HCIZ INTEGRAL FORMULA AS UNITARITY OF A CANONICAL MAP BETWEEN REPRODUCING KERNEL SPACES

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Abstract. In this article we prove that the Harish-Chandra-Itzykson-Zuber (HCIZ) integral formula is equivalent to the unitarity of a canonical map between invariant subspaces of Segal-Bargmann spaces. As a consequence, we provide two new proofs of the HCIZ integral formula and alternative proofs of related results.

Keywords. HCIZ integral, Segal Bargmann space, reproducing kernel, unitary map, Schur functions, complex Ginibre ensemble.

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1. Introduction

In [HC57] Harish-Chandra proved a formula for orbital integrals, see the expository article [McS21] and the references therein. The importance of such integrals for mathematical physics was first noted by Itzykson and Zuber [IZ80], the unitary integral is now known as the Harish-Chandra-Itzykson-Zuber (HCIZ) integral and has become an important identity in quantum field theory, random matrix theory, and algebraic combinatorics. It is usually written

$$\int_U e^{\mathrm{Tr}(uAu^{-1}B)}du = \left(\prod_{p=1}^{n-1} p!\right) \frac{\det\left[e^{a_i b_j}\right]_{i,j=1}^n}{\Delta(A)\Delta(B)} \tag{1}$$

where det is the determinant of a matrix, U is the group of n-by-n unitary matrices, A and B are fixed n-by-n diagonal matrices with eigenvalues $a_1 < \cdots < a_n$ and $b_1 < \cdots < b_n$ respectively, and

$$\Delta(A) = \prod_{i<j}(a_j - a_i)$$

is the Vandermonde determinant. In this article we link the HCIZ integral formula to the theory of Segal-Bargmann spaces thereby giving two new proofs of the integral formula. We first prove that the HCIZ integral formula is equivalent to the unitarity of a canonical map between invariant subspaces of Segal-Bargmann spaces. We then prove that the unitarity of the canonical map is equivalent to two known results: it is equivalent to a formula for the differentiation of conjugation invariant functions in terms of differentiation on the restriction to the Cartan algebra [HC57], and it is also equivalent to the orthonormality of scaled Schur functions of the spectrum [FR09]. Also, the orthonormailty of the Schur functions yields the character expansion of the HCIZ integral [IZ80] as the diagonalization of a reproducing kernel.

In Segal-Bargmann spaces [S60, B61] holomorphic and reproducing kernel techniques are available. Also, the orthonormal bases and annihilation and creation operators have a simple form, see [Ha00] and Chapter 4 of [N11]. Previous research on invariant subspaces of Segal-Bargmann spaces was done for example in [KL07] where rotationally invariant subspaces where studied. Also, in [Z02, Section 6] a unitarity result for the restriction to the Cartan algebra was given in the context of Dunkl theory.

The paper is organized as follows. In Section 2 we derive the reproducing kernels of invariant subspaces of two Segal-Bargmann spaces. We prove that the canonical map between these two spaces is unitary if and only if the HCIZ integral formula holds in Section 3. Finally, we provide alternative proofs of the HCIZ integral formula and other results in Section 4.

2. Invariant subspaces of Segal-Bargmann spaces

2.1. **Fixed point subspaces of Segal-Bargmann spaces.** In this section we recall results about Segal-Bargmann spaces and show properties of fixed point paces of these spaces. References are Section 2, 3.2 and 6.1 of the lecture notes [Ha00] and Chapter 4 of [N11].

Let V be a complex vector space of dimension $n \in \mathbb{N}$ with hermitian form $\langle \cdot, \cdot \rangle$. The Segal-Bargmann space $\mathcal{F}(V)$ consists of the holomorphic functions on V which are square integrable with respect to the measure $\pi^{-n} e^{-\langle z,z \rangle} dz$ where dz is the volume with respect to the inner product $\mathcal{R}](\langle\cdot,\cdot\rangle)$. This space is L^2-complete and its hermitian form is given by

$$\langle F, G\rangle = \pi^{-n}\int_V \overline{F(z)}G(z)e^{-\langle z,z\rangle}dz$$

for $F, G \in \mathcal{F}(V)$. For $a \in V$ the function

$$K_a(z) = e^{\langle z,a\rangle}$$

is called the coherent state with parameter a and it satisfies the reproducing property

$$F(a) = \langle K_a, F\rangle$$

for all $F \in \mathcal{F}(V)$. The function $K : V \times V \to \mathbb{C}$ given by

$$K(z, a) = K_a(z) = e^{\langle z,a\rangle}$$

is the reproducing kernel for the Segal-Bargmann space. We note that in $\mathcal{F}(V)$ multiplication by a variable is the adjoint of differentiation by the same variable. These are the creation and annihilation operators, see section 6.1 in [Ha00].

