Geodesic Convexity

Symmetric Spaces

and

Hilbert-Schmidt Operators

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ABSTRACT: A natural Riemannian structure is introduced on the set of positive invertible (unitized) Hilbert-Schmidt operators, in order to obtain several decomposition theorems by means of geodesically convex submanifolds. We also give an intrinsic (algebraic) characterization of such submanifolds, and we study the group of isometries. We show that any symmetric space of the noncompact type can be isometrically embedded in this manifold. We include a final section devoted to the study of the unitary orbits of a fixed operator and the diverse geometries that arise from endowing this orbit with different Riemanian metrics.

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1 Introduction

1.1 Hadamard manifolds

A Hadamard manifold is a Riemannian manifold which is simply connected, complete, and has nonpositive sectional curvature . From the topological viewpoint, it is a very simple object.

However, (see [Eb96]) for any manifold M of nonpositive sectional curvature, the higher homotopy groups ($\pi_k(M)$, $k \geq 2$) vanish, and M can be expressed as a quotient space of a Hadamard manifold (the universal covering of M) and a suitable deckgroup of isometries of the covering which is isomorphic to $\pi_1(M)$.

The geometry of nonpositevely curved spaces is indeed rich and has applications in many other branches of mathematics, such as harmonic maps ([Cor92], [GS92], [KS93], [MSY93]), 3-manifolds and Kleinian groups ([MS84], [Gab92], [Can93], [CJ94], [Min94], [McM96], [Otal96], [Gab97], [Otal98], [Min99], [Kap01], [GMT03]), structure theory and rigidity ([Ball85], [BBE85], [BBS85], [BS97], [EH90], [BB95], [Lee97]), high dimensional topology ([FH81], [FJ93], [CGM90]), hyperbolic groups and quasi conformal geometry ([Gro87], [Pan89], [BM91], [RS94], [Sela95], [Bow98a], [Bow98b], [BP99], [BP00], [HK98]), geometric and combinatorial group theory ([Gro87], [DJ91], [Sch95], [CD95], [BM97], [KL97a], [KL97b] and [Esk98]) and dynamics ([Cro90], [Otal90], [BCS95]).

The classical treatises [Hel62] by Sigurdur Helgason and [BGS85] by Wallman *et al.*, the introduction to the geometry of spaces of the noncompact type by Patrick Eberlein [Eb96], or the expository survey by the same author [Eb89] collect many of the relevant facts concerning the geometry of these objects, such as the Law of Cosines, orthogonal projections, convexity of the distance function, the construction of the boundary space, and rank rigidity theorems.

Let's focus briefly on six basic results which are valid (see [Hel62]) in any Hadamard manifold M of finite dimension:

1. The exponential map $\exp_p : T_pM \to M$ is a diffeomorphism for each $p \in M$.
2. For each pair $p, q \in M$ there exists a unique normal (*i.e.* unit speed), minimizing geodesic from p to q.
3. For any geodesic triangle in M whose sides are geodesics of length a, b and c, we have the Hyperbolic Law of Cosines, which states:
$$c^2 \geq a^2 + b^2 - 2ab\cos(\theta), \text{ where } \theta \text{ is the angle opposite to } c$$
4. The sum of the interior angles of any such triangle is at most π.
5. For any pair of geodesics α, β in M, the function
$$f(t) = \mathrm{dist}(\alpha(t), \beta(t))$$
is a real convex function.
6. Let C be a convex closed subset of M. Then for each $p \in M$ there exists a unique point $\Pi_{\mathsf{C}}(p) \in \mathsf{C}$ such that
$$\mathrm{dist}(p, \Pi_{\mathsf{C}}(p)) \leq \mathrm{dist}(p, q) \text{ for any } q \in \mathsf{C}$$
In the Riemannian context, the point $\Pi_{\mathsf{C}}(p)$ is called the *foot of the perpendicular* from p to C.

The notions of completeness as metric space and completeness in the geodesic sense are intimately related by Hopf-Rinow's theorem. Since this theorem is false in infinite dimensions (see [Atkin75], [Atkin87]), compactness of neighbourhoods of M seems to be relevant for these results to hold true. However, statements 1 through 6 are known to be valid in the setting of nonpositively curved spaces (which are metric spaces where some geodesic triangle comparison inequality is valid). In particular, the proof of existence of a unique distance-realizing point for any closed convex set (without assuming local compactness of neighbourhoods) can be found in [Jost97].

We will go in an alternate direction, in order to extend these results to a manifold Σ_∞ which is locally isomorphic to an infinite dimensional Hilbert space (in fact, the real part of a Banach algebra $\mathcal{B}$). The manifold Σ_∞ is simply connected, complete, and has nonpositive sectional curvature; moreover, $\Sigma_\infty = GL^+(\mathcal{B})$ is a symmetric space in the usual Riemannian sense. All the tools of the Riemannian geometry will be at hand, and we will be able to explore relationships between the Banach algebra and the geometry of the manifold.

For instance, we will prove that the unique minimizing geodesic that realices distance between a point and a convex set must be orthogonal to that set, obtaining in this way a decomposition theorem for operators, with many immediate applications.

The first result of the list will be apparent from the definition of Σ_∞; to prove the second, the third, the fourth and the fifth we will have to collect some facts from the existing literature of geometry on spaces of operators.

The space Σ_∞ is symmetric and nonpositively curved, and universal in this category in the sense that every symmetric space of the noncompact type can be (almost) isometrically embedded as a geodesically convex, closed submanifold.

Though we will not need it along this manuscript, it should be noted that the general classification theory of L^*-algebras (see [Sch60] and [Sch61] by J.R.Schue, [CGM90] by Mira, Martin and González, or [Neh93] by E. Neher) provides a general abstract framework for this manifold and its convex submanifods: the real part of any L^*-algebra can be naturally embedded as a convex closed submanifold of Σ_∞.

1.2 The main results

A few words about notation: we will use greek characters $\alpha, \beta, \delta, \dots$ to denote real and complex numbers, and capital characters $\Sigma, \Lambda, \Delta, \Omega, \dots$ to denote manifolds. The first characters of the alphabet $a, b, c, d, \cdots$ will be reserved for Hilbert-Schmidt operators and as usual, $p, q, r, s, \dots$ will be used for points (in Σ_∞); sometimes we will use capital letters $A, B, C, D, \dots$ to stress the fact that this points are positive invertible operators in the unitized Banach algebra of Hilbert-Schmidt operators. The capital letters $X, Y, Z, W, \dots$ will be used sometimes to denote selfadjoint operators (tangent vectors) in the aforementioned Banach algebra. German characters $\mathfrak{a}, \mathfrak{k}, \mathfrak{m}, \mathfrak{p}, \dots$ will be used as customary in Lie group theory to denote Lie algebras (or to denote certain subspaces of Lie algebras). Throughout, $\exp_p$ will denote the exponential of the Riemannian manifold at the point p, and we will use exp instead of $\exp_1$, which is the usual exponential of operators.

Now we outline the organization and main results of this work (previous results are mentioned as such, and new results are **Theorems I** through **VIII**):

In section 2, we introduce some notation and recall a few results we will need for the construction of a Hilbert manifold of infinite dimension Σ_∞, which is complete, simply connected and has nonpositive sectional curvature.

The ambient space for most of the computations is the Banach space with trace inner product $\mathcal{H}_\mathbb{R} = \{\lambda + a\}$, where λ is a real number and a is a selfadjoint Hilbert-Schmidt operator acting on a separable Hilbert space H. As a set, $\Sigma_\infty := \exp(\mathcal{H}_\mathbb{R})$. The exponential is an open mapping so Σ_∞ is open in $\mathcal{H}_\mathbb{R}$ (Proposition 2.3).

In section 3, we recall some definitions and facts about closed, geodesically convex subsets, which are the submanifolds where the projection theorem (Theorem II) applies. In particular, we have the following result

Result *Assume* $\mathfrak{m}$ *is a closed subspace such that* $[X, [X, Y]] \in \mathfrak{m}$ *whenever* $X, Y \in \mathfrak{m}$. *Then* $M = \exp(\mathfrak{m}) \subset \Sigma_\infty$ *with the induced metric is a closed, geodesically convex submanifold.*

This result is mainly due to Mostow [Mos55] (though Pierre de la Harpe sketches the proof for Hilbert-Schmidt operators in [Har72]), and it shows that there are plenty of this sets (see Corollary 3.13).
In particular, any closed abelian subalgebra of Hilbert-Schmidt operators provides an example of a convex submanifold. Other examples are provided by operators acting on fixed subspaces of H. In section 3.2.1 we give a list of convex sets; this list is exhaustive but by no means complete.

In section 3.3 we take Élie Cartan's viewpoint, and study convex submanifolds M as homogeneous symmetric spaces for the action of a convenient group G_M. This group is the smaller Lie group -inside the invertible operators of the Banach algebra- containing M. The main result is Theorem I below (3.30). Throughout, $GL(\mathcal{B})$ stands for the group of invertible elements in the Banach algebra $\mathcal{B}$ and $I_0(M)$ for the connected component of the identity of the group of isometries of M :

Theorem I *If $M = \exp(\mathfrak{m})$ is convex and closed, and $G_M \subset GL(\mathcal{H}_{\mathbb{C}})$ is the Lie subgroup with Lie algebra $\mathfrak{g}_M = \mathfrak{m} \oplus \overline{[\mathfrak{m},\mathfrak{m}]}$, then*

(a) *$P(G_M) = M$, so M is an homogeneous space for G_M.*

(b) *For any $g = |\, g \,|\, u_g$ (Cauchy polar decomposition) in G_M, we have*

$$|\, g \,| = \sqrt{gg^*} \in M \subset G_M,$$

and also $u_g \in K \subset G_M$ where K is the isotropy Lie subgroup

$$K = \{g \in G_M \,:\, gg^* = 1\} \text{ with Lie algebra } \mathfrak{k} = \overline{[\mathfrak{m},\mathfrak{m}]}$$

In particular, G_M has a polar decomposition

$$G_M \simeq M \times K = P(G_M) \times U(G_M)$$

(c) *$M = P(G_M) \simeq G_M/K$.*

(d) *M has nonpositive sectional curvature.*

(e) *For $g \in G_M$, consider $I_g(r) = grg^*$. Then $I : G_M \to I_0(M)$.*

(f) *Take $p, q \in M$, and set $g = p^{\frac{1}{2}}(p^{-\frac{1}{2}}qp^{-\frac{1}{2}})^{\frac{1}{2}}p^{-\frac{1}{2}} \in G_M$. Then I_g is an isometry in $I_0(M)$ which sends p to q, namely G_M acts transitively and isometrically on M.*

In section 4 we state and prove the main result about uniqueness and existence of the minimizing geodesic (4.8):

Theorem II *Let M be a geodesically convex, closed submanifold of Σ_∞. Then for every point $p \in \Sigma_\infty$, there exists a unique minimizing geodesic γ joining p to M such that*

$$Length(\gamma) = \operatorname{dist}(p, M)$$

Moreover, this geodesic is orthogonal to M, and if we call $\Pi_M : \Sigma_\infty \to M$ the map that assigns to p the endpoint of the minimizing geodesic starting at p, then Π_M is a Lipschitz (i.e. contractive) map.

Section 5 deals with a main application of the factorization theorem. When applied to the manifold of diagonal operators (5.2), provides a decomposition of positive operators as a product of a diagonal positive operator and the exponential of a codiagonal, selfadjoint operator:

Theorem III *Take any selfadjoint Hilbert-Schmidt operator A such that $1+A>0$. Then there exist a diagonal, strictly positive Hilbert-Schmidt perturbation of the identity D and a selfadjoint Hilbert-Schmidt operator V with null diagonal such that the following factorization holds:*

$$1+A=D\,\mathrm{e}^{V}D$$

Moreover, D and V are unique and the map $1+A\mapsto(D,V)$ is real analytic.

A straightforward application (5.3) of the last theorem is an alternative proof to an already known decomposition for matrices (Theorem 3 of the paper [Mos55] by G.D. Mostow, see also Theorem 1 of the paper [CPR91] by Corach, Porta and Recht)

Result *Fix a positive invertible matrix $A\in M_n^+$. Then there exist* unique *matrices $D,V\in M_n$, such that D is diagonal and strictly positive, V is symmetric and with null diagonal, and the following formula holds*

$$A=D\,\mathrm{e}^{V}D$$

Moreover, the maps $A\mapsto D$ and $A\mapsto V$ are real analytic.

In section 6, we discuss a foliation of codimension one of the total space by totally geodesic, closed leaves. The tangent space of each leaf is the Banach space of selfadjoint Hilbert-Schmidt operators (shortly, HS^h). The leaves are also parallel in the sense that geodesics that have minimal length among those which join them are orhogonal to both of them (Proposition 6.4).
We prove that sectional curvature is trivial along vertical 2-planes (Proposition 6.5), and also (Theorem 6.6) that Σ_∞ is isometric to the direct product of the complete and totally geodesic submanifolds $\Sigma_1=\exp(\mathrm{HS}^h)$ and Λ (the positive scalars), *i.e.*

$$\Sigma_\infty\simeq\Sigma_1\times\Lambda$$

The leaf Σ_1 contains the identity and its tangent space is the set of selfadjoint Hilbert-Schmidt operators, so whenever it is possible, we work inside Σ_1 to avoid the manipulation of scalars.

The intrinsic version of the decomposition theorem takes a simpler form; based upon the results of section 3, it reads:

Theorem IV *Assume $\mathfrak{m}\subset\mathrm{HS}^h$ is a closed subspace such that*

$$[X,[X,Y]]\in\mathfrak{m}\qquad \text{for any}\;\; X,Y\in\mathfrak{m}$$

Then for any $A\in\mathrm{HS}^h$ there is a unique decomposition of the form

$$\mathrm{e}^{A}=\mathrm{e}^{X}\,\mathrm{e}^{V}\,\mathrm{e}^{X}$$

where $X\in\mathfrak{m}$, and $V\in\mathrm{HS}^h$ is such that $tr(VZ)=0$ for any $Z\in\mathfrak{m}$.

We should stress that this is an infinite dimensional analogue of a theorem of G.D. Mostow for matrices [Mos55].

In section 6.2, we embed the space M_n^+ of positive invertible $n \times n$ matrices in Σ_1 (this embedding can also be found in [AV03]). This embedding is closed and geodesically convex; in (6.10) we only consider elements $p \in \Sigma_1$ and show another application of the main theorem:

Theorem V *If we identify M_n^+ with the first block of the matrix representation of the Hilbert-Schmidt operators (in any fixed orthornomal basis), for any positive invertible operator e^b (b is Hilbert-Schmidt and selfadjoint) there is a unique factorization of the form*

$$e^b = \begin{pmatrix} e^A & 0 \\ 0 & 1 \end{pmatrix} \exp\left\{ \begin{pmatrix} e^{-A} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbb{O}_{n\times n} & Y^* \\ Y & X \end{pmatrix} \right\}$$

In section 7, we sketch the proof of the inclusion of symmetric manifolds of the noncompact type in M_n^+ (this result is due to P. Eberlein, see [Eb85]). This result together with the embedding of M_n^+ in Σ_1 (which has been proved by Andruchow and Varela in [AV03]) gives us the following (see 7.3):

Result *For any symmetric manifold M of the noncompact type there is an embedding into Σ_1 which is a diffeomorphism betwen M and a closed, geodesically convex submanifold of Σ_1. This map preserves the metric tensor in the following sense: if we pull back the inner product of Σ_1 to M, then this inner product is a (positive) constant multiple of the inner product of M (on each irreducible de Rham factor of M). Assuming we identify M with its image, M factorizes Σ_∞* via *the contractive map Π_M.*

In section 8, for fixed $e^a \in \Sigma_1$, we consider the action of the full unitary group of $L(H)$ by means of the conjugation $g \mapsto ge^a g^*$, and also the action of the unitaries that are Hilbert-Schmidt perturbations of a scalar multiple of the identity. The orbit acting with either group is not necessarily the same set (Example 8.4). Throughout, $\mathcal{U}_\mathcal{B}$ stands for the unitary group of the involutive Banach algebra $\mathcal{B}$.

We discuss whether the orbit Ω can be given an analytic structure of submanifold; this question is partially answered by Theorem VI (8.5) and Theorem VII (8.2), which state:

Theorem VI *If the C^*-algebra generated by a and 1 is finite dimensional, then the orbit of e^a with the action of the Hilbert-Schmidt unitaries can be given an analytic submanifold structure.*

Theorem VII *The orbit of e^a under the action of the full unitary group of $L(H)$ can be given an analytic submanifold structure if and only if the C^*-algebra generated by a and 1 is finite dimensional.*

The results of section 8.2 are related to the study of the geodesics of the orbit Ω, with different Riemannian metrics. In section 8.2.1 we immerse the orbit in the Euclidean space of Hilbert-Schmidt operators and we give it the induced metric: we show that for any selfadjoint h, the curve

$$\gamma(t) = e^{ith} e^{a} e^{-ith}$$

is a geodesic of the orbit whenever $e^{a} - 1$ is an orthogonal projector and h is codiagonal in the representation associated to $e^{a} - 1$. This is Proposition 8.10.

We also show that, for any e^{a}, these curves are the usual geodesics of Σ_1 only if they are constant curves (this is Proposition 8.8); in particular, when the orbit is regarded as a submanifold of the Euclidean space of Hilbert-Schmidt operators, this submanifold is not geodesic in any of its points whenever $e^{a} - 1$ is an orthogonal projector.

In section 8.2.2 we take a peak at the geodesics of the orbit of e^{a} as a Riemannian submanifold $\Omega \subset \Sigma_1$; the main result is Theorem VIII below (8.13). Throughout $[\ ,\ ]$ stands for the usual commutator of operators, and these results are valid for the action of any of the groups $\mathcal{U}_{\mathsf{L}(\mathsf{H})}$ or $\mathcal{U}_{\mathcal{H}_{\mathbb{C}}}$ because they induce the same manifold in Σ_1 (this is proved in Lemma 8.6):

Theorem VIII *Assume* $e^{a} = 1 + A$ *with* A *an orthogonal projector, and* $\Omega \subset \Sigma_1$ *is the unitary orbit of* e^{a}. *Then*

1. Ω *is a Riemannian submanifold of* Σ_1.
2. $T_p\Omega = \{i[x,p] : x \in \mathrm{HS}^h\}$ *and* $T_p\Omega^{\perp} = \{x \in \mathrm{HS}^h : [x,p] = 0\}$.
3. *The action of the unitary group is isometric, namely*

   $$\mathrm{dist}^{\Omega}(upu^*, uqu^*) = \mathrm{dist}^{\Omega}(p,q)$$

   for any unitary operator $u \in \mathsf{L}(H)$.
4. *For any* $v = i[x,p] \in T_p\Omega$, *the exponential map is given by*

   $$\exp_p^{\Omega}(v) = e^{ighg^*} p\, e^{-ighg^*}$$

   where $p = ge^{a}g^*$ *and* h *is the codiagonal part of* g^*xg *(in the matrix representation of Proposition 8.10). In particular, the exponential map is defined in the whole tangent space.*
5. *If* $p = ge^{a}g^*$, $q = we^{a}w^*$, *and* h *is a selfadjoint, codiagonal operator such that* w^*ge^{ih} *commutes with* e^{a}, *then the curve* $\gamma(t) = e^{itghg^*} p e^{-itghg^*}$ *is a geodesic of* $\Omega \subset \Sigma_1$, *which joins* p *to* q.
6. *If we assume that* $h \in \mathrm{HS}^h$, *then* $L(\gamma) = \frac{\sqrt{2}}{2}\, \|h\|_2$
7. *The exponential map* $\exp_p^{\Omega} : T_p\Omega \to \Omega$ *is surjective.*

In the last section we end the exposition with some open questions and remarks.

