we obtain $\mid W^1 \mid = \mid W \mid + \mid B \mid - \mid \bar{B} \mid$. From (34) we derive $\Lambda(\bar{S} \cap \bar{T}) \subseteq \Lambda(\bar{T})$; hence $\mid W^1 \mid = \sum_{i=1}^{\ell} \sum_{j \geq 1} W_j^1(\lambda_i)$ and from (34) we obtain (49).
Analogously, if (e) is satisfied, then we obtain (50)-(52).

Sufficiency.

1., 2. We analyze together the cases 1. and 2, i.e., the cases when $B = \bar{B}$. There are two possibilities $C = \bar{C}$ and $C \neq \bar{C}$. If $C = \bar{C}$ assume that (43) holds, and if $C \neq \bar{C}$, assume that (43) and (44) hold. In both cases, as $d = \mid W \mid + \mid B \mid + \mid C \mid + B_1 = \mid \bar{W} \mid + \mid \bar{B} \mid + \mid \bar{C} \mid + \bar{B}_1$, we obtain $\mid W \mid + \mid C \mid = \mid \bar{W} \mid + \mid \bar{C} \mid$.
Define
$$\hat{B} = B = \bar{B}, \tag{55}$$
and
$$\hat{W}_j(\lambda) = \min\{W_j(\lambda), \bar{W}_j(\lambda)\}, \quad j \geq 1, \quad \lambda \in \bar{\mathbb{C}}.$$
Then $\hat{W}_j(\lambda) \geq \hat{W}_{j+1}(\lambda)$, for $j \geq 1$ and $\lambda \in \bar{\mathbb{C}}$, and from (43) we derive
$$\begin{array}{ll} W_j(\lambda) - 1 \leq \hat{W}_j(\lambda) \leq W_j(\lambda), & j \geq 1, \quad \lambda \in \bar{\mathbb{C}}, \\ \bar{W}_j(\lambda) - 1 \leq \hat{W}_j(\lambda) \leq \bar{W}_j(\lambda), & j \geq 1, \quad \lambda \in \bar{\mathbb{C}}. \end{array} \tag{56}$$
Define
$$\hat{W}(\lambda_i) = (\hat{W}_1(\lambda_i), \hat{W}_2(\lambda_i), \dots), \quad 1 \leq i \leq \ell,$$
$$\hat{W} = (\hat{W}(\lambda_1), \dots, \hat{W}(\lambda_\ell)).$$
We have $\mid \hat{W} \mid \leq \mid W \mid$ and $\mid \hat{W} \mid \leq \mid \bar{W} \mid$. Let $c = (n - d + B_1, C_1, C_2, \dots)$, $\bar{c} = (n - d + B_1, \bar{C}_1, \bar{C}_2, \dots)$ and let $X = \mid W \mid + \mid C \mid - \mid \hat{W} \mid - 1 = \mid \bar{W} \mid + \mid \bar{C} \mid - \mid \hat{W} \mid - 1$. Then $X \geq \mid C \mid - 1 \geq -1$ and $X \geq \mid \bar{C} \mid - 1$.
Let us see that $X \geq 0$. If $X = -1$, then $C = \bar{C} = 0$ and $\mid W \mid = \mid \bar{W} \mid = \mid \hat{W} \mid$; i.e., $\sum_{i=1}^{\ell} \sum_{j \geq 1} (\hat{W}_j(\lambda_i) - W_j(\lambda_i)) = \sum_{i=1}^{\ell} \sum_{j \geq 1} (\hat{W}_j(\lambda_i) - \bar{W}_j(\lambda_i)) = 0$, from where $\hat{W}_j(\lambda_i) = W_j(\lambda_i) = \bar{W}_j(\lambda_i))$ for $1 \leq i \leq \ell$ and $j \geq 1$; hence, $\hat{W} = W = \bar{W}$. Then $(W, B, C) = (\bar{W}, \bar{B}, \bar{C})$, which contradicts $S \overset{s.e.}{\not\sim} T$. Therefore $X \geq 0$.