Remark 2.1. *A general reproducing kernel space of holomorphic functions is a Hilbert space $\mathcal{H} \subseteq \mathcal{O}(W)$ such that the point evaluations are continuous. Since for a point $a \in W$ the evaluation ev_a is continuous by the Riesz representation theorem there is an element $M_a \in \mathcal{H}$ called the coherent state with parameter a such that $ev_a(F) = F(a) = \langle M_a, F\rangle$ for all $F \in \mathcal{H}$. The reproducing kernel is given by $M(x, y) = \langle M_y, M_x\rangle$.*

Let G be a compact topological group with normalized Haar measure and let $\rho : G \to U(V)$ and $\chi : G \to U(\mathbb{C})$ be unitary representations of G, where $U(W)$ denotes the unitary operators acting on a Hilbert space W. We can define a unitary representation

$$\pi : G \to \mathcal{F}(V)$$

given by

$$(\pi(g)F)(z) = \chi(g)F(\rho(g^{-1})z)$$

for $g \in G$, $F \in \mathcal{F}(V)$ and $z \in V$. This is the the representation obtained from the regular representation defined by rho by tensoring with the unitary character χ. It is unitary since the measure $\pi^{-n}e^{-\langle z,z\rangle}dz$ is invariant under the representation ρ. The fixed point space of this representation is

$$\mathcal{F}(V)^G = \{F \in \mathcal{F}(V) : \pi(g)F = F \text{ for all } g \in G\}.$$

The orthogonal projection P onto this fixed point space is given by

$$P(F) = \int_{g\in G} \pi(g)F dg,$$

where the integral is over the normalized Haar measure on G. The coherent states on $\mathcal{F}(V)^G$ are given by

$$S_a = PK_a \tag{2}$$

since for $\mathcal{F}(V)^G$ and $a \in V$ we have

$$F(a) = \langle K_a, F\rangle = \langle K_a, PF\rangle = \langle PK_a, F\rangle.$$

We denote by $S : V \times V \to \mathbb{C}$ the reproducing kernel on $\mathcal{F}(V)^G$.

Proposition 2.2. *The reproducing kernel on $\mathcal{F}(V)^G$ is given by*

$$S(z,a) = \left(\int_{g \in G} \pi(g) K_a dg \right)(z) = \int_{g \in G} \chi(g) e^{\langle \rho(g^{-1}z, a \rangle} dg$$

for $z, a \in V$.

Proof. Note that for $z, a \in V$

$$S(z,a) = PK_a(z) = \int_{g \in G} \chi(g) e^{\langle \rho(g^{-1}z, a \rangle} dg,$$

where we used the identity (2) for the coherent on the fixed point space. □

2.2. **Segal-Bargmann space of functions on complex matrices.** For $n \in \mathbb{N}$ let $\mathbb{C}^{n\times n} = M_n(\mathbb{C})$ be the space of $n \times n$ complex matrices with hermitian form $\langle x, y \rangle = \mathrm{Tr}(xy^*)$, where Tr is the trace. Let $U \subseteq M_n(\mathbb{C})$ be the group of unitary matrices and $\mathrm{GL}_n(\mathbb{C}) \subseteq M_n(\mathbb{C})$ the group of invertible matrices. We consider the Segal-Bargmann space $\mathcal{F}(\mathbb{C}^{n\times n})$.

Remark 2.3. *The Gaussian measure $\pi^{-n^2} e^{-\mathrm{Tr}(z^* z)} dz$ is the measure of the complex Ginibre ensemble, see* [M04, Chapter 15]. *An ensemble is a set endowed with a probability measure. In this ensemble the expectation of the modulus squared of the sum of eigenvalues is*

$$\mathbb{E}\,|\lambda_1(z) + \cdots + \lambda_n(z)|^2 = \langle \mathrm{Tr}(z), \mathrm{Tr}(z) \rangle = \left(\mathrm{Tr}\left(\frac{\partial}{\partial z} \right) \mathrm{Tr}(z) \right) \Big|_{z=0} = n,$$

where $\mathrm{Tr}(\frac{\partial}{\partial z}) = \frac{\partial}{\partial z_{11}} + \cdots + \frac{\partial}{\partial z_{nn}}$. One can compute the inner product directly knowing that scaled monomials form an orthonormal basis, but we used the fact that multiplication by z_{ij} is the adjoint of $\frac{\partial}{\partial z_{ij}}$. The expectation of the modulus squared of the product of eigenvalues is

$$\mathbb{E}\,|\lambda_1(z) \ldots \lambda_n(z)|^2 = \langle \det(z), \det(z) \rangle = \left(\det\left(\frac{\partial}{\partial z} \right) \det(z) \right) \Big|_{z=0} = n!.$$

Let ρ be the unitary representation of U on $\mathbb{C}^{n\times n}$ given by conjugation and let χ be the trivial character. In this case the subspace of fixed points are

$$\mathcal{F}(\mathbb{C}^{n\times n})^U = \{ F \in \mathcal{F}(\mathbb{C}^{n\times n}) : F(z) = F(u^{-1}zu) \text{ for all } u \in U \}.$$

We apply the "unitarian trick": for fixed $F \in \mathcal{F}(\mathbb{C}^{n\times n})^U$ and fixed $z \in \mathbb{C}^{n\times n}$ the map $\mathbb{C}^{n\times n} \to \mathbb{C}$ given by

$$x \mapsto F(z) - F(e^x z e^{-x})$$

is holomorphic and vanishes for skew-Hermitian x, so it vanishes for all $x \in \mathbb{C}^{n\times n}$. Therefore, the functions in $\mathcal{F}(\mathbb{C}^{n\times n})^U$ are actually the functions invariant under conjugation by all $g \in \mathrm{GL}_n(\mathbb{C})$, so they are functions of the spectrum. The reproducing kernel on $\mathcal{F}(\mathbb{C}^{n\times n})^U$ is given by

$$Q(z,a) = \int_U e^{\mathrm{Tr}(u^{-1}zua^*)} du \tag{3}$$

for $z, a \in \mathbb{C}^{n\times n}$.