.

1.3 Precedents

- Based in the classical construction of a Riemannian structure for the set M_n^+ of positive invertible matrices (the first published proof of the result seems to be [Mos55] by G.D. Mostow in the mid 50's), Andruchow and Varela show in a recent paper [AV03] how the algebra of Hilbert-Schmidt operators with the trace norm provide a convenient framework for the construction of a Riemannian manifold of infinite dimension Σ_∞ which is a Hadamard manifold in the classical (Riemannian) sense (see also [Har72] by P. de la Harpe). The present work is based upon this construction.

- The decomposition theorems have obvious precedents in the polar decomposition of operators, but we should also mention the splitting of the positive set of a matrix algebra (see [Mos55] by Mostow, [CPR91] by Corach *et al.*) and the paper by Porta and Recht [PR94] which deals with C^*-algebras and conditional expectations.

- In [Eb85], Patrick Eberlein shows that any symmetric manifold M of the noncompact type can be embedded in $P(\mathfrak{g})$ (the positive invertible operators acting in the Lie algebra of the group of isometries of M) as a closed, geodesically convex submanifold. This embedding is isometric in the following sense: if g^* is the pull back of the metric of $P(\mathfrak{g})$ on M, then g^* is a constant multiple of the metric of M on each irreducible de Rham factor of M.

- The relationship between the spectrum of an operator, and the existence of an homogeneous reductive structure for the orbit of that operator has been systematically studied through the years by diverse authors, including Andruchow, Deckard, Fialkow, Raeburn and Stojanoff in [DF79], [AFHS90], [Rae77], [AS89], [AS91], [Fial79] and [AS94]. In particular, [DF79] seems to be the first systematic approach to the subject.

- The geometry of the homogeneous reductive spaces which appear naturally in Banach and C^*-algebras has been extensively studied, and we should mention a few articles: Corach, Porta and Recht study the space of idempotents in ([PR87a], [PR87b], [CPR93b], [CPR90b]), the set of positive invertible operators is treated in the papers [CPR92], [CPR93a] and [AV03], and the space of relatively regular elements in a Banach Algebra in [CPR90a]. Generalized flags (grassmanians, spectral measures, etc) are also studied by Andruchow, Durán, Mata-Lorenzo, Recht, Stojanoff, and Wilkins in [ARS92], [DMR00], [DMR04a], [DMR04b] and [Wilk90]. Partial isometries are studied in [AC04], the sphere of a Hilbert module is treated in [ACS99], and weights on von Neumann algebras are studied by Andruchow and Varela in [AV99].

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2 The Main Objects Involved

The main framework of this manuscript is the von Neumann algebra $\mathsf{L}(H)$ of bounded operators acting on a complex, separable Hilbert space H.

2.1 Hilbert-Schmidt operators

Throughout, HS stands for the bilateral ideal of Hilbert-Schmidt operators of $\mathsf{L}(H)$. Recall that HS is a Banach algebra (without unit) when given the norm

$$\|a\|_2 = 2\, tr(a^*a)^{\frac{1}{2}} = 2\left(\sum_{i\geq 1} \langle ae_i, ae_i\rangle\right)^{\frac{1}{2}}$$

where $\{e_i\}_{i\in\mathbb{N}}$ is any given orthonormal basis of H. The reader can find many of the statements we will use about trace operators and trace ideals in [Simon89].

Inside $\mathsf{L}(H)$ we consider a certain kind of Fredholm operators, namely

$$\mathcal{H}_{\mathbb{C}} = \{a+\lambda : \ a \in \mathrm{HS},\ \lambda \in \mathbb{C}\},$$

the complex linear subalgebra consisting of Hilbert-Schmidt perturbations of scalar multiples of the identity. This algebra is not norm closed, in fact, its closure is the set of compact perturbations of scalar multiples of the identity.

There is a natural Hilbert space structure for this subspace, where scalar operators are orthogonal to Hilbert-Schmidt operators, which is given by the inner product

$$\langle a+\lambda, b+\beta\rangle_2 = 4tr(ab^*) + \lambda\overline{\beta}$$

This product is well defined and positive definite; $\mathcal{H}_{\mathbb{C}}$ is complete with this norm, due to the completeness of the HS operators with the trace inner product.

Remark 2.1. Another natural (but not quadratic) norm is given by the formula

$$\|a+\lambda\|_1 = 2\, tr(a^*a)^{\frac{1}{2}} + |\lambda|$$

With this norm $\mathcal{H}_{\mathbb{C}}$ becomes a Banach algebra, that is

$$\|(a+\lambda)(b+\beta)\|_1 \leq \|a+\lambda\|_1 \|b+\beta\|_1$$

However, we will use the norm defined by the inner product, that is

$$\|a+\lambda\|_2 = \sqrt{\|a\|_2^2 + |\lambda|^2} = \left(4tr(a^*a) + |\lambda|^2\right)^{\frac{1}{2}}$$

Both norms are equivalent, but $\|\cdot\|_2$ provides an Euclidean structure for $\mathcal{H}_{\mathbb{C}}$.

We also use the term Banach algebra for a normed algebra $\mathcal{B}$ where the sum and product are continuous operations; this is slightly different from the usual definition (see Rickart [Rick60] or Guichardet [Guich67]).
The model space that we are interested in is the real part of $\mathcal{H}_{\mathbb{C}}$:

$$\mathcal{H}_{\mathbb{R}} = \{a+\lambda : \ a^* = a,\ a \in \mathrm{HS},\ \lambda \in \mathbb{R}\},$$

which inherits the structure of real Banach space, and with the same inner product, becomes a real Hilbert space.

Remark 2.2. For this inner product, we have (by cyclicity of the trace)

$$\langle XY, Y^*X^*\rangle_2 = \langle YX, X^*Y^*\rangle_2 \qquad \text{for any } X,Y \in \mathcal{H}_{\mathbb{C}}\text{, and also}$$

$$\langle ZX, YZ\rangle_2 = \langle XZ, ZY\rangle_2 \qquad \text{for } X,Y \in \mathcal{H}_{\mathbb{C}} \text{ and } Z \in \mathcal{H}_{\mathbb{R}}$$

We will use HS^h to denote the closed subspace of selfadjoint Hilbert-Schmidt operators. Inside $\mathcal{H}_{\mathbb{R}}$, consider the subset

$$\Sigma_\infty := \{A > 0, A \in \mathcal{H}_{\mathbb{R}}\}$$

This is the set of invertible operators $a+\lambda$ such that $\sigma(a+\lambda) \subset (0,+\infty)$, with a selfadjoint and Hilbert-Schmidt, $\lambda \in \mathbb{R}$.
Note that, since a is compact, then $0 \in \sigma(a)$, which forces $\lambda > 0$ because

$$\sigma(a+\lambda) \subset (0,+\infty) \iff \sigma(a) \subset (-\lambda,+\infty)$$

Our main reference for standard facts about functional analysis, operator algebras and functional calculus is the four volume treatise of Functional Analysis by Michael Reed and Barry Simon, [RS79].

2.2 Some Basic Geometrical Facts

The following result is elementary, but we will give a proof anyway to get a taste of the nature of the objects involved, see also Corollary 3.14:

Proposition 2.3. Σ_∞ *is an open set of* $\mathcal{H}_\mathbb{R}$.

Proof. Consider the analytic exponential map $\exp : \mathcal{H}_\mathbb{C} \to \mathcal{H}_\mathbb{C}$ that assigns

$$A \mapsto \mathrm{e}^A = \sum \frac{A^n}{n!}$$

The restriction of exp to $\mathcal{H}_\mathbb{R}$ is well defined because for every Hilbert-Schmidt, selfadjoint operator and every real λ we can write $\mathrm{e}^{a+\lambda} = b+\beta$, where

$$b = \mathrm{e}^\lambda \sum \frac{a^k}{k!} \qquad \text{and} \qquad \beta = \mathrm{e}^\lambda$$

Obviously, β is real and b is selfadjoint; moreover b lies in HS because the latter is a bilateral ideal in $\mathrm{L}(H)$, and $b = a \cdot c$ for a bounded operator c.

We claim that $\Sigma_\infty = \exp(\mathcal{H}_\mathbb{R})$. One inclusion has already been proved. To prove the other, apply the functional calculus to the function $g(x) = \ln(x)$ and the operator $b+\beta \in \Sigma_\infty$. Since this operator is positive, the logarithm has the form of a series; an argument similar to the one we used for the exponential shows that $\ln(\beta+b) = \lambda + a$, with λ real and a Hilbert-Schmidt (and selfadjoint). This proves that the logarithm gives a local analytic inverse of exp, so exp maps onto.

The proof of our initial assertion follows from general results about Banach algebras and analytic maps: any analytic map (from a Banach algebra into itself) with an analytic local inverse is locally open, and as a consequence, $\Sigma_\infty = \exp(\mathcal{H}_\mathbb{R}) \subset \mathcal{H}_\mathbb{R}$ is open. □

For $p \in \Sigma_\infty$, we identify $T_p\Sigma_\infty$ with $\mathcal{H}_\mathbb{R}$, and endow this manifold with a (real) Riemannian metric by means of the formula

$$\langle X, Y \rangle_p = \langle p^{-1}X, Yp^{-1} \rangle_2 = \langle Xp^{-1}, p^{-1}Y \rangle_2$$

We collect some facts that can be found in [Wilk94], [AV03] and [CPR94]:

- With this metric Σ_∞ has nonpositive sectional curvature; moreover, the curvature tensor is given by the following commutant:

$$\mathcal{R}_p(X,Y)Z = -\frac{1}{4}\, p\left[\left[p^{-1}X, p^{-1}Y\right], p^{-1}Z\right] \tag{1}$$

- Covariant derivative is given by the expression

$$\nabla_X Y = X(Y) - \frac{1}{2}\left(Xp^{-1}Y + Yp^{-1}X\right) \tag{2}$$

where $X(Y)$ denotes derivation of the vector field Y in the direction of X (performed in the ambient space $\mathcal{H}_{\mathbb{R}}$).

- Euler's equation $\nabla_{\dot\gamma}\dot\gamma = 0$ reads

$$\ddot\gamma - \dot\gamma\gamma^{-1}\dot\gamma = 0, \tag{3}$$

and the unique geodesic joining $\gamma_{pq}(0) = p$ with $\gamma_{pq}(1) = q$ is given by the expression

$$\gamma_{pq}(t) = p^{\frac{1}{2}}\left(p^{-\frac{1}{2}}qp^{-\frac{1}{2}}\right)^t p^{\frac{1}{2}} \tag{4}$$

These curves look formally equal to the geodesics between positive definite matrices (regarded as a symmetric space).

- This geodesic is unique and realizes the distance: the manifold Σ_∞ turns out to be complete with this distance.

- The distance function $f : \mathbb{R} \to \mathbb{R}_{\geq 0}$, $f(t) = \mathrm{dist}(\gamma_1(t), \gamma_2(t))$ (γ_i are geodesics) is a convex function.

- The exponential map increases distance, that is

$$\|X - Y\|_2 \leq \mathrm{dist}(\mathrm{e}^X, \mathrm{e}^Y) = \|\ln(\mathrm{e}^{-X/2}\mathrm{e}^Y\mathrm{e}^{-X/2})\|_2$$

Remark 2.4. Throughout, $\|X\|_p^2 := \langle X, X\rangle_p$, namely

$$\|X\|_p^2 = \|p^{-1/2}Xp^{-1/2}\|_2 = \left\langle Xp^{-1}, p^{-1}X\right\rangle_2 = \left\langle p^{-1}X, Xp^{-1}\right\rangle_2,$$

which is the norm of tangent vectors $X \in T_p\Sigma_\infty$. We will use $\exp_p$ to denote the exponential map of Σ_∞. Note that $\exp_p(V) = p^{\frac{1}{2}}\,\mathrm{e}^{p^{-\frac{1}{2}}Vp^{-\frac{1}{2}}}\,p^{\frac{1}{2}}$, but rearranging the exponential series we get the simpler expressions

$$\exp_p(V) = p\,\mathrm{e}^{p^{-1}V} = \mathrm{e}^{Vp^{-1}}\,p$$

A straightforward computation also shows that for $p, q \in \Sigma_\infty$ we have

$$\exp_p^{-1}(q) = p^{\frac{1}{2}}\ln(p^{-\frac{1}{2}}\,q\,p^{-\frac{1}{2}})p^{\frac{1}{2}}$$

Lemma 2.5. *The metric in Σ_∞ is invariant for the action of the group of invertible elements: if g is an invertible operator in $\mathcal{H}_{\mathbb{C}}$, then $I_g(p) = gpg^*$ is an isometry of Σ_∞.*

Proof. Note that $d_r I_g(x) = gxg^*$ for any $x \in T_r\Sigma_\infty$, so

$$\|gxg^*\|^2_{grg^*} = \left\langle gxg^*(g^*)^{-1}r^{-1}g^{-1}, (g^*)^{-1}r^{-1}g^{-1}gxg^*\right\rangle_2 =$$

$$= \left\langle gxr^{-1}g^{-1}, (g^*)^{-1}r^{-1}xg^*\right\rangle_2 = \left\langle xr^{-1}, r^{-1}x\right\rangle_2 = \|x\|^2_r$$

where the third equality follows from Remark 2.2 (naming $X = gxr^{-1}$, $Y = g^{-1}$). □

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3 Convex Submanifolds

Convex sets are particulary useful in geometry, and play a major role in the theory of hyperbolic (*i.e.* nonpositevely curved) spaces.

3.1 Definitions

Definition 3.1. *A* geodesically convex *(also* totally convex*, or* convex*) set $M \subset \Sigma_\infty$ is a set such that, given any two points $p, q \in M$, the unique geodesic of Σ_∞ joining p with q lies entirely in M.*

Note that convex sets are connected. We refer the reader to Chapter IV, Section 5 of [SakT96] for a discussion of the different kinds of convex (strong, local, total) Riemannian objects. However, in our context, all definitions agree, because Σ_∞ is complete and for any two points there exists a unique normal (*i.e.* unit speed) geodesic joining them (which is clearly minimizing).

Definition 3.2. *A Riemannian submanifold $M \subset \Sigma_\infty$ (with the induced metric) is* geodesic *at $p \in M$ if geodesics of the ambient space starting at p, which have initial velocity in T_pM are also geodesics of M.*
We say that M is a totally geodesic manifold *if it is geodesic at any $p \in M$. This is equivalent to the fact that any geodesic of M is also a geodesic of the ambient space Σ_∞.*

Definition 3.3. *A Riemannian submanifold $M \subset \Sigma_\infty$ is* complete *at $p \in M$ if $\exp_p^M$ is defined in the whole tangent space and maps onto M. We say that M is a* complete manifold *if it is complete at any pont.*

Remark 3.4. Note that M is geodesic at p if and only if $\exp_p^M = \exp_p$. In particular $\exp_p^M$ is defined in the whole T_pM. So if M is geodesic at p, then M is complete at p if and only if for any point $q \in M$, there is a geodesic γ of M joining p to q (in other words, if $\exp_p^M = \exp_p$ maps *onto* M).

Remark 3.5. Σ_∞ is complete; moreover, $\exp_p$ is a diffeomorphism onto Σ_∞ for any p. The reader should be careful with other notions of completeness, because, as C.J. Atkin shows in [Atkin75] and [Atkin87], Hopf-Rinow theorem is not necessarily valid in this infinite dimensional context.

These previous notions are strongly related, as the following proposition shows:

Proposition 3.6. *Let $M \subset \Sigma_\infty$ be a Riemannian submanifold of Σ_∞ (with the induced metric). Then*

$$M \text{ geodesically convex } \iff M \text{ complete and totally geodesic}$$

Proof. The proof of ($\Leftarrow$) is trivial; let's prove ($\Rightarrow$). To see that M is complete, take $p, q \in M$. Then there exists a geodesic α of Σ_∞ joining p to q, $\alpha \subset M$. Among curves in M joining p to q, α is the shortest. So α is a critical point of the variational problem in M, hence a geodesic of M. To see that M is totally geodesic, take γ a geodesic of M joining p to q. By virtue of the convexity, there is a geodesic α of Σ_∞ joining p to q; by the preceeding argument α is also a geodesic of M. We can assume that q is close enough to p for the exponential map of M to be an isomorphism, and in this situation, geodesics are unique, so $\alpha = \gamma$ is a geodesic of Σ_∞. ☐

Remark 3.7. The reader should be aware of the fact that the concept of convexity is strong, and completely general (M does not need to have the induced submanifold metric, in fact, for the definition of geodesically convex to make sense, it is not necessary for M to have any manifold structure at all).

3.2 An intrinsic characterization of convexity

The following proposition relates (explicitly) curvature and commutativity in Σ_∞:

Proposition 3.8. *Assume $M \subset \Sigma_\infty$ is a flat submanifold. Assume further that M is geodesic at p. Then the tangent space at each $p \in M$ is $p-$abelian, namely*

$$p^{-\frac{1}{2}} X p^{-\frac{1}{2}} \text{ commutes with } p^{-\frac{1}{2}} Y p^{-\frac{1}{2}} \text{ for any pair } X, Y \in T_pM$$

Proof. Since M is geodesic at p, the curvature tensor is the restriction of the curvature tensor of Σ_∞. Set $x = p^{-\frac{1}{2}}Xp^{-\frac{1}{2}}$, $y = p^{-\frac{1}{2}}Yp^{-\frac{1}{2}}$. Then a straightforward computation shows that

$$\langle \mathcal{R}_p(X,Y)Y,X\rangle_p = -\frac{1}{4}\left\{\langle xy^2,x\rangle_2 - 2\langle yxy,x\rangle_2 + \langle y^2x,x\rangle_2\right\}$$

Now $x,y \in \mathcal{H}_\mathbb{R}$, so $x = \lambda + a$, $y = \beta + b$, and the equation reduces to

$$\langle \mathcal{R}_p(X,Y)Y,X\rangle_p = -\frac{1}{2}\left\{tr(a^2b^2) - tr((ab)^2)\right\} \tag{5}$$

The Cauchy-Schwarz inequality for the trace tells us that curvature at $p \in \Sigma_\infty$ is always nonpositive, and it is zero if and only if a and b commute. Hence whenever M is flat, x and y commute for any pair of tangent vectors $X,Y \in T_pM$ as stated. □

Definition 3.9. *We say that a subspace* $\mathfrak{m} \subset \mathcal{H}_\mathbb{R}$ *is a* Lie triple system *if* $[[a,b],c] \in \mathfrak{m}$ *for any* $a,b,c \in \mathfrak{m}$.