- If $C \neq \bar{C}$, by Lemma A.8, from (44), there exists a partition of nonnegative integers $\hat{C} = (\hat{C}_1, \hat{C}_2, \dots)$ such that $\mid \hat{C} \mid = X$ and
$$c \angle (n - d + B_1 + 1, \hat{C}_1, \hat{C}_2, \dots), \quad \bar{c} \angle (n - d + B_1 + 1, \hat{C}_1, \hat{C}_2, \dots). \tag{57}$$
where $c = (n - d + B_1, C_1, C_2, \dots)$ and $\bar{c} = (n - d + B_1, \bar{C}_1, \bar{C}_2, \dots)$.
- If $C = \bar{C}$, by Lemma A.2 there exists a partition of nonnegative integers $\hat{C} = (\hat{C}_1, \hat{C}_2, \dots)$ such that $\mid \hat{C} \mid = X$ and (57) holds.

From (57), $\hat{C}_1 \leq C_1 + 1$ and $| \hat{W} | + | \hat{B} | + | \hat{C} | + \hat{C}_1 = | \hat{W} | + | B | + X + \hat{C}_1 = | W | + | B | + | C | -1 + \hat{C}_1 \leq | W | + | B | + | C | + C_1$. By Lemma 3.11, $| W | + | B | + | C | + C_1 \leq n$. By the same Lemma, there exists a linear relation U in $\mathbb{C}^n$ such that the Weyr characteristic of U is $(\hat{W}, \hat{B}, \hat{C})$. Then $\dim U = | \hat{W} | + | \hat{B} | + | \hat{C} | + \hat{B}_1 = | W | + | B | + | C | + B_1 - 1 = d - 1$. From (55)-(57), by Theorem 7.2 there exists linear relations $\bar{S}$, $\bar{T}$ in $\mathbb{C}^n$ such that $\bar{S} \overset{s.e.}{\sim} S$, $\bar{T} \overset{s.e.}{\sim} T$ and $U \subseteq \bar{S} \cap \bar{T}$. As $\dim U = d - 1 \leq \dim(\bar{S} \cap \bar{T}) < \dim \bar{S} = d$, we have that $U = \bar{S} \cap \bar{T}$; hence $\dim \frac{\bar{S}}{\bar{S} \cap \bar{T}} = \dim \frac{\bar{T}}{\bar{S} \cap \bar{T}} = 1$.

3. Case $B \neq \bar{B}$ and $B_1 = \bar{B}_1$. Assume that (43), (45) and (46) hold. Let $Y' = \sum_{i \geq 1} \max\{B_i, \bar{B}_i\} - \max\{\bar{e}, \bar{e}'\}$ and $Z = B_1 = \bar{B}_1$. By Lemma A.8, there exists a partition of nonnonnegative integers $\hat{B}$ such that $(Z - 1, \hat{B}_1, \dots) \angle (Z, B_1, \dots), (Z-1, \hat{B}_1, \dots) \angle (Z, \bar{B}_1, \dots)$ and $| \hat{B} | = Y'$. As $\hat{B}_1 \leq Z - 1 = B_1 - 1 < B_1$, we have $B_1 = \bar{B}_1 = \hat{B}_1 + 1$; hence

$$\hat{B} \angle B, \quad \hat{B} \angle \bar{B}. \tag{58}$$

Define

$$\hat{C} = C = \bar{C}, \tag{59}$$

and $y = Y - Y'$. From (46), $y \geq 0$.
Fix $\lambda_0 \notin \{\lambda_1, \dots, \lambda_\ell\}$ and define

$$\begin{aligned} &\hat{W}_j(\lambda) = \max\{W_j(\lambda), \bar{W}_j(\lambda)\}, \quad j \geq 1, \quad \lambda_0 \neq \lambda \in \bar{\mathbb{C}}, \\ &\hat{W}_j(\lambda_0) = 1, \quad 1 \leq j \leq y, \\ &\hat{W}_j(\lambda_0) = 0, \quad j > y. \end{aligned}$$