2.3. **Segal-Bargmann space of alternating functions.** Let $\mathbb{C}^n$ be endowed with the hermitian form $\langle x, y\rangle = \sum_{i=1}^n x_i \overline{y_i}$ and let S_n be the symmetric group of degree n. Let ρ be the representation of S^n on C^n given by permutation of variables and let χ be alternating character sgn of S^n. Note that $\sigma(z_i) = (z_{\sigma^{-1}(i)})$. The fixed point space is the space $\mathcal{F}(\mathbb{C}^n)^{S_n} = \mathcal{F}(\mathbb{C}^n)_{\text{alt}}$ of alternating holomorphic functions:

$$\mathcal{F}(\mathbb{C}^n)_{\text{alt}} = \{F \in \mathcal{F}(\mathbb{C}^n) : F(z_1, \dots, z_n) = \operatorname{sgn}(\sigma) F(z_{\sigma^{-1}(1)}, \dots, z_{\sigma^{-1}(n)}) \text{ for all } \sigma \in S_n\}.$$

We denote by $R : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ the reproducing kernel on $\mathcal{F}(\mathbb{C}^n)_{\text{alt}}$ and note that it is given by

$$R(z, a) = \frac{1}{n!} \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) e^{\langle \sigma^{-1}(z), a\rangle}. \tag{4}$$

3. HCIZ integral formula as unitarity of a canonical map

In this section we prove the main result of the article. The next proposition gives a formulation of the HCIZ integral formula in terms of inner products in $\mathcal{F}(\mathbb{C}^{n\times n})^U$ and in $\mathcal{F}(\mathbb{C}^n)_{\text{alt}}$. We denote with $D \subseteq \mathbb{C}^{n\times n}$ the set of complex diagonal matrices, with D_{reg} the set of complex diagonal matrices with distinct eigenvalues, and with $D_{\mathbb{R}}$ the set of diagonal matrices with real entries. As before Δ denotes the Vandermonde determinant. We set the constant

$$c = \left(\prod_{p=1}^n p!\right)^{-\frac{1}{2}}.$$

Proposition 3.1. *The HCIZ integral formula is equivalent to*

$$\langle Q_x, Q_y\rangle_{\mathcal{F}(\mathbb{C}^{n\times n})^U} = \left\langle \frac{R_x}{c\Delta(\overline{x})}, \frac{R_y}{c\Delta(\overline{y})}\right\rangle_{\mathcal{F}(\mathbb{C}^n)_{\text{alt}}} \tag{5}$$

for all $x, y \in D_{\text{reg}}$.

Proof. The left hand side is

$$\langle Q_x, Q_y\rangle = Q_y(x) = \int_U e^{\operatorname{Tr}(u^{-1}xuy^*)} du,$$

where we used the reproducing property and the formula for the kernel given in (3). The right hand side is

$$\begin{aligned}\left\langle \frac{R_x}{c\Delta(\overline{x})}, \frac{R_y}{c\Delta(\overline{y})}\right\rangle &= \frac{1}{c^2 \Delta(x)\Delta(\overline{y})} R_y(x) \\ &= \left(\prod_{p=1}^n p!\right) \frac{1}{\Delta(x)\Delta(\overline{y})} R_y(x) \\ &= \left(\prod_{p=1}^n p!\right) \frac{1}{\Delta(x)\Delta(\overline{y})} \frac{1}{n!} \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) e^{\langle \sigma^{-1}(x), y\rangle},\end{aligned}$$

where in the third equality we used the formula for the reproducing kernel given in (4). The HCIZ formula (1) states that the left hand side and the right hand side agree when $x, y \in D_{\text{reg}} \cap D_{\mathbb{R}}$. Since both sides are holomorphic in x and anti-holomorphic in y we get equality for all $x, y \in D_{\text{reg}}$. The converse is straightforward. □

Theorem 3.2. *The HCIZ integral formula implies that the map* $\psi : \mathcal{F}(\mathbb{C}^{n\times n})^U \to \mathcal{F}(\mathbb{C}^n)_{\mathrm{alt}}$ *given by*

$$\psi(F)(x) = c\Delta(x)F|_D(x)$$

for $F \in \mathcal{F}(\mathbb{C}^{n\times n})^U$ *and* $x \in D$ *is well defined and unitary. Conversely, if this map is well defined and unitary, then the HCIZ integral formula holds. The map* ψ *satisfies*

$$\psi(Q_x) = \frac{R_x}{c\Delta(\overline{x})}$$

for any $x \in D_{\mathrm{reg}}$.

