KREIN-ŠMUL'JAN THEOREM REVISITED

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Abstract. We present a generalization of Krein-Šmul'jan theorem which involves several operators. Given bounded selfadjoint operators $A, B_1, \ldots, B_m$ acting on a Hilbert space $\mathcal{H}$, we provide sufficient conditions to determine whether there are $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$ such that $A + \sum_{i=1}^m \lambda_i B_i$ is a positive semidefinite operator.

1. Introduction

Along this paper $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ denotes a complex Hilbert space, and $\mathcal{L}(\mathcal{H})$ stands for the algebra of bounded linear operators in $\mathcal{H}$. An operator $A \in \mathcal{L}(\mathcal{H})$ is *positive semidefinite* if $\langle Ax, x \rangle \geq 0$ for all $x \in \mathcal{H}$; and it is *positive definite* if there exists $\alpha > 0$ such that $\langle Ax, x \rangle \geq \alpha \|x\|^2$ for every $x \in \mathcal{H}$.

Given bounded selfadjoint operators $A, B_1, \ldots, B_m$ acting on $\mathcal{H}$, the aim of this work is to determine whether there are $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$ such that the operator $A + \sum_{i=1}^m \lambda_i B_i$ is positive semidefinite. If $\geq$ denotes Löwner's partial order of selfadjoint operators, the problem can be restated as whether the inequality

$$A + \sum_{i=1}^{m} \lambda_i B_i \geq 0 \tag{1.1}$$

is feasible. If $\mathcal{H}$ is finite dimensional this is known as a *linear matrix inequality* (LMI), an area which has been thoroughly studied since the 1940's for its applications in System and Control theory, see [2] and the references therein.

Another reason that makes this problem interesting is that it is closely related to the existence of minimizers for quadratically constrained quadratic programming (QCQP) problems. A QCQP problem can be posed as:

$$\begin{array}{lll} \text{minimize} & f(x) = \langle Ax, x \rangle + 2\,\mathrm{Re}\,\langle y_0, x \rangle + \alpha_0 & \\ \text{subject to} & g_i(x) = \langle B_i x, x \rangle + 2\,\mathrm{Re}\,\langle y_i, x \rangle \leq \alpha_i, & i = 1, \ldots, m, \end{array}$$

where the optimization variable x varies in $\mathcal{H}$, and the data consists of bounded selfadjoint operators $A, B_1, \ldots, B_m$ acting in $\mathcal{H}$, vectors $y_i \in \mathcal{H}$ and scalars $\alpha_i \in \mathbb{R}$, for $i = 0, 1, ..., m$. Note that the Hessian of such a quadratic function is constant. In particular, the Hessian of $f, g_1, \ldots, g_m$ are given by the selfadjoint operators $A, B_1, \ldots, B_m$, respectively. Hence, if x_0 is a minimizer of the above problem then there exist $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$ such that (1.1) holds, see e.g. [14, 1].

For a finite dimensional space, studies on the simplest case (i.e. $m = 1$) can be traced back to works of Finsler [7], Hestenes [8], and Calabi [3]. But the result characterizing the feasibility of $A + \lambda B \geq 0$ in an arbitrary Hilbert space is known as the Krein-Smul'jan theorem [10, 11], see also [12, 13]. Given a selfadjoint operator $B \in \mathcal{L}(\mathcal{H})$ we say that B is indefinite if it is not semidefinite i.e. there exist $x_+, x_- \in \mathcal{H}$ such that $\langle Bx_+, x_+ \rangle > 0$ and $\langle Bx_-, x_- \rangle < 0$.

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Theorem 1.1. *If $B \in \mathcal{L}(\mathcal{H})$ is indefinite, then there exists $\lambda \in \mathbb{R}$ such that $A + \lambda B \geq 0$ if and only if*

$$\langle Ax, x \rangle \geq 0 \qquad \textit{whenever} \qquad \langle Bx, x \rangle = 0.$$

In this case,

$$\frac{\langle Ay, y \rangle}{\langle By, y \rangle} \leq \frac{\langle Az, z \rangle}{\langle Bz, z \rangle}$$

for every $y, z \in \mathcal{H}$ such that $\langle By, y \rangle < 0$ and $\langle Bz, z \rangle > 0$. Also, if

$$\lambda_- := -\inf_{\langle Bx,x \rangle > 0} \frac{\langle Ax, x \rangle}{\langle Bx, x \rangle} \qquad \textit{and} \qquad \lambda_+ := -\sup_{\langle Bx,x \rangle < 0} \frac{\langle Ax, x \rangle}{\langle Bx, x \rangle}, \tag{1.2}$$

then $\lambda_- \leq \lambda_+$ and $\{\lambda \in \mathbb{R} : \; A + \lambda B \geq 0\} = [\lambda_-, \lambda_+]$.

To the best of our knowledge, there is no such a result for an inequality which involves several variables like (1.1). Even in the finite dimensional setting, there are only a few results. Among them, it is worthwhile mentioning the works by Dines [4, 5] and Hestenes and McShane [9].

The paper is organized as follows. Section 2 starts with a discussion about weakly indefinite sets of selfadjoint operators. We show that this notion is only sufficient to prove a generalization of Krein-Smul'jan theorem in the case of pairs $\{B_1, B_2\}$. For finite sets $\{B_1, \ldots, B_m\}$ with $m > 2$ it is necessary to impose some extra condition, named strongly indefiniteness. After discussing what strongly indefiniteness means, in Theorem 4.6 we state a generalization of Krein-Smul'jan theorem. Finally, in Section 5 we give a sufficient condition on $\{B_1, \ldots, B_m\}$ to be strongly indefinite, which is inspired by the results of Hestenes and McShane in [9].

2. Weakly indefinite sets of selfadjoint operators

We start with a definition which is mainly motivated by [5, 9].

Definition 2.1. A set of selfadjoint operators $\{B_1, \ldots, B_m\}$ is *weakly indefinite* if

$$\sum_{i=1}^{m} \mu_i B_i \text{ is indefinite for every } (\mu_1, \ldots, \mu_m) \in \mathbb{R}^m \setminus \{0\}.$$

If $\{B_1, B_2, \ldots, B_m\}$ is weakly indefinite, then any subset of it is also weakly indefinite. In particular, B_i is indefinite for every $i = 1, 2, \ldots, m$. Also, if $\{B_1, \ldots, B_m\}$ is weakly indefinite then it is a linearly independent set.

