Norm inequalities for the spectral spread of Hermitian operators

Pedro Massey, Demetrio Stojanoff and Sebastian Zarate *
Centro de Matemática, FCE-UNLP, La Plata and IAM-CONICET, Argentina

Abstract

In this work we introduce a new measure of dispersion of the eigenvalues of a Hermitian (self-adjoint) compact operator, that we call spectral spread. Then, we obtain several inequalities for unitarily invariant norms involving the spectral spread of self-adjoint compact operators, that are related with Bhatia-Davis's and Corach-Porta-Recht's work on Arithmetic-Geometric mean inequalities, Zhan's inequality for the difference of positive compact operators, Tao's inequalities for anti-diagonal blocks of positive compact operators and Kittaneh's commutator inequalities for positive operators.

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Contents

*Partially supported by CONICET (PICT ANPCyT 1505/15) and Universidad Nacional de La Plata (UNLP 11X829) e-mail addresses: massey@mate.unlp.edu.ar , demetrio@mate.unlp.edu.ar , seb4.zarate@gmail.com

1 Introduction

The development of inequalities involving compact operators is a central topic in operator theory. The literature related with this research area is vast (see [1, 2, 5, 8, 9, 16, 17, 18, 19, 20, 21, 27, 28] just to mention a few works strictly related with our present research). Therefore obtaining new inequalities, involving new concepts, is of interest in itself. In this work we obtain several inequalities for unitarily invariant norms of compact operators in terms of a new notion related with the spectral dispersion of a compact Hermitian (self-adjoint) operator.

In order to put our results into context, we first describe the following fundamental inequalities for singular values. We begin by considering the (non-commutative) Arithmetic-Geometric mean inequality (AG) for operators: given two compact operators $A, B \in K(\mathcal{H})$, Bhatia and Kittaneh showed in [8] that

$$2\, s_i(A\, B^*) \leq s_i(A^*A + B^*B) \ , \qquad \text{for} \qquad i \in \mathbb{N} \, , \tag{1}$$

where $s(C) = (s_i(C))_{i\in\mathbb{N}}$ denotes the sequence of singular values of a compact operator $C \in K(\mathcal{H})$ arranged in non-increasing order. As a consequence we get that given bounded operators $S, C \in B(\mathcal{H})$ such that $C^*C + S^*S \leq I$ and a positive compact operator $E \in K(\mathcal{H})$ then

$$2\, s_i(S\, E\, C^*) \leq s_i(E) \ , \qquad \text{for} \qquad i \in \mathbb{N} \, , \tag{2}$$

by taking $A = SE^{1/2}$ and $E^{1/2}C^* = B^*$ in Eq. (1). Corach, Porta and Recht, motivated by their study of the geometry in the context of operator algebras, obtained in [10] the following inequality with respect to a unitarily invariant norm $N(\cdot)$

$$N(T) \leq \frac{1}{2} N(STS^{-1} + S^{-1}TS) \, , \tag{3}$$

where $T \in K(\mathcal{H})$ is a compact operator and $S \in B(\mathcal{H})$ is self-adjoint and invertible bounded operator. Later on, Bhatia and Davis showed in [7] the following AG-type inequality with respect to a unitarily invariant norm

$$N(A^*XB) \leq \frac{1}{2} N(AA^*X + XBB^*) \, , \tag{4}$$

where $X \in K(\mathcal{H})$ is a compact operator and A, $B \in B(\mathcal{H})$ are bounded operators. It turns out that inequalities in Eqs. (3) and (4) are equivalent (by simple substitutions).

These AG-type inequalities (both for singular values and for unitarily invariant norms) have been studied and extended in different contexts ([1, 4, 5]); it turns out that they are related with deep geometric properties of operators [2, 10, 11, 12]. On the other hand, the AG inequalities are related with some other fundamental inequalities for singular values. Indeed, in [27] (see also [28, 29]) Zhan showed that if $C, D \in K(\mathcal{H})$ are compact and positive operators, then

$$s_i(C - D) \leq s_i(C \oplus D) \ , \qquad \text{for} \qquad i \in \mathbb{N} \, , \qquad \text{where} \qquad C \oplus D = \begin{bmatrix} C & 0 \\ 0 & D \end{bmatrix} . \tag{5}$$

It has been shown that Eq. (5) is equivalent to AG inequality (1) (see [5, 28, 29]).

It is interesting to notice that the inequality in Eq. (5) can be put into the context of commutator inequalities for compact operators (see [15, 16, 18, 19]). Indeed, Kittaneh obtained the following generalization of Eq. (5) in [19]: given positive compact operators $C, D \in K(\mathcal{H})^+$ and a bounded operator $X \in B(\mathcal{H})$, then

$$s_i(CX - XD) \le \|X\|\, s_i(C \oplus D) \ , \qquad \text{for} \qquad i \in \mathbb{N} \, , \tag{6}$$

where $\|\cdot\|$ denotes the operator (or spectral) norm.

There is yet another inequality related with Eqs. (1) (5) and (6), that we would like to consider. Indeed, Tao showed in [26] that given a positive compact operator $F \in K(\mathcal{H} \oplus \mathcal{K})$ represented as a block matrix

$$F = \begin{bmatrix} F_1 & G \\ G^* & F_2 \end{bmatrix} \quad \text{then} \quad 2\, s_i(G) \le s_i(F) \ , \quad \text{for} \quad i \in \mathbb{N} \, . \tag{7}$$

Moreover, it turns out that Eq. (7) is equivalent to Eq. (1) and Eq. (5) (see [26]).