Then $\hat{W}_j(\lambda) \geq \hat{W}_{j+1}(\lambda)$, for $j \geq 1$ and $\lambda \in \bar{\mathbb{C}}$, and from (43) we derive

$$\begin{aligned} W_j(\lambda) \leq \hat{W}_j(\lambda) \leq W_j(\lambda) + 1, \quad j \geq 1, \quad \lambda \in \bar{\mathbb{C}}, \\ \bar{W}_j(\lambda) \leq \hat{W}_j(\lambda) \leq \bar{W}_j(\lambda) + 1, \quad j \geq 1, \quad \lambda \in \bar{\mathbb{C}}. \end{aligned} \tag{60}$$

Let

$$\hat{W}(\lambda) = (\hat{W}_1(\lambda), \hat{W}_2(\lambda), \dots), \quad \lambda \in \bar{\mathbb{C}},$$

and $\hat{W} = (\hat{W}(\lambda_0), \hat{W}(\lambda_1), \dots, \hat{W}(\lambda_\ell))$.
We have $| \hat{W} | = \sum_{i=1}^{\ell} \sum_{j \geq 1} \max\{W_j(\lambda_i), \bar{W}_j(\lambda_i)\} + y = | W | + | B | - Y'$ and $| \hat{W} | + | \hat{B} | + | \hat{C} | + \hat{C}_1 = | W | + | B | + | C | + C_1$. By Lemma 3.11, $| W | + | B | + | C | + C_1 \leq n$. By the same Lemma, there exists a linear relation U in $\mathbb{C}^n$ such that the Weyr characteristic of U is $(\hat{W}, \hat{B}, \hat{C})$. Then $\dim U = | \hat{W} | + | \hat{B} | + | \hat{C} | + \hat{B}_1 = | W | + | B | + | C | + B_1 - 1 = d - 1$. From (58)-(60), by Theorem 7.2 there exists linear relations $\bar{S}$, $\bar{T}$ in $\mathbb{C}^n$ such that $\bar{S} \overset{s.e.}{\sim} S$, $\bar{T} \overset{s.e.}{\sim} T$ and $U \subseteq \bar{S} \cap \bar{T}$. As $\dim U = d - 1 \leq \dim(\bar{S} \cap \bar{T}) < \dim \bar{S} = d$, we have that $U = \bar{S} \cap \bar{T}$; hence $\dim \frac{\bar{S}}{\bar{S} \cap \bar{T}} = \dim \frac{\bar{T}}{\bar{S} \cap \bar{T}} = 1$.

4. Case $B_1 \neq \bar{B}_1$.
Assume that (47)-(49) hold. Define

$$\hat{B} = \bar{B}, \tag{61}$$

$$\hat{C} = C. \tag{62}$$

From (48) we obtain

$$\hat{B} \angle B, \tag{63}$$

and

$$\bar{c} \angle \hat{c}, \tag{64}$$

where $\bar{c} = (n - d + \bar{B}_1, \bar{C}_1, \bar{C}_2, \dots)$ and $\hat{c} = (n - d + \bar{B}_1 + 1, \hat{C}_1, \hat{C}_2 \dots)$. For $\lambda \in \bar{\mathbb{C}}$ and $j \geq 1$,

$$m_j(\lambda) = \max\{W_j(\lambda), \bar{W}_j(\lambda) - 1\}, \quad M_j(\lambda) = \min\{W_j(\lambda) + 1, \bar{W}_j(\lambda)\}.$$

