Restricted orbits of closed range operators

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Abstract. We survey results on orbits of closed range operators defined by the action of Banach-Lie groups associated to symmetrically-normed ideals. Then, we adapt these results to describe the geometric structure of the set of all frames for subspaces of a Hilbert space that are equivalent by means of invertible operators associated to a symmetrically-normed ideal.

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1. Introduction

Let $\mathcal{H}$ be a separable complex infinite-dimensional Hilbert space, $\mathcal{B}(\mathcal{H})$ be the algebra of bounded linear operators on $\mathcal{H}$ and $\mathcal{CR} \subset \mathcal{B}(\mathcal{H})$ the set of closed range operators. Let $\mathfrak{S}$ be a proper symmetrically-normed ideal of $\mathcal{B}(\mathcal{H})$ equipped with a norm $\| \cdot \|_{\mathfrak{S}}$ (e.g. p-Schatten ideals $\mathfrak{S}_p$, for $1 \leq p \leq \infty$). We denote by $\mathcal{G}\ell(\mathcal{H})$ and $\mathcal{U}(\mathcal{H})$ the groups of invertible and unitary operators acting on $\mathcal{H}$. We consider the *restricted invertible group associated to* $\mathfrak{S}$ defined by

$$\mathcal{G}\ell_{\mathfrak{S}} := \{ G \in \mathcal{G}\ell(\mathcal{H}) : G - I \in \mathfrak{S} \} .$$

Similarly, we consider the *restricted unitary group associated to* $\mathfrak{S}$ defined by

$$\mathcal{U}_{\mathfrak{S}} := \{ U \in \mathcal{U}(\mathcal{H}) : U - I \in \mathfrak{S} \} .$$

It is well known that $\mathcal{G}\ell_{\mathfrak{S}}$ and $\mathcal{U}_{\mathfrak{S}}$ are real Banach-Lie group with the metric $d_{\mathfrak{S}}(X,Y) = \|X - Y\|_{\mathfrak{S}}$, for $X, Y \in \mathcal{G}\ell_{\mathfrak{S}}$ or $X, Y \in \mathcal{U}_{\mathfrak{S}}$, respectively. The product group $\mathcal{G}\ell_{\mathfrak{S}} \times \mathcal{G}\ell_{\mathfrak{S}}$ acts on $\mathcal{CR}$ as follows

$$L : \mathcal{G}\ell_{\mathfrak{S}} \times \mathcal{G}\ell_{\mathfrak{S}} \times \mathcal{CR} \to \mathcal{CR}, \quad L(G, K, B) = GBK^{-1}.$$

Thus, we have the following orbits

$$\mathcal{O}_{\mathfrak{S}}(B) = \{ GBK^{-1} : G, K \in \mathcal{G}\ell_{\mathfrak{S}} \}. \tag{1.1}$$

This action was previously considered in [17] for the full invertible group instead of restricted groups. The study of the orbits induced by restricted groups as in (1.1) naturally shows up in relation to continuity and differentiability properties of the Moore-Penrose and the polar decomposition of perturbations by operator ideals ([16]).

Another action one might consider is obtained by only moving the range of the operators

$$L : \mathcal{G}\ell_{\mathfrak{S}} \times \mathcal{CR} \to \mathcal{CR}, \quad L(G, B) = GB.$$

The corresponding orbits are given by

$$\mathcal{LO}_{\mathfrak{S}}(B) = \{GB : G \in \mathcal{G}\ell_{\mathfrak{S}}\}. \tag{1.2}$$

Spatial characterizations of the orbits (1.1) and (1.2) were given in [15] for a more general class of operator ideals than symmetrically-normed ideals. These orbits are indeed examples of homogeneous space related to operator ideals; we refer to [2, 5, 7, 9, 23–26, 28] for several interesting examples of this type of infinite-dimensional homogeneous spaces.

It is worth mentioning that the definition of the orbits in (1.2) is related to frame theory. For $\mathcal{S} \subset \mathcal{H}$ a closed subspace of $\mathcal{H}$, recall that a sequence $\mathcal{F} = \{f_n\}_{n\geq 1} \subset \mathcal{S}$ is a *frame for* $\mathcal{S}$ if there exist constants $0 < \alpha \leq \beta$ such that

$$\alpha\|f\|^2 \leq \sum_{n\geq 1} |\langle f\,,\, f_n\rangle|^2 \leq \beta\|f\|^2, \tag{1.3}$$

for all $f \in \mathcal{S}$. In this case, there always exist (possibly many) frames $\mathcal{F}^\# = \{f_n^\#\}_{n\geq 1}$ for $\mathcal{S}$ that are in duality with $\mathcal{F}$, i.e. such that

$$f = \sum_{n\geq 1} \langle f\,,\, f_n\rangle\, f_n^\# = \sum_{n\geq 1} \langle f\,,\, f_n^\#\rangle\, f_n$$