It is natural to ask whether the inequalities in Eqs. (2), (5), (6) or (7) hold in the more general case in which E, A, B, C, D and F are self-adjoint compact operators. It turns out that all these singular value inequalities fail in this more general setting. Motivated by our previous work [24, 25] (in the finite dimensional case) we first introduce a new notion that we call the *spectral spread of a self-adjoint operator* that lies in the algebra $\mathcal{A} = K(\mathcal{H}) + \mathbb{C}\, I$ formed by compact perturbations of multiples of the identity operator acting on $\mathcal{H}$. Then, we obtain inequalities in terms of submajorization (which can be regarded as inequalities for unitarily invariant norms) with respect to the spectral spread, in the general context of compact self-adjoint operators. We regard these new inequalities as natural substitutes of the singular value inequalities mentioned above (that are stronger, but only valid for positive compact operators). We point out in a series of examples that these inequalities (relative to the spectral spread) cannot be improved to the entry-wise case as in the positive case.

In order to describe the spectral spread of a compact self-adjoint operator $A \in K(\mathcal{H})$, we consider its full spectral scale, that is the (unique) sequence $\lambda(A) = (\lambda_i(A))_{i \in \mathbb{Z}_0}$ (here $\mathbb{Z}_0 = \mathbb{Z} \setminus \{0\}$) such that its terms are the eigenvalues of A (or zero), counting multiplicities and arranged so that

$$\lambda_{-i}(A) \le \lambda_{-i-1}(A) \le 0 \le \lambda_{i+1}(A) \le \lambda_i(A) \quad \text{for} \quad i \in \mathbb{N} \, .$$

Then, the spectral spread of A, noted $\mathrm{Spr}^+(A) \in c_0(\mathbb{N})$, is given by

$$\mathrm{Spr}^+(A) = (\, \lambda_i(A) - \lambda_{-i}(A)\,)_{i \in \mathbb{N}} \, .$$

Notice that $\mathrm{Spr}^+(A)$ is a non-increasing non-negative sequence that measures the dispersion of the eigenvalues of A. After recalling some well known facts and elementary results related with spectral scales and singular values (see Section 2) we obtain several inequalities for unitarily invariant norms, associated to a symmetrically normed ideal in $B(\mathcal{H})$ (see Sections 3 and 4). In order to do this, we obtain submajorization relations ($\prec_w$) between singular values of compact operators (which is a well known technique, see [14]).

Indeed, in Section 3 we give new proofs of the infinite dimensioinal version of two inequalities proved in [24]: we show that given a self-adjoint compact operator $F \in K(\mathcal{H} \oplus \mathcal{K})$ represented as a block matrix

$$F = \begin{bmatrix} F_1 & G \\ G^* & F_2 \end{bmatrix} \quad \text{then} \quad 2\, s(G) \prec_w \mathrm{Spr}^+(F) \, . \tag{8}$$

We regard Eq. (8) as a natural substitute-generalization of Eq. (7), since $s(F) = \mathrm{Spr}^+(F)$ for positive compact operators. As a consequence of this fact, we show that if $C, D \in K(\mathcal{H})$ are compact self-adjoint operators, then

$$s(C - D) \prec_w \mathrm{Spr}^+(C \oplus D)\,, \tag{9}$$

that we regard as a natural substitute of Eq. (5). In Section 4 we first consider submajorization relations involving singular values of commutators and the spectral spread; we show that given self-adjoint compact operators $A\,,\ B \in K(\mathcal{H})$ we have that

$$s(AB - BA) \prec_w \frac{1}{2}\,\mathrm{Spr}^+(A \oplus A)\,\cdot\,\mathrm{Spr}^+(B \oplus B) \tag{10}$$

where $\cdot$ means the entry-wise product of these sequences. This inequality is not known even for matrices (we give a proof for this case in the Appendix). Using (10) we give a proof of the infinite dimensional version of an inequality proved in [24]: given self-adjoint compact operators $C,\ D \in K(\mathcal{H})$ and a compact operator $X \in K(\mathcal{H})$, we have that

$$s(CX - XD) \prec_w \mathrm{Spr}^+(C \oplus D)\,\cdot\,s(X)\ . \tag{11}$$

We consider Eq. (11) as a natural substitute of Eq. (6).

We point out that analogues of some of these inequalities (e.g. Eqs. (8), (9) and (11)) were previously proved for the spectral spread of matrices in [24]. As we explain in the Appendix, the spectral spread of matrices is different from the spectral spread of compact operators, so we need new proofs. On the other hand, by adapting some of our present arguments we obtain some inequalities which are new for the spread of matrices (see the Appendix).

We also explore the relation between the spectral spread and AG-type inequalities for unitarily invariant norms of selfadjoint compact operators. We show that for bounded operators $S, C \in B(\mathcal{H})$ such that $C^*C + S^*S = P$ where P is an orthogonal projection, and a self-adjoint compact operator $E \in K(\mathcal{H})$ then

$$2\,s(SEC^*) \prec_w \mathrm{Spr}^+(E)\,. \tag{12}$$

This inequality, that is a substitute of Eq. (2), allows us to derive other forms of AG-type inequalities involving the spectral spread related with Eq. (4): indeed, given $A,\ B \in B(\mathcal{H})$ and $E \in K(\mathcal{H})$ a self-adjoint compact operator, we show that

$$s(A\,E\,B^*) \prec_w \frac{1}{2}\,\mathrm{Spr}^+\big(\,(A^*A + B^*B)^{1/2}\,E\,(A^*A + B^*B)^{1/2}\,\big)\ . \tag{13}$$

Finally, we show that the inequalities in Eqs. (8), (9), (10), (11) and (13) are all equivalent. In the Appendix we also discuss, in some detail, versions of these inequalities related with the spectral spread of self-adjoint matrices introduced in [24].