Then $m_j(\lambda) \geq m_{j+1}(\lambda)$ and $M_j(\lambda) \geq M_{j+1}(\lambda)$ for $j \geq 1$ and $\lambda \in \bar{\mathbb{C}}$. Let $m(\lambda) = (m_1(\lambda), \dots,)$ and $M(\lambda) = (M_1(\lambda), \dots,)$ for $\lambda \in \bar{\mathbb{C}}$. With this notation, condition (49) becomes

$$\sum_{i=1}^{\ell} \mid m(\lambda_i) \mid \leq x \leq \sum_{i=1}^{\ell} \mid M(\lambda_i) \mid .$$

From (47), we have

$$m_j(\lambda) \leq M_j(\lambda), \quad j \geq 1, \quad \lambda \in \bar{\mathbb{C}};$$

hence $\mid m(\lambda_i) \mid \leq \mid M(\lambda_i) \mid$ for $1 \leq i \leq \ell$. From Lemma A.9, there exist integers $x(\lambda_1), \dots, x(\lambda_\ell)$ such that

$$\sum_{i=1}^{\ell} x(\lambda_i) = x \text{ and } \mid m(\lambda_i) \mid \leq x(\lambda_i) \leq \mid M(\lambda_i) \mid, \quad 1 \leq i \leq \ell. \tag{65}$$

From (47) we have $\Lambda(S) \subseteq \Lambda(T)$; hence $\Lambda(T) = \{\lambda_1, \dots, \lambda_\ell\}$. For $1 \leq i \leq \ell$, let $\bar{n}_i = \max\{j :: \bar{W}_j(\lambda_i) > 0\}$. Then $m_j(\lambda_i) = M_j(\lambda_i) = 0$ for $j > \bar{n}_i$ and

$$\mid m(\lambda_i) \mid = \sum_{j=1}^{\bar{n}_i} m_j(\lambda_i), \quad \mid M(\lambda_i) \mid = \sum_{j=1}^{\bar{n}_i} M_j(\lambda_i).$$

Again by Lemma A.9, from (65), for $1 \leq i \leq \ell$, there exist integers $\hat{W}_1(\lambda_i) \geq \cdots \geq \hat{W}_{\bar{n}_i}(\lambda_i)$ such that

$$\sum_{j=1}^{\bar{n}_i} \hat{W}_j(\lambda_i) = x(\lambda_i) \text{ and } m_j(\lambda_i) \leq \hat{W}_j(\lambda_i) \leq M_j(\lambda_i), \quad 1 \leq j \leq \bar{n}_1. \tag{66}$$

Define $\hat{W}(\lambda_i) = (\hat{W}_1(\lambda_i), \dots)$, for $1 \leq i \leq \ell$, $\hat{W} = (\hat{W}(\lambda_1), \dots, \hat{W}(\lambda_\ell))$ and $\hat{W}(\lambda) = (0, \dots)$ if $\lambda \notin \{\lambda_1, \dots, \lambda_\ell\}$. From (66) we have

$$W_j(\lambda) \leq \hat{W}_j(\lambda) \leq W_j(\lambda) + 1, \quad j \geq 1, \quad \lambda \in \bar{\mathbb{C}}, \tag{67}$$

and

$$\bar{W}_j(\lambda) - 1 \leq \hat{W}_j(\lambda) \leq \bar{W}_j(\lambda), \quad j \geq 1, \quad \lambda \in \bar{\mathbb{C}}. \tag{68}$$