Proof. Define the map

$$\phi : \operatorname{span}\{Q_x\}_{x\in D_{\mathrm{reg}}} \to \operatorname{span}\left\{\frac{R_x}{c\Delta(\overline{x})}\right\}_{x\in D_{\mathrm{reg}}}$$

by

$$\sum_{i=1}^{m} \alpha_i Q_{x_i} \mapsto \sum_{i=1}^{m} \alpha_i \frac{R_{x_i}}{c\Delta(\overline{x_i})},$$

where $\alpha_i \in \mathbb{C}$. Equation (5) in Proposition 3.1 implies that the map is well defined and an isometry, therefore it extends to an isometry between

$$\overline{\operatorname{span}}\{Q_x\}_{x\in D_{\mathrm{reg}}} \quad \text{and} \quad \overline{\operatorname{span}}\left\{\frac{R_x}{c\Delta(\overline{x})}\right\}_{x\in D_{\mathrm{reg}}},$$

where $\overline{\operatorname{span}}(A)$ denotes the closure of $\operatorname{span}(A)$.

We now prove that $\overline{\operatorname{span}}\{Q_x\}_{x\in D_{\mathrm{reg}}} = \mathcal{F}(\mathbb{C}^{n\times n})^U$. If this does not hold take an $F \in \mathcal{F}(\mathbb{C}^{n\times n})^U$ which is orthogonal to $\operatorname{span}\{Q_x\}_{x\in D_{\mathrm{reg}}}$. This means that

$$F(x) = \langle Q_x, F\rangle = 0$$

for all $x \in D_{\mathrm{reg}}$. Since F is invariant by conjugation of the variables it vanishes on all $g \in \mathrm{GL}_n(\mathbb{C})$ with n distinct eigenvalues. Since these matrices are dense in $\mathbb{C}^{n\times n}$ we conclude that $F = 0$. The fact that

$$\overline{\operatorname{span}}\left\{\frac{R_x}{c\Delta(\overline{x})}\right\}_{x\in D_{\mathrm{reg}}} = \mathcal{F}(\mathbb{C}^n)_{\mathrm{alt}}$$

is proved similarly. Therefore ϕ defines a unitary map from $\mathcal{F}(\mathbb{C}^{n\times n})^U$ onto $\mathcal{F}(\mathbb{C}^n)_{\mathrm{alt}}$. For $F \in \mathcal{F}(\mathbb{C}^{n\times n})^U$ and $x \in D_{\mathrm{reg}}$ we get

$$\begin{aligned} F(x) =& \langle Q_x, F\rangle = \langle \phi(Q_x), \phi(F)\rangle = \left\langle \frac{R_x}{c\Delta(\overline{x})}, \phi(F)\right\rangle \\ &= \frac{1}{c\Delta(x)}\langle R_x, \phi(F)\rangle = \frac{1}{c\Delta(x)}\phi(F)(x), \end{aligned}$$

where the first and last equalities follow from the reproducing property, the second from the unitarity of ϕ, and the third from the definition of ϕ. Therefore

$$\phi(F)(x) = c\Delta(x)F(x)$$

for $x \in D_{\mathrm{reg}}$. Hence, the map ϕ is equal to the map ψ and the first assertion of the theorem is proved.

To prove the second assertion assume that ψ is well defined and unitary. Then for $F \in \mathcal{F}(\mathbb{C}^{n\times n})^U$ and $x \in D_{\mathrm{reg}}$ we have

$$F(x) = \langle Q_x, F\rangle = \langle \psi(Q_x), \psi(F)\rangle = \langle \psi(Q_x), c\Delta F|_D\rangle.$$

Also

$$\langle R_x, \psi(F)\rangle = \langle R_x, c\Delta F|_D\rangle = c\Delta(x)F(x),$$

hence

$$F(x) = \left\langle \frac{R_x}{c\Delta(\overline{x})}, \psi(F)\right\rangle.$$

Therefore for fixed $x \in D_{\text{reg}}$

$$\langle \psi(Q_x), \psi(F)\rangle = \left\langle \frac{R_x}{c\Delta(\overline{x})}, \psi(F)\right\rangle$$

for all $F \in \mathcal{F}(\mathbb{C}^{n\times n})^U$, and since ψ is unitary this implies that

$$\psi(Q_x) = \frac{R_x}{c\Delta(\overline{x})}.$$

From this property and Proposition 3.1 the HCIZ formula follows. □

Remark 3.3. *Note that Theorem 3.2 implies that the map* $\mathbb{C}\left[\mathbb{C}^{n\times n}\right]^U \to \mathbb{C}\left[\mathbb{C}^n\right]_{\text{alt}}$ *given by* $F \mapsto \Delta.F|_D$ *is a linear isomorphism, where* $\mathbb{C}\left[V\right]$ *denotes the polynomial functions of a complex vector space* V*. Also note that multiplication by* Δ *defines a linear isomorphism* $M_\Delta : \mathbb{C}\left[\mathbb{C}^n\right]_{\text{sym}} \to \mathbb{C}\left[\mathbb{C}^n\right]_{\text{alt}}$ *from the symmetric to the alternating polynomials. Therefore the restriction map*

$$\text{res} : \mathbb{C}\left[\mathbb{C}^{n\times n}\right]^U \to \mathbb{C}\left[\mathbb{C}^n\right]_{\text{sym}} \textit{ given by } \text{res}(F) = F|_D$$

is a linear isomorphism. This is a special case of the Chevalley restriction theorem, see [C55].