Given a selfadjoint operator $B \in \mathcal{L}(\mathcal{H})$, denote by $Q(B)$ the set of neutral vectors for the quadratic form induced by B, $Q(B) = \{x \in \mathcal{H} : \; \langle Bx, x \rangle = 0\}$. Given a set of selfadjoint operators $\{B_1, \ldots, B_m\}$ for brevity we write $Q_i = Q(B_i)$ for each $i = 1, \ldots, m$. Also, we consider the sets of vectors which are positive (negative) with respect to the quadratic form induced by B_i:

$$\mathcal{P}_i^+ = \{\, x \in \mathcal{H} : \langle B_i x, x \rangle > 0 \,\} \quad \text{and} \quad \mathcal{P}_i^- = \{\, x \in \mathcal{H} : \langle B_i x, x \rangle < 0 \,\}.$$

It is well known that B is indefinite if and only if $Q(B) \setminus N(B) \neq \{0\}$, i.e. if there exists $x \in \mathcal{H}$ such that

$$\langle Bx, x \rangle = 0 \qquad \text{and} \qquad Bx \neq 0.$$

The next result presents a sufficient condition to guarantee the weakly indefiniteness of $\{B_1, \ldots, B_m\}$.

Proposition 2.2. *Given selfadjoint operators $B_1, \ldots, B_m \in \mathcal{L}(\mathcal{H})$, if there exists $x \in \mathcal{H}$ such that*

$$x \in \textstyle\bigcap_{j=1}^m Q_j \quad \textit{and} \quad \{B_1 x, \ldots, B_m x\} \textit{ is linearly independent in } \mathcal{H}$$

then $\{B_1, \ldots, B_m\}$ is weakly indefinite.

Proof. It suffices to show that $Q\big(\sum_{j=1}^m \lambda_j B_j\big) \setminus N\big(\sum_{j=1}^m \lambda_j B_j\big) \neq \{0\}$ for every $(\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^m \setminus \{0\}$. Given $(\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^m \setminus \{0\}$, note that $\big(\sum_{j=1}^m \lambda_j B_j\big)x$ is not trivial because $\{B_1 x, \ldots, B_m x\}$ is linearly independent. Then,

$$x \in \big(\textstyle\bigcap_{j=1}^m Q_j\big) \setminus N\big(\sum_{j=1}^m \lambda_j B_j\big) \subseteq Q\big(\sum_{j=1}^m \lambda_j B_j\big) \setminus N\big(\sum_{j=1}^m \lambda_j B_j\big),$$

and since $(\lambda_1, \ldots, \lambda_m)$ was arbitrary the proof is complete. □

Given $x \in \mathcal{H}$, note that $\{B_1 x, \ldots, B_m x\}$ is linearly independent if and only if $x \notin N(\sum_{j=1}^m \lambda_j B_j)$ for every $(\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^m \setminus \{0\}$.

The sufficient condition presented above is not necessary to guarantee weakly indefiniteness of a set of operators, because it imposes that $\bigcap_{i=1}^m Q_i \neq \{0\}$. In Example 3.4 below we present a set $\{B_1, B_2, B_3, B_4\}$ which is weakly indefinite but $\bigcap_{i=1}^4 Q_i = \{0\}$.

Lemma 2.3. *Given two indefinite selfadjoint operators $B_1, B_2 \in \mathcal{L}(\mathcal{H})$, the family $\{B_1, B_2\}$ is weakly indefinite if and only if B_i is indefinite in Q_j for $j \neq i$. In this case $Q_1 \cap Q_2 \neq \{0\}$.*

Proof. Assume, for example, that B_1 is indefinite in Q_2 but there exists $(\lambda_1, \lambda_2) \neq (0,0)$ such that $\lambda_1 B_1 + \lambda_2 B_2 \geq 0$. If $\lambda_1 = 0$ then $\lambda_2 B_2 \geq 0$ leading to a contradiction. If $\lambda_1 > 0$ then $B_1 + \frac{\lambda_2}{\lambda_1} B_2 \geq 0$. In particular $B_1 \geq 0$ in Q_2, which is a contradiction to our assumption. If $\lambda_1 < 0$ then it is easy to see that $B_1 \leq 0$ in Q_2, which leads to another contradiction.

Conversely, suppose that $\{B_1, B_2\}$ is weakly indefinite and that B_1 is definite in Q_2. If $B_1 \geq 0$ in Q_2 then, by Theorem 1.1, there exists $\lambda \in \mathbb{R}$ such that $B_1 + \lambda B_2 \geq 0$, which is a contradiction to $\{B_1, B_2\}$ being indefinite. If $B_1 \leq 0$ in Q_2, consider $-B_1$. By symmetry, B_2 is indefinite in Q_1.

To see that $Q_1 \cap Q_2 \neq \{0\}$, take $y \in \mathcal{P}_1^- \cap Q_2$ and $z \in \mathcal{P}_1^+ \cap Q_2$, and choose $\theta \in [0, \pi)$ such that $\mathrm{Re}\,\big\langle B_2 y, e^{i\theta} z \big\rangle = 0$. Consider

$$\gamma(t) = t\,y + (1-t)e^{i\theta}\,z, \qquad t \in [0,1].$$

Since $y, z \in Q_2$, for $t \in [0,1]$

$$\langle B_2\gamma(t), \gamma(t) \rangle = t^2 \langle B_2 y, y \rangle + (1-t)^2 \langle B_2 z, z \rangle + 2\,t\,(1-t)\,\mathrm{Re}\,\big\langle B_2 y, e^{i\theta} z \big\rangle = 0.$$

Hence, $\gamma\big([0,1]\big) \subseteq Q_2$ and the real valued function

$$f(t) = \langle B_1\gamma(t), \gamma(t) \rangle, \qquad t \in [0,1],$$

satisfies $f(0) = \langle B_1 y, y \rangle < 0$ and $f(1) = \langle B_1 z, z \rangle > 0$. Thus, there exists $t_0 \in (0,1)$ such that $f(t_0) = 0$. This implies that $\gamma(t_0) \in Q_1 \cap Q_2$. Also, $\gamma(t_0) \neq 0$ because $\{y, z\}$ is a linearly independent set. □

In the following we denote by Ω the feasibility set for inequality (1.1), i.e.

$$\Omega = \Omega\big(A, (B_i)_{i=1}^m\big) := \left\{ (\lambda_1, \lambda_2, \ldots, \lambda_m) \in \mathbb{R}^m \,:\, A + \sum_{i=1}^m \lambda_i B_i \geq 0 \right\}. \tag{2.1}$$

It is easy to check that Ω is a closed convex subset of $\mathbb{R}^m$.

The next proposition characterizes the feasibility of (1.1) for $m = 2$. Its proof follows the lines of one given in [6].