From (66) and (65), $\mid \hat{W} \mid = \sum_{i=1}^{\ell} \mid \hat{W}(\lambda_i) \mid = \sum_{i=1}^{\ell} x(\lambda_i) = x$; hence $\mid \hat{W} \mid + \mid \hat{B} \mid + \mid \hat{C} \mid + \hat{C}_1 = \mid W \mid + \mid B \mid + \mid C \mid + C_1$. As in the case 3., by Lemma 3.11, $\mid W \mid + \mid B \mid + \mid C \mid + C_1 \leq n$. By the same Lemma, there exists a linear relation U in $\mathbb{C}^n$ such that the Weyr characteristic of U is $(\hat{W}, \hat{B}, \hat{C})$. Then $\dim U = \mid \hat{W} \mid + \mid \hat{B} \mid + \mid \hat{C} \mid + \hat{B}_1 = \mid W \mid + \mid B \mid + \mid C \mid + B_1 - 1 = d - 1$. On one hand, from (67), (63) and (62) and on the other hand, from (68), (61) and (64), by Theorem 7.2 there exists linear relations $\bar{S}$, $\bar{T}$ in $\mathbb{C}^n$ such that $\bar{S} \overset{s.e.}{\sim} S$, $\bar{T} \overset{s.e.}{\sim} T$ and $U \subsetneq \bar{S} \cap \bar{T}$. As $\dim U = d - 1 \leq \dim(\bar{S} \cap \bar{T}) < \dim \bar{S} = d$, we have that $U = \bar{S} \cap \bar{T}$; hence $\dim \frac{\bar{S}}{\bar{S} \cap \bar{T}} = \dim \frac{\bar{T}}{\bar{S} \cap \bar{T}} = 1$.
If (50)-(52) hold, the proof is analogous.

$\square$

Appendix A. Auxiliary results to prove Theorem 7.4

Lemma Appendix A.1 ([2, Lemma 5.5])**.** *Let $X \geq 0$ be a nonnegative integer and let $\mathbf{a} = (a_1, \dots, a_m)$ be a finite sequence of nonnegative integers. Then there exists a finite sequence of nonnegative integers $\mathbf{g} = (g_1, \dots, g_{m+1})$ such that $\mid \mathbf{g} \mid = X$ and $\mathbf{g} \prec' \mathbf{a}$.*

From Lemmas A.1 and 5.9, we obtain Lemma A.2.

Lemma Appendix A.2. *Let $X \geq 0$ be a nonnegative integer and let A be a partition. Then there exists a partition G such that $\mid G \mid = X$ and*

$$A \angle (A_1 + 1, G_1, G_2, \dots).$$

Lemma Appendix A.3 ([2, Lemma 5.8])**.** *Let $X, Y \geq 0$ be nonnegative integers and let $\mathbf{c} = (c_1, \dots, c_m)$, $\mathbf{d} = (d_1, \dots, d_m)$ be finite sequences of nonnegative integers such that $\mathbf{c} \neq \mathbf{d}$. Let $\ell = \max\{i : c_i \neq d_i\}$, $f = \max\{i \in \{1, \dots, \ell\} : c_i < d_{i-1}\}$ and $f' = \max\{i \in \{1, \dots, \ell\} : d_i < c_{i-1}\}$.*

1. *There exists a finite sequence of nonnegative integers $\mathbf{g} = (g_1, \dots, g_{m+1})$ such that $\mid \mathbf{g} \mid = X$, $\mathbf{g} \prec' \mathbf{c}$ and $\mathbf{g} \prec' \mathbf{d}$ if and only if*

$$X \leq \sum_{i=1}^{m} \min\{c_i, d_i\} + \max\{c_f, d_{f'}\}.$$

2. *If $f > 1$ and $f' > 1$, there exists a finite sequence of nonnegative integers $\mathbf{e} = (e_1, \ldots, e_{m-1})$ such that $|\, \mathbf{e} \,| = Y$, $\mathbf{c} \prec' \mathbf{e}$ and $\mathbf{d} \prec' \mathbf{e}$ if and only if*

$$Y \geq \sum_{i=1}^{m} \max\{c_i, d_i\} - \max\{c_f, d_{f'}\}.$$

3. *If $f = 1$ or $f' = 1$, there exists a finite sequence of nonnegative integers $\mathbf{e} = (e_1, \ldots, e_{m-1})$ such that $|\, \mathbf{e} \,| = Y$, $\mathbf{c} \prec' \mathbf{e}$ and $\mathbf{d} \prec' \mathbf{e}$ if and only if*

$$\begin{array}{l} Y = \sum_{i=1}^{m} \max\{c_i, d_i\} - \max\{c_f, d_{f'}\}, \\ \textit{or} \\ Y \geq \sum_{i=1}^{m} \max\{c_i, d_i\} - \max\{c_{f+1}, d_{f'+1}\}. \end{array}$$