for all $f \in \mathcal{S}$. The previous facts can be interpreted by saying that frames are stable and generally redundant spanning sequences of a given subspace, which are the key properties that make them useful for many applications (see [10–12, 19]). Given a frame $\mathcal{F}$ for the closed subspace $\mathcal{S}$ we consider its synthesis operator, denoted by $T_{\mathcal{F}} : \ell^2(\mathbb{N}) \to \mathcal{H}$, which is uniquely determined by

$$T_{\mathcal{F}}(e_n) = f_n\,, \quad n \geq 1,$$

where $\{e_n\}_{n\geq 1}$ is the canonical orthonormal basis of $\ell^2(\mathbb{N})$. Along this work, for simplicity of notation we identify $\mathcal{H} \simeq \ell^2(\mathbb{N})$, and consider $\{e_n\}_{n\geq 1}$ as a fixed orthonormal basis of $\mathcal{H}$. It is clear that we can identify $\mathcal{F}$ with its synthesis operator $T_{\mathcal{F}}$. It turns out that $T_{\mathcal{F}}$ has closed range, and therefore we can study frames for closed subspaces of $\mathcal{H}$ by studying closed range operators $\mathcal{CR}$. This point of view has already been exploited in [18] for the study of frames for $\mathcal{H}$.

Now consider two frames $\mathcal{F} = \{f_n\}_{n\geq 1}$ and $\mathcal{G} = \{g_n\}_{n\geq 1}$ for subspaces $\mathcal{S}$ and $\mathcal{T}$, respectively, and $\mathfrak{S}$ an operator ideal (not necessarily symmetrically-normed). The frames $\mathcal{F}$ and $\mathcal{G}$ are said to be $\mathfrak{S}$-*equivalent* (or $\mathfrak{S}$-*unitarily equivalent*) if there exists an $G \in \mathcal{G}\ell_{\mathfrak{S}}$ (resp. $U \in \mathcal{U}_{\mathfrak{S}}$) such that $Gf_n = g_n$

(resp. $Uf_n = g_n$), for all $n \geq 1$. This definition was introduced and linked to the orbits (1.2) in [15], where we showed that the orbit $\mathcal{LO}_{\mathfrak{S}}(T_{\mathcal{F}})$ describes the set of all synthesis operators of frames for subspaces that are $\mathfrak{S}$-equivalent to $\mathcal{F}$. We point out that the previous definition of $\mathfrak{S}$-equivalence extends the usual equivalence of frames for the full underlying Hilbert space in [4], the weakly equivalence of frames of subspaces defined in [21], and the $\mathfrak{S}$-equivalence between bases introduced in [8].

This paper is organized as follows. In Section 2 we briefly describe some geometric results on the orbits of closed range operators of the form (1.1). In Section 3 we give the corresponding geometric results for orbits as in (1.2), which by the above remarks are connected with the classes of $\mathfrak{S}$-equivalent frames.

2. Orbits of closed range operators

Let $\mathcal{H}$ be an infinite-dimensional (complex separable) Hilbert space, and let $\mathcal{B}(\mathcal{H})$ be the algebra of bounded operators on $\mathcal{H}$. For an operator $A \in \mathcal{B}(\mathcal{H})$ we write $N(A)$ and $R(A)$ for the nullspace and range of A, respectively. The orthogonal projection onto a closed subspace $\mathcal{S}$ is denoted by $P_{\mathcal{S}}$. The set of all closed range operators is given by

$$\mathcal{CR} = \{A \in \mathcal{B}(\mathcal{H}) \, : \, R(A) \text{ is a closed subspace}\}.$$

An operator $B \in \mathcal{B}(\mathcal{H})$ is said to be the *Moore-Penrose inverse* of $A \in \mathcal{B}(\mathcal{H})$ if it satisfies that $ABA = A$, $BAB = B$, $(AB)^* = AB$ and $(BA)^* = BA$. If the Moore-Penrose exists, then it is uniquely determined, and we denote it by $B = A^\dagger$. It is not difficult to check that $A \in \mathcal{B}(\mathcal{H})$ admits a Moore-Penrose inverse if and only if $A \in \mathcal{CR}$. Furthermore, $AA^\dagger = P_{R(A)}$ and $A^\dagger A = P_{N(A)^\perp}$, whenever $A \in \mathcal{CR}$.

As we mentioned above, there is an action of the full group of invertible operators on $\mathcal{CR}$ defined by

$$L : \mathcal{G\ell}(\mathcal{H}) \times \mathcal{G\ell}(\mathcal{H}) \times \mathcal{CR} \to \mathcal{CR}, \;\; L(G, K, B) = GBK^{-1}.$$

This action was studied in [17], with emphasis in metric and topological properties of the orbits

$$\mathcal{O}(A) = \{GAK^{-1} : G, K \in \mathcal{G\ell}(\mathcal{H})\}.$$

The following metrics were introduced. For $A, B \in \mathcal{CR}$, set $d_N(A, B) = \|A - B\| + \|P_{N(A)} - P_{N(B)}\|$, and $d_R(A, B) = \|A - B\| + \|P_{R(A)} - P_{R(B)}\|$, where here $\| \cdot \|$ denotes the usual operator norm. The notation d_X will be used to refer to any of the metrics d_N or d_R. We write $(\mathcal{O}(A), d_X)$ for the orbit considered with the topology given by d_X.