Theorem 2.4. *Given selfadjoint operators $A, B_1, B_2 \in \mathcal{L}(\mathcal{H})$, assume that $\{B_1, B_2\}$ is weakly indefinite. Then,*

$$A \geq 0 \quad \text{in} \quad Q_1 \cap Q_2 \qquad \text{if and only if} \qquad \Omega \neq \varnothing.$$

Proof. The fact that $\Omega \neq \varnothing$ trivially implies $A \geq 0$ in $Q_1 \cap Q_2$. To prove the converse, assume that $A \geq 0$ in $Q_1 \cap Q_2$. By Lemma 2.3, B_1 is indefinite in Q_2. Hence, fixing $y \in \mathcal{P}_1^- \cap Q_2$, $z \in \mathcal{P}_1^+ \cap Q_2$ and choosing $\theta \in [0, \pi)$ so that $\operatorname{Re}\left\langle B_2 y, e^{i\theta} z \right\rangle = 0$, consider

$$\gamma_\pm(t) = t\, y \pm (1-t) e^{i\theta}\, z, \qquad t \in [0,1].$$

Note that $\gamma_\pm\big([0,1]\big) \subseteq Q_2$ and take $t_\pm \in (0,1)$ as in the proof of Lemma 2.3 such that $\gamma_\pm(t_\pm) \in Q_1 \cap Q_2$. Now, we have the equations

$$a \left\langle B_1 y, y \right\rangle + \tfrac{1}{a} \left\langle B_1 z, z \right\rangle + 2 \operatorname{Re}\left\langle B_1 y, e^{i\theta} z \right\rangle = \tfrac{1}{t_+(1-t_+)} \left\langle B_1 \gamma_+(t_+), \gamma_+(t_+) \right\rangle = 0,$$

$$b \left\langle B_1 y, y \right\rangle + \tfrac{1}{b} \left\langle B_1 z, z \right\rangle - 2 \operatorname{Re}\left\langle B_1 y, e^{i\theta} z \right\rangle = \tfrac{1}{t_-(1-t_-)} \left\langle B_1 \gamma_-(t_-), \gamma_-(t_-) \right\rangle = 0.$$

where $a := \frac{t_+}{1-t_+}$ and $b := \frac{t_-}{1-t_-}$ are positive. Then, adding these two we get

$$(a+b) \left\langle B_1 y, y \right\rangle + \left(\tfrac{1}{a} + \tfrac{1}{b}\right) \left\langle B_1 z, z \right\rangle = 0,$$

or equivalently,

$$a\, b = -\frac{\left\langle B_1 z, z \right\rangle}{\left\langle B_1 y, y \right\rangle}. \tag{2.2}$$

Now, since $\left\langle A\gamma_\pm(t_\pm), \gamma_\pm(t_\pm) \right\rangle \geq 0$, in the same fashion we get that

$$0 \leq \frac{1}{a\, b} \left\langle Az, z \right\rangle + \left\langle Ay, y \right\rangle.$$

Combining this with (2.2) yields

$$\frac{\left\langle Ay, y \right\rangle}{\left\langle B_1 y, y \right\rangle} \leq \frac{\left\langle Az, z \right\rangle}{\left\langle B_1 z, z \right\rangle},$$

for arbitrary $y \in \mathcal{P}_1^- \cap Q_2$ and $z \in \mathcal{P}_1^+ \cap Q_2$. Therefore,

$$\sup_{y \in \mathcal{P}_1^- \cap Q_2} \frac{\left\langle Ay, y \right\rangle}{\left\langle B_1 y, y \right\rangle} \leq \inf_{z \in \mathcal{P}_1^+ \cap Q_2} \frac{\left\langle Az, z \right\rangle}{\left\langle B_1 z, z \right\rangle}.$$

If $\lambda_1 \in \mathbb{R}$ is such that $-\inf_{z \in \mathcal{P}_1^+ \cap Q_2} \frac{\langle Az, z \rangle}{\langle B_1 z, z \rangle} \leq \lambda_1 \leq -\sup_{y \in \mathcal{P}_1^- \cap Q_2} \frac{\langle Ay, y \rangle}{\langle B_1 y, y \rangle}$ then

$$\left\langle (A + \lambda_1 B_1)x, x \right\rangle \geq 0 \quad \text{for every } x \in (\mathcal{P}_1^- \cap Q_2) \cup (\mathcal{P}_1^+ \cap Q_2).$$

Considering that $\left\langle Ax, x \right\rangle \geq 0$ for every $x \in Q_1 \cap Q_2$, we then have that

$$\left\langle (A + \lambda_1 B_1)x, x \right\rangle \geq 0 \quad \text{for every } x \in Q_2.$$

Finally, by Theorem 1.1 there exists $\lambda_2 \in \mathbb{R}$ such that $A + \lambda_1 B_1 + \lambda_2 B_2 \geq 0$, i.e. $(\lambda_1, \lambda_2) \in \Omega$. ☐

Corollary 2.5. *Given three indefinite selfadjoint operators $B_1, B_2, B_3 \in \mathcal{L}(\mathcal{H})$, the family $\{B_1, B_2, B_3\}$ is weakly indefinite if and only B_j is indefinite in $\bigcap_{i \neq j} Q_i$ for every $j = 1, 2, 3$.*

Proof. It is analogous to the proof of Lemma 2.3, using Theorem 2.4 instead of Theorem 1.1. ☐

3. Indefinite sets

Definition 3.1. Given selfadjoint operators $B_1, \ldots, B_m \in \mathcal{L}(\mathcal{H})$, $m \geq 2$, the set $\{B_1, \ldots, B_m\}$ is *indefinite* if B_j is indefinite in $\bigcap_{i \neq j} Q_i$ for every $j = 1, \ldots, m$.

Note that the above definition imposes that $\bigcap_{i \neq j} Q_i \neq \{0\}$ for any $j = 1, \ldots, m$.

Lemma 3.2. *Assume that $\{B_1, \ldots, B_m\}$ is indefinite. Then, $\{B_1, \ldots, B_m\}$ is weakly indefinite.*

Proof. The proof is similar to that corresponding to Lemma 2.3. Suppose that there exists $\mu \in \mathbb{R}^m \setminus \{0\}$ such that $\sum_{i=1}^m \mu_i B_i \geq 0$ and $\mu_j \neq 0$ for some $j = 1, \ldots, m$. If $\mu_j > 0$ then $B_j + \sum_{i \neq j} \frac{\mu_i}{\mu_j} B_i \geq 0$, so that $B_j \geq 0$ in $\bigcap_{i \neq j} Q_i$, leading to a contradiction. A similar argument holds if $\mu_j < 0$. $\square$

Remark 3.3. Consider an indefinite set $\{B_1, \ldots, B_m\}$. If $\bigcap_{i=1}^m Q_i = \{0\}$ then it is a maximal indefinite set.

In fact, given any selfadjoint $B \in \mathcal{L}(\mathcal{H})$ such that $B \neq B_i$ for $i = 1, \ldots, m$, by definition, a necessary condition for the set $\{B_1, \ldots, B_m, B\}$ to be indefinite is that $\bigcap_{i=1}^m Q_i \neq \{0\}$.