Equivalently,

$$Y = \sum_{i=2}^{m} \max\{c_i, d_i\} \text{ or } Y \geq \max\{c_1, d_1\} + \sum_{i=3}^{m} \max\{c_i, d_i\}.$$

Lemma Appendix A.4 ([3, Lemma 4.7])**.** *Given two finite sequence of nonnegative integers $\mathbf{a} = (a_1, \ldots, a_m)$ and $\mathbf{b} = (b_1, \ldots, b_m)$, let $x_i = \min\{a_i, b_i\}$, $1 \leq i \leq m$. Let $r = \overline{\mathbf{a}}$, $s = \overline{\mathbf{b}}$, and $y_i = \min\{r_i, s_i\}$, $i \geq 1$. Then*

$$(y_1, \ldots) = \overline{(x_1, \ldots, x_m)}.$$

Analogously we can prove Lemma A.5.

Lemma Appendix A.5. *Given two finite sequence of nonnegative integers $\mathbf{a} = (a_1, \ldots, a_m)$ and $\mathbf{b} = (b_1, \ldots, b_m)$, let $X_i = \max\{a_i, b_i\}$, $1 \leq i \leq m$. Let $r = \overline{\mathbf{a}}$, $s = \overline{\mathbf{b}}$, and $Y_i = \max\{r_i, s_i\}$, $i \geq 1$. Then*

$$(Y_1, \ldots) = \overline{(X_1, \ldots, X_m)}.$$

Lemma Appendix A.6 ([3, Lemma 4.6])**.** *Given two sequences of nonnegative integers $\mathbf{a} = (a_1, \ldots, a_m)$ and $\mathbf{b} = (b_1, \ldots, b_m)$ such that $\mathbf{a} \neq \mathbf{b}$, let*

$$\ell = \max\{i \ : \ a_i \neq b_i\},$$

$$f = \max\{i \in \{1, \ldots, \ell\} \ : \ a_i < b_{i-1}\}, \quad f' = \max\{i \in \{1, \ldots, \ell\} \ : \ b_i < a_{i-1}\}.$$

Let $r = \overline{\mathbf{a}}$, $s = \overline{\mathbf{b}}$, $r_0 = s_0 = m$,

$$x = \min\{i \ : \ r_i \neq s_i\},$$

$$e = \min\{i \geq x - 1 : \ s_{i+1} \geq r_{i+1}\}, \quad e' = \min\{i \geq x - 1 : \ r_{i+1} \geq s_{i+1}\}.$$

Then

$$e = a_f, \quad e' = b_{f'}.$$

Remark Appendix A.7. *From Lemmas A.5 and A.6, we have* $\sum_{i\geq 1}\max\{r_i,s_i\}-\max\{e,e'\}=\sum_{i=1}^{m}\max\{a_i,b_i\}-\max\{a_f,b_{f'}\}$. *If* $f\leq f'$, *then* $\max\{a_f,b_f\}\geq\max\{a_f,b_{f'}\}$ *and, if* $f\geq f'$, *then* $\max\{a_{f'},b_{f'}\}\geq\max\{a_f,b_{f'}\}$. *Therefore*

$$\sum_{i\geq 1}\max\{r_i,s_i\}-\max\{e,e'\}=\sum_{i=1}^{m}\max\{a_i,b_i\}-\max\{a_f,b_{f'}\}\geq 0.$$

From Lemmas A.3-A.6 and 5.9, we obtain Lemma A.8.