Remark 3.10. Note that whenever a,b,c are selfadjoint operators, $d = [a,[b,c]]$ is also a selfadjoint operator. So, for any algebra of operators $\mathfrak{a} \subset \mathcal{H}_\mathbb{C}$, $\mathfrak{m} = \mathfrak{Re}(\mathfrak{a})$ is a Lie triple system in $\mathcal{H}_\mathbb{R}$. This is also true for a Lie algebra of operators $\mathfrak{a}$.

Remark 3.11. Assume $M \subset \Sigma_\infty$ is a submanifold such that $1 \in M$, and M is geodesic at $p = 1$. Then T_1M is a Lie triple system, because the curvature tensor at $p = 1$ is the restriction to T_1M of the curvature tensor of Σ_∞, and $\mathcal{R}_1(X,Y)Z = -\frac{1}{4}[[X,Y],Z]$.

This particular condition on the tangent space turns out to be strong enough to ensure convexity; this result is standard:

Theorem 3.12. *Assume* $\mathfrak{m} \subset \mathcal{H}_\mathbb{R}$ *is a closed subspace, set* $M = \exp(\mathfrak{m}) \subset \Sigma_\infty$ *with the induced topology and Riemannian metric.*

If $\mathfrak{m}$ *is a Lie triple system, then* $p,q \in M \Rightarrow qpq \in M$

Proof. As Pierre de la Harpe pointed out, the proof of G.D. Mostow for matrices in [Mos55] can be translated to Hilbert-Schmidt operators without any modification: we give a sketch of the proof here. Assume $p = e^X$, $q = e^Y$, and consider the curve $e^{\alpha(t)} = e^{tY} \cdot e^X \cdot e^{tY}$. Then it can be proved that $\dot\alpha(t) = G(\alpha(t))$ for a map G that maps $\mathfrak{m}$ into $\mathfrak{m}$ (this is nontrivial). Since $\alpha(0) = X \in \mathfrak{m}$ and G is a Lipschitz map by the uniqueness of the solutions of ordinary differential equations we have $\alpha \subset \mathfrak{m}$. Hence $e^{\alpha(1)} = qpq \in M$ and the claim follows. □

Corollary 3.13. *Assume* $M = \exp(\mathfrak{m}) \subset \Sigma_\infty$ *as above, and* $\mathfrak{m}$ *is a Lie triple system. Then* M *is geodesically convex.*

Proof. Take $p,q \in M$. Then $p = e^X$, $q = e^Y$ with $X,Y \in \mathfrak{m}$. If we set $r = e^{-X/2}e^Ye^{-X/2}$, then $r \in M$ because $e^{-X/2}$ and e^Y are in M. Moreover, $Z = \ln(r) \in \mathfrak{m}$. But the unique geodesic of Σ_∞ joining p to q is

$$\gamma(t) = e^{X/2}e^{tZ}e^{X/2}, \quad \text{so} \quad \gamma \subset M \quad \square$$

Corollary 3.14. *Assume $\mathfrak{m} \subset \mathcal{H}_{\mathbb{R}}$ is a closed abelian subalgebra of operators. Then the manifold $M = \exp(\mathfrak{m}) \subset \Sigma_\infty$ is a closed, convex and flat Riemannian submanifold. Moreover, $M \subset \mathfrak{m}$ is open and M is an abelian Banach-Lie group.*

Proof. The first assertion follows from the fact that $\mathfrak{m}$ is a Lie triple system; since curvature is given by commutators, M is flat. Since $\mathfrak{m}$ is a closed subalgebra, $e^X = \sum \frac{X^n}{n!} \in \mathfrak{m}$ for any $X \in \mathfrak{m}$, so $M \subset \mathfrak{m}$. Note that every $p = e^X \in M$ has a unique analytic logarithm (because it is a positive operator), and since $\mathfrak{m}$ is closed, an argument identical to the one we used in the proof of Proposition 2.3 proves that M is open in $\mathfrak{m}$. □

Corollary 3.15. *Assume $M = \exp(\mathfrak{m})$ is closed and flat. If M is geodesic at $p = 1$, then M is a convex submanifold. Moreover, M is an abelian Banach-Lie group and $M \subset \mathfrak{m}$ is open.*

Proof. If M is geodesic at $p = 1$, $T_1M = \mathfrak{m}$ is abelian (by Proposition 3.8). □

The definition of symmetric space we adopt is the usual definition for Riemannian manifolds, see the book [Hel62] by Sigurdur Helgason:

Definition 3.16. *A Hilbert manifold M is called a* globally symmetric space *if each point $p \in M$ is an isolated fixed point of an involutive isometry $s_p : M \to M$. The map s_p is called the* geodesic symmetry at *p.*

Theorem 3.17. *Assume $M = \exp(\mathfrak{m})$ is closed and geodesically convex. Then M is a symmetric space; the geodesic symmetry at $p \in M$ is given by $s_p(q) = pq^{-1}p$ for any $q \in M$. In particular, Σ_∞ is a symmetric space.*

Proof. First observe that, for $p = e^X$, $q = e^Y$, $s_p(q) = e^X e^{-Y} e^X$, which shows that s_p maps M into M. To prove that s_p is an isometry, take a geodesic α_V of M such that $\alpha(0) = q$ and $\dot\alpha(0) = V$. Then $\alpha(t) = qe^{tq^{-1}V}$ and

$$d_q(s_p)(V) = (s_p \circ \alpha_V)^{\cdot}\big|_{t=0} = -pq^{-1}Vq^{-1}p$$

Since M has the induced metric, $\|pq^{-1}Vq^{-1}p\|^2_{pq^{-1}p} = \|V\|^2_q$ by Lemma 2.5 (take $g = pg^{-1}$). In particular, $d_ps_p = -id$, so p is an isolated fixed point of s_p for any $p \in M$. □

As we see from Theorem 3.12 and its corollaries, Σ_∞ (as any symmetric space) contains plenty of convex sets; in particular

Remark 3.18. We can embed isometrically any k-dimensional plane in Σ_∞ as a geodesically convex, closed submanifold: take an orthonormal set of k commuting operators (for instance, fix an orthonormal basis $\{e_i\}_{i\in\mathbb{M}}$ of H, and take $p_i = e_i \otimes e_i$, $i = 1, \cdots, k$), now take the exponential of this set . In the languaje of symmetric spaces, what we are saying is that $rank(\Sigma_\infty) = +\infty$.

Following the usual notation for symmetric spaces, we set $I_0(M)$=the connected component of the identity of the group of isometries of M.

Remark 3.19. Assume $1 \in M \subset \Sigma_\infty$ is closed and convex. Then, since any isometry φ is uniquely determined by its value at $1 \in M$ and its differential $d_1\varphi$, $I_0(M)$ carries a natural structure of Banach-Lie group (this result was proved by J. Eells in the mid 60's, [Eells66]). Moreover, the Lie algebra of $I_0(M)$ identifies naturally whith the Killing vectors of M. We can be more precise in this context: take $\varphi \in I_0(M)$, and consider

$$\overline{\varphi}\,(q) = \varphi(1)^{1/2} \cdot \varphi(q) \cdot \varphi(1)^{1/2}$$

note that $d_1\overline{\varphi}$ is a unitary operator of $T_1M = \mathfrak{m}$ (which carries a natural Hilbert-space structure), so there is an inclusion $J : I_0(M) \hookrightarrow M \times \mathcal{U}_{L(\mathfrak{m})}$ given by $\varphi \mapsto (\varphi(1), d_1\overline{\varphi})$. We will see later that the unitaries of the form $x \mapsto gxg^*$ (inner automorphisms) are enough to act transitively on M (g must be in G_M, see Theorem 3.30).

Theorem 3.20. *Assume $M = \exp(\mathfrak{m})$ is closed and geodesically convex. Then $I_0(M)$ acts transitively on M.*

Proof. Take $p = e^X$, $q = e^Y$ two points in M. Take $\gamma(t) = pe^{tp^{-1}V}$ the geodesic joining p to q. Note that $p = \gamma(1) = pe^{p^{-1}V} = e^{Vp^{-1}}p$. Consider the curve of isometries $\varphi_t = s_{\gamma(t/2)} \circ s_p$. Since $\varphi_0 = id$, $\varphi_t \subset I_0(M)$. Now

$$\varphi_1(p) = e^{\frac{1}{2}Ve^{-X}} e^X e^{-X} e^{\frac{1}{2}Ve^{-X}} e^X = e^{Ve^{-X}} e^X = q$$

which proves that $I_0(M)$ acts transitively on M. □

Remark 3.21. If $M = \exp(\mathfrak{m})$ is closed and convex, in particular it is geodesic at p for any $p \in M$, so $T_pM = \exp_p^{-1}(M) = \{p^{\frac{1}{2}} \ln(p^{-\frac{1}{2}} q p^{-\frac{1}{2}}) p^{\frac{1}{2}} : q \in M\}$ (see Remark 2.4). This observation together with Theorem 3.12 proves the identification

$$T_pM = p^{\frac{1}{2}}\,(T_1M)\,p^{\frac{1}{2}} = p^{\frac{1}{2}}\,\mathfrak{m}\,p^{\frac{1}{2}}$$

From the previous identifications of the tangent space follow easily (see Remark 2.2) that an operator $V \in \mathcal{H}_{\mathbb{R}}$ is orthogonal to M at p (that is, $V \in T_pM^\perp$) if and only if

$$\left\langle p^{-\frac{1}{2}}\,Z\,p^{-\frac{1}{2}}\,,\,V \right\rangle_2 = \left\langle p^{-\frac{1}{2}}\,V\,p^{-\frac{1}{2}}, Z \right\rangle_2 = 0 \quad \text{for any} \quad Z \in \mathfrak{m}$$

In particular, $T_1M^\perp = \mathfrak{m}^\perp = \{V \in \mathcal{H}_{\mathbb{R}} \,:\, \langle V, Z\rangle_2 = 0 \text{ for any } Z \in \mathfrak{m}\}$.

Remark 3.22. Note that when $\mathfrak{m}$ is a closed abelian *subalgebra* of operators, $p^{\frac{1}{2}} = e^{X/2} \in \mathfrak{m}$ and also the map $Y \mapsto Y \cdot p^{\frac{1}{2}}$ is an isomorphism of $\mathfrak{m}$; so $T_pM = \mathfrak{m} = T_1M$ in this case (for any $p \in M$). This also follows easily from Corollary 3.14. In particular,

$$T_pM^\perp = T_1M^\perp = \mathfrak{m}^\perp \quad \text{for any } p \in M$$

Remark 3.23. Assume $M \subset \Sigma_\infty$ is geodesically convex. Then, if γ is the geodesic joining p to q, the isometry $\varphi_t = s_{\gamma(t/2)} \circ s_p$ translates along the curve γ, namely

$$\varphi_t(\gamma(s)) = p\,e^{\frac{t}{2}p^{-1}V} \cdot p^{-1} \cdot p\,e^{sp^{-1}V} \cdot p^{-1} \cdot p\,e^{\frac{t}{2}p^{-1}V} =$$

$$= p\, e^{\frac{t}{2}p^{-1}V} \cdot e^{sp^{-1}V} \cdot\, e^{\frac{t}{2}p^{-1}V} = p\, e^{(s+t)p^{-1}V} = \gamma(s+t)$$

Now take any tangent vector $W \in T_{\gamma(s)}M$, and set

$$W(t) := (d\varphi_t)_{\gamma(s)}(W) = e^{\frac{t}{2}Vp^{-1}} \cdot W \cdot e^{\frac{t}{2}p^{-1}V}$$

Then $W(t)$ is the parallel translation of W from $\gamma(s)$ to $\gamma(s+t)$; namely $\nabla_{\dot\gamma} W \equiv 0$ (this follows from a straightforward computation using equation (2))

We conclude that the map $(d\varphi_t)_{\gamma(s)} : T_{\gamma(s)}M \to T_{\gamma(s+t)}M$ gives parallel translation along γ, namely $(d\varphi_t)_{\gamma(s)} = P_s^{t+s}(\gamma)$. In particular, since $q = \gamma(1) = p^{\frac{1}{2}}\, e^{p^{-1/2}Vp^{1/2}}\, p^{\frac{1}{2}}$,

$$W \mapsto p^{\frac{1}{2}}(p^{-\frac{1}{2}}qp^{-\frac{1}{2}})^{\frac{1}{2}}p^{-\frac{1}{2}} \cdot W \cdot p^{-\frac{1}{2}}(p^{-\frac{1}{2}}qp^{-\frac{1}{2}})^{\frac{1}{2}}p^{\frac{1}{2}}$$

gives parallel translation from T_pM to T_qM.

3.2.1 A few examples of convex sets

We list several convex submanifolds of Σ_∞; for some of them we present on this manuscript an explicit factorization theorem. The general factorization theorems (Theorem 4.10 and Theorem 4.11) apply for any of these:

1. For any subspace $\mathfrak{s} \subset \mathcal{H}_{\mathbb{R}}$, the subspace

   $$\mathfrak{m}_{\mathfrak{s}} = \{X \in \mathcal{H}_{\mathbb{R}} \,:\, [X,Y] = 0\, \forall\, Y \in \mathfrak{s}\}$$

   is a Lie triple system.

2. In particular, for any $Y \in \mathcal{H}_{\mathbb{R}}$,

   $$\mathfrak{m}_Y = \{X \in \mathcal{H}_{\mathbb{R}} \,:\, [X,Y] = 0\}$$

   is a Lie triple system.

3. The family of operators in $\mathcal{H}_{\mathbb{R}}$ which act as endomorphisms of a closed subspace $S \subset H$ form a Lie triple system in $\mathcal{H}_{\mathbb{R}}$.

4. Any norm closed abelian subalgebra of $\mathcal{H}_{\mathbb{R}}$ is a Lie triple system, in particular

   (a) The diagonal operators (see section 5). This is a maximal abelian closed subspace of $\mathcal{H}_{\mathbb{R}}$, hence the manifold Δ (which is the exponential of this set) is a maximal flat submanifold of Σ_∞.

   (b) The scalar manifold $\Lambda = \{\lambda \cdot 1\, \lambda > 0\}$

   (c) For fixed $a \in HS^h$, the real part of the closed algebra generated by a, which is the closure in the 2-norm of the set of polynomials in a.

5. The real part of any Lie subalgebra of $\mathcal{H}_{\mathbb{C}}$ is a Lie triple system (in particular: the real part of any Banach subalgebra).

6. Any real Banach-Lie algebra $\mathfrak{g}$ with a compatible Riemannian product invariant by inner automorphisms has a complexification which leads to the structure of an L^*-algebra, and any L^*-algebra can be embedded as a closed Lie subalgebra of HS (see [CGM90] and [Neh93]).

7. If G is a simply connected semisimple locally compact Lie group, then any irreducible unitary representation of $C^*(G)$ into $\mathcal{L}(H)$ maps $C_K(G)$ (the continous functions with compact support) into HS (see [Bag69]). This inclusion is also true for any irreducible subrepresentation of the left regular representation of a unimodular group G.

3.3 Convex manifolds as homogeneous spaces

Definition 3.24. *A Banach-Lie group is an algebraic group G together with a compatible Banach manifold structure. If G is a Banach-Lie group, we say $K \subset G$ is a* Lie subgroup *if K is an algebraic subgroup of G which is also a submanifold (henceforth a closed subgroup) of G.*

We recall (without proof, see for instance [Lang95] or [Lar80]) a result for quotients of Banach-Lie groups:

Theorem 3.25. *Let G be an analytic Banach-Lie group, and K a Banach-Lie subgroup. Then on the left cosets space G/K there exists a unique analytic manifold strutcture such that the projection is a submersion. The canonical action $G \times G/K \to G/K$ is analytic.*

For any Banach algebra $\mathcal{B}$, we will denote $GL(\mathcal{B})$ the group of invertible elements. Note that this group has a natural structure of manifold as an open set of the algebra, so $GL(\mathcal{B})$ is always a Banach-Lie group with Lie algebra $\mathcal{B}$.

Remark 3.26. The group $GL(\mathcal{H}_{\mathbb{C}})$, having the homotopy type of the inductive limit of the groups $GL(n, \mathbb{C})$ (see [Har72], section II.6) is connected; moreover, there is an homotopy equivalence

$$GL(\mathcal{H}_{\mathbb{C}}) \simeq S^1 \times S^1 \times SU(\infty)$$

Here $SU(\infty)$ stands for the inductive limit of the groups $SU(n, \mathbb{C})$

The following result is standard in finite dimensions (see for instance, [Hel62]); we say that G is a selfadjoint subgroup (shortly, $G^* = G$) if $g^* \in G$ whenever $g \in G$. Note that G is selfadjoint iff $\mathfrak{g}^* = \mathfrak{g}$, where the latter denotes the Lie algebra of G.

Theorem 3.27. *Fix a connected Lie subgroup $G \subset GL(\mathcal{H}_{\mathbb{C}})$ such that $G^* = G$. Let P be the analytic map*

$$P : GL(\mathcal{H}_{\mathbb{C}}) \to GL(\mathcal{H}_{\mathbb{C}}) \quad \textit{where} \quad g \mapsto gg^* = |\, g \,|^2$$

If K denotes the isotropy group of P (namely $K = P^{-1}(1) \cap G$ with the induced analytic structure), then $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$, where $\mathfrak{k}$ is the Lie algebra of K and $\mathfrak{p}$ are the selfadjoint elements of $\mathfrak{g}$. In particular, K is a Lie subgroup of G.