Example 3.4. In what follows we give an example of a maximal indefinite set. Consider the operators $B_1, \ldots, B_4$ acting on $\mathbb{C}^4$ which are represented by

$$B_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix},$$

$$B_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad B_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

These four matrices satisfy that B_j is indefinite in $\bigcap_{i \neq j} Q_i$ for $j = 1, 2, 3, 4$. Indeed,

$$(1-\sqrt{2},-1,1-\sqrt{2},1) \in \mathcal{P}_1^- \cap \bigcap_{i \neq 1} Q_i, \qquad (3-\sqrt{2},1+2\sqrt{2},-1+5\sqrt{2},7) \in \mathcal{P}_1^+ \cap \textstyle\bigcap_{i \neq 1} Q_i,$$

$$(1-\sqrt{2},1,-1+\sqrt{2},1) \in \mathcal{P}_2^- \cap \bigcap_{i \neq 2} Q_i, \qquad (1+\sqrt{2},1,-1-\sqrt{2},1) \in \mathcal{P}_2^+ \cap \textstyle\bigcap_{i \neq 2} Q_i,$$

$$(1-\sqrt{2},-1,-1+\sqrt{2},1) \in \mathcal{P}_3^- \cap \bigcap_{i \neq 3} Q_i, \qquad (1-\sqrt{2},-1+2\sqrt{2},3-\sqrt{2},1) \in \mathcal{P}_3^+ \cap \textstyle\bigcap_{i \neq 3} Q_i,$$

$$(1-5\sqrt{2},-1-2\sqrt{2},-3+\sqrt{2},7) \in \mathcal{P}_4^- \cap \bigcap_{i \neq 4} Q_i, \qquad (-1+\sqrt{2},1,-1+\sqrt{2},1) \in \mathcal{P}_4^+ \cap \textstyle\bigcap_{i \neq 4} Q_i.$$

Nevertheless, $\bigcap_{i=1}^4 Q_i = \{0\}$ because the system of equations

$$\begin{cases} |x_1|^2 + |x_2|^2 + 2\,\mathrm{Re}(x_3 x_4) & = & 0 \\ 2\,\mathrm{Re}(x_1 x_2) + |x_3|^2 + |x_4|^2 & = & 0 \\ |x_1|^2 + 2\,\mathrm{Re}(x_2 x_3) + |x_4|^2 & = & 0 \\ 2\,\mathrm{Re}(x_1 x_4) + |x_2|^2 + |x_3|^2 & = & 0 \end{cases}$$

admits only the trivial solution.

Lemma 3.5. *Let $\{B_1, \ldots, B_m\}$ be an indefinite set. Take $y \in \mathcal{P}_k^- \cap \bigcap_{i \neq k} Q_i$ and $z \in \mathcal{P}_k^+ \cap \bigcap_{i \neq k} Q_i$ for some $k = 1, \ldots, m$ and consider $\mathcal{S} := \{\alpha y + \beta z \,:\, \alpha, \beta \in \mathbb{R}\}$. Then,*

$$\textit{either} \qquad \mathcal{S} \subseteq \textstyle\bigcap_{i \neq k} Q_i \qquad \textit{or} \qquad \mathcal{S} \cap Q_k \cap Q_l = \{0\} \qquad \textit{for some } l \neq k.$$

Proof. Assume that there exists $l \neq k$ such that $\mathcal{S} \nsubseteq Q_l$. This is equivalent to $\operatorname{Re}\langle B_l y, z\rangle \neq 0$. Then for any $\alpha, \beta \in \mathbb{R}$, $\alpha \neq 0$, $\beta \neq 0$, $\alpha y + \beta z \notin Q_l$. Since $y, z \notin Q_k$ we get that $\mathcal{S} \cap Q_k \cap Q_l = \{0\}$. □

The following lemma shows that, under suitable hypotheses, proving that the notions of indefinite and weakly indefinite sets coincide is equivalent to generalizing Krein-Šmul'jan theorem.

Lemma 3.6. *Given $B_1, \ldots, B_m \in \mathcal{L}(\mathcal{H})$, assume that $\{B_1, \ldots, B_m\}$ is linearly independent, $B_j \not\equiv 0$ in $\bigcap_{i\neq j} Q_i$ for every $j = 1, \ldots, m$, and $\{B_j\}_{j\in J}$ is weakly indefinite for every $J \subset \{1, \ldots, m\}$ with $|J| = m-1$. Then, the following statements are equivalent:*

i) there exists $j = 1, \ldots, m$ such that $B_j \geq 0$ in $\bigcap_{i\neq j} Q_i$ if and only if $B_j + \sum_{i\neq j} \lambda_i B_i \geq 0$ for some $(\lambda_j)_{j\neq i} \in \mathbb{R}^{m-1}$;

ii) $\{B_1, \ldots, B_m\}$ is indefinite if and only if $\{B_1, \ldots, B_m\}$ is weakly indefinite.

Proof. Assume that *i*) holds and also that $\{B_1, \ldots, B_m\}$ is not indefinite, i.e. there exists $j = 1, \ldots, m$ such that B_j is semidefinite in $\bigcap_{i\neq j} Q_i$. If $B_j \geq 0$ in $\bigcap_{i\neq j} Q_i$ then, by *i*), there exists $(\lambda_i)_{i\neq j} \in \mathbb{R}^{m-1}$ such that $B_j + \sum_{i\neq j} \lambda_i B_i \geq 0$. If $B_j \leq 0$ in $\bigcap_{i\neq j} Q_i$ then $-B_j \geq 0$ in $\bigcap_{i\neq j} Q_i$ and, by *i*), there exists $(\mu_i)_{i\neq j} \in \mathbb{R}^{m-1}$ such that $-B_j + \sum_{i\neq j} \mu_i B_i \geq 0$. Therefore, $\{B_1, \ldots, B_m\}$ is not weakly indefinite. The converse implication is always true, see Lemma 3.2. Thus, i) implies ii).

Conversely, assume that *ii*) holds and also that there exists $j = 1, \ldots, m$ such that $B_j \geq 0$ in $\bigcap_{i\neq j} Q_i$. Then, by *ii*), $\{B_1, \ldots, B_m\}$ is neither indefinite nor weakly indefinite. Hence, there exists $(\lambda_i)_{i\neq j} \in \mathbb{R}^{m-1}$ such that $B_j + \sum_{i\neq j} \lambda_i B_i$ is semidefinite. Since $B_j \not\equiv 0$ in $\bigcap_{i\neq j} Q_i$, there exists $x \in \bigcap_{i\neq j} Q_i$ such that $\langle B_j x, x\rangle > 0$. Hence,

$$\left\langle \big(B_j + \sum_{i\neq j} \lambda_i B_i\big)x, x \right\rangle = \langle B_j x, x\rangle > 0,$$

which proves that $B_j + \sum_{i\neq j} \lambda_i B_i \geq 0$. The converse implication is immediate. Therefore, *ii*) implies *i*). □

4. Strongly indefinite sets

Given a set $\{B_1, \ldots, B_m\}$ of selfadjoint operators, our aim is to impose condition(s) onto it in order to prove a generalization of Krein-Smul'jan theorem of the form: if $A \in \mathcal{L}(\mathcal{H})$ is selfadjoint then,

$$A \geq 0 \quad \text{in} \quad \textstyle\bigcap_{i=1}^m Q_i \qquad \text{if and only if} \qquad \Omega \neq \varnothing. \tag{4.1}$$

Remark 4.1. If (4.1) holds for a weakly indefinite set $\{B_1, \ldots, B_m\}$ then $\bigcap_{i=1}^m Q_i \neq \{0\}$.