Theorem 2.1. *([17]) For $A \in \mathcal{CR}$, the map induced by the action*

$$\pi_A : \mathcal{G\ell}(\mathcal{H}) \times \mathcal{G\ell}(\mathcal{H}) \to (\mathcal{O}(A), d_X), \;\; \pi_A(G, K) = GAK^{-1}$$

is continuous and admits continuous local cross sections. In particular, the orbit $(\mathcal{O}(A), d_X)$ is a (topological) homogeneous space of $\mathcal{G\ell}(\mathcal{H}) \times \mathcal{G\ell}(\mathcal{H})$.

It is interesting to observe that the metrics d_N or d_R that appear in the statement are needed to obtain the continuity of the local cross sections of π_A. Indeed, these sections are constructed by using the Moore-Penrose inverse; although the Moore-Penrose (MP) inverse is a non continuous map between closed range operators in general, these metrics allow to control the projections onto the nullspaces or ranges, a fact that guarantees the continuity of the MP inverse.

Now we turn to the restricted versions of orbits of closed range operators. Recall that an *operator ideal* is a two-sided ideal $\mathfrak{S}$ of $\mathcal{B}(\mathcal{H})$ (see [20, 22, 27]). Throughout, we assume that every operator ideal is proper (i.e. $\{0\} \neq \mathfrak{S} \neq \mathcal{B}(\mathcal{H})$). Operator ideals are closed under the operator adjoint ($A^* \in \mathfrak{S}$ whenever $A \in \mathfrak{S}$). Another useful property is the following: $\mathcal{F} \subseteq \mathfrak{S} \subseteq \mathcal{K}$, for any operator ideal $\mathfrak{S}$, where $\mathcal{F} = \mathcal{F}(\mathcal{H})$ and $\mathcal{K} = \mathcal{K}(\mathcal{H})$ denote the ideals of finite-rank operators and compact operators on $\mathcal{H}$, respectively.

In the sequel, we will deal only with the following class of operator ideals. A *symmetrically-normed ideal* is an operator ideal $\mathfrak{S}$ endowed with a norm $\|\cdot\|_{\mathfrak{S}}$ satisfying $\|ABC\|_{\mathfrak{S}} \leq \|A\|\|B\|_{\mathfrak{S}}\|C\|$, for all $A, C \in \mathcal{B}(\mathcal{H})$ and $B \in \mathfrak{S}$; and $\|B\|_{\mathfrak{S}} = \|B\|$, for every rank-one operator B. We also assume that $(\mathfrak{S}, \|\cdot\|_{\mathfrak{S}})$ is a Banach space. The p-Schatten ideals $\mathfrak{S}_p$ $(1 \leq p \leq \infty)$ are well-known examples of symmetrically-normed ideals, whose norms are given by $\|A\|_p = \mathrm{Tr}(|A|^p)^{1/p} = (\sum_{n\geq 1} s_n^p(A))^{1/p}$, $p \geq 1$; and for $p = \infty$, $\mathfrak{S}_\infty = \mathcal{K}$ endowed with the usual operator norm $\|A\|_\infty = \|A\| = s_1(A)$. Here $s(A) = \{s_n(A)\}_{n\geq 1}$ is the sequence of singular values of A arranged in non-increasing order and counting multiplicities.

For a symmetrically-normed ideal $\mathfrak{S}$, the restricted invertible group associated to $\mathfrak{S}$ is given by

$$\mathcal{G}\ell_{\mathfrak{S}} := \mathcal{G}\ell(\mathcal{H}) \cap (I + \mathfrak{S}).$$

$\mathcal{G}\ell_{\mathfrak{S}}$ is real Banach-Lie group with the metric $d_{\mathfrak{S}}(G, K) = \|G - K\|_{\mathfrak{S}}$, for $G, K \in \mathcal{G}\ell_{\mathfrak{S}}$, whose Lie algebra can be identified with the ideal $\mathfrak{S}$ ([5, 24]). As we have explained in Section 1, the product group $\mathcal{G}\ell_{\mathfrak{S}} \times \mathcal{G}\ell_{\mathfrak{S}}$ acts on $\mathcal{CR}$ inducing the following restricted orbits: for $B \in \mathcal{CR}$,

$$\mathcal{O}_{\mathfrak{S}}(B) = \{GBK^{-1} : G, K \in \mathcal{G}\ell_{\mathfrak{S}}\}.$$

A natural generalization of Theorem 2.1 in the restricted setting would involve a version of the metrics d_N and d_R with respect to the norm $\|\cdot\|_{\mathfrak{S}}$. However, for restricted orbits we have at hand the notion of essential codimension (see [1,3]), which will play a crucial role in the geometric description of the restricted orbits.