2 Preliminaries

In this section we introduce the basic notation and definitions used throughout our work.

Notation and terminology. We let $\mathcal{H}$ be an infinite dimensional separable Hilbert space and we denote by $B(\mathcal{H})$ the algebra of linear operators over $\mathcal{H}$. In this case, $K(\mathcal{H}) \subset B(\mathcal{H})$

denotes the ideal of compact operators and $\mathcal{U}(\mathcal{H}) \subset B(\mathcal{H})$ denotes the group of unitary operators acting on $\mathcal{H}$. In what follows $B(\mathcal{H})^{sa}$ and $K(\mathcal{H})^{sa}$ denote the real subspaces of self-adjoint and compact and self-adjoint operators, respectively. Also, we denote by $B(\mathcal{H})^+$ the cone of positive operators and $K(\mathcal{H})^+ = B(\mathcal{H})^+ \cap K(\mathcal{H})$.

We write $\mathbb{I}_k = \{1, ..., k\} \subset \mathbb{N}$, for $k \in \mathbb{N}$, and $\mathbb{I}_\infty = \mathbb{N}$.

2.1 Submajorization of sequences in c_0

In what follows we let $\mathbb{M}$ denote $\mathbb{N}$ or $\mathbb{Z}_0 \overset{\text{def}}{=} \mathbb{Z} \setminus \{0\}$. We consider the space $c_0(\mathbb{M})$ of sequences $a = (a_n)_{n\in\mathbb{M}}$ such that for every $\epsilon > 0$, the set $\{n \in \mathbb{M} : |a_n| > \epsilon\}$ is finite. We will need both $\mathbb{N}$ and $\mathbb{Z}_0$ as index sets to describe the (ordered rearrangements of) singular values and eigenvalues of self-adjoint compact operators, respectively (as in [23]). Given $k \in \mathbb{N}$, we denote by $\mathcal{P}_k(\mathbb{M})$ the set of finite subsets of $\mathbb{M}$ with exactly k elements.

Definition 2.1. Let $a = (a_n)_{n\in\mathbb{M}} \in c_0(\mathbb{M}) \cap \mathbb{R}^{\mathbb{M}}$ be a real sequence. We define the following rearranged sequences associated to a:

1. If $a \in \mathbb{R}^{\mathbb{M}}_{\geq 0}$, we define $a^\downarrow \in c_0(\mathbb{N}) \cap \mathbb{R}^{\mathbb{N}}_{\geq 0}$ uniquely determined by the (recursive) conditions

$$a_1^\downarrow = \max_{k\in\mathbb{M}} a_k \quad \text{and} \quad \sum_{k=1}^{n} a_k^\downarrow = \max_{\mathbb{F}\in\mathcal{P}_n(\mathbb{M})} \sum_{i\in\mathbb{F}} a_i \quad \text{for every} \quad n \in \mathbb{N}\,.$$

   For example, if $a_1^\downarrow = a_r$, then $a_2^\downarrow = \max_{k\neq r} a_k$ and so on. Hence $a_{n+1}^\downarrow \leq a_n^\downarrow$ for every $n \in \mathbb{N}$.

2. If $a \in \mathbb{R}^{\mathbb{M}}_{\leq 0}$, we define $a^\uparrow = -(-a)^\downarrow \in \mathbb{R}^{\mathbb{N}}_{\leq 0}$. Hence

$$\sum_{k=1}^{n} a_k^\uparrow = \min_{\mathbb{F}\in\mathcal{P}_n(\mathbb{M})} \sum_{i\in\mathbb{F}} a_i \quad \text{and} \quad a_{n+1}^\uparrow \geq a_n^\uparrow \quad \text{for every} \quad n \in \mathbb{N}\,.$$

3. For a general $a \in c_0(\mathbb{M}) \cap \mathbb{R}^{\mathbb{M}}$ we define a^+ and $a^- \in c_0(\mathbb{M}) \cap \mathbb{R}^{\mathbb{M}}$ given by:

$$a_k^+ = \begin{cases} a_k & \text{if} \quad a_k \geq 0 \\ 0 & \text{if} \quad a_k < 0 \end{cases} \quad \text{for} \quad k \in \mathbb{M}\,,$$

   and $a^- = -(-a)^+ = a - a^+$, the negative part of a.

4. Finally we define $a^{\uparrow\downarrow} \in c_0(\mathbb{Z}_0)$ given by

$$a_n^{\uparrow\downarrow} = \begin{cases} (a^-)^\uparrow_{-n} & \text{if} \quad n < 0 \\ (a^+)^\downarrow_n & \text{if} \quad n > 0 \end{cases} \quad \text{for} \quad n \in \mathbb{Z}_0\,.$$

   We remark that even if a has finite positive (or negative) entries, the positive and negative parts of $a^{\uparrow\downarrow}$ are (by construction) infinite sequences. △

Remark 2.2. Notice that the re-arrangements of sequences $a \in c_0(\mathbb{M})$ given in Definition 2.1 do not have zero entries when a has infinitely many positive and negative entries. Then, in general, $a^{\uparrow\downarrow}$ is not a permutation of a. On the other hand, in case that $a \in c_0(\mathbb{M})$ has only k positive values, then $a_i^{\uparrow\downarrow} = 0$ for $k < i \in \mathbb{N}$. △