Proof. Note that $\sigma(g) = g^*$ is involutive so its differential at $g = 1$ gives an involution Θ of $\mathfrak{g}$ that induces the desired splitting of the Lie algebra of G. Now K is a Lie subgroup because the Lie algebra splits conveniently. □

Remark 3.28. For $M = \exp(\mathfrak{m})$ a geodesically convex closed manifold in Σ_∞, consider

$$[\mathfrak{m},\mathfrak{m}] = span\{[a,b] : a,b \in \mathfrak{m}\} = \left\{ \sum_{i\in F} [a_i,b_i] \ : \ a_i,b_i \in \mathfrak{m}; F \text{ a finite set} \right\}$$

Note that all the operators in $[\mathfrak{m},\mathfrak{m}]$ are skewadjoint. Set $\mathfrak{g}_M = \mathfrak{m} \oplus \overline{[\mathfrak{m},\mathfrak{m}]}$. Then $\mathfrak{g}_M$ is a closed Lie subalgebra of $\mathcal{H}_{\mathbb{C}}$ because $\mathfrak{m}$ is a Lie triple system (see [Hel62]). Since $\mathcal{H}_{\mathbb{C}}$ is a Hilbert space and $\mathfrak{g}_M$ is closed, the Lie algebra splits: it follows that $\mathfrak{g}_M$ is integrable (see [Lang95]). Let $G_M \subset GL(\mathcal{H}_{\mathbb{C}})$ be the Lie subgroup which is the component of the identity of the Lie group which has $\mathfrak{g}_M$ as Lie algebra. Note that $M \subset G_M$ and $G_M^* = G_M$ since $(a + [b,c])^* = a + [c,b]$ for any $a,b,c \in \mathfrak{m}$. It is also clear that $\mathfrak{k} = \overline{[\mathfrak{m},\mathfrak{m}]}$ (in the notation of Theorem 3.27). G_M is the smallest Lie group containing M.

The elements of M are indeed the positive elements of G_M, and the elements of K the unitary operators of G_M; we prove it below. Note that when $\mathfrak{m}$ is abelian, $\mathfrak{g}_M = \mathfrak{m}$ and also $G_M = M \subset \mathfrak{m}$ is an open set.

Lemma 3.29. *With the notation of the above Remark and the hypothesis of Theorem 3.27, $P(G_M) \subset M$.*

Proof. Since $\mathfrak{g}_M$ splits, there are neighbourhoods of zero $U_{\mathfrak{m}} \subset \mathfrak{m}$ and $U_{\mathfrak{k}} \subset \mathfrak{k} = \overline{[\mathfrak{m},\mathfrak{m}]}$ such that the map $X_{\mathfrak{m}} + Y_{\mathfrak{k}} \mapsto e^{X_{\mathfrak{m}}} e^{Y_{\mathfrak{k}}}$ is an isomorphism from $U_{\mathfrak{m}} \oplus U_{\mathfrak{k}}$ onto an open neighbourhood V_M of $1 \in G_M$. Clearly, the group generated by V_M is open (and closed) in G_M, so $\langle V_M \rangle = G_M$. So, for any $g \in G_M$, $g = (e^{x_1} e^{y_1})^{\alpha_1} \cdots (e^{x_n} e^{y_n})^{\alpha_n}$ for some selfadjoint $x_i \in U_{\mathfrak{m}}$, some skewadjoint $y_i \in U_{\mathfrak{k}}$, and some $\alpha_i = \pm 1$.

Now $e^x e^y e^x \in M$ whenever $x,y \in \mathfrak{m}$ (see Theorem 3.12), so an inspection of the expression for $P(g) = gg^*$ shows that $P(g)$ will be in M if we can prove that $e^y e^x e^{-y} \in M$ whenever $x \in \mathfrak{m}$ and $y \in \mathfrak{k}$ (namely, if we can prove that $kMk^* \subset M$ for any $k \in K$). It will be enough to show this is valid for $x \in \mathfrak{m}$ and $y = \sum_i [a_i,b_i] \in [\mathfrak{m},\mathfrak{m}]$ because M is closed. We assert that this is true, but to avoid cumbersome notations we write the proof for $y = [a,b]$. The proof of the general case is identical and therefore we omit it.

Consider the map $F : \mathcal{H}_{\mathbb{R}} \to \mathcal{H}_{\mathbb{R}}$ given by $F(z) = [[a,b],z]$. Since F maps $\mathfrak{m}$ into $\mathfrak{m}$, the flow of F in $\mathfrak{m}$ stays in $\mathfrak{m}$, so the ordinary differential equation $\dot{x}(t) = F(x(t))$ has unique solution in $\mathfrak{m}$ if $x(0) \in \mathfrak{m}$ is given (see [Lang95]). Take $\alpha(t) = e^{t[a,b]} x\, e^{-t[a,b]}$. Then $\alpha(0) = x \in \mathfrak{m}$; moreover

$$\dot{\alpha}(t) = e^{t[a,b]} [[a,b],x]\, e^{-t[a,b]} = [[a,b], e^{t[a,b]} x\, e^{-t[a,b]}] = F(\alpha(t))$$

which proves that $\alpha(t) \subset \mathfrak{m}$ for any $t \geq 0$. In particular,

$$\alpha(1) = e^{[a,b]} x\, e^{-[a,b]} \in \mathfrak{m}$$

Taking the exponential and using that $e^{[a,b]}$ is a unitary operator, we get

$$e^{[a,b]}e^{x}\,e^{-[a,b]} \in M,$$

which finishes the proof. □

The previous lemma will be used to prove the first of the following results:

Theorem 3.30. *If $M = \exp(\mathfrak{m})$ is convex and closed, and $G_M \subset GL(\mathcal{H}_{\mathbb{C}})$ is the Lie subgroup with Lie algebra $\mathfrak{g}_M = \mathfrak{m} \oplus \overline{[\mathfrak{m},\mathfrak{m}]}$, then*

(a) $P(G_M) = M$, so M is an homogeneous space for G_M.

(b) For any $g = \mid g \mid u_g$ (Cauchy polar decomposition) in G_M, we have

$$\mid g \mid = \sqrt{gg^*} \in M \subset G_M,$$

and also $u_g \in K \subset G_M$ where K is the isotropy Lie subgroup

$$K = \{g \in G_M \,:\, gg^* = 1\} \text{ with Lie algebra } \mathfrak{k} = \overline{[\mathfrak{m},\mathfrak{m}]}$$

In particular, G_M has a polar decomposition

$$G_M \simeq M \times K = P(G_M) \times U(G_M)$$

(c) $M = P(G_M) \simeq G_M/K$.

(d) M has nonpositive sectional curvature.

(e) For $g \in G_M$, consider $I_g(r) = grg^*$. Then $I : G_M \to I_0(M)$.

(f) Take $p,q \in M$, and set $g = p^{\frac{1}{2}}(p^{-\frac{1}{2}}qp^{-\frac{1}{2}})^{\frac{1}{2}}p^{-\frac{1}{2}} \in G_M$. Then I_g is an isometry in $I_0(M)$ which sends p to q, namely G_M acts transitively and isometrically on M.

Proof. Since any $p \in M$ is the exponential of some $x \in \mathfrak{m}$, we get $p = P(e^{x/2})$, which proves that $M \subset P(G_M)$; the other inclusion is given by Lemma 3.29.

To prove (b), note that $P(G_M) = M = \exp(\mathfrak{m})$; namely for any $g \in G_M$, $gg^* = P(g) \in M$; hence $gg^* = e^{x_0}$ for some $x_0 \in \mathfrak{m}$ and then also $\mid g \mid = e^{x_0/2} \in M \subset G_M$. Now $u_g = \mid g \mid^{-1} \cdot g \in G_M$ also (and clearly $u_g \in K$).

Statement (c) follows from Theorem 3.27, Remark 3.28 and statement (b).

The assertion in (d) follows from (a) and the fact that M is totally geodesic, together with equation (5) in the proof of Proposition 3.8.

To prove (e) , note that I_g is an isometry of M because M has the induced metric so Lemma 2.5 applies; from Lemma 3.29 we deduce that $kMk^* \subset M$ for any $k \in K$; from Theorem 3.12 and statement (b) follows easily that I_g maps M into M; since G_M is connected, we have the assertion.

Statement (f) follows from statement (e) and the proof of Theorem 3.20. □

From the classification of L^*-algebras (see [Neh93] and [CGM90]) follows that

$$\overline{[\mathrm{HS},\mathrm{HS}]} = \mathrm{HS} \quad \text{and} \quad \overline{[\mathrm{HS}^h,\mathrm{HS}^h]} = i\mathrm{HS}^h,$$

so taking $\mathfrak{m} = \mathcal{H}_{\mathbb{R}} = \mathbb{R} \oplus \mathrm{HS}^h$ we get $\mathfrak{k} = i\mathrm{HS}^h$, so $\mathfrak{g} = \mathbb{R} \oplus \mathrm{HS} = \mathcal{H}_{\mathbb{C}}/i\mathbb{R}$, hence

$$G_{\Sigma_\infty} = GL(\mathcal{H}_{\mathbb{C}})/S^1 = \{\alpha + a \,;\, \alpha \in \mathbb{R}_{>0},\ a \in \mathrm{HS} \text{ and } -\alpha \notin \sigma(a)\}$$

Clearly $P(G_{\Sigma_\infty}) = P(GL(\mathcal{H}_{\mathbb{C}})) = \Sigma_\infty$ since any positive invertible operator has an invertible square root. On the other hand it is also obvious that the isotropy group K equals $\mathcal{U}_{\mathcal{H}_{\mathbb{C}}}$ (the unitary group of $\mathcal{H}_{\mathbb{C}}$, see section 8), so

Corollary 3.31. *There is an analytic isomorphism given by polar decomposition*

$$\Sigma_\infty \simeq GL(\mathcal{H}_{\mathbb{C}})/\mathcal{U}_{\mathcal{H}_{\mathbb{C}}}$$

The manifold of positive invertible operators Σ_∞ is an homogeneous space for the group of invertible operators $GL(\mathcal{H}_{\mathbb{C}})$, which acts isometrically and transitively on Σ_∞.

4 Factorization theorems for convex manifolds

Combining the usual theory of Hadamard manifolds with some ad-hoc techniques for the infinite-dimensional context, we shall prove that given a geodesically convex closed submanifold M of Σ_∞, there is a unique geodesic γ joining p and M, such that the length of γ is exactly the distance between p and M.

4.1 Geodesic Projection

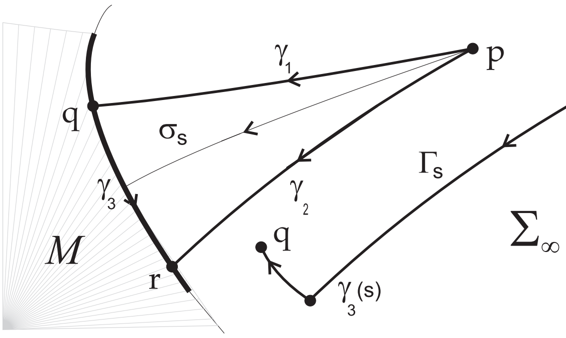
We will make extensive use of the first and second variation formulas for curves in Riemannian Manifolds; we refer the reader to [Lang95].

Proposition 4.1. *Let M be a geodesically convex subset of Σ_∞. Then there is at most one geodesic γ joining p and M such that $L(\gamma) = \operatorname{dist}(p,M)$. In other words, there is at most one point $q \in M$ such that* $\operatorname{dist}(p,q) = \operatorname{dist}(p,M)$.

Proof. Suppose there are two such points, q and $r \in M$, joined by a geodesic $\gamma_3 \in M$, such that $L(\gamma_1) = \operatorname{dist}(p,q) = L(\gamma_2) = \operatorname{dist}(p,r) = d(p,M)$. We construct a proper variation of $\gamma \equiv \gamma_1$, which we call Γ_s.
The construction follows the figure in next page, where

$$\sigma_s(t) = \sigma(s,t) = p^{\frac{1}{2}} \left[p^{-\frac{1}{2}} q^{\frac{1}{2}} \left(q^{-\frac{1}{2}} r q^{-\frac{1}{2}} \right)^s q^{\frac{1}{2}} p^{-\frac{1}{2}} \right]^t p^{\frac{1}{2}}$$

is the minimal geodesic joining p with $\gamma_3(s)$.



$$\Gamma(s,t) = \begin{cases} \sigma(s,t) & \text{if} \quad 0 \leq t \leq 1 \\ \gamma_3\,(s(2-t)) & \text{if} \quad 1 \leq t \leq 2 \end{cases}$$

Note that

$$\gamma(t) = \Gamma(0,t) = \begin{cases} \sigma(t,0) & \text{if} \quad 0 \leq t \leq 1 \\ q & \text{if} \quad 1 \leq t \leq 2 \end{cases} = \begin{cases} \gamma_1(t) & \text{if} \quad 0 \leq t \leq 1 \\ q & \text{if} \quad 1 \leq t \leq 2 \end{cases}$$

so

$$\dot\gamma(t) = \begin{cases} \dot\gamma_1(t) & \text{if} \quad 0 \leq t \leq 1 \\ 0 & \text{if} \quad 1 \leq t \leq 2 \end{cases}$$

Also note that the variation vector field (which is a piecewise Jacobi field for the curve γ) is given by equations

$$V(t) = \frac{\partial \Gamma}{\partial s}(t,0) = \begin{cases} \frac{\partial \sigma}{\partial s}(t,0) & \text{if} \quad 0 \leq t \leq 1 \\ \\ (2-t)\dot\gamma_3(0) & \text{if} \quad 1 \leq t \leq 2 \end{cases}$$

If $\Delta_i\dot\gamma$ denotes the jump of the tangent vector field to γ at t_i, namely $\dot\gamma(t_i^+) - \dot\gamma(t_i^-)$, and Γ is a proper variation of γ, then the first variation formula for $\gamma : [a,b] \to \Sigma_\infty$ reads

$$||\dot\gamma||\ \frac{d}{ds}|_{s=0} L\,(\Gamma_s) = -\int_a^b \langle V(t), D_t\dot\gamma(t)\rangle\, dt\ - \sum_{i=1}^{k-1} \langle V(t_i), \Delta_i\dot\gamma\rangle$$

where D_t stands for the covariant derivative.

In this case, $D_t\dot\gamma$ is null in the whole $[0,2]$, because γ consists (piecewise) of geodesics. The jump points are $t_0 = 0$, $t_1 = 1$ and $t_2 = 2$, so the formula reduces to

$$||\dot\gamma||\ \frac{d}{ds}|_{s=0} L\,(\Gamma_s) = -\,\langle V(1), \Delta_1\dot\gamma\rangle$$

Thus, we get

$$\langle \dot\gamma_3(0), \dot\gamma_1(1)\rangle = \frac{d}{ds}|_{s=0} L\,(\Gamma_s)\, ||\dot\gamma||$$

Recall that $\gamma_3 \subset M$, and that γ_1 is minimizing. Then the right hand term is zero, which proves that γ_1 and γ_3 are orthogonal at q. Similarly, γ_2 and γ_3 are orthogonal at r.

Henceforth, the sum of the three inner angles of this geodesic triangle is *at least* π.

If we can prove that the sum cannot exceed π (see the Lemma below), it will follow that the angle at p must be null, which proves that γ_1 and γ_2 are the same geodesic, and uniqueness follows. □

To prove the upper bound we need a hyperbolic triangle comparison lemma, which is known to be valid in finite dimensional manifolds; however in our context it is a consequence of the fact that the exponential map increases distance, which can be viewed as a particular case of the theory developed by McAlpin in his thesis [McA65] (as cited in the book [Lang95] by Lang), or as a delicate computation for compact operators (see the paper [AV03]):

Lemma 4.2. *Let p,q and r be the vertices of a geodesic triangle in Σ_∞. Then the sum of the inner angles of this triangle cannot exceed π.*

Proof. Naming the sides of the triangle as in the geodesic triangle figure of Proposition 4.1, if we introduce the notation $L(\gamma_i) = l_i$, α_i =angle opposite to γ_i , then the length of the Euclidean geodesic (*i.e.* segment) joining $V = \exp_p^{-1}(q)$ and $W = \exp_p^{-1}(r)$ cannot exceed l_3 (see [AV03], Corollary 3.5).
Thus we get the following known formula for manifols of nonpositive curvature (expanding $\|W-V\|_p = \langle W-V, W-V\rangle_p$, and using that $\|W\|_p = l_2$, $\|V\|_p = l_1$)

$$l_3^2 \geq l_2^2 + l_1^2 - 2l_1 l_2 \cos(\alpha_3) \tag{6}$$

If we construct an Euclidean triangle of side lengths l_i, the Cosine Law tells us that

$$l_3^2 = l_2^2 + l_1^2 - 2l_1 l_2 \cos(\Delta_3)$$

being Δ_i the angle opposite to l_i. The two formulas put together prove that $\cos(\alpha_i) \geq \cos(\Delta_i)$, which implies (noting that all angles are less than π) that $\alpha_i \leq \Delta_i$. Therefore,

$$\alpha_1 + \alpha_2 + \alpha_3 \leq \Delta_1 + \Delta_2 + \Delta_3 = \pi$$

This inequality also finishes the proof of the uniqueness of the minimizing geodesic. □

Now, we consider the problem of the existence of the minimizing geodesic. We can rephrase the problem in the following way:

Theorem 4.3. *Let M be a geodesically convex submanifold of Σ_∞, and p a point not in M. Then existence of a minimizing geodesic joining p with M such that $L(\gamma) = \text{dist}(p, M)$ is equivalent to the existence of a geodesic joining p with M, with the property that γ is orthogonal to M.*

Proof. In fact, the existence of such a geodesic is equivalent to the existence of a point $q_p \in M$ such that $\text{dist}(p, M) = \text{dist}(p, q_p)$, and we will show that if $q \in M$ is a point such that γ_{qp} is orthogonal to M at q, then $\text{dist}(q, p) = \text{dist}(D, p)$. The other implication follows from the uniqueness theorem.
For this, consider the geodesic triangle generated by p, q and d, where d is any point in M different form q. As γ_{qp} is orthogonal to $T_q M$, it is, in particular, orthogonal to γ_{qd}. Then, by virtue of the hyperbolic Cosine Law (equation (6)), we have

$$L(\gamma_{dp})^2 \geq L(\gamma_{qp})^2 + L(\gamma_{qd})^2 > L(\gamma_{qp})^2$$

which implies $\text{dist}(q, p) < \text{dist}(d, p)$. □

We conclude that the existence problem is equivalent to the question:

- Is NM, the normal bundle of M, diffeomorphic to the whole Σ_∞, *via* the exponential map?

The answer to the local version of this question is yes, by virtue of the inverse function theorem. The reader can find the Banach space version of such theorem in [Lang95].