Remark 3.4. *In the last remark we used the linear isomorphism* $M_\Delta : \mathbb{C}\left[\mathbb{C}^n\right]_{\text{sym}} \to \mathbb{C}\left[\mathbb{C}^n\right]_{\text{alt}}$*. There are more general ways to formulate this kind of spaces and maps as follows. Assume the context of the first part of Section 2. Let* $\rho : G \to \mathrm{U}(V)$ *be a unitary representation and let* $\chi : G \to \mathrm{U}(\mathbb{C})$ *be a unitary character. We define*

$$\mathbb{C}\left[V\right]^\chi = \{F \in \mathbb{C}\left[V\right] : F(\rho(g)z) = \chi(g)F(z) \textit{ for all } g \in G\}.$$

If χ_1, χ_2*, and* χ *are unitary characters and* ϕ_χ *is a non-zero polynomial function in* $\mathbb{C}\left[V\right]^\chi$*, the multiplication by* ϕ_χ *defines a linear injection*

$$M_{\phi_\chi} : \mathbb{C}\left[V\right]^{\chi_1} \to \mathbb{C}\left[V\right]^{\chi\cdot\chi_2}.$$

If $\mathbb{C}\left[V\right]^{\chi_1}$ *and* $\mathbb{C}\left[V\right]^{\chi_2}$ *are endowed by the* L^2 *norms defined by the measures*

$$|\phi_\chi(z)|^2\pi^{-n}e^{-\langle z,z\rangle}dz \textit{ and } \pi^{-n}e^{-\langle z,z\rangle}dz$$

then M_{ϕ_χ} *is an isometry which can be extended to the completions.*

4. Differentiation of polynomials and orthonormal bases

In this section we give alternative proofs of the HCIZ integral and other results. For a polynomial F we define $F^*(z) = \overline{F(\overline{z})}$, that is, the coefficients of F^* are the complex conjugates of the coefficients of F. An alternative proof of the HCIZ integral formula is given by the following

Proposition 4.1. *The unitarity of the map* ψ *in Theorem 3.2 implies that for polynomials* $F, G \in \mathcal{F}(\mathbb{C}^{n\times n})^U$ *the equation*

$$\Delta(x).F\left(\frac{\partial}{\partial z}\right)G(x) = \left(F\Big|_D\left(\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n}\right)(\Delta.G|_D)\right)(x) \tag{6}$$

holds for all $x \in D$. Conversely, if formula (6) holds for all polynomials $F, G \in \mathcal{F}(\mathbb{C}^{n\times n})^U$, then the map ψ is unitary.

Proof. We denote by $M_F G = F.G$ the multiplication operator. For a polynomial $F \in \mathcal{F}(\mathbb{C}^{n\times n})^U$ it is easy to check that

$$M_F = \psi^{-1} \circ M_{F|_D} \circ \psi.$$

Therefore, by applying adjoints

$$F^*\left(\frac{\partial}{\partial z}\right) = (M_F)^* = \psi^{-1} \circ (M_{F|_D})^* \circ \psi = \psi^{-1} \circ F^*\Big|_D \left(\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n}\right) \circ \psi.$$

Here we used the unitarity of ψ and the fact that multiplication by a variable is the adjoint of differentiation by the same variable. We evaluate

$$\left(F^*\Big|_D \left(\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n}\right) \circ \psi\right) G = cF^*\Big|_D \left(\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n}\right) (\Delta.G|_D).$$

Applying ψ^{-1} and evaluating at $x \in D_{\mathrm{reg}}$ is the same as multiplying by

$$\frac{1}{c\Delta(x)},$$

so we get the formula for F^*. Since $F = (F^*)^*$ the first claim follows.

We denote for simplicity $\partial = \frac{\partial}{\partial z}$. To prove the second claim we need to verify that for all polynomials $F, G \in \mathcal{F}(\mathbb{C}^{n\times n})^U$ the equality

$$\langle F^*, G\rangle = \langle \psi(F^*), \psi(G)\rangle = \langle c\Delta F^*|_D, c\Delta G|_D\rangle$$

holds. By the definition of the inner product in terms of differentiation at $z = 0$ we have

$$\langle F^*, G\rangle = F(\partial)G|_{z=0}$$

and

$$\langle c\Delta F^*|_D, c\Delta G|_D\rangle = c^2\Delta(\partial)F|_D(\partial)(\Delta G|_D)|_{z=0}.$$

Therefore, we need to verify that

$$F(\partial)G|_{z=0} = c^2\Delta(\partial)F|_D(\partial)(\Delta G|_D)|_{z=0},$$

which by formula (6) is equivalent to

$$\frac{1}{\Delta}F|_D(\partial)(\Delta G|_D)\Big|_{z=0} = c^2\Delta(\partial)F|_D(\partial)(\Delta G|_D)|_{z=0}.$$

We set

$$F|_D(\partial)(\Delta G|_D) = \Delta.(d + H),$$

where $d \in \mathbb{C}$ and H is a symmetric polynomial on $\mathbb{C}^n$ without constant term.