Definition 2.3. Let $a = (a_n)_{n\in\mathbb{M}_1} \in c_0(\mathbb{M}_1)$, $b = (b_n)_{n\in\mathbb{M}_2} \in c_0(\mathbb{M}_2)$ be real sequences, where $\mathbb{M}_j = \mathbb{N}$ or $\mathbb{M}_j = \mathbb{Z}_0$, for $j = 1, 2$ (but possibly $\mathbb{M}_1 \neq \mathbb{M}_2$). We say that a is submajorized by b and write

$$a \prec_w b \quad \text{if} \quad \sum_{k=1}^{n} a_k^{\uparrow\downarrow} \le \sum_{k=1}^{n} b_k^{\uparrow\downarrow} \quad \text{for every} \quad n \in \mathbb{N}\,.$$

We say that a is majorized by b and write

$$a \prec b \quad \text{if} \quad a \prec_w b \quad \text{and} \quad -a \prec_w -b\,. \qquad \triangle$$

Remark 2.4. With the notation of Definition 2.3 and using Definition 2.1 it is straightforward to show that

$$a \prec_w b \quad \text{if} \quad \max_{\mathbb{F}\in\mathcal{P}_n(\mathbb{M}_1)} \sum_{i\in\mathbb{F}} a_i^+ \le \max_{\mathbb{F}\in\mathcal{P}_n(\mathbb{M}_2)} \sum_{i\in\mathbb{F}} b_i^+ \ , \quad n \in \mathbb{N}\,.$$

Moreover, we also get that $a \prec b$ if

$$\max_{\mathbb{F}\in\mathcal{P}_n(\mathbb{M}_1)} \sum_{i\in\mathbb{F}} a_i^+ \le \max_{\mathbb{F}\in\mathcal{P}_n(\mathbb{M}_2)} \sum_{i\in\mathbb{F}} b_i^+ \quad \text{and} \quad \min_{\mathbb{F}\in\mathcal{P}_n(\mathbb{M}_1)} \sum_{i\in\mathbb{F}} a_i^- \ge \min_{\mathbb{F}\in\mathcal{P}_n(\mathbb{M}_2)} \sum_{i\in\mathbb{F}} b_i^- \ , \quad n \in \mathbb{N}\,.$$

$\triangle$

We end this section with the following constructions. First, given sequences $a = (a_n)_{n\in\mathbb{N}} \in c_0(\mathbb{N})$ and $b = (b_n)_{n\in\mathbb{N}} \in c_0(\mathbb{N})$ we let $(a, b) \in c_0(\mathbb{Z}_0)$ be the sequence determined by

$$(a\,,\,b)_n = \begin{cases} a_{-n} & \text{if} \quad n < 0 \\ b_n & \text{if} \quad n > 0 \end{cases} \quad \text{for} \quad n \in \mathbb{Z}_0\,. \tag{14}$$

For example, if $c \in c_0(\mathbb{M}) \cap \mathbb{R}^{\mathbb{M}}$ then $c^{\uparrow\downarrow} = (c^{\uparrow}, c^{\downarrow})$ (see Definition 2.1). Finally, given $a = (a_n)_{n\in\mathbb{N}} \in \ell^\infty(\mathbb{M})$ and $b = (b_n)_{n\in\mathbb{M}} \in c_0(\mathbb{M})$ we let

$$a \cdot b \in c_0(\mathbb{M}) \quad \text{be given by} \quad (a \cdot b)_n = a_n\, b_n \ , \quad \text{for} \quad n \in \mathbb{M}\,, \tag{15}$$

where $\mathbb{M} = \mathbb{N}$ or $\mathbb{M} = \mathbb{Z}_0$.

2.2 Compact self-adjoint operators and majorization

Definition 2.5. Let $A \in K(\mathcal{K}, \mathcal{H})$ be a compact transformation. Then we consider

1. The non-increasing sequence $s(A) = (s_n(A))_{n\in\mathbb{N}} \in c_0(\mathbb{N})$ of its singular values.

   If we assume further that $\mathcal{H} = \mathcal{K}$ and $A = A^*$, then we consider:

2. The sequence of its eigenvalues (counting multiplicity and in any order) $\mu(A) \in c_0(\mathbb{N})$;

3. The ordered sequence $\lambda(A) \overset{\text{def}}{=} \mu(A)^{\uparrow\downarrow} \in c_0(\mathbb{Z}_0)$, called the full spectral scale of A (also called "complete system of eigenvalues" by Markus in [23]). $\triangle$

Remark 2.6. Let $A \in K(\mathcal{H})^{sa}$ be a compact self-adjoint operator, and let $\mu(A) \in c_0(\mathbb{N})$ and $\lambda(A) \in c_0(\mathbb{Z}_0)$ be as in Definition 2.5. First notice that if we let A^+, $A^- \in K(\mathcal{H})^+$ the positive and negative parts of A, then $A = A^+ - A^-$ and

$$\lambda_i(A) = \lambda_i(A^+) \quad \text{and} \quad \lambda_{-i}(A) = -\lambda_i(A^-) \quad \text{for every} \quad i \in \mathbb{N} \, . \tag{16}$$

Let $|\mu(A)| \in c_0(\mathbb{N})$ (respectively $|\lambda(A)| \in c_0(\mathbb{Z}_0)$) denote the sequence obtained by replacing the values of $\mu(A)$ (respectively $\lambda(A)$) with their absolute values, so that $|\mu(A)|$ and $|\lambda(A)|$ are non-negative sequences. Then,

$$s(A) = |\mu(A)|^\downarrow \quad \text{and similarly} \quad s(A) = |\lambda(A)|^\downarrow \, . \tag{17}$$