Let $P, Q \in \mathcal{B}(\mathcal{H})$ be two orthogonal projections such that the operator $QP|_{R(P)} : R(P) \to R(Q)$ is Fredholm. In this case, (P, Q) is known as a Fredholm pair and the index of this Fredholm operator

$$\begin{aligned}[P : Q] &:= \mathrm{Ind}(QP|_{R(P)} : R(P) \to R(Q)) \\ &= \dim(N(Q) \cap R(P)) - \dim(R(Q) \cap N(P))\end{aligned}$$

is called the *essential codimension* (or *Fredholm index of the pair*). It is straightforward to check that (P, Q) is a Fredholm pair whenever $P - Q \in \mathcal{K}$.

Before we present the spatial characterization of restricted orbits, we recall that the right polar decomposition of $A \in \mathcal{B}(\mathcal{H})$ is given by $A = V_A|A|$, where $|A| = (A^*A)^{1/2}$ is the operator modulus and V_A is the unique partial isometry that further satisfies the condition $N(V_A) = N(A)$.

Theorem 2.2. *([15]) Let $A, B \in \mathcal{CR}$ and $\mathfrak{S}$ be a symmetrically-normed ideal. The following conditions are equivalent:*

i) $B = GAK^{-1}$, *for some* $G, K \in \mathcal{G}\ell_{\mathfrak{S}}$;
ii) $A - B \in \mathfrak{S}$ *and* $[P_{N(A)} : P_{N(B)}] = 0$;
iii) $|A| - |B| \in \mathfrak{S}$, $V_A - V_B \in \mathfrak{S}$ *and* $[V_A^*V_A : V_B^*V_B] = 0$.

We remark that the first two items in previous result remain equivalent for general operator ideals, and they are also equivalent to the third item for arithmetic mean closed operator ideals.

The essential codimension is linked to the continuity of the Moore-Penrose inverse as a map between closed range operators. In particular, one has that the map

$$\mu : \mathcal{O}_{\mathfrak{S}}(A) \to \mathcal{O}_{\mathfrak{S}}(A^\dagger), \;\; \mu(B) = B^\dagger, \tag{2.1}$$

turns out to be continuous for the relative topology induced by the inclusions $\mathcal{O}_{\mathfrak{S}}(A) \subseteq A + \mathfrak{S}$ and $\mathcal{O}_{\mathfrak{S}}(A^\dagger) \subseteq A^\dagger + \mathfrak{S}$ (see [16, Thm. 3.9]). This fact makes it possible to construct the continuous local cross sections for the action, and furthermore, one can endow the restricted orbits with a differentiable structure. We refer to [5, 29] for the setting of Banach manifolds.

Theorem 2.3. *([16]) Let $A \in \mathcal{CR}$ and let $\mathfrak{S}$ be a symmetrically-normed ideal. Then $\mathcal{O}_{\mathfrak{S}}(A)$ is a real analytic homogeneous space of $\mathcal{G}\ell_{\mathfrak{S}} \times \mathcal{G}\ell_{\mathfrak{S}}$. Furthermore, $\mathcal{O}_{\mathfrak{S}}(A)$ is also a real analytic submanifold of $A + \mathfrak{S}$ with the same differential structure, whose tangent space at $B \in \mathcal{O}_{\mathfrak{S}}(A)$ is given by*

$$(T\mathcal{O}_{\mathfrak{S}}(A))_B = \{XB - BY : X, Y \in \mathfrak{S}\}.$$

The previous result makes it possible to study differentiability properties of the Moore-Penrose inverse, as a map in (2.1) between Banach manifolds. It was proved that this map is real bianalytic. Furthermore, other restricted orbits consisting in positive operators and partial isometries can be used to study the maps given by the operator modulus and partial isometry in the polar decomposition. The precise statements and proofs can be found in [16]. Concerning differentiability properties of the Moore-Penrose inverse in the general setting of Banach algebras, we mention the theory of Banach-Lie groupoids [6]. On the other hand, we mention that the problem of global symmetric approximation of frames can be solved by using orbits like in (1.1) (see [14]).

3. Orbits related to $\mathfrak{S}$-equivalence of frames

We first recall the following spatial characterization.

Proposition 3.1. *([15]) Let $A, B \in \mathcal{CR}$ and $\mathfrak{S}$ be a symmetrically-normed ideal. Then the following conditions are equivalent:*

i) *There exists $G \in \mathcal{G}\ell_{\mathfrak{S}}$ such that $GA = B$;*
ii) *$A - B \in \mathfrak{S}$ and $N(A) = N(B)$.*
iii) *$|A| - |B| \in \mathfrak{S}$, $V_A - V_B \in \mathfrak{S}$ and $V_A^* V_A = V_B^* V_B$, where $A = V_A|A|$ and $B = V_B|B|$ are the right polar decompositions of A and B, respectively.*

If any of the conditions above hold, then $[P_{R(A)} : P_{R(B)}] = 0$.