Hence, we have to show that

$$\frac{1}{\Delta}\Delta.(d + H)\Big|_{z=0} = c^2\Delta(\partial)(\Delta.(d + H))|_{z=0}.$$

The left hand side is d and the right hand side is

$$c^2 d\Delta(\partial)\Delta|_{z=0} + c^2\Delta(\partial)(\Delta.H)|_{z=0}.$$

We have

$$\Delta(\partial)\Delta|_{z=0} = \left(\prod_{p=1}^{n} p!\right) = \frac{1}{c^2}.$$

Since Δ is homogeneous of degree $n(n-1)/2$ and $\Delta.H$ is a sum of monomials of higher degree we get

$$\Delta(\partial)(\Delta.H)|_{z=0} = 0.$$

Therefore the right hand side is also d. □

Formula (6) was obtained by Harish-Chandra in [HC57] for semi-simple Lie groups. His proof of (1) is based on this result, see [HC57] and Theorem 3.8 in [McS21].

Remark 4.2. *If $F \in \mathcal{F}(\mathbb{C}^{n\times n})^U$ is a polynomial then*

$$F\left(\frac{\partial}{\partial z}\right) Q_a = F(\overline{a}) Q_a$$

for any $a \in \mathbb{C}^{n\times n}$. This follows from

$$\left\langle F\left(\frac{\partial}{\partial z}\right) Q_a, G\right\rangle = \langle Q_a, F^*.G\rangle = F^*(a)G(a) = F^*(a)\langle Q_a, G\rangle = \langle \overline{F^*(a)} Q_a, G\rangle$$

for any polynomial $G \in \mathcal{F}(\mathbb{C}^{n\times n})^U$. By the same argument

$$F\left(\frac{\partial}{\partial z}\right) R_a = F(\overline{a}) R_a$$

for any $a \in \mathbb{C}^n$ and any symmetric polynomial F on $\mathbb{C}^n$.

Let $\mathbb{N}_0$ stand for the non negative integers. For $\mu \in \mathbb{N}_0^n$ we denote the monomials as usual with $z^\mu = z_1^{\mu_1} \dots z_n^{\mu_n}$ and we use the notation $\mu! = \mu_1! \dots \mu_n!$. In the Segal-Bargmann space $\mathcal{F}(\mathbb{C}^n)$ an orthonormal basis of the space is

$$\left(\frac{1}{\sqrt{\mu!}} z^\mu\right)_{\mu \in \mathbb{N}_0^n},$$

see [Ha00, Section 3.2]. The set Π of partitions is defined as

$$\Pi = \{\lambda \in \mathbb{N}_0^n : \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n\}.$$

We set $\delta = (n-1, n-2, \dots, 0)$ and $\mathbf{1} = (1, 1, \dots, 1)$. For $\lambda \in \Pi$, if $(x_1, \dots, x_n)$ are the eigenvalues of $x \in \mathrm{GL}_n(\mathbb{C})$ we define

$$\chi_\lambda(x) = s_\lambda(x_1, \dots, x_n),$$

where s_λ is a Schur polynomial. These polynomials are defined by

$$s_\lambda(x_1, \dots, x_n) = \frac{a_{\lambda+\delta}(x_1, \dots, x_n)}{a_\delta(x_1, \dots, x_n)},$$

where

$$a_\mu(x_1, \dots, x_n) = \det\left[x_i^{\mu_j}\right]_{i,j=1}^n.$$

Note that $a_\delta(x_1, \dots, x_n) = \Delta(x_1, \dots, x_n)$ is the Vandermonde determinant.

Remark 4.3. *The irreducible polynomial representations of the general linear group $\mathrm{GL}_n(\mathbb{C})$ are labelled by Young diagrams, which we think of as vectors $\lambda \in \Pi$. The character of the λ-representation is given by χ_λ, the Schur polynomial evaluated at the eigenvalues of an invertible matrix.*

For $\lambda \in \Pi$ we denote

$$d_\lambda(z) = \frac{1}{\sqrt{n!(\lambda+\delta)!}} a_{\lambda+\delta}(z) \in \mathcal{F}(\mathbb{C}^n)_{\mathrm{alt}}.$$

Proposition 4.4. *An orthonormal basis of the space $\mathcal{F}(\mathbb{C}^n)_{\mathrm{alt}}$ is given by*

$$(d_\lambda)_{\lambda\in\Pi}\,.$$

Proof. It is straightforward to check that this set is an orthonormal set. To check that it is a basis assume that $F \in \mathcal{F}(\mathbb{C}^n)_{\mathrm{alt}}$ and that $\langle F, a_{\lambda+\delta}\rangle = 0$ for all $\lambda \in \Pi$. Take a $\mu \in \mathbb{N}_0^n$ and assume that all the μ_i are different. Then $z^\mu = \sigma(z^{\lambda+\delta})$ for a $\sigma \in S_n$ and a $\lambda \in \Pi$. We consider as in Section 2 the orthogonal projection P onto $\mathcal{F}(\mathbb{C}^n)_{\mathrm{alt}}$. Then

$$\langle F, z^\mu\rangle = \langle PF, z^\mu\rangle = \langle F, Pz^\mu\rangle = \left\langle F, \mathrm{sgn}(\sigma)\frac{1}{n!}a_{\lambda+\delta}\right\rangle = 0.$$