On the other hand, we also have that

$$s(A) = \big(\, (\lambda_j(A))_{j\in\mathbb{N}} \, , \, (\lambda_j(-A))_{j\in\mathbb{N}} \,\big)^\downarrow \in c_0(\mathbb{N}) \, . \tag{18}$$

In what follows, for $1 \leq k \leq \infty$, we let

$$\mathcal{P}_k(\mathcal{H}) = \{P \in B(\mathcal{H}) \,:\, P^2 = P^* = P \,,\ \operatorname{rk}(P) = k\} \,,$$

denote the subset of $B(\mathcal{H})$ of all the orthogonal projections of rank k. △

The following results are known and correspond to spectral relations between compact operators. Since we will use these facts along our work, we collect the detailed statements below (see [3, 14, 23]).

Theorem 2.7. Let A, $B \in K(\mathcal{H})^{sa}$ be self-adjoint compact operators. Then we have the following:

1. Interlacing inequalities: if $P \in P_k(\mathcal{H})$ $(1 \leq k \leq \infty)$ then

$$\lambda_j(A) \geq \lambda_j(PAP) \quad \text{and} \quad \lambda_{-j}(A) \leq \lambda_{-j}(PAP) \quad \text{for} \quad j \in \mathbb{I}_k \, .$$

2. For every $k \in \mathbb{N}$,

$$\sum_{i=1}^{k} \lambda_i(A) = \max_{P\in\mathcal{P}_k} \operatorname{tr}(P\,A\,P) \quad \text{and} \quad \sum_{i=1}^{k} \lambda_{-i}(A) = \min_{P\in\mathcal{P}_k} \operatorname{tr}(P\,A\,P) \, .$$

3. Weyl's inequality for the spectral scale: $\lambda(A+B) \prec \lambda(A) + \lambda(B)$.

4. Lidskii's inequality [23, Thm. 5.1]: $\lambda(A) - \lambda(B) \prec \lambda(A - B)$. □

We finally recall Weyl's additive and multiplicative inequalities for singular values: for arbitrary A, $B \in K(\mathcal{H})$, with the notation of Eq. (15), we have that

$$s(A+B) \prec_w s(A) + s(B) \quad \text{and} \quad s(AB) \prec_w s(A) \,\cdot\, s(B) \, . \tag{19}$$

We also recall the following well known inequalities: Given $A \in K(\mathcal{H})$, then

$$s_j(D\,A\,E) \leq \|D\|\,\|E\|\,s_j(A) \quad \text{for every} \quad j \in \mathbb{N} \quad \text{and} \quad D, E \in B(\mathcal{H}) \, . \tag{20}$$

Remark 2.8. Submajorization relations play a central role in the study of *symmetrically normed operator ideals* (see [14] for a detailed exposition). A symmetrically normed operator ideal $\mathcal{C}$ of $B(\mathcal{H})$ is a two-sided ideal with a symmetric norm $N(\cdot)$, i.e, a norm with the following additional properties:

1. Given $A \in \mathcal{C}$ and $D, E \in B(\mathcal{H})$, then $N(D\,A\,E) \leq \|D\|\,N(A)\,\|E\|$;
2. For any operator A such that $\text{rk}(A) = 1$, $N(A) = \|A\| = s_1(A)$.

Moreover, any symmetric norm results unitarily invariant as a consequence of item 1; that is, if $U, V \in \mathcal{U}(\mathcal{H})$ then $N(UAV) = N(A)$, for every $A \in \mathcal{C}$. Any such norm induces a gauge symmetric function g_N for sequences, such that $N(A) = g_N(s(A)\,)$.

In this context, we have that: given $B \in \mathcal{C}$ and $A \in K(\mathcal{H})$ then

$$s(A) \prec_w s(B) \implies A \in \mathcal{C} \quad \text{and} \quad N(A) = g_N(s(A)\,) \leq g_N(s(B)\,) = N(B)\ . \tag{21}$$

For the sake of simplicity, in what follows we will call such a norm $N(\cdot)$ a *unitarily invariant norm*. As examples of unitarily invariant norms we mention the Schatten p-norms, for $1 \leq p < \infty$ associated to the Schatten ideals in $B(\mathcal{H})$. △

3 Spectral spread for self-adjoint operators

In this section we introduce and develop the first properties of the spectral spread for self-adjoint operators in the algebra $\mathcal{A}(\mathcal{H}) = K(\mathcal{H}) \oplus \mathbb{C}\,I$. Then, we show that the spectral spread of a self-adjoint operator $A \in \mathcal{A}(\mathcal{H})$ is an upper bound for the singular values of any of its off diagonal blocks (with respect to the submajorization preorder). This last fact plays a central role in the rest of our work.

3.1 Definition and basic properties

In the rest of this work we consider the unital C^*-subalgebra $\mathcal{A}(\mathcal{H}) = K(\mathcal{H}) \oplus \mathbb{C}\,I \subseteq B(\mathcal{H})$ and its self-adjoint part $\mathcal{A}(\mathcal{H})^{sa} = \mathcal{A}(\mathcal{H}) \cap B(\mathcal{H})^{sa}$. In order to simplify our exposition, in what follows we construct the full spectral scale of an arbitrary operator $A \in \mathcal{A}(\mathcal{H})^{sa}$, in terms of the compact case considered in Definition 2.5.