Again we remark that the first two items in the previous result are equivalent for general operator ideals, and the third item is equivalent to the first two in the case of arithmetic mean closed operator ideals. Next we consider the differential structure of these orbits.

Theorem 3.2. *Let $A \in \mathcal{CR}$ and let $\mathfrak{S}$ be a symmetrically-normed ideal. Then $\mathcal{LO}_{\mathfrak{S}}(A)$ is a real analytic homogeneous space of $\mathcal{G}\ell_{\mathfrak{S}}$ and a real analytic submanifold of $A+\mathfrak{S}$. Both differentiable structures coincide and the tangent space at $B \in \mathcal{LO}_{\mathfrak{S}}(A)$ is given by*

$$(T\mathcal{LO}_{\mathfrak{S}}(A))_B = \{XB : X \in \mathfrak{S}\}.$$

Proof. We adapt the arguments given in [16, Thm. 3.14]. In order to prove the real analytic homogeneous space structure, we observe that the isotropy group at A, which is given by

$$K = \{G \in \mathcal{G}\ell_{\mathfrak{S}} : GA = A\},$$

is clearly a Banach-Lie group in the same topology of $\mathcal{G}\ell_{\mathfrak{S}}$. Denoting by $P = AA^\dagger$, we have that $GA = A$ if and only if $GP = P$. Then the Lie algebra of K can be identified with the closed subspace

$$\mathfrak{k} = \{X \in \mathfrak{S} : XP = 0\}.$$

The projection $\mathcal{E} : \mathfrak{S} \to \mathfrak{S}$, $\mathcal{E}(X) = X(I - P)$, is continuous with range $\mathfrak{k}$. Hence K is a Banach-Lie subgroup of $\mathcal{G}\ell_{\mathfrak{S}}$, and then $\mathcal{LO}_{\mathfrak{S}}(A) \simeq G/K$ is a real analytic homogeneous space, and the map $\pi_A : \mathcal{G}\ell_{\mathfrak{S}} \to \mathcal{LO}_{\mathfrak{S}}(A)$ defined by $\pi_A(G) = GA$ is a real analytic submersion (see [29, Thm. 8.19]).

Now we prove the submanifold structure. First we show that the relative topology inherited from $A + \mathfrak{S}$ coincides with the quotient topology in the orbit. For it is enough to prove that the map $\pi_A : \mathcal{G}\ell_{\mathfrak{S}} \to \mathcal{LO}_{\mathfrak{S}}(A)$ has continuous local cross sections, when $\mathcal{LO}_{\mathfrak{S}}(A)$ is considered with the relative topology. By a standard translation argument, we may only construct such section in a neighborhood of A. For $r > 0$ take the ball $\mathcal{W} = \{B \in \mathcal{LO}_{\mathfrak{S}}(A) : \|B - A\|_{\mathfrak{S}} < r\}$, and define the map

$$\sigma : \mathcal{W} \to \mathcal{G}\ell_{\mathfrak{S}}, \quad \sigma(B) = BA^\dagger + (I - P_{R(B)})(I - P_{R(A)}).$$

It holds that $\pi_A(\sigma(B)) = \sigma(B)A = B$, because $N(A) = N(B)$. By the same condition on the nullspaces we have the continuity of the Moore-Penrose inverse, and then the continuity of the map $B \mapsto BB^\dagger = P_{R(B)}$ with the norm $\|\cdot\|_{\mathfrak{S}}$ (see [16, Thm. 3.4]). Hence σ is continuous. Next note

$$\sigma(B) = B(A^\dagger - B^\dagger) + (P_{R(B)} - P_{R(A)})P_{R(A)} + I.$$

From this expression it follows that $\sigma(B) \in \mathcal{G}\ell_{\mathfrak{S}}$. Indeed, we have $A^\dagger - B^\dagger \in \mathfrak{S}$ by Proposition 3.1 together with Wedin's Formula (see [30]):

$$A^\dagger - B^\dagger = -\,A^\dagger(A-B)B^\dagger + (A^*A)^\dagger(A^* - B^*)(I - BB^\dagger) \\ + (I - A^\dagger A)(A^* - B^*)(BB^*)^\dagger\,.$$

It follows that $P_{R(B)} - P_{R(A)} = BB^\dagger - AA^\dagger = B(B^\dagger - A^\dagger) - (A-B)A^\dagger \in \mathfrak{S}$. Hence $\sigma(B) - I \in \mathfrak{S}$. Furthermore, we find that $\|\sigma(B) - I\| < 1$ by taking $r > 0$ small enough. This can be done by using the estimate $\|X\| \leq \|X\|_{\mathfrak{S}}$, for $X \in \mathfrak{S}$. Hence $\sigma(B)$ is invertible.