If there are $i \neq j$ such that $\mu_i = \mu_j$ take as σ the transposition of i and j. Then

$$\langle F, z^\mu\rangle = \langle \sigma(F), z^\mu\rangle = \langle F, \sigma(z^\mu)\rangle = -\langle F, z^\mu\rangle,$$

so that $\langle F, z^\mu\rangle = 0$. We proved that the inner product of F with all the elements of the orthonormal basis of $\mathcal{F}(\mathbb{C}^n)$ vanish, so $F = 0$. □

For $\lambda \in \Pi$ we denote

$$e_\lambda(z) = \sqrt{\frac{\delta!}{(\lambda+\delta)!}}\chi_\lambda(z) \in \mathcal{F}(\mathbb{C}^{n\times n})^U.$$

Another proof of the HCIZ integral formula is given by

Proposition 4.5. *The unitarity of the map ψ in Theorem 3.2 implies that $(e_\lambda)_{\lambda\in\Pi}$ is an orthonormal basis of $\mathcal{F}(\mathbb{C}^{n\times n})^U$. Conversely, the fact that $(e_\lambda)_{\lambda\in\Pi}$ is an orthonormal basis implies the unitarity of ψ.*

Proof. We check that for $\lambda \in \Pi$

$$\psi(e_\lambda) = ca_\delta e_\lambda|_D = d_\lambda.$$

The proposition follows. □

The fact that $(e_\lambda)_{\lambda\in\Pi}$ is orthonormal was proved in [FR09, Proposition 2]. Since $\mathcal{F}(\mathbb{C}^{n\times n})^U$ and $\mathcal{F}(\mathbb{C}^n)_{\mathrm{alt}}$ are reproducing kernel spaces their kernels can be diagonalized in terms of orthonormal bases. Proposition 1.6 in [Hi08] yields the character expansion of the HCIZ integral formula. This expansion appeared in formula (3.20) in [IZ80] and it was derived using the Frobenius identity and the Schur orthogonality relations.

Proposition 4.6. *The kernel of $\mathcal{F}(\mathbb{C}^n)_{\mathrm{alt}}$ can be written as*

$$R(x,y) = \sum_{\lambda\in\Pi} d_\lambda(x)\overline{d_\lambda(y)},$$

where the sum converges absolutely and uniformly on compact subsets of $\mathbb{C}^n \times \mathbb{C}^n$. The kernel of $\mathcal{F}(\mathbb{C}^{n\times n})^U$ can be written as

$$Q(x,y) = \sum_{\lambda\in\Pi} e_\lambda(x)\overline{e_\lambda(y)},$$

that is

$$\int_U e^{\mathrm{Tr}(u^{-1}xuy^*)}du = \sum_{\lambda\in\Pi}\frac{\delta!}{(\lambda+\delta)!}\chi_\lambda(x)\overline{\chi_\lambda(y)},$$

where the sum converges absolutely and uniformly on compact subsets of $\mathbb{C}^{n\times n}\times\mathbb{C}^{n\times n}$.

A partial converse to this proposition is

Proposition 4.7. *If the expansion*

$$Q(x,y) = \sum_{\lambda\in\Pi} e_\lambda(x)\overline{e_\lambda(y)}$$

holds, then $(e_\lambda)_{\lambda\in\Pi}$ *is an orthonormal basis of* $\mathcal{F}(\mathbb{C}^{n\times n})^U$.

Proof. The e_λ with $|\lambda| = \lambda_1 + \cdots + \lambda_n = m \in \mathbb{N}_0$ are polynomials of degree m and are orthogonal to $e_{\lambda'}$ with $|\lambda'| = m' \neq m$. Therefore, for λ' such that $|\lambda'| = m \in \mathbb{N}_0$ we have

$$e_{\lambda'}(a) = \langle Q(z,a), e_{\lambda'}(z)\rangle = e_{\lambda'}(a)\langle e_{\lambda'}, e_{\lambda'}\rangle + \sum_{\lambda:|\lambda|=m,\lambda\neq\lambda'} e_\lambda(a)\langle e_\lambda, e_{\lambda'}\rangle.$$

For a λ'' with $|\lambda''| = m$ such that $\lambda'' \neq \lambda'$ choose an $a \in D$ such that $e_{\lambda''}(a) \neq 0$ and $e_\lambda(a) = 0$ for all other λ with $|\lambda| = m$. We conclude that

$$e_{\lambda''}(a) = e_{\lambda''}(a)\langle e_{\lambda''}, e_{\lambda'}\rangle = 0,$$

so the orthonormality of $(e_\lambda)_{\lambda\in\Pi}$ follows. To prove that this set is a basis let $F \in \mathcal{F}(\mathbb{C}^{n\times n})^U$ be written as a sum $F = \sum_{m\in\mathbb{N}_0} P_m$ of polynomials P_m of degree m. If $\langle e_\lambda, F\rangle = 0$ for all $\lambda \in \Pi$, then for $m \in \mathbb{N}_0$ we have $\langle e_\lambda, P_m\rangle = 0$ for all λ with $|\lambda| = m$. Therefore, for $a \in D$ we have

$$P_m(a) = \langle Q_a, P_m\rangle = \left\langle \sum_{\lambda:|\lambda|=m} \overline{e_\lambda(a)} e_\lambda(z), P_m(z)\right\rangle = 0,$$

so all the P_m vanish and $F = 0$. □

The next proposition computes the coefficients of the expansion of an invariant holomorphic function in terms of the characters χ_λ.