Remark 3.1. Given $A \in \mathcal{A}(\mathcal{H})^{sa}$ we notice that there exists a unique $\mu \in \mathbb{R}$ such that

$$A_0 \stackrel{\text{def}}{=} A - \mu\,I \in K(\mathcal{H})^{sa}\,.$$

In what follows, $A_0 \in K(\mathcal{H})^{sa}$ will always denote the unique compact operator associated to $A \in \mathcal{A}(\mathcal{H})^{sa}$ as above.

We can construct the full spectral scale for A in terms of the full spectral scale of $A_0 \in K(\mathcal{H})^{sa}$ (see Definition 2.5) as follows: we consider the full spectral scale of A given by

$$\lambda(A) \stackrel{\text{def}}{=} \lambda(A_0) + \mu\,\mathbb{1} \in \ell^\infty(\mathbb{Z}_0)\,, \tag{22}$$

where $\mathbb{1} \in \ell^\infty(\mathbb{Z}_0)$ is the sequence with all its entries equal to 1. Notice that

$$\lambda_{-n}(A) \leq \lambda_{-(n+1)}(A) \leq \mu \leq \lambda_{n+1}(A) \leq \lambda_n(A) \quad , \quad \text{for every} \quad n \in \mathbb{N}\ . \qquad \triangle$$

The following notion is motivated by the spectral spread of self-adjoint matrices introduced in [22].

Definition 3.2. Given $A \in \mathcal{A}(\mathcal{H})^{sa}$ we define the *full spectral spread* of A, denoted $\operatorname{Spr}(A) \in c_0(\mathbb{Z}_0)$ as the sequence: If $A = A_0 + \mu\, I$ with $A_0 \in K(\mathcal{H})^{sa}$, then

$$\operatorname{Spr}(A) \stackrel{\text{def}}{=} (\,\lambda_i(A) - \lambda_{-i}(A)\,)_{i\in\mathbb{Z}_0} = \lambda(A_0) + \lambda(-A_0)\,. \tag{23}$$

We also consider the *spectral spread* of A, that is the non-negative part of $\operatorname{Spr}(A) \in c_0(\mathbb{N})$:

$$\operatorname{Spr}^+(A) \stackrel{\text{def}}{=} \big(\operatorname{Spr}_i(A)\big)_{i\in\mathbb{N}}\,. \tag{24}$$

△

It is clear that the spectral spread of an operator in $\mathcal{A}(\mathcal{H})^{sa}$ is a vector valued measure of the dispersion of its eigenvalues. We point out that although related, the spectral spread introduced in [22] differs from the notion introduced in Definition 3.2 above (see Section 5.1).

In the next result we collect several basic properties about the spectral spread in $\mathcal{A}(\mathcal{H})^{sa}$.

Proposition 3.3. *Let $A, B \in \mathcal{A}(\mathcal{H})^{sa}$. The following properties holds:*

1. $\operatorname{Spr}(A) \in c_0^{\uparrow\downarrow}(\mathbb{Z}_0)$ *is symmetric* ($\operatorname{Spr}_{-j}(A) = -\operatorname{Spr}_j(A)$) *and* $\operatorname{Spr}^+(A) \in c_0^{\downarrow}(\mathbb{N}) \cap \mathbb{R}^{\mathbb{N}}_{\geq 0}$.
2. *The spectral spread is invariant under real translations i.e., for every $c \in \mathbb{R}$,*
$$\operatorname{Spr}(A + c\, I) = \operatorname{Spr}(A)\ .$$
3. *For $c \in \mathbb{R}$ we have that* $\operatorname{Spr}(c\, A) = |c| \operatorname{Spr}(A)$. *In particular,* $\operatorname{Spr}^+(A) = \operatorname{Spr}^+(-A)$.

Proof. Items 1. and 2. are direct consequences of Eq. (23) in Definition 3.2. Item 3. is a consequence of the following fact: given $A \in \mathcal{A}(\mathcal{H})^{sa}$ then $\lambda(cA) = c\,\lambda(A)$ if $c \geq 0$ and $\lambda(cA) = c\,(\lambda_{-i}(A))_{i\in\mathbb{Z}_0}$ if $c < 0$. □

The following result describes several relations between the spectral spread and singular values of self-adjoint compact operators.

Proposition 3.4. *Let $A,\ B \in K(\mathcal{H})^{sa}$.*

1. *The following entry-wise inequalities hold:*
$$\operatorname{Spr}^+_i(A) = |\lambda_i(A)| + |\lambda_{-i}(A)| \leq 2 s_i(A) \quad \textit{for every} \quad i \in \mathbb{N}\,. \tag{25}$$
In the positive case, we have that:
$$A \in K(\mathcal{H})^+ \implies \operatorname{Spr}^+_i(A) = \lambda_i(A) = s_i(A) \quad \textit{for every} \quad i \in \mathbb{N}\,. \tag{26}$$
2. *Let $A \oplus A \in K(\mathcal{H} \oplus \mathcal{H})^{sa}$ be given by $A \oplus A = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}$. Then*
$$\frac{1}{2} \operatorname{Spr}^+(A \oplus A) \prec_w s(A) \stackrel{(17)}{=} |\lambda(A)|^{\downarrow}\,. \tag{27}$$

3. *Conversely, we have that* $s(A) \prec_w \mathrm{Spr}^+(A)$.

4. *If* $\lambda(A) \prec \lambda(B)$ *then* $\mathrm{Spr}^+(A) \prec_w \mathrm{Spr}^+(B)$.