Indeed, given a weakly indefinite set $\{B_1, \ldots, B_m\}$ (where m is less than the dimension of the real subspace of selfadjoint operators in $\mathcal{H}$) suppose that $\bigcap_{i=1}^m Q_i = \{0\}$ and consider any selfadjoint operator $B \in \mathcal{L}(\mathcal{H})$ such that $\{B_1, \ldots, B_m, B\}$ is a linearly independent set. Since both $B \geq 0$ and $-B \geq 0$ in $\bigcap_{i=1}^m Q_i = \{0\}$, if (4.1) holds then there exist $(\lambda_1, \ldots, \lambda_m), (\mu_1, \ldots, \mu_m) \in \mathbb{R}^m \setminus \{0\}$ such that

$$B + \sum_{i=1}^m \lambda_i B_i \geq 0 \qquad \text{and} \qquad -B + \sum_{i=1}^m \mu_i B_i \geq 0.$$

Then, there exists $j = 1, \ldots, m$ such that $\mu_j \neq -\lambda_j$, otherwise, $B + \sum_{i=1}^m \lambda_i B_i = 0$ which is a contradiction to $\{B_1, \ldots, B_m, B\}$ being linearly independent.

Hence, adding the above inequalities we get that $\sum_{i=1}^m (\lambda_i + \mu_i) B_i \geq 0$ which is a contradiction to $\{B_1, \ldots, B_m\}$ being a weakly indefinite set.

Example 3.4 presents an indefinite set $\{B_1, B_2, B_3, B_4\}$ such that $\bigcap_{i=1}^4 Q_i = \{0\}$. Hence, for $m \geq 3$ assuming that $\{B_1, \ldots, B_m\}$ is weakly indefinite, or even indefinite, is not enough as a suitable hypothesis for generalizing Theorem 1.1.

If $\{B_1, \ldots, B_m\}$ is a weakly indefinite set with $m \geq 3$ then, by Corollary 2.5, any trio $\{B_i, B_j, B_k\}$ is an indefinite set. In particular B_i is indefinite in $Q_j \cap Q_k$, i.e. there always exist $x_+ \in \mathcal{P}_i^+ \cap Q_j \cap Q_k$ and $x_- \in \mathcal{P}_i^- \cap Q_j \cap Q_k$.

Also, by Lemma 3.5, if $x_\pm \in \mathcal{P}_i^\pm \cap Q_j \cap Q_k$ and $\mathcal{S} := \{\alpha x_- + \beta x_+ : \alpha, \beta \in \mathbb{R}\}$ then either $\mathcal{S} \subseteq Q_j \cap Q_k$ or $\mathcal{S} \cap Q_i \cap Q_j = \{0\}$ or $\mathcal{S} \cap Q_i \cap Q_k = \{0\}$.

Definition 4.2. Given selfadjoint operators $B_1, \ldots, B_m \in \mathcal{L}(\mathcal{H})$, $m \geq 2$, the set $\{B_1, \ldots, B_m\}$ is *strongly indefinite* if

i) $\{B_1, \ldots, B_m\}$ is weakly indefinite;
ii) given $i, j, k = 1, \ldots, m$, if $x_\pm \in \mathcal{P}_i^\pm \cap Q_j \cap Q_k$ then there exists $\theta \in [0, \pi)$ such that B_j and B_k are definite in $\{\alpha x_- + \beta e^{i\theta} x_+ : \alpha, \beta \in \mathbb{R}\}$.

Given a selfadjoint operator $B \in \mathcal{L}(\mathcal{H})$, assume that $\{y, z\}$ is a linearly independent set in $Q(B)$. Then, B is definite in $\{\alpha y + \beta z : \alpha, \beta \in \mathbb{R}\}$ if and only if $\operatorname{Re} \langle By, z \rangle = 0$. In fact, if $x_\pm = ty \pm (1-t)z$ for some $t \in \mathbb{R}$ then

$$\langle Bx_\pm, x_\pm \rangle = \pm 2t(1-t) \operatorname{Re} \langle By, z \rangle,$$

and these two real numbers have the same sign if and only if $\operatorname{Re} \langle By, z \rangle = 0$. Therefore, item ii) in Definition 4.2 can be alternatively stated as:

ii') given $i, j, k = 1, \ldots, m$, if $x_\pm \in \mathcal{P}_i^\pm \cap Q_j \cap Q_k$ then there exists $\theta \in [0, \pi)$ such that $\operatorname{Re} \left\langle B_j x_+, e^{i\theta} x_- \right\rangle = \operatorname{Re} \left\langle B_k x_+, e^{i\theta} x_- \right\rangle = 0$.

Remark 4.3. If $\{B_1, \ldots, B_m\}$ is strongly indefinite, then it is immediate that $\{B_i\}_{i \in \mathcal{F}}$ is strongly indefinite for every $\mathcal{F} \subseteq \{1, \ldots, m\}$.

From now on we assume that $m > 2$. Given $y, z \in \mathcal{H}$, $y \neq z$, consider

$$[y, z] := \left\{ ty + (1-t)z \,:\, t \in [0, 1] \right\},$$

and $(y, z) := \left\{ ty + (1-t)z \,:\, t \in (0, 1) \right\}$.

Proposition 4.4. *Let $\{B_1, \ldots, B_m\}$ be a strongly indefinite set. Given $i = 1, \ldots, m$, if there exists $x_\pm \in \mathcal{P}_i^\pm \cap \bigcap_{j \neq i} Q_j$ then there exists $\theta \in [0, \pi)$ such that*

$$[x_-, \pm e^{i\theta} x_+] \subseteq \textstyle\bigcap_{j \neq i} Q_j.$$

Moreover, there exists $y_i^\pm \in (x_-, \pm e^{i\theta} x_+)$ such that

$$y_i^\pm \in \bigcap_{i=1}^m Q_i \setminus N\left(\textstyle\sum_{j=1}^m \mu_j B_j\right) \tag{4.2}$$

for every $(\mu_1, \ldots, \mu_m) \in \mathbb{R}^m$ with $\mu_i \neq 0$.