Lemma Appendix A.8. *Let* $X,Y\geq 0$ *be nonnegative integers and let* A, B *be partitions such that* $A\neq B$. *Let* $x=\min\{i\ :\ A_i\neq B_i\}$,

$$e=\min\{i\geq x-1:\ B_{i+1}\geq A_{i+1}\},\quad e'=\min\{i\in\{i\geq x-1:\ A_{i+1}\geq B_{i+1}\}.$$

Let Z *be an integer such that* $Z\geq\max\{A_1,B_1\}$.

1. *There exists a partition* G *such that* $\mid G\mid=X$,

   $$(Z,A_1,\dots)\angle(Z+1,G_1,\dots)\ \text{and}\ (Z,B_1,\dots)\angle(Z+1,G_1,\dots)$$

   if and only if

   $$X\leq\sum_{i\geq 1}\min\{A_i,B_i\}+\max\{e,e'\}.$$

2. *If there exists a partition* E *such that* $\mid E\mid=Y$,

   $$(Z-1,E_1,\dots)\angle(Z,A_1,\dots)\ \text{and}\ (Z-1,E_1,\dots)\angle(Z,B_1,\dots),$$

   then

   $$Y\geq\sum_{i\geq 1}\max\{A_i,B_i\}-\max\{e,e'\}.$$

3. *If*

   $$Y=\sum_{i\geq 1}\max\{A_i,B_i\}-\max\{e,e'\}.$$

   then there exists a partition E *such that* $\mid E\mid=Y$,

   $$(Z-1,E_1,\dots)\angle(Z,A_1,\dots)\ \text{and}\ (Z-1,E_1,\dots)\angle(Z,B_1,\dots).$$

Lemma Appendix A.9. *Let* $m_1,\dots,m_n$, $M_1,\dots,M_n$ *and* x *be integers such that*

$$\sum_{i=1}^{n}m_i\leq x\leq\sum_{i=1}^{n}M_i\ \text{and}\ m_i\leq M_i,\quad 1\leq i\leq n.$$

Then, there exist integers $x_1,\dots,x_n$ *such that*

$$\sum_{i=1}^{n}x_i=x\ \text{and}\ m_i\leq x_i\leq M_i,\quad 1\leq i\leq n.$$

And, if $m_1\geq\cdots\geq m_n$ *and* $M_1\geq\cdots\geq M_n$, *then* $x_1\geq\cdots\geq x_n$.

Proof. Let $k = \min\{j \geq 0 \,:\, x \leq \sum_{i=1}^{j} M_i + \sum_{i=j+1}^{n} m_i\}$. Then $k \leq n$.

If $k = 0$, then $x = \sum_{i=1}^{n} m_i$. Define $x_i = m_i$, $1 \leq i \leq n$. Then $x_1, \ldots, x_n$ satisfy the conditions.

If $k > 0$, then

$$\sum_{i=1}^{k-1} M_i + \sum_{i=k}^{n} m_i < x \leq \sum_{i=1}^{k} M_i + \sum_{i=k+1}^{n} m_i. \tag{A.1}$$

Define

$$\begin{aligned} &x_i = M_i, \quad 1 \leq i \leq k-1,\\ &x_k = x - \textstyle\sum_{i=1}^{k-1} M_i - \sum_{i=k+1}^{n} m_i,\\ &x_i = m_i, \quad k+1 \leq i \leq n. \end{aligned}$$

It is clear that $\sum_{i=1}^{n} x_i = x$ and that $m_i \leq x_i \leq M_i$ for $1 \leq i \leq k-1$ and $k+1 \leq i \leq n$. From (69), we obtain $m_k < x_k \leq M_k$.

If $m_1 \geq \cdots \geq m_n$ and $M_1 \geq \cdots \geq M_n$, then $x_1 \geq \cdots \geq x_{k-1}$ and $x_{k+1} \geq \cdots \geq x_n$. Moreover, $x_{k-1} = M_{k-1} \geq M_k \geq x_k > m_k \geq m_{k+1} = x_{k+1}$. □

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