Lemma 4.4. *For every point $q \in M$, there exists an open neighbourhood $V_q \subset \Sigma_\infty$, and an open neighbourhood NM_q in the normal bundle of M, such that V_q is diffeomorphic to NM_q* via *the exponential map $E : NM \to \Sigma_\infty$, which assigns $(q,V) \mapsto \exp_q(V)$.*

Proof. First recall that the exponential map $(q,V) \mapsto \exp_q(V)$ is given by the expression (Remark 2.4)

$$\exp_q(V) = q\, e^{q^{-1}V} = e^{V q^{-1}} q$$

Now observe that $T(NM)$ is isomorphic to $T\Sigma_\infty$, because for every $(q,V) \in NM$, the tangent space $T_{(q,V)}NM$ can be decomposed in the tangent space T_qM and the normal space $T_qM^\perp$, and $T_qM \oplus T_qM^\perp = T_q\Sigma_\infty$. We will show that the differential of E at any point $(q,0)$ is the identity (in particular, an isomorphism) and the inverse map theorem will provide us with the desired neighbourhoods.
Let's consider the curve $\alpha(t) = (\delta_q(t), tv)$, where $\delta_q(0) = q$ and $\dot{\delta}(0) = W \in T_qM$. Then

$$(DE)_{(p,0)}(W,V) = (E \circ \alpha)^{\cdot}_{t=0}$$

But

$$(E \circ \alpha)^{\cdot}(t) = \dot{\delta}(t) e^{t\delta^{-1}(t)V} + \delta(t) e^{t\delta^{-1}(t)V} \left[\delta^{-1}(t)V + t\dot{\delta}^{-1}(t)V\right]$$

so

$$(DE)_{(p,0)}(W,V) = (E \circ \alpha)^{\cdot}_{t=0} = W + V \quad \square$$

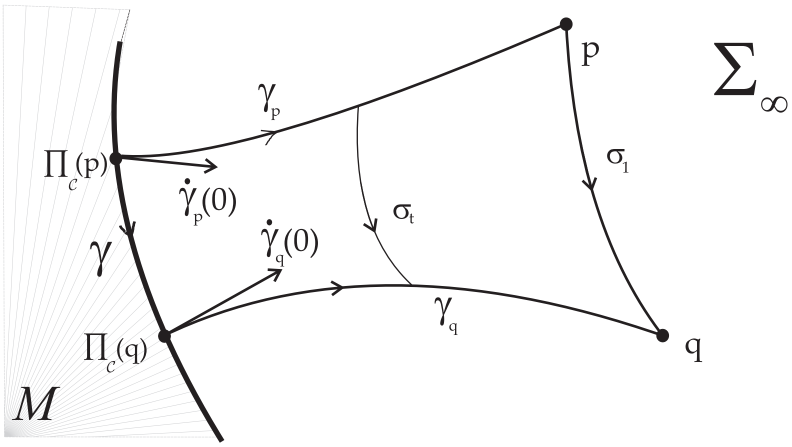
Remark 4.5. The preceeding result says that the map E is, in fact, an open mapping, so we can think of its image, which we will call $B_M = E(NM)$, as a "tubular neighbouhood" of M. In particular, B_M is an open neighbourhood of M in Σ_∞, and putting together the preceeding results, we can conclude that for every point $p \in B_M$ there exists a unique minimizing geodesic which realizes the distance between the operator p and the manifold M.
It must also be noted (this is a consequence of the previous results of this section) that B_M can be described by the following property: a point $p \in B_M$ *if and only if* there exists a point $q \in M$ such that $\mathrm{dist}(q,p) = \mathrm{dist}(M,p)$.

With the above construction, we have proved the existence of a map $\Pi_M : B_M \to M$, which assigns the unique point $q \in M$ such that $\mathrm{dist}(q,p) = \mathrm{dist}(M,p)$ to any point $p \in B_M$. Observe that this map is obtained *via* a geodesic which joins p and M, and this geodesic is orthogonal to M. We call $\Pi_M(p)$ the *foot of the perpendicular* from p to M.

Theorem 4.6. *The map Π_M is a contraction, namely*

$$\mathrm{dist}(\Pi_M(p), \Pi_M(q)) \leq \mathrm{dist}(p,q)$$

Proof. We may assume that $p,q \notin M$, and that $\Pi_M(p) \neq \Pi_M(q)$.
If γ_p is a geodesic that joins $\Pi_M(p)$ to p and γ_q joins $\Pi_M(q)$ to q, set $f(t) = d(\gamma_p(t), \gamma_q(t))$. Note that $f(0) = d(\Pi_M(p), \Pi_M(q))$ and $f(1) = \text{dist}(p,q)$. We also know that $f''(t) \geq 0$ (this is because distance is a convex function; this can be deduced from the variation formulas, or it can be found in [CPR94]). If we can prove that $f'(0) \geq 0$, it will follow that f is monotone increasing, and we will have proved the assertion.
Take a variation $\sigma(t,s)$, being $\sigma_t(s)$ the unique geodesic joining $\gamma_p(t)$ to $\gamma_q(t)$. Then $\sigma(t,0) = \gamma_p(t)$, $\sigma(t,1) = \gamma_q(t)$, $\sigma(0,s) = \gamma(s) =$ the geodesic joining $\Pi_M(p)$ to $\Pi_M(q)$ (which is contained in M by virtue of the convexity), and finally $\sigma(1,s) =$ the geodesic joining p to q. This construction is better shown in the following figure:



Note that $f(t) = L(\sigma_t)$. We apply the first variation formula to this particular σ, to get

$$\|\dot\gamma\| \frac{d}{dt}|_{t=0} L(\sigma_t) = -\int_0^1 \langle V(s), D_s\dot\gamma(s)\rangle \, ds + \langle V(1), \dot\gamma(1)\rangle - \langle V(0), \dot\gamma(0)\rangle$$

The fact that γ is a geodesic and observation of the figure above reduces the formula to

$$\|\dot\gamma\| \, f'(0) = -\langle V(1), -\dot\gamma(1)\rangle + \langle -V(0), \dot\gamma(0)\rangle$$

Looking at the figure, it is also obvious that $V(0) = \dot\gamma_p(0)$, $V(1) = \dot\gamma_q(0)$. Recalling that the angles at M are right angles, we get $f'(0) = 0$. ☐

Remark 4.7. In the preceeding proof, $f'(0) = 0$ implies that it is exactly in this geodesic joining the projections that the distance between the projecting geodesics is minimal. This is related with the fact that Σ_∞ is a symmetric manifold.

Now we are ready to prove the main result of this section:

Theorem 4.8. *Let M be a geodesically convex, closed submanifold of Σ_∞. Then for every point $p \in \Sigma_\infty$, there is a unique minimizing geodesic γ joining p to M such that $L(\gamma) = \text{dist}(p,M)$.*
Moreover, this geodesic is orthogonal to M, and if we call $\Pi_M : \Sigma_\infty \to M$ to the map that assigns the endpoint of the minimizing geodesic starting at p, then Π_M is a contraction.

Proof. The theorem will hold true when we prove that $B_M = \Sigma_\infty$. But since Σ_∞ is connected and B_M is open, equality will immediately follow if we can prove that B_M is also closed.
For this, take a point $p \in \overline{B_M}$. Then there exist points $q_n \in M$, $V_n \in T_{q_n}D^\perp$ such that

$$p = \lim_n p_n = \lim_n \exp_{q_n}(V_n)$$

Now observe that $q_n = \Pi_M(p_n)$, so $\operatorname{dist}(q_n, q_m) \leq \operatorname{dist}(p_n, p_m)$. But $\{p_n\}$ converges to p, so it is a Cauchy sequence. It follows that $\{q_n\}$ is Cauchy, and, since M is closed (and then complete), there must exist a point $q \in M$ such that $q = \lim_n q_n$. We assert that $\operatorname{dist}(p,q) = \operatorname{dist}(p,M)$. For this, observe that

$$\operatorname{dist}(p,q_n) \leq \operatorname{dist}(p,p_n) + \operatorname{dist}(p_n,q_n)$$

and $\operatorname{dist}(p_n,q_n) = \operatorname{dist}(p_n,M)$, so

$$\operatorname{dist}(p,q_n) \leq \operatorname{dist}(p,p_n) + \operatorname{dist}(p_n,M)$$

Taking limits gets us to the inequality $\operatorname{dist}(p,q) \leq \operatorname{dist}(p,M)$, which tells us that the distance between p and M is given by $\operatorname{dist}(p,q)$. This concludes the proof. □

Note that Σ_∞ decomposes as a direct product: with the contraction Π_M, we can decompose Σ_∞ by picking, for fixed p,

1. the unique point $q = \Pi_M(p)$ such that $\operatorname{dist}(p,q) = \operatorname{dist}(p,M)$
2. a vector V_p normal to T_qM such that the ambient geodesic with this initial velocity starting at q passes through p.

Note that $V_p = \exp^{-1}_{\Pi_M(p)}(p)$, and also $\|V_p\|_p = \operatorname{dist}(p,M)$.
Since the exponential map is an analytic function of both of its variables (recall that, for any $q \in \Sigma_\infty$, and any $V \in \mathcal{H}_\mathbb{R}$, $\exp_q(V) = q e^{q^{-1}V}$), we get

Theorem 4.9. *The map $p \mapsto (\Pi_M(p), V_p)$ is the inverse of the exponential map $(q,V_q) \mapsto \exp_q(V_q)$, and it is, in fact, a real-analytic isomorphism between the manifolds NM and Σ_∞.*

This is a remarkable global analogue of the (linear) orthogonal decomposition of tangent spaces; we can restate the theorem in a different way if we recall that all points and tangent vectors are operators:

Theorem 4.10. *Fix a closed, geodesically convex submanifold M of Σ_∞. Take any operator $A \in \Sigma_\infty$. Then there exist operators $C, V \in \Sigma_\infty$ such that $C \in M$, $V \in T_CM^\perp$, and the following formula holds:*

$$A = C e^{C^{-1}V} \tag{7}$$

Moreover, C and V are unique, and the map $A \mapsto (C,V)$ (which maps $\Sigma_\infty \to NM$) is a real analytic isomorphism between manifolds.

Naming $B = C^{\frac{1}{2}}$, $W = C^{-\frac{1}{2}} V C^{-\frac{1}{2}}$, equation (7) reads $A = B e^W B$ for unique B, W.

Using the tools of section 3, we can restate the theorem in terms of intrinsic operator equations (see [Mos55] for the finite dimensional analogue):

Theorem 4.11. *Assume $\mathfrak{m} \subset \mathcal{H}_{\mathbb{R}}$ is a closed Lie triple system. Then for any operator $A \in \mathcal{H}_{\mathbb{R}}$, there exist unique operators $X \in \mathfrak{m}$, $V \in \mathfrak{m}^{\perp}$ such that the following decomposition holds:*

$$e^A = e^X e^V e^X$$

.

5 Some applications of the factorization theorems

5.1 Preliminaries

We will use the factorization theorem in several ways; for convenience we first state the following lemma, which we will be useful later on several ocasions:

Lemma 5.1. *(the generalized exponential formula): for the exponential map in Σ_∞, we have*

$$\exp_{\alpha+a}(\beta+b) = (\alpha+a)\,e^{(\alpha+a)^{-1}(\beta+b)} = (\alpha+a)[1+(\alpha+a)^{-1}(\beta+b)+\cdots]$$
$$= (\alpha+a)\left[1+\frac{\beta}{\alpha}+\frac{1}{2}\left(\frac{\beta}{\alpha}\right)^2+\cdots+k\right] = \alpha\, e^{\beta/\alpha}+k$$

where k is a Hilbert-Schmidt operator.

We need some remarks before we proceed with the main applications. Fix an orthonormal basis of H.

1. The diagonal manifold $\Delta \subset \Sigma_\infty$ we define below is closed and geodesically convex.

$$\Delta = \{d + \alpha : d \text{ is a diagonal, Hilbert-Schmidt operator}\}$$

This is due to the fact that the diagonal operators form a closed abelian subalgebra (see Propositions 3.6 and Corollary 3.14).

2. If $p_0 \in \Delta$, then $T_{p_0}\Delta = \{\alpha + d;\ \alpha \in \mathbb{R},\ d \text{ a diagonal operator}\} = T_1\Delta$ by Remarks 3.21 and 3.22.

3. Consider the map $A \mapsto A^D =$ the diagonal part of A. Then

   (a) For Hilbert-Schmidt operators we have $A^D = \sum_i p_i A p_i$ where convergence is in the 2-norm (and hence in the operator norm); here $p_i = \langle e_i, \cdot \rangle e_i$ is the orthogonal projection in the real line generated by e_i

   (b) $(A^D)^D = A^D$ and $tr\left(A^D A\right) = tr((A^D)^2)$

4. The scalar manifold $\Lambda = \{\lambda.1 : \lambda > 0\}$, is geodesically convex and closed in Σ_∞, with tangent space $\mathbb{R} \cdot 1$

5. A vector $V = \mu + u \in T_{p_0}\Delta^\perp$ if and only if $\mu = 0$ and $u^D = 0$. This follows from: Remarks 3.21 and 3.22, the fact that $\mu + u^D \in T_{p_0}\Delta$, and Remark (3) of this list.

5.2 The Factorizaton Theorems

Theorem 5.2. *(infinite dimensional diagonal factorization): Take any selfadjoint Hilbert-Schmidt operator a. Then there exist real $\lambda > 0$, a positive invertible Hilbert-Schmidt diagonal operator d and a Hilbert-Schmidt selfadjoint operator with null diagonal V such that the following formula holds:*

$$a + \lambda = (d+\lambda)e^{(d+\lambda)^{-1}V} = (d+\lambda)^{\frac{1}{2}} e^{(d+\lambda)^{-\frac{1}{2}}V(d+\lambda)^{-\frac{1}{2}}} (d+\lambda)^{\frac{1}{2}}$$

Moreover (for fixed λ) d and V are unique and the map $a + \lambda \mapsto (d, V)$ (wich maps $\Sigma_\infty \to N\Delta$) is a real analytic isomorphism between manifolds.

Proof. Take $\lambda = \|a\|_\infty + \varepsilon$, for any $\varepsilon > 0$. Then $p = a + \lambda \in \Sigma_\infty$, and $\Pi_\Delta(p) = d + \alpha$. This is the d we need. Now pick the unique $V \in T_{d+\alpha}\Delta^\perp$ such that $\exp_{d+\alpha}(V) = p$. This V has the desired form because of Remark (5) above. As a consequence of the 'exponential formula' (Lemma 5.1), $\alpha = \lambda$, for in this case, $\beta = 0$. □

This theorem can be rephrased saying that, given a selfadjoint Hilbert-Schmidt operator a, for any λ that makes $a + \lambda > 0$, one has a unique factorization

$$a + \lambda = D\, e^W D$$

where $D = (\lambda + d)^{\frac{1}{2}} > 0$ is diagonal and $W = D^{-1}VD^{-1}$ is symmetric and has null diagonal.

We now observe that, for finite (strictly positive) matrices, we could choose $\lambda = 0$ (in a sense we will make precise) because any matrix has finite spectrum. With this observation in mind, we can state and prove a finite dimensional analogue of the factorization theorem, which has a simpler form. We should remark that (in this particular case) this result is **exactly** Theorem 3 of [Mos55] by G.D. Mostow (see also [CPR91] by Corach, Porta and Recht). The only thing to remark is that the geometric intrepretation of the splitting is crystal clear in this context, because the diagonal matrix D is the closest diagonal matrix to A, and V is the initial direction of the geodesic starting at D which joins D to A. We will use the standard notation

$$M_n^+ = \{M \in \mathbb{C}^{n\times n} : M^t = M^\dagger,\, \sigma(M) \subset (0,+\infty)\}$$

The dagger (†) stands for complex conjugation of the coefficients of M.
We will use M_n to denote the tangent space of M_n^+ at Id; recall that M_n^+ is open in M_n, and also that M_n can be identified with the hermitian matrices of $\mathbb{R}^{n\times n}$.

Theorem 5.3. *(finite dimensional diagonal factorization): Fix a positive invertible matrix $A \in M_n^+$. Then there exist* unique *matrices $D, V \in M_n$, such that D is diagonal and strictly positive, V is symmetric and with null diagonal, which make the following formula hold:*

$$A = D\, e^V D$$

Moreover, the maps $A \mapsto D$ and $A \mapsto V$ are real analytic.

Proof. We will prove the result using block products. For this, choose an orthonormal basis of H, and write Hilbert-Schmidt operators as infinite matrices. In this way we can embed M_n^+ in Σ_∞, by means of the map that sends A to the first $n \times n$ block:

$$A \to \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} = a+1, \text{ where } a = \begin{pmatrix} A-1 & 0 \\ 0 & 0 \end{pmatrix} \in \mathrm{HS}^h$$

Note that $a+1 = A + P_{(KerA)^\perp}$, and that $a+1 > 0$ because $A > 0$.
Using the infinite dimensional theorem, we can factorize $a+1 = de^V d$, where $V = d^{-1}Wd^{-1}$ and W is orthogonal to the diagonal submanifold $\Delta \subset \Sigma_\infty$. Obviously,

$$d = \begin{pmatrix} D & 0 \\ 0 & D_\infty \end{pmatrix}$$

but note that $V = \ln\left(d^{-1}(a+1)d^{-1}\right)$ so

$$V = \ln \begin{pmatrix} D^{-1}AD^{-1} & 0 \\ 0 & D_\infty^{-2} \end{pmatrix} = \begin{pmatrix} \ln(D^{-1}AD^{-1}) & 0 \\ 0 & \ln(D_\infty^{-2}) \end{pmatrix}$$

which shows that the desired V (the first block of V) has the desired properties, as V has them. Now $a+1 = de^V d$ reads

$$\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} D & 0 \\ 0 & D_\infty \end{pmatrix} \exp\left\{ \begin{pmatrix} V & 0 \\ 0 & \ln(D_\infty^{-2}) \end{pmatrix} \right\} \begin{pmatrix} D & 0 \\ 0 & D_\infty \end{pmatrix} =$$

$$= \begin{pmatrix} D & 0 \\ 0 & D_\infty \end{pmatrix} \begin{pmatrix} e^V & 0 \\ 0 & D_\infty^{-2} \end{pmatrix} \begin{pmatrix} D & 0 \\ 0 & D_\infty \end{pmatrix} = \begin{pmatrix} D\,e^V D & 0 \\ 0 & 1 \end{pmatrix}$$

and comparing the first blocks, we have the claim. □

Remark 5.4. In [AV03] Andruchow and Varela prove that there is a natural, flat embedding of $M = M_n^+$ into Σ_∞ (Proposition 4.1 and Remark 4.2). This embedding makes M_n^+ a closed, geodesically convex submanifold of Σ_∞. We will postpone a projection theorem for this submanifold for the sake of simplicity. See Theorem 6.10

6 A foliation of codimension one

In this section we describe a foliation of the total manifold, and show how to translate the results from previous sections to a particular leaf (the submanifold Σ_1). We begin with a description of the leaves.