We prove that tangent spaces are closed and complemented in $\mathfrak{S}$ (tangent space of $A + \mathfrak{S}$). As we have showed above, π_A is a submersion, so the tangent space at A is identified with

$$(T\mathcal{LO}_{\mathfrak{S}}(A))_A = \{XA : X \in \mathfrak{S}\}.$$

This is a closed subspace, and it is the range of the following continuous projection $\mathcal{F} : \mathfrak{S} \to \mathfrak{S}$, $\mathcal{F}(Z) = ZA^\dagger A$. Hence tangent spaces are closed and complemented.

The previous facts imply that the inclusion map $\iota : \mathcal{LO}_{\mathfrak{S}}(A) \to A + \mathfrak{S}$ is a real analytic immersion and a homeomorphism onto its image. According to [29, Prop. 8.5] we obtain that $\mathcal{LO}_{\mathfrak{S}}(A)$ is a real analytic submanifold of $A + \mathfrak{S}$, and the manifold structure as homogeneous space coincides with the submanifold structure. □

As we have observed in Section 1, frames $\mathcal{F} = \{f_n\}_{n\geq 1}$ for closed subspaces $\mathcal{S} \subset \mathcal{H}$ can be identified with their synthesis operators $T_{\mathcal{F}}$, which are closed range operators. Furthermore, every $\mathfrak{S}$-equivalent frame $\mathcal{G} = \{g_n\}_{n\geq 1}$ to $\mathcal{F}$ has the form $g_n = GT_{\mathcal{F}}e_n = Gf_n$, for all $n \geq 1$, for some $G \in \mathcal{G}\ell_{\mathfrak{S}}$. Hence the (equivalence) class of all frames for closed subspaces $\mathcal{G}$ that are $\mathfrak{S}$-equivalent with $\mathcal{F}$ can be identified with the orbit $\mathcal{LO}_{\mathfrak{S}}(T_{\mathcal{F}})$. This means that Theorem 3.2 provides with a manifold structure for the set of all the frames for closed subspaces that are $\mathfrak{S}$-equivalent to a given frame. We mention that previous results on the differential structure of the set of all frames for a Hilbert space can be found in [18].

There are some well-known constructions in frame theory that have their operator theory counterparts. For example, given a frame $\mathcal{F} = \{f_n\}_{n\geq 1}$ for a closed subspace $\mathcal{S} \subset \mathcal{H}$ then we can consider the *canonical dual* frame $\mathcal{F}^\# = \{S_{\mathcal{F}}^\dagger f_n\}_{n\geq 1} = \{f_n^\#\}_{n\geq 1}$, where $S_{\mathcal{F}} = T_{\mathcal{F}}T_{\mathcal{F}}^* \in \mathcal{B}(\mathcal{H})$ is the so-called *frame operator* of $\mathcal{F}$. Since $T_{\mathcal{F}}$ is a closed range operator, the so is $S_{\mathcal{F}}$, and therefore $S_{\mathcal{F}}^\dagger$ is well defined. It turns out that $\mathcal{F}^\#$ is also a frame for $\mathcal{S}$ such that the canonical reconstruction formulas hold:

$$f = \sum_{n\geq 1} \langle f\,,\, f_n\rangle\, f_n^\# = \sum_{n\geq 1} \langle f\,,\, f_n^\#\rangle\, f_n$$

for all $f \in \mathcal{S}$. Also, we can consider the *associated Parseval frame* $\tilde{\mathcal{F}} = \{(S_{\mathcal{F}}^{1/2})^\dagger f_n\}_{n\geq 1} = \{\tilde{f}_n\}_{n\geq 1}$. Since $T_{\mathcal{F}}$ is a closed range operator, then so is the operator square root $S_{\mathcal{F}}^{1/2}$, and therefore $\tilde{\mathcal{F}}$ is well defined. It turns out that $\tilde{\mathcal{F}}$ is a frame for the closed subspace $\mathcal{S}$ such that its frame operator $S_{\tilde{\mathcal{F}}} = P_{\mathcal{S}}$ so that $\tilde{\mathcal{F}}$ coincides with its canonical dual. Hence, the corresponding reconstruction formulas become

$$f = \sum_{n\geq 1} \langle f \, , \, \tilde{f}_n\rangle \, \tilde{f}_n$$

for all $f \in \mathcal{S}$. Notice that the previous identity resembles the representation of $f \in \mathcal{S}$ with respect to an orthonormal basis of $\mathcal{S}$; yet, $\tilde{\mathcal{F}}$ is generally (linearly) redundant and therefore it is not an orthonormal basis.