Proof. Suppose that $x_\pm \in \mathcal{P}_i^\pm \cap \bigcap_{l \neq i} Q_l$ for some fixed $i = 1, \ldots, m$. Now choose $j \in \{1, \ldots, m\} \setminus \{i\}$. Considering $k_1, k_2 \in \{1, \ldots, m\} \setminus \{i\}$, there exist $\theta_1, \theta_2 \in [0, \pi)$ such that

$$\operatorname{Re}\left(e^{-i\theta_1} \langle B_j x_-, x_+ \rangle\right) = 0 = \operatorname{Re}\left(e^{-i\theta_1} \langle B_{k_1} x_-, x_+ \rangle\right), \text{ and}$$
$$\operatorname{Re}\left(e^{-i\theta_2} \langle B_j x_-, x_+ \rangle\right) = 0 = \operatorname{Re}\left(e^{-i\theta_2} \langle B_{k_2} x_-, x_+ \rangle\right).$$

This implies that $\theta_2 = \theta_1 + n\pi$ for some $n \in \mathbb{N}$, and consequently

$$\operatorname{Re} \left\langle B_{k_2} x_-, e^{i\theta_1} x_+ \right\rangle = \pm \operatorname{Re} \left\langle B_{k_2} x_-, e^{i\theta_2} x_+ \right\rangle = 0.$$

Since k_2 was arbitrary, it then holds that

$$\operatorname{Re} \left\langle B_k x_-, e^{i\theta_1} x_+ \right\rangle = 0 \qquad \text{for every } k \neq i.$$

Therefore, $[x_-, \pm e^{i\theta}x_+] \subseteq Q_k$ for every $k \neq i$ because $x_\pm \in \bigcap_{k\neq i} Q_k$.

Finally, following the same procedure as in the proof of Proposition 2.3, there exists $t \in (0,1)$ such that

$$y_i^+ := tx_- + (1-t)e^{i\theta}x_+ \in \textstyle\bigcap_{i=1}^m Q_i \setminus \{0\}.$$

Given $(\mu_1, \dots, \mu_m) \in \mathbb{R}^m$ with $\mu_i \neq 0$, consider $B := B_i + \sum_{j\neq i} \frac{\mu_j}{\mu_i} B_j$ and assume that $By_i^+ = 0$. Then, $Bx_- = -\frac{1-t}{t}e^{i\theta}Bx_+$. Since $x_\pm \in \bigcap_{j\neq i} Q_i$, we have that

$$\begin{aligned} 0 = \langle By_i^+, y_i^+ \rangle &= t^2 \langle Bx_-, x_- \rangle + (1-t)^2 \langle Bx_+, x_+ \rangle + 2t(1-t)\operatorname{Re}\langle Bx_-, e^{i\theta}x_+ \rangle \\ &= t^2 \langle Bx_-, x_- \rangle + (1-t)^2 \langle Bx_+, x_+ \rangle - 2(1-t)^2 \operatorname{Re}\langle Bx_+, x_+ \rangle \\ &= t^2 \langle B_i x_-, x_- \rangle - (1-t)^2 \operatorname{Re}\langle B_i x_+, x_+ \rangle < 0, \end{aligned}$$

leading to a contradiction. Therefore, $y_i^+ \notin N(B)$. A similar argument proves the existence of y_i^-. □

Remark 4.5. By the proof of Theorem 2.4, every weakly indefinite set $\{B_1, B_2\}$ is strongly indefinite. Also, if $\{B_1, \dots, B_m\}$ is a strongly indefinite set and there exists $j = 1, \dots, m$ such that B_j is indefinite in $\bigcap_{i\neq j} Q_i$, then $\bigcap_{i=1}^m Q_i \neq \{0\}$ (see Proposition 4.4). Hence, $\{B_1, B_2, B_3, B_4\}$ from Example 3.4 is an indefinite set which is not strongly indefinite.

The following result generalizes Krein-Smul'jan theorem for $m \geq 3$.

Theorem 4.6. *Given selfadjoint operators $A, B_1, \dots, B_m \in \mathcal{L}(\mathcal{H})$, assume that $\{B_1, B_2, \dots, B_m\}$ is strongly indefinite. Then,*

$$A \geq 0 \quad \text{in} \quad \textstyle\bigcap_{i=1}^m Q_i \qquad \text{if and only if} \qquad \Omega \neq \varnothing.$$

Proof. The fact that $\Omega \neq \varnothing$ implies $A \geq 0$ in $\bigcap_{i=1}^m Q_i$ is trivial. We prove the converse by induction on m. The case for $m = 2$ operators B_1, B_2 follows readily from Theorem 2.4.

For the inductive step fix $n \in \mathbb{N}$, $n \geq 3$, and assume the statement holds for $m = n-1$. Now consider selfadjoint operators $B_1, B_2, \dots, B_n \in \mathcal{L}(\mathcal{H})$ such that $\{B_1, B_2, \dots, B_n\}$ is strongly indefinite.

First, let us show that $B_j \not\equiv 0$ in $\bigcap_{i\neq j} Q_i$ for every $j = 1, \dots, n$. Indeed, if there exists $j = 1, \dots, n$ such that $B_j \equiv 0$ in $\bigcap_{i\neq j} Q_i$ then, by inductive hypothesis, there exists $(\lambda_i)_{i\neq j} \in \mathbb{R}^{n-1}$ such that $B_j + \sum_{i\neq j} \lambda_i B_i \geq 0$, which is a contradiction. Then $\{B_1, \dots, B_n\}$ is indefinite (by Remark 3.6) and, by Proposition 4.4, $\bigcap_{i=1}^n Q_i \neq \{0\}$.

Now, assume that $A \geq 0$ in $\bigcap_{i=1}^n Q_i$. Since B_n is indefinite in $\bigcap_{i=1}^{n-1} Q_i$, take $y \in \mathcal{P}_n^- \cap \bigcap_{i=1}^{n-1} Q_i$ and $z \in \mathcal{P}_n^+ \cap \bigcap_{i=1}^{n-1} Q_i$. Again, by Proposition 4.4, there exist $\theta \in [0, \pi)$ and $t_\pm \in (0,1)$ such that $x_\pm := t_\pm y \pm (1-t_\pm)e^{i\theta}z \in \bigcap_{i=1}^n Q_i$. Then $\langle Ax_\pm, x_\pm \rangle \geq 0$.