6.1 The leaves Σ_λ

Recall that we use HS^h (hermitian Hilbert-Schmidt operators) to denote the closed vector space of operators in $\mathcal{H}_\mathbb{R}$ with no scalar part. We define the following family of submanifolds:

$$\Sigma_\lambda = \{a + \lambda \in \Sigma_\infty,\ a \in \mathrm{HS}^h \text{ and } \lambda > 0 \text{ fixed }\}$$

We first observe that, by virtue of the 'exponential formula' (Lemma 5.1), for any real $\lambda > 0$ and any $p \in \Sigma_\lambda$, there is an identification *via* the inverse exponential map at p, $T_p\Sigma_\lambda = \mathrm{HS}^h$.

Observe that $\Sigma_\lambda \cap \Sigma_\beta = \emptyset$ when $\lambda \neq \beta$, since $a + \lambda = b + \beta$ implies $a - b = \beta - \lambda$.
In this way, we can decompose the total space by means of these leaves,

$$\Sigma_\infty = \dot{\bigcup_{\lambda>0}} \Sigma_\lambda$$

Theorem 6.1. *The leaves Σ_λ are closed and geodesically convex submanifolds of Σ_∞.*

Proof. That Σ_λ is closed is a consequence of the fact that the projection to Λ is a contraction, and as consequence, continuous map. One must only observe that $\Sigma_\lambda = \Pi_\Lambda^{-1}(\lambda)$.
That Σ_λ is geodesically convex follows from inspection of the formula for a geodesic joining $a+\lambda$, $b+\lambda$. □

Remark 6.2. Take $\delta + c \in T_{a+\lambda}\Sigma_\lambda{}^\perp$. Since $T_{a+\lambda}\Sigma_\lambda$ can be identified with HS^h, condition

$$\langle \delta + c, d \rangle_{a+\lambda} = 0 \qquad \forall \quad d \in \mathrm{HS}^h$$

immediately translates into

$$tr\left[(a+\lambda)^{-1}\left[(\delta+c)(a+\lambda)^{-1} - \frac{\delta}{\lambda}\right]d\right] = 0 \qquad \forall \quad d \in \mathrm{HS}^h$$

This says that $T_{a+\lambda}\Sigma_\lambda^\perp = span(a+\lambda)$; shortly $T_p\Sigma_\lambda^\perp = span(p)$ for any $p \in \Sigma_\lambda$.

Proposition 6.3. *Fix real $\alpha, \lambda > 0$. Set $\Pi_{\alpha,\lambda} = \Pi_{\Sigma_\lambda}|_{\Sigma_\alpha} : \Sigma_\alpha \to \Sigma_\lambda$. Then*

1. *$\Pi_{\alpha,\lambda}(p) = \frac{\lambda}{\alpha}p$, so $\Pi_{\alpha,\lambda}(p)$ commutes with p*

2. *$\Pi_{\alpha,\lambda}$ is an isometric bijection between Σ_α and Σ_λ, with inverse $\Pi_{\lambda,\alpha}$.*

3. *$\Pi_{\alpha,\lambda}$ gives parallel translation along 'vertical' geodesics joining both leaves.*

Proof. Notice that for a point $b+\alpha \in \Sigma_\alpha$ to be the endpoint of the geodesic γ starting at $a+\lambda \in \Sigma_\lambda$ such that $L(\gamma) = \mathrm{dist}(b+\alpha, \Sigma_\lambda)$, we must have

$$b+\alpha = \exp_{a+\lambda}(x+c) = \exp_{a+\lambda}(k.(a+\lambda)) = \mathrm{e}^k(a+\lambda)$$

because $x+c \in T_{a+\lambda}\Sigma_\lambda{}^\perp$. From Lemma 5.1, we deduce that $k = \ln\left(\frac{\alpha}{\lambda}\right)$, and $a = \frac{\lambda}{\alpha}b$. So, $b+\alpha = \frac{\alpha}{\lambda}(a+\lambda)$ and also

$$\gamma(t) = (a+\lambda)\left(\tfrac{\alpha}{\lambda}\right)^t$$

Now it is obvious that $\Pi_\lambda(b+\alpha) = \frac{\lambda}{\alpha}(b+\alpha)$ and commutes with $b+\alpha$.
To prove that Π is isometric, observe that

$$\mathrm{dist}(\Pi_{\alpha,\lambda}(p), \Pi_{\alpha,\lambda}(q)) = \mathrm{dist}(\frac{\lambda}{\alpha}p, \frac{\lambda}{\alpha}q) = \mathrm{dist}(p,q)$$

by inspection of the geodesic equation (3) of section 2.
That Π gives parallel translation along γ follows from $q = \frac{\lambda}{\alpha}p$ and Remark 3.23. □

Proposition 6.4. *The leaves Σ_α, Σ_λ are also parallel in the following sense: any minimizing geodesic joining a point in one of them with its projection in the other is orthogonal to both of them. For any $b+\alpha \in \Sigma_\alpha$,*

$$\mathrm{dist}(b+\alpha, \Sigma_\lambda) = \mathrm{dist}(\Sigma_\alpha, \Sigma_\lambda) = \left| \ln\left(\tfrac{\alpha}{\lambda}\right) \right|$$

In particular, the distance between α, λ in the scalar manifold Λ is given by the Haar measure of the open interval (α, β) on $\mathbb{R}_{>0}$. (This was remarked by E. Vesentini in his paper [Ves76]).

Proof. It is a straightforward computation that follows from the previous results. □

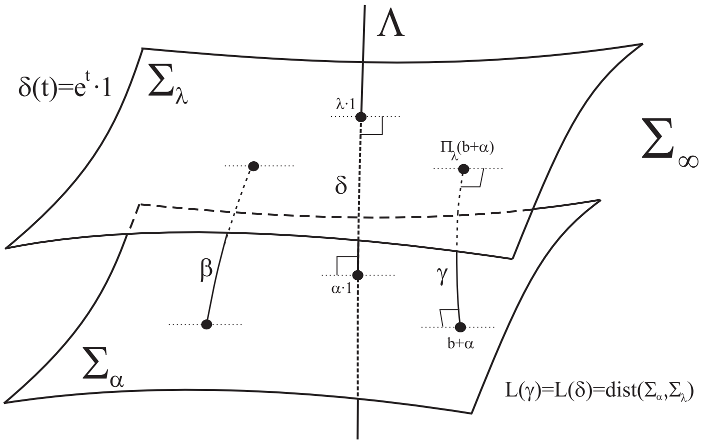


Figure 1: The geodesics γ and δ are minimizing, the geodesic β is not

Since Σ_∞ is a symmetric space, curvature is preserved when we parallel-translate bidimensional planes; note also that vertical planes are abelian, so

Proposition 6.5. *For any point $p \in \Sigma_\lambda$, sectional curvature of vertical 2-planes is trivial.*

Proof. We know that p generates $T_p\Sigma_\lambda^\perp$; take any other vector $V \in T_p\Sigma_\lambda = \mathrm{HS}^h$. Equation (1) of section 2 says

$$\langle \mathcal{R}_p(p,V)V, p\rangle_p = -\frac{1}{4}\left\langle \left[\left[p^{-1}p, p^{-1}V\right] p^{-1}V\right], pp^{-1}\right\rangle_2 = 0 \quad \square$$

Theorem 6.6. *The map $T : \Sigma_\infty \to \Sigma_1 \times \Lambda$, which assigns*

$$a+\alpha \mapsto \left(\frac{1}{\alpha}(a+\alpha), \alpha\right)$$

is bijective and isometric (Σ_1 and Λ have the induced submanifold metric). In other words, there is a Riemannian isomorphism

$$\Sigma_\infty \simeq \Sigma_1 \times \Lambda$$

Proof. Another straightforward computation. □

The previous theorems show that the geometry of Σ_∞ is essentially the geometry of Σ_1; in particular, the factorization theorem inside Σ_1 has a simpler form; we state it below

Theorem 6.7. *Fix a closed, geodesically convex submanifold M of Σ_1. For any $a+1 \in \Sigma_1$, there is a selfadjoint Hilbert-Schmidt operator d such that $d+1 \in M$, and a selfadjoint Hilbert-Schmidt operator V, such that $V \in T_{d+1}M^{\perp}$, which make the following formula hold:*

$$1+a = [1+d]\ e^{(1+d)^{-1}V}$$

Moreover d and V are unique, and the map $1+a \mapsto (1+d, V)$ (which maps Σ_1 to NM) is a real analytic isomorphism between manifolds. Equivalently,

$$1+a = [1+d]^{\frac{1}{2}}\ e^{(1+d)^{-\frac{1}{2}} V (1+d)^{-\frac{1}{2}}}\ [1+d]^{\frac{1}{2}}$$

The intrinsic version of the theorem reads (see Theorem 3.13):

Theorem 6.8. *Assume $\mathfrak{m} \subset \mathrm{HS}^h$ is a closed subspace such that*

$$[X,[X,Y]] \in \mathfrak{m} \quad \text{for any} \quad X,Y \in \mathfrak{m}$$

Then for any $A \in \mathrm{HS}^h$ there is a unique decomposition of the form

$$\mathrm{e}^A = \mathrm{e}^X\, \mathrm{e}^V\, \mathrm{e}^X$$

where $X \in \mathfrak{m}$ and $V \in \mathrm{HS}^h$ is such that $tr(VZ) = 0$ for any $Z \in \mathfrak{m}$.

6.2 The embedding of M_n^+ in Σ_1

We are ready to state and prove a projection theorem for $M = M_n^+$.

First note that we can embed $M_n^+ \hookrightarrow \Sigma_1$ for any $n \in \mathbb{N}$ (see the proof of Theorem 5.3). Fix an orthonormal basis $\{e_n\}_{n\in\mathbb{B}}$ of H, set $p_{ij} = e_i \otimes e_j$, and identify M_n with the set

$$\mathcal{T} = \left\{ \sum_{i,j=1}^{n} a_{ij}\, p_{ij} \ :\ a_{ij} = a_{ji} \in \mathbb{R} \right\} \subset \mathrm{HS}^h$$

In this way, we can identify isometrically (Proposition 4.1 of [AV03]) the manifolds M_n^+ with the set

$$\mathcal{P} = \left\{ \mathrm{e}^T : T \in \mathcal{T} \right\} \subset \Sigma_1$$

and the tangent space at each $\mathrm{e}^T \in \mathcal{P}$ is $\mathcal{T}$. $\mathcal{P}$ is closed and geodesically convex in Σ_1 by Corollary 3.13

Let's call $S = \mathrm{span}(e_1, \cdots, e_n)$, $S^{\perp} = \mathrm{span}(e_{n+1}, e_{n+2} \cdots)$. The operator P_S is the orthogonal projection to S and $Q_S = 1 - P_S$ is the orthogonal projection to $S^{\perp}$.

Using matrix blocks, for any operator $A \in \mathsf{L}(S)$, we identify

$$\mathcal{T} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathcal{P} = \begin{pmatrix} \mathrm{e}^A & 0 \\ 0 & 1 \end{pmatrix}$$

Remark 6.9. There is a direct sum decomposition of $\mathrm{HS}^h = \mathcal{T} \oplus \mathcal{J}$ where operators in $J \in \mathcal{J}$ are such that $P_S J P_S = 0$. A straightforward computation using the matrix-block representation shows that $tr(ab) = 0$ for any $a \in \mathcal{T}, b \in \mathcal{J}$, which says $\mathcal{T}^\perp = \mathcal{J}$.
So the manifolds $\exp(\mathcal{J})$ and $\mathcal{P} = \exp(\mathcal{T})$ are orthogonal at 1, the unique intersection point.

Theorem 6.10. *(projection to positive invertible $n \times n$ matrices) : Set $\mathcal{P} \simeq M_n^+ \subset \Sigma_1$ with the above identification. Then for any positive invertible operator $e^b \in \Sigma_1$, ($b \in \mathrm{HS}^h$) there is a unique factorization of the form*

$$e^b = \begin{pmatrix} e^A & 0 \\ 0 & 1 \end{pmatrix} \exp\left\{ \begin{pmatrix} e^{-A} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbb{O}_{n\times n} & Y^* \\ Y & X \end{pmatrix} \right\}$$

where $e^a = e^A P_S + Q_S \in \mathcal{P} \simeq M_n^+$, ($a \in \mathcal{T}$), $X^* = X$ acts on the subspace $S^\perp$ and $Y \in \mathsf{L}(S, S^\perp)$.

An equivalent expression for the factorization is

$$\mathrm{e}^b = \begin{pmatrix} \mathrm{e}^{A/2} & 0 \\ 0 & 1 \end{pmatrix} \exp\left\{ \begin{pmatrix} \mathbb{O}_{n\times n} & \mathrm{e}^{-A/2} Y^* \\ Y \mathrm{e}^{-A/2} & X \end{pmatrix} \right\} \begin{pmatrix} \mathrm{e}^{A/2} & 0 \\ 0 & 1 \end{pmatrix}$$

Yet another form is the following: for any $p \in \Sigma_1$ exist unique $V \in \mathrm{HS}^h$ such that $P_S V P_S = 0$, and unique $q \in \Sigma_1$ such that $P_S q Q_S = Q_S q P_S = 0$ and $Q_S q Q_S = Q_S$ which make the following equation valid

$$p = q\, \mathrm{e}^V q$$

Proof. From previous theorems and the observations we made, we know that

$$\mathrm{e}^b = \begin{pmatrix} \mathrm{e}^{A/2} & 0 \\ 0 & 1 \end{pmatrix} \exp\left\{ \begin{pmatrix} \mathrm{e}^{-A/2} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} V_{11} & V_{21}^* \\ V_{21} & V_{22} \end{pmatrix} \begin{pmatrix} \mathrm{e}^{-A/2} & 0 \\ 0 & 1 \end{pmatrix} \right\} \begin{pmatrix} \mathrm{e}^{A/2} & 0 \\ 0 & 1 \end{pmatrix}$$

for some $A \in \mathsf{L}(S)$ and some $V \in \mathrm{HS}^h$. That $V_{11} = 0$ follows from the fact (see Remark 6.9) that $\mathcal{T}^\perp = \mathcal{J}$, and $V \in T_{\mathrm{e}^a}\mathcal{P}^\perp$ iff $tr(\mathrm{e}^{-A} B \mathrm{e}^{-A} V_{11}) = 0$ for any $B \in \mathcal{T}$. This says that V has the desired form. □

Remark 6.11. Since V is orthogonal to $\mathcal{P}$ at any point, in particular it is orthogonal to $\mathcal{P}$ at 1; so 1 is the foot of the perpendicular from e^V to $\mathcal{P}$, or, in other words, 1 is the point in $\mathcal{P}$ closer to e^V; the distance between 1 and e^V is exactly $\|V\|_2 = tr(V^2)$.
In the notation of Theorem 6.10, $\mathrm{e}^a = 1$ if and only if $A = 0$, if and only if $V = b$, and we conclude that for any $b \in \mathrm{HS}^h$ such that $P_S b P_S = 0$, the point in $\mathcal{P}$ closer to e^b is 1. This is nothing but Remark 6.9 in disguise.

For any $b \in \mathrm{HS}^h$, it holds true that the operator

$$\mathrm{e}^a = \mathrm{e}^A P_{S_n} + P_{S_n^\perp} = \begin{pmatrix} \mathrm{e}^A & 0 \\ 0 & 1 \end{pmatrix} = \exp\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$$

is the 'first block' $n \times n$ matrix which is closer to e^b in Σ_∞, and with a slight abuse of notation for the traces of $\mathsf{L}(S_n)$ and $\mathsf{L}(S_n^\perp)$, we have

$$\operatorname{dist}(\mathcal{P}, \mathrm{e}^b) = \operatorname{dist}(\mathrm{e}^a, \mathrm{e}^b) = \left\| \begin{pmatrix} \mathbb{O}_{n\times n} & Y^* \\ Y & X \end{pmatrix} \right\|_{\mathrm{e}^a} = \sqrt{\|Y\,\mathrm{e}^{-A/2}\|_2^2 + \|X\|_2^2}$$

7 The embedding of nonpositively curved spaces

7.1 A Classical Result

In a series of notes devoted to the geometry of manifolds of nonpositive sectional curvature (in particular, [Eb85]), Patrick Eberlein puts together a result which 'does not seem to be stated in the literature in precisely this form' (*sic*).

Eberlein shows that every symmetric (real, finite dimensional) manifold M of noncompact type can be realized isometrically as a complete, totally geodesic sumanifold of $M_n^+(\mathbb{R})$, where $n = dim(M)$, with the precaution that one multiplies the metric on each irreducible de Rham factor of M by a suitable constant.

If $I_0(M)$ denotes the connected component of the isometry group of M that contains the identity, then $G = I_0(M)$ is a Lie group given the compact-open topology; if $\mathfrak{g}$ is the Lie algebra of G, the idea of this result is based in the representation of M into $End(\mathfrak{g})$. In the following paragraphs we outline the main tools an ennunciate the result.

We state the de Rham decomposition theorem; for a proof see Theorem 6.11 of Chapter III in [SakT96]

Theorem 7.1. *Let M be a complete simply connected Riemannian manifold. Then M is isometric to the Riemannian direct product $M_0 \times M_1 \times \cdots \times M_k$, where M_0 is Euclidean space and the other M_i are complete simply connected irreducible Riemannian manifolds. Moreover, this decomposition is unique up to order.*

Let $(M, \langle\ ,\ \rangle_M)$ be a symmetric space of noncompact type (*i.e.* simply connected, with no Euclidean de Rham factor and nonpositive sectional curvature). For these manifolds, $G = I_0(M)$ is a semisimple Lie group (see [Eb85]).

Fix a point $p \in M$. Since M is symmetric, the geodesic symmetry s_p generates an involutive automorphism σ_p of $I_0(M)$, where $\sigma_p(g) = s_p \circ g \circ s_p$. The differential of this map gives an involutive Lie algebra automorphism (see section 3 of this manuscript, [Eb85], or [Hel62]) $\Theta_p = d_p\sigma_p : \mathfrak{g} \to \mathfrak{g}$; this map is characterized by the equation

$$\sigma_p\left(e^{tX}\right) = e^{t\,\Theta_p(X)} \qquad \text{for all } X \in \mathfrak{g} \text{ and all } t \in \mathbb{R}$$

and gives a canonical decomposition of $\mathfrak{g}$ where $\mathfrak{m}$ identifies with the tangent space T_pM and $\mathfrak{k} = Fix(\Theta_p)$.

Let's denote with $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ the *Killing form* of G, which maps

$$(X,Y) \mapsto trace(adX \circ adY)$$

We define an inner product on $\mathfrak{g}$ using the Killing form:

$$\langle X,Y \rangle_{\mathfrak{g}} = -B\left[\Theta_p(X), Y\right] = -trace\left(ad\Theta_p(X) \circ adY\right)$$

Now we ennunciate a few facts that we prove only partially, because they can be deduced from the general theory of representations (see, for instance, secion 6.2, Chapter IV of [SakT96]), or can be found in Eberlein's paper [Eb85]. See also section 3.2 of this manuscript.