From an operator theoretic point of view, given $\mathcal{F} = \{f_n\}_{n\geq 1}$ as above, we identify it with its synthesis operator $T_{\mathcal{F}}$. Similarly, we identify $\mathcal{F}^\# = \{f_n^\#\}_{n\geq 1}$ and $\tilde{\mathcal{F}} = \{\tilde{f}_n\}_{n\geq 1}$ with their synthesis operators that are given by $T_{\mathcal{F}^\#} = S_{\mathcal{F}}^\dagger T_{\mathcal{F}}$ and $T_{\tilde{\mathcal{F}}} = (S_{\mathcal{F}}^{1/2})^\dagger T_{\mathcal{F}}$, respectively. If we consider the left polar decomposition $T_{\mathcal{F}} = |T_{\mathcal{F}}^*|\, V^{\mathcal{F}}$, where $|T_{\mathcal{F}}^*| = (T_{\mathcal{F}}\, T_{\mathcal{F}}^*)^{1/2} = S_{\mathcal{F}}^{1/2}$ and $V^{\mathcal{F}}$ is a partial isometry that further satisfies $R(V^{\mathcal{F}}) = R(T_{\mathcal{F}})$, then $T_{\mathcal{F}^\#} = (S_{\mathcal{F}}^{1/2})^\dagger V^{\mathcal{F}} = (T_{\mathcal{F}}^\dagger)^*$ and $T_{\tilde{\mathcal{F}}} = V^{\mathcal{F}}$. Furthermore, if $\mathcal{G}$ is $\mathfrak{S}$-equivalent to $\mathcal{F}$ in terms of a symmetrically norm ideal $\mathfrak{S}$, then their corresponding canonical duals (respectively, associated Parseval frames) are also $\mathfrak{S}$-equivalent.

Next we show the last claim from the operator theory point of view. Hence, we consider $A \in \mathcal{CR}$ and $B \in \mathcal{LO}_{\mathcal{G}}(A)$. Arguing as in the proof of Theorem 3.2, Wedin's formula shows that $A^\dagger - B^\dagger \in \mathfrak{S}$; hence, $(A^\dagger)^* - (B^\dagger)^* = (A^\dagger - B^\dagger)^* \in \mathfrak{S}$. On the other hand, $N((B^\dagger)^*) = R(B^\dagger)^\perp = N(B)$ and similarly, $N((A^\dagger)^*) = N(A)$; since $N(A) = N(B)$ then $(B^\dagger)^* \in \mathcal{LO}_{\mathfrak{S}}((A^\dagger)^*)$, by the equivalence between items $i)$ and $ii)$ in Proposition 3.1. On the other hand, consider the left polar decompositions $A = |A^*|\, V^A$ and $B = |B^*|\, V^B$, where $|A^*| = (AA^*)^{1/2}$, $|B^*| = (BB^*)^{1/2}$ and V^A, V^B are the unique partial isometries that further satisfy $R(V^A) = R(A)$ and $R(V^B) = R(B)$. It is easy to see that $V^A = (V_{A^*})^*$, where $A^* = V_{A^*}|A^*|$ is the right polar decomposition of A^*; similarly, $V^B = (V_{B^*})^*$, where $B^* = A^* G^*$, for $G^* \in \mathcal{G}\ell_{\mathfrak{S}}$. Thus, $B^* \in \mathcal{O}_{\mathfrak{S}}(A^*)$ and hence $V^A - V^B = (V_{A^*} - V_{B^*})^* \in \mathfrak{S}$, by Theorem 2.2, and $N(V^A) = R(V_{A^*})^\perp = R(V_{B^*})^\perp = N(V^B)$. By [13], we get that

$$V^B \in \mathcal{LU}_{\mathfrak{S}}(V^A) := \{UV^A : \ U \in \mathcal{U}_{\mathfrak{S}}\}\,,$$

where $\mathcal{U}_{\mathfrak{S}}$ denotes the restricted unitary group associated to $\mathfrak{S}$, introduced in Section 1. In [13] it was shown that $\mathcal{LU}_{\mathfrak{S}}(V^A)$ is a real analytic homogeneous space of $\mathcal{U}_{\mathfrak{S}}$ and a submanifold of $V^A + \mathfrak{S}$. The previous remarks suggest the study of the following maps: given $A \in \mathcal{CR}$ let

$$\varphi : \mathcal{LO}_{\mathfrak{S}}(A) \to \mathcal{LO}_{\mathfrak{S}}((A^\dagger)^*) \quad \text{and} \quad \psi : \mathcal{LO}_{\mathfrak{S}}(A) \to \mathcal{LU}_{\mathfrak{S}}(V^A)$$

given by $\varphi(B) = (B^\dagger)^*$ and $\psi(B) = V^B$, for $B \in \mathcal{LO}_{\mathfrak{S}}(A)$; here V^A, V^B denote the partial isometries in the left polar decomposition $A = |A^*|V^A$ and $B = |B^*|V^B$. Notice that the remarks above show that these maps are well defined. Moreover, these maps are smooth in the following sense

Theorem 3.3. *With the previous notation the maps φ and ψ are real analytic between the corresponding real analytic manifolds.*