Following a procedure similar to the one in the proof of Theorem 2.4 we then get that

$$\frac{\langle Ay, y \rangle}{\langle B_n y, y \rangle} \leq \frac{\langle Az, z \rangle}{\langle B_n z, z \rangle},$$

for arbitrary $y \in \mathcal{P}_n^- \cap \bigcap_{i=1}^{n-1} Q_i$ and $z \in \mathcal{P}_n^+ \cap \bigcap_{i=1}^{n-1} Q_i$. Hence, there exists $\lambda_n \in \mathbb{R}$ such that

$$\langle (A + \lambda_n B_n)x, x \rangle \geq 0 \quad \text{for every } x \in \big(\mathcal{P}_n^- \cap \textstyle\bigcap_{i=1}^{n-1} Q_i\big) \cup \big(\mathcal{P}_n^+ \cap \bigcap_{i=1}^{n-1} Q_i\big).$$

Considering that $\langle Ax, x \rangle \geq 0$ for every $x \in \bigcap_{i=1}^n Q_i$, we have that

$$\langle (A + \lambda_n B_n)x, x \rangle \geq 0 \qquad \text{for every } x \in \textstyle\bigcap_{i=1}^{n-1} Q_i.$$

Then, applying the inductive hypothesis to $A' := A + \lambda_n B_n$ and the strongly indefinite set $\{B_1, B_2, \ldots, B_{n-1}\}$, there exists $(\lambda_1, \lambda_2, \ldots, \lambda_{n-1}) \in \mathbb{R}^{n-1}$ such that

$$(A + \lambda_n B_n) + \sum_{i=1}^{n-1} \lambda_i B_i \geq 0,$$

i.e. $(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \Omega$, completing the proof. □

Corollary 4.7. *Given selfadjoint operators $B_1, \ldots, B_m \in \mathcal{L}(\mathcal{H})$, if $\{B_1, \ldots, B_m\}$ is strongly indefinite then $\{B_1, \ldots, B_m\}$ is indefinite.*

Proof. Suppose that $\{B_1, \ldots, B_m\}$ is strongly indefinite and there exists $i = 1, \ldots, m$ such that B_i is definite in $\bigcap_{j \neq i} Q_j$. Let us assume that $B_i \geq 0$ in $\bigcap_{j \neq i} Q_j$. Note that $\{B_1, \ldots, B_{i-1}, B_{i+1}, \ldots, B_m\}$ is also strongly indefinite and, by Theorem 4.6, there exists $(\lambda_j)_{j \neq i} \in \mathbb{R}^{m-1}$ such that $B_i + \sum_{j \neq i} \lambda_j B_j \geq 0$, which is a contradiction to $\{B_1, \ldots, B_m\}$ being weakly indefinite. □

Corollary 4.8. *Given selfadjoint operators $B_1, \ldots, B_m \in \mathcal{L}(\mathcal{H})$, the following conditions are equivalent:*

i) $\{B_1, \ldots, B_m\}$ *is strongly indefinite.*

ii) *(a)* $\{B_1, \ldots, B_m\}$ *is indefinite;*

(b) if $x_\pm \in \mathcal{P}_i^\pm \cap Q_j \cap Q_k$ then there exists $\theta \in [0, \pi)$ such that

$$\operatorname{Re} \langle B_j x_+, e^{i\theta} x_- \rangle = \operatorname{Re} \langle B_k x_+, e^{i\theta} x_- \rangle = 0.$$

iii) *(a) for each $i = 1, \ldots, m$ there exists $x_i \in \bigcap_{j=1}^m Q_j \setminus N(\sum_{j=1}^m \mu_j B_j)$ for every choice of $(\mu_1, \ldots, , \mu_m) \in \mathbb{R}^m$ with $\mu_i \neq 0$;*

(b) if $x_\pm \in \mathcal{P}_i^\pm \cap Q_j \cap Q_k$ then there exist $\theta \in [0, \pi)$ and $x_\theta \in (x_-, e^{i\theta} x_+)$ such that

$$x_\theta \in Q_i \cap Q_j \cap Q_k.$$

Proof. i)→ii) If $\{B_1, \ldots, B_m\}$ is strongly indefinite then, by Corollary 4.7, the set $\{B_1, \ldots, B_m\}$ is indefinite. We have already mentioned that *(b)* is equivalent to the second condition in Definition 4.2.

ii)→iii) Item (a) follows from the fact that $\{B_1, \ldots, B_m\}$ is indefinite and Proposition 4.4. Fix $i, j, k \in \{1, 2, \ldots, m\}$ and take $x_\pm \in \mathcal{P}_i^\pm \cap Q_j \cap Q_k$. Since $\{B_1, \ldots, B_m\}$ is indefinite, $\{B_i, B_j, B_k\}$ is also indefinite, and the result follows from Proposition 4.4.

iii)→i) To see that $\{B_1, \ldots, B_m\}$ is weakly indefinite, consider $(\mu_1, \ldots, \mu_m) \in \mathbb{R}^m$ and suppose that $\mu_i \neq 0$ for some $i = 1, \ldots, m$. Then, by (a), there exists $x_i \in \bigcap_{j=1}^m Q_j \setminus N(\sum_{j=1}^m \mu_j B_j)$. Hence, $x_i \in Q(\sum_{j=1}^m \mu_j B_j) \setminus N(\sum_{j=1}^m \mu_j B_j)$, which implies that $\sum_{j=1}^m \mu_j B_j$ is indefinite. Since $(\mu_1, \ldots, \mu_m) \in \mathbb{R}^m \setminus \{0\}$ was arbitrary, we have that $\{B_1, \ldots, B_m\}$ is weakly indefinite.

Fix $i, j, k \in \{1, 2, \ldots, m\}$ and take $x_\pm \in \mathcal{P}_i^\pm \cap Q_j \cap Q_k$. By $iii)$ there exist $\theta \in [0, \pi)$ and $t_0 \in (0, 1)$ such that $x_\theta := t_0 x_- + (1 - t_0) e^{i\theta} x_+ \in Q_i \cap Q_k \cap Q_k$. This implies that $\operatorname{Re} \langle B_j x_-, e^{i\theta} x_+ \rangle = \operatorname{Re} \langle B_k x_-, e^{i\theta} x_+ \rangle = 0$, which in turn implies that B_j and B_k are definite in $\{\alpha x_- + \beta e^{i\theta} x_+ : \alpha, \beta \in \mathbb{R}\}$. □

5. A sufficient condition for strongly indefiniteness

Given a set of selfadjoint operators $\{B_1, \ldots, B_m\}$ in $\mathcal{L}(\mathcal{H})$, we now present a sufficient condition to guarantee that it is a strongly indefinite set. It is inspired by previous works by Hestenes and McShane for the (real) finite dimensional case. Given symmetric matrices $A, B_1, \ldots, B_m \in \mathbb{R}^{n \times n}$, assume that $\{B_1, \ldots, B_m\}$ is weakly indefinite. In [9] the authors included the following additional condition: for every subspace L of $\mathbb{R}^n$ such that $L \cap (\bigcap_{i=1}^m Q_i) = \{0\}$ there exists $(\mu_1, \ldots, \mu_m) \in$

$\mathbb{R}^m \setminus \{0\}$ such that $\sum_{i=1}^m \mu_i B_i$ is positive definite in the subspace L. Under these assumptions they showed that, if A is positive definite in $\bigcap_{i=1}^m Q_i$ then there exists $(\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^m$ such that $A + \sum_{i=1}^m \lambda_i B_i$ is positive definite.