Proposition 4.8. *For an* $F \in \mathcal{F}(\mathbb{C}^{n\times n})^U$ *written as*

$$F = \sum_{\lambda\in\Pi} f_\lambda \chi_\lambda$$

the coefficients f_λ *are given by*

$$f_\lambda = \Big(\text{ coefficient of } z^{\lambda+\delta} \text{ in } \Delta.F|_D\Big)$$

for $\lambda \in \Pi$.

Proof. We have the expansion of F in terms of the orthonormal basis

$$F = \sum_{\lambda\in\Pi} \langle e_\lambda, F\rangle e_\lambda.$$

By the unitarity of ψ the Fourier coefficients are

$$\langle e_\lambda, F\rangle = \langle \psi(e_\lambda), \psi(F)\rangle = \langle d_\lambda, c\Delta.F|_D\rangle$$

for $\lambda \in \Pi$. Note that

$$d_\lambda(z) = \frac{1}{\sqrt{n!(\lambda+\delta)!}} a_{\lambda+\delta}(z) = \frac{1}{\sqrt{n!(\lambda+\delta)!}} n! P(z^{\lambda+\delta}) = \sqrt{\frac{n!}{(\lambda+\delta)!}} P(z^{\lambda+\delta}),$$

where P is the orthogonal projection to the alternating functions as in Section 2. Therefore

$$\langle d_\lambda, c\Delta.F|_D\rangle = \sqrt{\frac{n!}{(\lambda+\delta)!}} \left\langle P(z^{\lambda+\delta}), c\Delta.F|_D\right\rangle = \sqrt{\frac{n!}{(\lambda+\delta)!}} \left\langle z^{\lambda+\delta}, c\Delta.F|_D\right\rangle.$$

Also $\langle z^{\lambda+\delta}, c\Delta.F|_D\rangle = (\lambda+\delta)!(\text{ coefficient of } z^{\lambda+\delta} \text{ in } c\Delta.F|_D)$. Some calculations yield

$$f_\lambda = \sqrt{\prod_{p=1}^{n} p!}\left(\text{ coefficient of } z^{\lambda+\delta} \text{ in } c\Delta.F|_D\right),$$

and since $\sqrt{\prod_{p=1}^{n} p!} = \frac{1}{c}$ the proposition follows. □

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References

[B61] V. Bargmann, *On a Hilbert space of analytic functions and an associated integral transform.* Comm. Pure Appl. Math. 14 (1961), 187–214.

[C55] C. Chevalley, *Invariants of finite groups generated by reflections.* Amer. J. Math. 77 (1955), 778–782.

[FR09] P. J. Forrester, E. M. Rains, *Matrix averages relating to Ginibre ensembles.* J. Phys. A 42 (2009), no. 38, 385205, 13 pp. MR2540392

[Ha00] B. C. Hall, *Holomorphic methods in analysis and mathematical physics.* First Summer School in Analysis and Mathematical Physics (Cuernavaca Morelos, 1998), 1–59, Contemp. Math., 260, Aportaciones Mat., Amer. Math. Soc., Providence, RI, 2000.

[Hi08] J. Hilgert, *Reproducing kernels in representation theory.* Symmetries in complex analysis, 1–98, Contemp. Math., 468, Amer. Math. Soc., Providence, RI, 2008.

[HC57] Harish-Chandra. *Differential operators on a semisimple Lie algebra.* American Journal of Mathematics, 79:87-120, 1957.

[IZ80] C. Itzykson, J.-B. Zuber, *The planar approximation. II.* Journal of Mathematical Physics, 21:411-421, 1980.

[KL07] A. Kaewthep, W. Lewkeeratiyutkul, *A pointwise bound for rotation-invariant holomorphic functions that are square integrable with respect to a Gaussian measure.* Taiwanese J. Math. 11 (2007), no. 5, 1443–1455.

[McS21] C. McSwiggen, *The Harish-Chandra integral: an introduction with examples.* Enseign. Math. 67 (2021), no. 3-4, 229–299.

[M04] Mehta, Madan Lal. Random matrices. Third edition. Pure and Applied Mathematics (Amsterdam), 142. Elsevier/Academic Press, Amsterdam, 2004.

[N11] Y. A. Neretin, *Lectures on Gaussian integral operators and classical groups.* EMS Series of Lectures in Mathematics. European Mathematical Society (EMS), Zürich, 2011.

[S60] I. E. Segal, *Mathematical problems of relativistic physics. With an appendix by George W. Mackey.* Proceedings of the Summer Seminar, Boulder, Colorado, 1960, Vol. II. Lectures in Applied Mathematics. American Mathematical Society, Providence, RI, 1963.

[Z02] G. Zhang, *Branching coefficients of holomorphic representations and Segal-Bargmann transform.* J. Funct. Anal. 195 (2002), no. 2, 306–349.