- The fact that G is semisimple ensures that B is nondegenerate.
- By definition, Ad_g is the differential at $Id \in G$ of the g-inner automorphism α_g, that is, the map which sends $\phi \mapsto g\phi g^{-1}$; since this map fixes Id, its differential is an endomorphism of $\mathfrak{g}$.
- G acts transitively on M, by means of the symmetries s_{p_q}, where p_q is the middle point of the minimal geodesic joining p to q.
- This inner product makes $\mathfrak{m} \perp \mathfrak{k}$; adX is symmetric relative to this inner product for any $X \in \mathfrak{m}$, and adX is skew-symmetric for any $X \in \mathfrak{k}$.
- Recall that $adX(Z) = [X,Z]$. Then $adX = \frac{d}{dt}\Big|_{t=0} \alpha\left(e^{tX}\right)$ and also $Ad_{e^X} = e^{\,adX}$.
- $tr(adX) = 0$ for any $X \in \mathfrak{g}$. This is due to the following:
    1. We can span $\mathfrak{g}$ with a basis $\{E_i\}$, such that $\Theta_p(E_i) = \varepsilon_i E_i$, $\varepsilon_i = \pm 1$ and $B[E_i, E_j] = \varepsilon_i \delta_{ij}$, so $\left\langle E_i, E_j \right\rangle_{\mathfrak{g}} = \delta_{ij}$
    2. $\langle adX(E_i), E_i \rangle_{\mathfrak{g}} = -B\left[adX(E_i), \Theta_p(E_i)\right] = -\varepsilon_i B\left[adX(E_i), E_i\right]$
    3. $tr(adX) = \sum_{i=1}^{n} \langle adX(E_i), E_i \rangle_{\mathfrak{g}} = -\sum_{i=1}^{n} \varepsilon_i B\left[adX(E_i), E_i\right]$
    4. $B\left[adZ(X), Y\right] = -B\left[X, adZ(Y)\right]$ (this can be deduced using the Jacobi identity twice)

- $Ad_G \subset GL(\mathfrak{g})$, in fact $Ad_G \subset SL(\mathfrak{g})$. This a consecuence of:

  1. The image of the exponential map $e : \mathfrak{g} \to G$ generates G; in other words
  $$G = \bigcup_n e(\mathfrak{g})^n$$
  2. $det(e^A) = e^{trA}$ for any linear operator A
  3. The two previous observations

- If we denote with a dagger the adjoint whith respect to the inner product introduced above, then $Ad_G^\dagger \subset SL(\mathfrak{g})$ also.

- The inner product $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ is invariant under Ad_K, that is
$$\langle Ad_k X, Ad_k Y \rangle_{\mathfrak{g}} = \langle X, Y \rangle_{\mathfrak{g}} \quad \text{for any} \quad k \in K$$
(K is the isotropy group of p).

- If $q \in M$ is such that $q = g_1(p) = g_2(p)$ ($g_i \in G$) then calling $u = g_1^{-1} g_2$, u is in the isotropy group K of p, and using that the inner product is Ad_K-invariant, we get
$$Ad_{g_1}^\dagger Ad_{g_1} = Ad_{g_2}^\dagger Ad_{g_2}$$

- Moreover, $Ad : G \to SL(\mathfrak{g})$ is injective. This is a consecuence of the fact that M has no Euclidean de Rham factor (see [Wolf64]).

Theorem 7.2. *Fix a point p in any symmetric (real, finite dimensional) manifold M of noncompact type. Then the map $F_p : M \to GL^+(\mathfrak{g})$ given by*
$$q = g(p) \mapsto Ad_g^\dagger Ad_g$$
is a diffeomorphism with a closed, totally geodesic submanifold of $GL^+(\mathfrak{g})$
Moreover, if we pull back the inner product on $GL^+(\mathfrak{g})$ to M, this inner product differs only by a constant positive factor from the inner product of M, on each irreducible de Rham factor of M.

Proof. That $Ad_g^\dagger Ad_g$ is positive and invertible in $End(\mathfrak{g})$, and the map is well defined is a consequence of the previous observations.
The proof of the theorem can be found in Eberlein's survey, Proposition 19 of [Eb85]. □

7.2 A New Result

Now fix an orthonormal basis $\{X_1, \cdots, X_n\}$ of $\mathfrak{g}$ and identify X_i with e_i in $\mathbb{R}^n$. Then we obtain an embedding $\mathcal{F}_p : M \to \Sigma_\infty$ which is the composition of the previous map, the identification of $GL(\mathfrak{g})$ with $GL(n, \mathbb{R})$, and the isometric, closed and geodesically convex embedding of $M_n^+(\mathbb{R})$ in Σ_∞ (see section 6.2 of this manuscript and also section 4 of [AV03] by Andruchow *et al.*).

In this way, we can identify M with a subset of the first $n \times n$ block in the matrix representation of Σ_∞; in the notation of section 6.2, M can be identified with a closed and geodesically convex submanifold of $\mathcal{P}$; remember that operators in $\mathcal{P}$ have a matrix representation of the form

$$\begin{pmatrix} e^A & 0 \\ 0 & 1 \end{pmatrix}$$

Theorem 7.3. *For any (finite dimensional, real) symmetric manifold M of the noncompact type (that is, with no Euclidean de Rham factor, simply connected and with nonpositive sectional curvature), there is an embedding $\mathcal{F}_M : M \longrightarrow \Sigma_\infty$ which is a diffeomorphism betwen M and a closed, geodesically convex submanifold of Σ_∞. This map preserves the metric tensor in the following sense: if we pull back the inner product on Σ_∞ to M, then this inner product is a (positive) constant multiple of the inner product of M (on each irreducible de Rham factor of M). Moreover, $\mathcal{F}_M(M) \subset \Sigma_1$.*

This theorem together with the general factorization theorem says that, for any finite dimensional symmetric manifold M of the noncompact type, we can project operators in Σ_∞ using the contraction Π_M (assuming we identify M with its image $\mathcal{F}_M(M)$).

8 Unitary orbits

There is a distinguished leaf in the foliation we defined in Section 6, namely Σ_1, which contains the identity. Moreover, $\Sigma_1 = \exp(\mathrm{HS}^h)$. We will focus on this submanifold since the nontrivial part of the geometry of Σ_∞ is, by Theorem 6.6 contained in the leaves. We won't have to deal with the scalar part of tangent vectors, and some computations will be less involved.

8.1 The action of the unitary groups $\mathcal{U}_{\mathcal{H}_\mathbb{C}}$ and $\mathcal{U}_{\mathrm{L}(H)}$

We are interested in the orbit of an element $1+a \in \Sigma_1$ by means of the action of some group of unitaries.

We first consider the group of unitaries of the complex Banach algebra of 'unitized' Hilbert-Schmidt operators. To be precise, let's call

$$\mathcal{U}_{\mathcal{H}_\mathbb{C}} = \{g = \lambda + a : a \in \mathrm{HS}, \lambda \in \mathbb{C}, g^* = g^{-1}\}$$

It is apparent from the definition that $|\lambda| = 1$ (so we can write $\lambda = e^{i\theta}$), and also that a must be a normal operator; this definition can be restated (naming $g = a + \lambda = u + iv + e^{i\theta}$) in the form of the following operator equation:

$$(u + \cos(\theta))^2 + (v + \sin(\theta))^2 = 1$$

It will be apparent from the definition of the action that we will be always able to choose $\theta = 0$, so $g = 1 + x$ with x a normal operator and $\sigma(x) \subset S^1 - 1$ (here -1 denotes translation in the complex plane).

The Lie algebra of this Lie group consists of the operators of the form $i(x + r1)$ where x is a Hilbert-Schmidt, selfadjoint operator, and r is a real number, that is

$$T_1\left(\mathcal{U}_{\mathcal{H}_\mathbb{C}}\right) = i\mathcal{H}_\mathbb{R} = \{a + \lambda : \ a^* = -a \text{ and } \lambda \in \ i\,\mathbb{R}\}$$

Since these are the antihermitian operators of the unitized Hilbert-Schmidt algebra, we have $T_1\left(\mathcal{U}_{\mathcal{H}_\mathbb{C}}\right) = \mathcal{H}_\mathbb{C}^{ah}$. But we mentioned early that it will be enough to consider unitaries $\lambda + x$ with $\lambda = 1$; in this case, with a slight abuse of notation, we have an identification

$$T_1\left(\mathcal{U}_{\mathcal{H}_\mathbb{C}}\right) = i\mathrm{HS}^h$$

Remark 8.1. The problem of determining whether a set in Σ_1 can be given the structure of submanifold (or not) can be translated into the tangent space by taking logarithms; to be precise, note that

$$\exp(UaU^*) = Ue^aU^*$$

for any $a \in \mathrm{HS}^h$ and any unitary U, and that this map is an analytic isomorphism between Σ_1 and its tangent space. We will state the problem in this context.

We fix an element a in the tangent space (that is, $a \in \mathrm{HS}^h$) and make the unitary group act *via* the map

$$\pi_a : \mathcal{U}_{\mathcal{H}_\mathbb{C}} \to \mathrm{HS}^h \qquad g \mapsto gag^*$$

- When is the orbit (which we will denote S_a) of a selfadjoint Hilbert-Schmidt operator a submanifold of HS^h?

The answer to this question can be partially answered in terms of the spectrum:

Theorem 8.2. *If the algebra $C^*(a)$ generated by a and 1 is finite dimensional, then the orbit $S_a \subset \mathrm{HS}^h$ can be given an analytic submanifold structure.*

Proof. We give the tools for constructing the proof, and refer the reader to [AS89] and [AS91]. A local section for the map π_a is a pair (U_a, φ_a) where U_a is an open neighbourhood of a in HS^h and φ_a is an analytic map from U_a to $\mathcal{U}_{\mathcal{H}_\mathbb{C}}$ such that:

- $\varphi_a(a) = 1$
- φ_a restricted to $U_a \cap S_a$ is a section for π_a, that is

$$\pi_a \circ \varphi_a|_{U_a \cap S_a} = id_{U_a \cap S_a}$$

A section for π_a provides us with sufficient conditions to give the orbit the structure of immerse submanifold of HS^h (see Propostion 2.1 of [AS89]). The section φ_a can be constructed by means of the finite rank projections in the matrix algebra where

$C^*(a)$ is represented. The finite dimension of the algebra is key to the continuity (and furthermore analyticity) of all the maps involved. To fix some notation, as in Theorem 1.3 of [AS91], suppose $n = \sum_{i=1}^{p} n_i = dim C^*(a)$ and τ is the *-isomorphism

$$\tau : C^*(a) \to M_{n_1}(\mathbb{C}) \oplus M_{n_2}(\mathbb{C}) \oplus \cdots \oplus M_{n_p}(\mathbb{C})$$

Consider the set of systems of projections (here $p_i^2 = p_i = p_i^*, p_i p_j = 0$ for any $i \neq j$):

$$P_n = \{(p_1, \cdots, p_n) \in \mathcal{H}_{\mathbb{C}}^n : \sum_{i=1}^{m} p_i = 1\}$$

Denote $e^i_{jk} \in M_{n_i}(\mathbb{C})$ the elementary matrix with 1 in the (j,k)-entry and zero elsewhere, but embedded in the direct sum; take $p^i_{jk}(X)$ the polynomial which makes $e^i_{jk} = p^i_{jk}(\tau(a))$, and consider the following element in HS^h: $\mathrm{e}^i_{jk} = p^i_{jk}(a)$
There is a neighbourhood U_a of a in HS^h such that $1 - \left[\mathrm{e}^i_{11} - p^i_{11}(x)\right]^2$ has strictly positive spectrum, because $r(x) = \|x\| \leq \|x\|_2$ and $\mathcal{H}_{\mathbb{C}}$ is a Banach algebra (here $r(x)$ denotes spectral radius). The map

$$\varphi_a(x) = \sum_{i=1}^{p} \sum_{j=i}^{n_i} p^i_{j1}(x) E^i_{11} \left[1 - \left(E^i_{11} - p^i_{11}(x)\right)^2\right]^{-\frac{1}{2}} E^i_{1j}$$

is a cross section for π_a, and it is analytic from $U_a \subset \mathrm{HS}^h \to \mathcal{U}_{\mathcal{H}_{\mathbb{C}}}$ since the p^i_{jk} are multilinear and all the operations are taken inside the Banach algebra $\mathcal{H}_{\mathbb{C}}$ (the computation that proves that φ_a is in fact a cross section for π_a is straightforward and can be found in the article by Andruchow *et al.*, [AFHS90]). □

Remark 8.3. At first sight, it is not obvious if this strong restriction (on the spectrum of a) is necessary for S_a to be a submanifold of HS^h. The main difference with the work done so far by Deckard and Fialkow in [DF79], Raeburn in [Rae77], and Andruchow *et al.* in [AS89], [AS91] is that the Hilbert-Schmidt operators (with any norm equivalent to the $\|\cdot\|_2$-norm) are not a C^*-algebra. A remarkable byproduct of Voiculescu's theorem [Voic76] says that, for the unitary orbit of an operator a with the action of the full group of unitaries of $\mathsf{L}(H)$, it is indeed necessary that a has finite spectrum. For the time being, we don't know if this is true for the algebra $\mathcal{B} = \mathcal{H}_{\mathbb{C}}$.

Let's examine what happens when we act with the full group $\mathcal{U}_{\mathsf{L}(H)}$ by means of the same action. For convenience let's fix the notation

$$\mathfrak{S}_a = \{UaU^* \,:\, U \in \mathcal{U}_{\mathsf{L}(H)}\}$$

We will develop an example that shows that the two orbits (S_a and $\mathfrak{S}_a$) are, in general, not equal when the spectrum of a is infinite. Since a is compact and selfadjoint, we can assume that a is a diagonal operator; that is, there's an orthonormal basis $\{e_k\}$ of H such that

$$a = \sum_k \alpha_k \, e_k \otimes e_k, \qquad \text{where} \qquad \sum_k |\alpha_k|^2 = tr(a^*a) < +\infty$$

Example 8.4. Take $H = l_2(\mathbb{Z})$, $S \in \mathsf{L}(H)$ the right shift ($Se_k = e_{k+1}$). Then S is a unitary operator with $S^*e_k = e_{k-1}$. Pick any a of the form

$$a = \sum_{k\in\mathbb{Z}} r_k\, e_k \otimes e_k \qquad \text{and} \qquad \sum_k |\, r_k\,|^2 < +\infty$$

where all the r_k are different. (For instance, $r_k = \frac{1}{|k|+1}$ would do). Obviously, $a \in \mathrm{HS}^h$. We affirm that there is no Hilbert-Schmidt unitary such that $SaS^* = waw^*$

Proof. To prove this, suppose that there is an $w \in \mathcal{U}_{\mathcal{H}_\mathbb{C}}$ such that $SaS^* = waw^*$. From this equation we deduce that S^*w commutes with a, and given the particular a and the fact that S^*w is unitary, we have

$$S^*w = \sum_{k\in\mathbb{Z}} \omega_k\, e_k \otimes e_k \qquad \text{with } \;|\,\omega_k\,| = 1$$

because $C^*(a)$ is maximal abelian. Multiplying by S we get to

$$w = \sum_{k\in\mathbb{Z}} \omega_k\, (Se_k) \otimes e_k = \sum_{k\in\mathbb{Z}} \omega_k\, e_{k+1} \otimes e_k$$

or, in other terms, $we_k = \omega_k e_{k+1}$. Since w is a compact perturbation of a scalar operator, w must have a nonzero eigenvector x, with eigenvalue $\alpha = \mathrm{e}^{i\theta}$ (since w is also unitary); comparing coefficients the equation $\alpha x = wx$ reads

$$\alpha x_k = \omega_{k-1} x_{k-1}, \qquad \text{where } x = \sum_k x_k e_k$$

This is impossible because $x \in l_2(\mathbb{Z})$, but the previous equation leads to $|\, x_k\,| = |\, x_j\,|$ for any $k, j \in \mathbb{Z}$. □

As we see from the previous example, the two orbits do not coincide in general. For the action of the full group of unitaries we have the following:

Theorem 8.5. *The set $\mathfrak{S}_a \subset \mathrm{HS}^h$ (the orbit of the Hilbert-Schmidt operator a under the action of the full unitary group $\mathcal{U}_{\mathsf{L}(H)}$) can be given an analytic submanifold structure if and only if the C^*-algebra generated by a and 1 is finite dimensional.*

Proof. The 'only if' part goes in the same lines of the proof of the previous theorem but being careful about the topologies involved, since now we must take an open set $U_a \subset \mathrm{HS}^h$ such that the map $\phi : U_a \to \mathcal{U}_{\mathsf{L}(H)}$ is analytic. But this can be done since the polynomials p^i_{jk} are now taken from U_a to $\mathsf{L}(H)^n$, and the maps $+$ and $\cdot$ are analytic since $\|x.y\|_{\mathsf{L}(H)} \le \|x\|_2 \|y\|_2$.
The relevant part of this theorem is the 'if' part. Suppose we can prove that the orbit $\mathfrak{S}_a$ is closed in $\mathsf{L}(H)$. Then Voiculescu's theorem (see [Voic76], Proposition 2.4) would tell us that $C^*(a)$ is finite dimensional. This is a deep result about $*-$representations, and the argument works in the context of $\mathsf{L}(H)$, but not in $\mathcal{H}_\mathbb{C}$ because the latter is not a C^*-algebra.
To prove that $\mathfrak{S}_a$ is closed in $\mathsf{L}(H)$, we first prove that it is closed in $\mathcal{H}_\mathbb{C}$. To do this, observe that if $\mathfrak{S}_a$ is an analytic submanifold of HS^h, then $\mathfrak{S}_a$ must be locally

closed in the $\|\cdot\|_2$ norm. Since the action of the full unitary group is isometric, the neighbourhood can be chosen uniformly, that is, there is an $\varepsilon > 0$ such that for all $c \in \mathfrak{S}_a$, the set $N_c = \{d \in \mathfrak{S}_a : \|c-d\|_2 \leq \varepsilon\}$ is closed in HS^h (with the 2-norm, of course). This is another way of saying that $\mathfrak{S}_a$ is closed in HS^h.
Now suppose $a_n = u_n a u_n^* \to y$ in $\mathsf{L}(H)$. We claim that $\|a_n - y\|_2 \to 0$, which follows from a dominated convergence theorem for trace class operators (see [Simon89], Theorem 2.17). The theorem states that whenever $\|a_n - y\|_\infty \to 0$ and $\mu_k(a_n) \leq \mu_k(a)$ for some $a \in \mathrm{HS}$, and all k (here $\mu_k(x)$ denotes the non zero eigenvalues of $|x|$), then $\|a_n - y\|_2 \to 0$.
Observe that $|a_n| = u_n |a| u_n^*$ so we have in fact equality of eigenvalues. This proves that $\mathfrak{S}_a$ is closed in $\mathsf{L}(H)$ since it is closed in HS^h. □