Hypotheses (HM). *Given $m \geq 3$ and $B_1, \ldots, B_m \in \mathcal{L}(\mathcal{H})$, assume that the set $\{B_1, \ldots, B_m\}$ is weakly indefinite. Assume also that if $\mathcal{S}$ is a real subspace of $\mathcal{H}$ with $\dim \mathcal{S} = 2$ and $\{i, j, k\} \subset \{1, \ldots, m\}$ is a trio such that*

$$\mathcal{S} \cap (Q_i \cap Q_j \cap Q_k) = \{0\}$$

then there exists $(\lambda_i, \lambda_j, \lambda_k) \in \mathbb{R}^3 \setminus \{0\}$ such that $\lambda_i B_i + \lambda_j B_j + \lambda_k B_k \geq 0$ in $\mathcal{S}$ and $\lambda_i B_i + \lambda_j B_j + \lambda_k B_k \neq 0$ in $\mathcal{S}$.

If the set $\{B_1, \ldots, B_m\}$ satisfies Hypotheses (HM) then it is immediate that $\{B_i\}_{i \in \mathcal{F}}$ also satisfies Hypotheses (HM) for every $\mathcal{F} \subseteq \{1, \ldots, m\}$ with $|\mathcal{F}| \geq 3$.

Proposition 5.1. *Given $m \geq 3$ and $B_1, \ldots, B_m \in \mathcal{L}(\mathcal{H})$, assume that $\{B_1, \ldots, B_m\}$ satisfies Hypotheses (HM). Then, $\{B_1, \ldots, B_m\}$ is a strongly indefinite set.*

Proof. We prove the result by induction on m. First, assume that $m = 3$. By Corollary 2.5, $\{B_1, B_2, B_3\}$ is indefinite. Fix $i \in \{1, 2, 3\}$ and consider $x_\pm \in \mathcal{P}_i^\pm \cap Q_j \cap Q_k$. On the one hand, if $k = j$ then $x_\pm \in \mathcal{P}_i^\pm \cap Q_j$ and we can choose $\theta \in [0, \pi)$ such that $\operatorname{Re} \left\langle B_j x_+, e^{i\theta} x_- \right\rangle = 0$. Hence, in this case we have that $B_j = B_k$ is zero in $\mathcal{S} = \{\alpha x_+ + \beta e^{i\theta} x_- : \ \alpha, \beta \in \mathbb{R}\}$.

On the other hand, if $k \neq j$ choose $\theta \in [0, \pi)$ such that $\operatorname{Re} \left\langle B_j x_+, e^{i\theta} x_- \right\rangle = 0$ and consider the real subspace

$$\mathcal{S} = \{\alpha x_+ + e^{i\theta} \beta x_- \ : \ \alpha, \beta \in \mathbb{R}\} \subseteq Q_j.$$

If $\mathcal{S} \cap Q_i \cap Q_j \cap Q_k = \{ 0 \}$, then there exist $\lambda_i, \lambda_j, \lambda_k \in \mathbb{R}$ such that $B := \lambda_i B_i + \lambda_j B_j + \lambda_k B_k \geq 0$ (and non zero) in $\mathcal{S}$. Thus,

$$0 \leq \langle Bx_, x_- \rangle = \lambda_i \langle B_i x_-, x_- \rangle \qquad \text{and} \qquad 0 \leq \langle Bx_+, x_+ \rangle = \lambda_i \langle B_i x_+, x_+ \rangle .$$

But $\langle B_i x_-, x_- \rangle < 0$ and $\langle B_i x_+, x_+ \rangle > 0$ implies that $\lambda_i = 0$. Since $B_j\big|_{\mathcal{S}} = 0$ we get that $B\big|_{\mathcal{S}} = \lambda_k B_k\big|_{\mathcal{S}}$. Then

$$0 \leq \left\langle B(x_+ + e^{i\theta} x_-), x_+ + e^{i\theta} x_- \right\rangle = 2\lambda_k \operatorname{Re} \left\langle B_k x_+, e^{i\theta} x_- \right\rangle,$$
$$0 \leq \left\langle B(x_+ - e^{i\theta} x_-), x_+ - e^{i\theta} x_- \right\rangle = -2\lambda_k \operatorname{Re} \left\langle B_k x_+, e^{i\theta} x_- \right\rangle,$$

and consequently either $\lambda_k = 0$ or $\operatorname{Re} \left\langle B_k x_+, e^{i\theta} x_- \right\rangle = 0$. But if $\lambda_k = 0$ then $B\big|_{\mathcal{S}} = 0$, leading to a contradiction. Hence, $\operatorname{Re} \left\langle B_k x_+, e^{i\theta} x_- \right\rangle = 0$ and condition ii') is verified. Therefore, $\{B_1, B_2, B_3\}$ is strongly indefinite.

For the inductive step fix $n \in \mathbb{N}$, $n \geq 4$, and assume the statement holds for $m = n - 1$ operators. Now consider $B_1, \ldots, B_n \in \mathcal{L}(\mathcal{H})$ satisfying the hypotheses.

Hence, by inductive hypothesis, $\{B_1, \ldots, B_{k-1}, B_{k+1}, \ldots, B_n\}$ is strongly indefinite for $k = 1, \ldots, n$. Then, by Remark 3.6, $\{B_1, \ldots, B_n\}$ is indefinite.

Now take three different indices $i, j, k \in \{1, \ldots, n\}$. If none of them is equal to n, by inductive hypothesis, item ii) in the definition of strongly indefiniteness is satisfied. Assume that $k = n$ and take $y \in \mathcal{P}_n^- \cap \bigcap_{l=1}^{n-1} Q_l$, $z \in \mathcal{P}_n^+ \cap \bigcap_{l=1}^{n-1} Q_l$. Then, choose $\theta \in [0, \pi)$ such that $\operatorname{Re} \left\langle B_i y, e^{i\theta} z \right\rangle = 0$ and consider the real subspace $\mathcal{S} = \{\alpha y + e^{i\theta} \beta z \ : \ \alpha, \beta \in \mathbb{R}\} \subseteq Q_i$. If $\mathcal{S} \cap Q_i \cap Q_j \cap Q_n = \{ 0 \}$ then, following the same procedure as in the previous step, $\operatorname{Re} \left\langle B_j y, e^{i\theta} z \right\rangle = 0$ and condition ii') is verified. Therefore, $\{B_1, \ldots, B_n\}$ is strongly indefinite. □

Corollary 5.2. *Given $B_1, \ldots, B_m \in \mathcal{L}(\mathcal{H})$ assume that $\{B_1, \ldots, B_m\}$ satisfies Hypotheses (HM). If $A \in \mathcal{L}(\mathcal{H})$ is selfadjoint, then*

$$A \geq 0 \quad \textit{in} \quad \bigcap\nolimits_{i=1}^m Q_i \qquad \textit{if and only if} \qquad \Omega \neq \varnothing.$$

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