5. *If* $P = P^* = P^2$ *then, for every* $i \in \mathbb{N}$,

$$\mathrm{Spr}_i^+(PAP) \le \mathrm{Spr}_i^+(A) \implies \mathrm{Spr}^+(PAP) \prec_w \mathrm{Spr}^+(A) \ . \tag{28}$$

6. *With the order of the eigenvalues defined in 2.5, it holds that*

$$\lambda(A \oplus 0_{\mathcal{H}}) = \lambda(A) \implies \mathrm{Spr}^+(A \oplus 0_{\mathcal{H}}) = \mathrm{Spr}^+(A) \ . \tag{29}$$

7. *Additive Spread inequality:*

$$\mathrm{Spr}(A) - \mathrm{Spr}(B) \prec \mathrm{Spr}(A - B) \prec \mathrm{Spr}(A) + \mathrm{Spr}(B) \ .$$

Proof. Since $s(A) = |\lambda(A)|^{\downarrow}$ (see Definition 2.1 and Remark 2.6), it follows that

$$\max\{|\lambda_i(A)|, |\lambda_{-i}(A)|\} \le s_i(A) \quad \text{for every} \quad i \in \mathbb{N} \ . \tag{30}$$

This shows item 1. In order to show item 2. notice that

$$\sum_{i=1}^{n} \mathrm{Spr}_i^+(A \oplus A) = \begin{cases} 2 \sum\limits_{i=1}^{k} \mathrm{Spr}_i^+(A) & \text{if} \quad n = 2k \\ \\ 2 \sum\limits_{i=1}^{k} \mathrm{Spr}_i^+(A) + \mathrm{Spr}_{k+1}^+(A) & \text{if} \quad n = 2k+1 \end{cases} .$$

Recall that $\sum\limits_{i=1}^{n} s_i(A) = \max\{\sum\limits_{i \in F} |\lambda_i(A)| \ : \ F \subset \mathbb{Z}_0 \, , \ |F| = n\}$. Then, using that

$$\mathrm{Spr}_{k+1}^+(A) = |\lambda_{k+1}(A)| + |\lambda_{-(k+1)}(A)| \le 2 \, \max\{|\lambda_{k+1}(A)|, |\lambda_{-(k+1)}(A)|\}$$

and that

$$2 \sum_{i=1}^{k} \mathrm{Spr}_i^+(A) = 2 \sum_{i=1}^{k} |\lambda_i(A)| + |\lambda_{-i}(A)| \ ,$$

we can easily prove the majorization relation in Eq. (25).

To prove item 3. notice that since $s(A) \stackrel{(17)}{=} |\lambda(A)|^{\downarrow}$ then, for $k \in \mathbb{N}$ we have that

$$\sum_{i=1}^{k} s_i(A) = \sum_{i=1}^{k_1} \lambda_i(A) + \sum_{i=1}^{k_2} |\lambda_{-i}(A)|$$

for some $0 \le k_1$, k_2 such that $k_1 + k_2 = k$. Hence,

$$\sum_{i=1}^{k} s_i(A) \le \sum_{i=1}^{k} \lambda_i(A) + |\lambda_{-i}(A)| = \sum_{i=1}^{k} \mathrm{Spr}_i^+(A) \, .$$

To show 4., fix $k \in \mathbb{N}$. Since $\lambda(A) \prec \lambda(B)$, then

$$\sum_{i=1}^k \lambda_i(A) \le \sum_{i=1}^k \lambda_i(B) \quad \text{and} \quad \sum_{i=1}^k \lambda_{-i}(A) \ge \sum_{i=1}^k \lambda_{-i}(B)\,.$$

Therefore,

$$\sum_{i=1}^k \mathrm{Spr}_i(A) = \sum_{i=1}^k \lambda_i(A) - \lambda_{-i}(A) \le \sum_{i=1}^k \lambda_i(B) - \lambda_{-i}(B) = \sum_{i=1}^k \mathrm{Spr}_i(B)\,.$$

To show item 5., apply the interlacing inequalities of Eq. (28),

$$\lambda_j(A) \ge \lambda_j(PAP) \ge 0 \quad \text{and} \quad \lambda_{-j}(A) \le \lambda_{-j}(PAP) \le 0 \quad \text{for} \quad j \in \mathbb{I}_k\,,$$

where $1 \le k = \mathrm{rk}(P) \le \infty$. In particular, we now see that $\mathrm{Spr}^+_j(PAP) \le \mathrm{Spr}^+_j(A)$, for $j \in \mathbb{I}_k$. Notice that for $j \notin \mathbb{I}_k$ we have that $\lambda_j(PAP) = \lambda_{-j}(PAP) = 0$ and then we also get that $\mathrm{Spr}^+_j(PAP) = 0 \le \mathrm{Spr}^+_j(A)$. The submajorization relation $\mathrm{Spr}^+(PAP) \prec_w \mathrm{Spr}^+(A)$ now follows directly from these facts. Item 6. follows because appending zero eigenvalues is ignored in the order of $\lambda(A)$ given in Definition 2.5.