Proof. In what follows, we consider $\mathcal{O}_{\mathfrak{S}}(A)$ and $\mathcal{O}_{\mathfrak{S}}((A^\dagger)^*)$ with their real analytic manifold structures, as considered in Theorem 2.3. As we have observed the map in (2.1) is real analytic (see [16, Theorem 3.18]), and using the fact that the operator adjunction is a real linear operator, we conclude that the map $\tilde{\mu} : \mathcal{O}_{\mathfrak{S}}(A) \to \mathcal{O}_{\mathfrak{S}}((A^\dagger)^*)$ given by $\tilde{\mu}(B) = (B^\dagger)^*$, for $B \in \mathcal{O}_{\mathfrak{S}}(A)$, is well defined and real analytic. Similarly, if we let $\mathcal{U}_{\mathfrak{S}}(V^A) = \{UV^AW^* : U\,,\, W \in \mathcal{U}_{\mathfrak{S}}\}$, then $\mathcal{U}_{\mathfrak{S}}(V^A)$ is a real analytic homogeneous space of the product group $\mathcal{U}_{\mathfrak{S}} \times \mathcal{U}_{\mathfrak{S}}$ (see [13]). By [16, Theorem 4.17] we get that the map $v : \mathcal{O}_{\mathfrak{S}}(A^*) \to \mathcal{U}_{\mathfrak{S}}(V_{A^*})$ given by $v(B) = V_B$, for $B \in \mathcal{O}_{\mathfrak{S}}(A^*)$, is real analytic; here $B = V_B\,|B|$ denotes the right polar decomposition of B. Clearly, the map $\mathcal{A} : \mathcal{O}_{\mathfrak{S}}(A) \to \mathcal{O}_{\mathfrak{S}}(A^*)$ given by $\mathcal{A}(B) = B^*$, for $B \in \mathcal{O}_{\mathfrak{S}}(A)$, is well defined and real analytic. Therefore, the map $\tilde{v} : \mathcal{O}_{\mathfrak{S}}(A) \to \mathcal{U}_{\mathfrak{S}}(V^A)$ given by $\tilde{v}(B) = (v(B^*))^* = (V_{B^*})^* = V^B$, for $B \in \mathcal{O}_{\mathfrak{S}}(A)$, is real analytic.

Note that the map φ in the statement coincides with the restriction $\tilde{\mu}|_{\mathcal{LO}_{\mathfrak{S}}(A)} : \mathcal{LO}_{\mathfrak{S}}(A) \to \mathcal{LO}_{\mathfrak{S}}((A^\dagger)^*)$. So it is enough to show that $\mathcal{LO}_{\mathfrak{S}}(A)$ is a submanifold of $\mathcal{O}_{\mathfrak{S}}(A)$. This can be shown by using the charts given by the homogenous spaces structure of these orbits, or by the same technique used in Theorem 3.2. Alternatively, observe that $(T\mathcal{LO}_{\mathfrak{S}}(A))_A = \{XA :\ X \in \mathfrak{S}\} \subset (T\mathcal{O}_{\mathfrak{S}}(A))_A$ is a closed subspace (see the expression of the second tangent space in Theorem 2.3). Moreover, notice that $(T\mathcal{O}_{\mathfrak{S}}(A))_A \subset \mathfrak{S}$ is an invariant space of the projection $\mathcal{F} : \mathfrak{S} \to \mathfrak{S}$ given by $\mathcal{F}(Z) = ZA^\dagger A$ in the proof of Theorem 3.2. Furthermore, the range of the restriction of $\mathcal{F}$ to $(T\mathcal{O}_{\mathfrak{S}}(A))_A$ also coincides with $(T\mathcal{LO}_{\mathfrak{S}}(A))_A$. Thus, $(T\mathcal{LO}_{\mathfrak{S}}(A))_A$ is a closed complemented subspace of $(T\mathcal{O}_{\mathfrak{S}}(A))_A$. Then the inclusion map between these orbits is an immersion, and since both orbits can be considered in the relative topology of $A+\mathfrak{S}$, we get that $\mathcal{LO}_{\mathfrak{S}}(A)$ is a submanifold of $\mathcal{O}_{\mathfrak{S}}(A)$. Similarly, since ψ coincides with the restriction $\tilde{v}|_{\mathcal{LO}_{\mathfrak{S}}(A)} : \mathcal{LO}_{\mathfrak{S}}(A) \to \mathcal{LU}_{\mathfrak{S}}(V^A)$, one can follow the same steps, to prove that ψ is real analytic. ☐

Again we may interpret the previous result in terms of frame theory. Roughly speaking, it means that the map that assigns a frame $\{f_n\}_{n\geq 1}$ to its canonical dual frame $\{f_n^{\#}\}_{n\geq 1}$, turns out to be real analytic between the corresponding $\mathfrak{S}$-equivalent frames. Similarly, the map sending $\{f_n\}_{n\geq 1}$ to its associated Parseval frame $\{\tilde{f}_n\}_{n\geq 1}$ is real analytic between the corresponding $\mathfrak{S}$-equivalent frames and $\mathfrak{S}$-unitarily equivalent Parseval frames.

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