We proved that, when the spectrum of a is finite, S_a and $\mathfrak{S}_a$ are submanifolds of Σ_1. But more can be said: S_a and $\mathfrak{S}_a$ are the *same subset* of HS^h (compare with Example 8.4):

Lemma 8.6. *If $a \in \mathrm{HS}^h$ has finite spectrum, then the orbit under both unitary groups are the same submanifold.*

Proof. The main idea behind the proof is the fact that, when $\sigma(a)$ is finite, a and gag^* act on a finite dimensional subspace of H (for any $g \in \mathcal{U}_{\mathsf{L(H)}}$). To be more precise, let's call $S = Ran(a)$, $V = Ran(b)$, where $b = gag^*$. Note that $V = g(S)$ so S and V are isomorphic, finite dimensional subspaces of H. Naming $T = S + V$ this is another finite dimensional subspace of H, and clearly a and b act on T, since they are both selfadjoint operators. For the same reason, there exist unitary operators $P, Q \in \mathsf{L(T)}$ and diagonal operators $D_a, D_b \in \mathsf{L(T)}$ such that

$$a = PD_aP^*, \quad b = QD_bQ^*$$

But $\sigma(b) = \sigma(gag^*) = \sigma(a)$, so $D_a = D_b := D$. This proves that $b = QP^*aPQ^*$ (the equality should be interpreted in T). Now take P_T the orthogonal projector in $\mathsf{L(H)}$ with rank T, and set $u = 1 + (QP^* - 1_T)P_T$ (note the slight abuse of notation). Then clearly $u \in \mathcal{U}_{\mathcal{H}_\mathbb{C}}$ and $uau^* = b$. □

8.2 Riemannian structures for the orbit Ω

Suppose that there is, in fact, a submanifold structure for $\mathfrak{S}_a$ (resp. S_a). Then the tangent map ($= d_1\pi_a$) has image

$$\{va - av : v \in \mathcal{B}^{ah}\},$$

where $\mathcal{B}$ stands for $\mathsf{L}(H)$ (resp. $\mathcal{H}_\mathbb{C}$). So, in this case

$$T_aS_a(\text{ or } T_a\mathfrak{S}_a) = \{va - av : v \in \mathcal{B}^{ah}\}$$

We can go back to the manifold Σ_1 *via* the usual exponential of operators; we will use the notation

$$\Omega = e^{S_a} \quad \text{or} \quad \Omega = e^{\mathfrak{S}_a}$$

without further distinction. Note that $\Omega = \{ue^a u^* \,:\, u \in \mathcal{U}_{\mathcal{B}}\} \subset \Sigma_1$ and we can identify

$$T_{e^a}\Omega = \{ve^a - e^a v : v \in \mathcal{B}^{ah}\} = \{\, i(he^a - e^a h) : h \in \mathcal{B}^h\}$$

Remark 8.7. For any $p \in \Omega$, we have

$$T_p\Omega = \{vp - pv : v \in \mathcal{B}^{ah}\} \quad \text{and} \quad T_p\Omega^{\perp} = \{X \in \mathrm{HS}^h : [X, p] = 0\}$$

These two identifications follow from the definition of the action, and the equality

$$\langle x, vp - pv\rangle_p = 4tr\left[(p^{-1}x - xp^{-1})V\right]$$

The submanifold Ω is connected: the curves indexed by $w \in \mathcal{B}^{ah}$,

$$\gamma_w(t) = e^{tw} e^a e^{-tw}$$

join e^a to $ue^a u^*$, assuming that $u = e^w$.

We can ask whether the curves γ_w will be the familiar geodesics of the ambient space (equation (4) of section 2). Of course they are trivial geodesics if a and w commute. We will prove that this is the only case, for any a:

Proposition 8.8. *For any $a \in \mathrm{HS}^h$, the curve γ_w is a geodesic of Σ_1 if and only if w commutes with a. In this case the curve reduces to the point e^a.*

Proof. The (ambient) covariant derivative for γ_w (equations (2) and (3) of section 2) simplifies up to $we^a we^{-a} = e^a we^{-a} w$ or, writing $w = ih$ (h is selfadjoint)

$$he^a he^{-a} = e^a he^{-a} h \tag{8}$$

Consider the Hilbert space $(H, \langle\,,\,\rangle_a)$ with inner product

$$\langle x, y\rangle_a = \left\langle e^{-a/2}x, e^{-a/2}y\right\rangle,$$

where $\langle\,,\,\rangle$ is the inner product of H. The norm of an operator X is given by

$$\|X\|_a = \sup_{\|z\|_a=1} \|Xz\|_a = \sup_{\|e^{-a/2}z\|=1} \|e^{-a/2}Xz\|_\infty = \|e^{-a/2}Xe^{a/2}\|_\infty$$

because $e^{-a/2}$ is an isomorphism of H. This equation also shows that the Banach algebras $(\mathrm{L}(H), \|\cdot\|_\infty)$ and $\mathcal{B} = (\mathrm{L}(H), \|\cdot\|_a)$ are topologically isomorphic and, as a byproduct, $\sigma_{\mathcal{B}}(h) \subset \mathbb{R}$. From the very definition it also follows easily that $\mathcal{B}$ is indeed a C*-algebra. A similar computation shows that $X^{*\mathcal{B}} = e^a X^* e^{-a}$. Note that e^a is $\mathcal{B}$-selfadjoint, moreover, it is $\mathcal{B}$-positive. We can restate equation (8) as

$$hh^{*\mathcal{B}} = h^{*\mathcal{B}}h,$$

This equations says that h is $\mathcal{B}$-normal, so a theorem of Weyl and von Neumann (see [Dav96]) says it can be aproximated by diagonalizable operators with the same spectrum;

since h has real spectrum, h turns out to be $\mathcal{B}$-selfadjoint . That h is $\mathcal{B}$-selfadjoint reads, by definition, $e^a h e^{-a} = h^{*\mathcal{B}} = h$; this proves that a and h (and also a and w) commute. □

8.2.1 Ω as a Riemannian submanifold of HS^h

We've shown earlier that the orbit of an element $a \in \mathrm{HS}^h$ has a structure of analytic submanifold of HS^h (which is a flat Riemannian manifold) if and only if $\Omega = e^a$ has a structure of analytic submanifold of Σ_1.
Since the inclusion $\Omega \subset \mathrm{HS}^h$ is an analytic embedding, we can ask whether the curves

$$\gamma_w(t) = e^{tw} e^a e^{-tw}$$

will be geodesics of Ω as a Riemannian submanifold of HS^h (with the induced metric). For this, we notice that the geodesic equation reads $\ddot{\gamma}_w(t) \perp T_{\gamma_w(t)}\Omega$, and we use the elementary identities $\dot{\gamma} = w\gamma - \gamma w$, $\ddot{\gamma} = w^2\gamma - 2w\gamma w + \gamma w^2$; we get to the following necessary and sufficient condition using the caracterization of the normal space at $\gamma(t)$ of the previous section:

$$w^2\gamma^2 - 2w\gamma w\gamma + 2\gamma w\gamma w - \gamma^2 w^2 = 0$$

But observing that $e^{-wt}\gamma^{\pm 1} e^{wt} = e^{\pm a}$, this equation transforms in the operator condition

$$w^2 e^{2a} - 2we^a we^a + 2e^a we^a w - e^{2a} w^2 = 0 \tag{9}$$

Let's fix some notation: set $e^a = 1 + A$ with $A \in \mathrm{HS}^h$; then the tangent space at e^a can be thought of as the subspace

$$T_{e^a}\Omega = \{\, i(Ah - hA) : h \in \mathcal{B}^h\} \subset \mathrm{HS}^h$$

and its orthogonal complement in HS^h is (see Remark 8.7)

$$T_{e^a}\Omega^{\perp} = \{\, X \in \mathcal{B}^h : [X, A] = 0\}$$

It should be noted that both subspaces are closed by hypothesis. Then equation (9) can be restated as

$$h^2A^2 - 2hAhA + 2AhAh - A^2h^2 = 0 \tag{10}$$

where h is the hermitian generating the curve

$$\gamma(t) = 1 + e^{ith} A e^{-ith} = e^{ith} e^a e^{-ith}$$

Let's consider the case when $A^2 = A$:

Remark 8.9. If $A^2 = A$, A must be a finite rank orthogonal projector (since $A = e^a - 1$ and a is a Hilbert-Schmidt operator). Hence, $\sigma(a)$ must be a finite set, and in this case (Remark 8.6) the orbit with the full unitary group and the orbit with the Hilbert-Schmidt unitary group are the same set.

To solve the problem of the geodesics completely, we review the work of Corach, Porta and Recht ([PR96] or, more specifically [CPR93a]); we follow the idea of section 4 of that article and put the result in context.

Observe that when A is a projector, we have a matrix decomposition of the tangent space of Σ_1, namely $\mathrm{HS}^h = A_0 \oplus A_1$, where

$$A_0 = \left\{ \begin{pmatrix} x_{11} & 0 \\ 0 & x_{22} \end{pmatrix} \right\} \quad \text{and} \quad A_1 = \left\{ \begin{pmatrix} 0 & x_{12} \\ x_{21} & 0 \end{pmatrix} \right\}$$

In this decomposition, $x_{11} = AhA$, $x_{22} = (1-A)h(1-A)$ are selfadjoint operators (since h is) and also $x_{12}^* = x_{21} = (1-A)hA$ for the same reason.

Theorem 8.10. *Whenever $A = \mathrm{e}^a - 1$ is a projector, any curve of the form $\gamma(t) = e^{ith} e^a e^{-ith}$ with h selfadjoint and codiagonal is a geodesic of $\Omega \subset \mathrm{HS}^h$*

Proof. Note that $A_0 = T_{\mathrm{e}^a}\Omega^{\perp}$, and $A_1 = T_{\mathrm{e}^a}\Omega$; note also that equation (10) translates in this context to $x_{11}x_{12} = x_{12}x_{22}$, a condition which is obviously fullfilled by $h \in A_1$. □

Remark 8.11. Equation (10) translates exactly in 'h_0 commutes with h_1' whenever $h = h_0 + h_1 \in \mathrm{HS}^h$, and we have

$$[A_0,A_1] \subset A_1 \quad [A_0,A_0] \subset A_0 \quad [A_1,A_1] \subset A_0$$

Since the orbit under both unitary groups coincide (Remark 8.9), assume that we are acting with $G = \mathcal{U}_{\mathcal{B}}$; since the tangent space at the identity of this group can be identified with $\mathcal{B}^{ah}$, the above commutator relationships say that $iA_0 \oplus iA_1$ is a Cartan decomposition of the Lie algebra $\mathfrak{g} = \mathcal{B}^{ah}$. It is apparent that iA_0 is the vertical space, and iA_1 is the horizontal space (see section 7). Moreover,

$$A_0 \cdot A_0 \subset A_0 \qquad A_1 \cdot A_1 \subset A_0 \qquad A_0 \cdot A_1 \subset A_1 \qquad A_1 \cdot A_0 \subset A_1$$

Corollary 8.12. *If $\mathrm{e}^a - 1$ is an orthogonal projector, there is no point $p \in \Omega$ such that Ω is geodesic at p.*

8.2.2 Ω as a Riemannian submanifold of Σ_1

In this section we give Ω the induced Riemannian metric as a submanifold of Σ_1, and discuss shortly the form of the geodesics and the sectional curvature.

Recall that covariant derivative in the ambient space is given by $\nabla_{\dot\gamma}\dot\gamma = \ddot\gamma - \dot\gamma\gamma^{-1}\dot\gamma$ and the orthogonal space to $p \in \Omega$ are the operators commuting with p, so $\nabla_{\dot\gamma}\dot\gamma \perp T_\gamma\Omega$ if and only if

$$\ddot\gamma\gamma - \gamma\ddot\gamma + \gamma\dot\gamma\gamma^{-1}\dot\gamma - \dot\gamma\gamma^{-1}\dot\gamma\gamma = 0 \tag{11}$$

This is an odd equation; we know that any curve in Ω starting at $p = \mathrm{e}^a$ must be of the form $\gamma(t) = g(t)\mathrm{e}^a g(t)^*$ for some curve of unitary operators g.

For the particular curves $\gamma(t) = \mathrm{e}^{ith}\mathrm{e}^a\mathrm{e}^{-ith}$, $h(t) = ith$, so $\dot h(t) = ih$, and $\ddot h(t) \equiv 0$; equation (11) reduces to the operator equation

$$h\mathrm{e}^a h\mathrm{e}^{-a} + h\mathrm{e}^{-a}h\mathrm{e}^a = \mathrm{e}^{-a}h\mathrm{e}^a h + \mathrm{e}^a h\mathrm{e}^{-a}h \tag{12}$$

or $X^* = X$, where $X = he^a he^{-a} + he^{-a} he^a$.

Recall that the unitary groups $\mathcal{U}_{\mathsf{L(H)}}$ and $\mathcal{U}_{\mathcal{H}_\mathbb{C}}$ induce the same manifold $\Omega \subset \Sigma_1$.

Theorem 8.13. *Assume* $e^a = 1 + A$ *with* A *an orthogonal projector, and* $\Omega \subset \Sigma_1$ *is the unitary orbit of* e^a. *Then (throughout* $[\ ,\ ]$ *stands for the usual commutator of operators)*

(1) Ω *is a Riemannian submanifold of* Σ_1.

(2) $T_p\Omega = \{i[x,p] : x \in \mathrm{HS}^h\}$ *and* $T_p\Omega^\perp = \{x \in \mathrm{HS}^h : [x,p] = 0\}$.

(3) The action of the unitary group is isometric, namely

$$\mathrm{dist}^\Omega(upu^*, uqu^*) = \mathrm{dist}^\Omega(p,q)$$

for any unitary operator $u \in \mathsf{L}(H)$.

(4) For any $v = i[x,p] \in T_p\Omega$, *the exponential map is given by*

$$\exp_p^\Omega(v) = e^{ighg^*} p\, e^{-ighg^*}$$

where $p = ge^a g^*$ *and* h *is the codiagonal part of* g^*xg *(in the matrix representation of Proposition 8.10). In particular, the exponential map is defined in the whole tangent space.*

(5) If $p = ge^a g^*$, $q = we^a w^*$, *and* h *is a selfadjoint, codiagonal operator such that* $w^* g e^{ih}$ *commutes with* e^a, *then the curve* $\gamma(t) = e^{itghg^*} p e^{-itghg^*}$ *is a geodesic of* $\Omega \subset \Sigma_1$, *which joins* p *to* q.

(6) If we assume that $h \in \mathrm{HS}^h$, *then* $L(\gamma) = \frac{\sqrt{2}}{2}\|h\|_2$

(7) The exponential map $\exp_p^\Omega : T_p\Omega \to \Omega$ *is surjective.*

Proof. Statements (1) and (2) are a consequence of Remark 8.9 and Theorems 8.2 and 8.5. Statement (3) is obvious because the action of the unitary group is isometric for the 2-norm (see Lemma 2.5). To prove statement (4), take $x \in \mathrm{HS}^h$, and set

$$v = i[x,p] = i(xgAg^* - gAg^*x) = ig[g^*xg, e^a]g^*$$

Observe that

$$e^{-a} = (1+A)^{-1} = 1 - \frac{1}{2}A$$

Rewriting equation (12), we obtain

$$h^2A - Ah^2 + 2AhAh - 2hAhA = 0$$

Now if $y = g^*xg$, take $h =$ the codiagonal part of y; clearly $hA - Ah = yA - Ay$, so

$$\gamma_1(t) = e^{ith} e^a e^{-ith}$$

is a geodesic of Ω starting at $r = e^a$ with initial speed $w = i[y, e^a] = g^* v g$ (see Proposition 8.10). Now consider $\gamma = g\gamma_1 g^*$. Clearly γ is a geodesic of Ω starting at $p = g e^a g^*$ with initial speed v. To prove (5), note that

$$\gamma(t) = g e^{iht} e^a e^{-iht} g^* = e^{itghg^*} g e^a g^* e^{itghg^*} = e^{itghg^*} p e^{itghg^*}$$

which shows that $\gamma(0) = p$ and $\gamma(1) = q$ because $w^* g e^{ih} e^a = e^a w^* g e^{ih}$. To prove (6), we can assume that $p = e^a$, and then

$$L(\gamma)^2 = \|[h, p]\|_p^2 = \|[h, e^a]\|_{e^a}^2 = 4 \cdot tr(2he^a he^{-a} - 2h^2)$$

Now write h as a matrix operator $[0, Y^*, Y, 0] \in A_1$ (see Proposition 8.10), to obtain

$$tr(2he^a he^{-a} - 2h^2) = tr(Y^* Y) = \tfrac{1}{2} tr(h^2),$$

hence $L(\gamma)^2 = 2\,tr(h^2) = \frac{1}{2}\|h\|_2^2$ as stated. The assertion in (7) can be deduced from folk results (see [Br93]) because $q = we^a w^*$ and $p = ge^a g^*$ are finite rank projectors acting on a finite dimensional space (see the proof of Lemma 8.6). □

9 Concluding remarks

Remark 9.1. Theorem 8.2 doesn't answer whether is it necessary that the spectrum of a should be finite for the orbit to be a submanifold, when we act with $\mathcal{U}_{\mathcal{H}_\mathbb{C}}$ (see Remark 8.3). The problem can be stated in a very simple form:

- Choose any involutive Banach algebra with identity $\mathcal{B}$, take $a \in \mathcal{B}$ such that $a^* = a$, and denote $\mathcal{U}_\mathcal{B} = \{u \in \mathcal{B} : u^* = u^{-1}\}$, the unitary group of $\mathcal{B}$.
- Name S_a the image of the map $\pi_a : \mathcal{U}_\mathcal{B} \to \mathcal{B}$ which assigns $u \mapsto uau^*$
- Is the condition "a has finite spectrum" necessary for the set $S_a \subset \mathcal{B}$ to be closed?

Remark 9.2. The standard representation of L(H) (acting on the Hilbert-Schmidt operators by left or right product) induces a morphism of the latter operators into the state space of L(H). Geometry of states seems to be at hand.

Remark 9.3. In several recent papers (the latest at the moment we write these lines is [CGM]), R. Cirelli, M. Gatti and A. Manià propose a delinearization program for quantum mechanics based in identifying the pure state space with a convenient homogeneous manifold (the infinite projective space). The manifold Σ_∞ seems to be another convenient setting for a delinearization program.

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