To show item 7. notice that by Lidskii's additive inequality and item 3. of Lemma 5.7

$$\begin{aligned}\mathrm{Spr}(A) - \mathrm{Spr}(B) &= \lambda(A) - \lambda(B) + \lambda(-A) - \lambda(-B) \\ &\overset{\text{Lidskii}}{\prec} \lambda(A-B) + \lambda(-(A-B)) = \mathrm{Spr}(A-B)\ .\end{aligned}$$

For the other inequality we need to use Weyl's additive inequality. Notice that $\mathrm{Spr}(A-B) = \lambda(A-B) + \lambda(-(A-B))$. Therefore

$$\begin{aligned}\mathrm{Spr}(A-B) &= \lambda(A-B) + \lambda(B-A) \\ &\overset{\text{Weyl}}{\prec} \lambda(A) + \lambda(-B) + \lambda(B) + \lambda(-A) = \mathrm{Spr}(A) + \mathrm{Spr}(B)\ ,\end{aligned}$$

where we have used again item 3. of Lemma 5.7. □

Remark 3.5. In what follows we obtain several inequalities involving the spectral spread of operators. Some of these results are formally analogous to those obtained in [24] for the spectral spread of self-adjoint matrices. We point out that the spectral spread of a self-adjoint matrix A differs from the spectral spread of the operator obtained by embedding A as a finite rank operator $\tilde{A} \in K(\mathcal{H})$ (see Section 5.1). For example, Eq's. (26) and (29) and the inequality of item 3. Proposition 3.4 (i.e. $s(A) \prec_w \mathrm{Spr}^+(A)$) fail in the matrix context. Then, the following results are new and they need new proofs in the operator context. △

3.2 A key inequality for the spectral spread

We can now state the main result of this section. The next submajorization inequality, which is related to Tao's inequalities in [26], plays a key role in the rest of this work.

Theorem 3.6. *Let $A \in \mathcal{A}(\mathcal{H} \oplus \mathcal{K})^{sa}$ be such that $A = \begin{bmatrix} A_1 & B \\ B^* & A_2 \end{bmatrix} \begin{matrix} \mathcal{H} \\ \mathcal{K} \end{matrix}$ is the block representation for A. Then, $B \in K(\mathcal{K}\,,\,\mathcal{H})$ (by construction) and*

$$2\,s(B) \prec_w \mathrm{Spr}^+(A)\,. \tag{31}$$

Proof. Consider $U = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{matrix} \mathcal{H} \\ \mathcal{K} \end{matrix} \in \mathcal{U}(\mathcal{H} \oplus \mathcal{K})$. Then $UA - AU \in K(\mathcal{H} \oplus \mathcal{K})$ and

$$s(UA - AU) = s(A - U^*AU) \overset{(17)}{=} |\lambda(A - U^*AU)|^{\downarrow} .$$

Let $A_0 \in K(\mathcal{H})^{sa}$ be associated to A as in Remark 3.1. By Weyl's inequality in Theorem 2.7, we have that

$$\lambda(A - U^*A\,U) = \lambda(A_0 - U^*A_0\,U) \prec \lambda(A_0) + \lambda(-U^*A_0\,U) = \operatorname{Spr}(A_0) = \operatorname{Spr}(A) .$$

Using item 2. from Lemma 5.7 we deduce that

$$s(UA - AU) = |\lambda(A - U^*A\,U)|^{\downarrow} \prec_w |\operatorname{Spr}(A)|^{\downarrow} = \big(\operatorname{Spr}^+(A)\,,\,\operatorname{Spr}^+(A)\,\big)^{\downarrow} ,$$

where $\big(\operatorname{Spr}^+(A)\,,\,\operatorname{Spr}^+(A)\big) \in c_0(\mathbb{Z}_0)$ is constructed as in Eq. (14). Straightforward computations, Proposition 5.8 and Eq. (17) show that

$$UA - AU = \begin{bmatrix} 0 & 2\,B \\ -2\,B^* & 0 \end{bmatrix} \begin{matrix} \mathcal{H} \\ \mathcal{K} \end{matrix} \implies s(UA - AU) = 2\,\big(\,s(B)\,,\,s(B)\,\big)^{\downarrow} ,$$

and we conclude that

$$2\,\big(\,s(B)\,,\,s(B)\,\big)^{\downarrow} \prec_w \big(\operatorname{Spr}^+(A)\,,\,\operatorname{Spr}^+(A)\,\big)^{\downarrow} \implies 2\,s(B) \prec_w \operatorname{Spr}^+(A) . \qquad \square$$

The following result is related with Zhan's inequality for the singular values of the difference of positive compact operators.

Theorem 3.7. *Let $A, B \in K(\mathcal{H})^{sa}$. Then, if we denote $A \oplus B = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \in K(\mathcal{H} \oplus \mathcal{H})^{sa}$,*

$$s(A - B) \prec_w \operatorname{Spr}^+(A \oplus B) , \tag{32}$$

Hence, for a unitarily invariant norm N,

$$N(A - B) \leq g_N\big(\operatorname{Spr}^+(A \oplus B)\big) \tag{33}$$

Proof. Consider the operators

$$Z = \frac{1}{\sqrt{2}} \begin{bmatrix} I & I \\ -I & I \end{bmatrix} \in \mathcal{U}(\mathcal{H} \oplus \mathcal{H}) \quad \text{and} \quad C = \frac{1}{2} \begin{bmatrix} A + B & A - B \\ A - B & A + B \end{bmatrix} \in K(\mathcal{H} \oplus \mathcal{H})^{sa} .$$

Notice that $Z\,C\,Z^* = A \oplus B \in K(\mathcal{H} \oplus \mathcal{H})$. Then by Theorem 3.6,

$$s(A - B) \prec_w \operatorname{Spr}^+(C) = \operatorname{Spr}^+(A \oplus B) . \tag{34}$$

Finally, by Eq. (21) we know that (32) $\implies$ (33). $\square$

Remark 3.8. In [27], Zhan obtained the inequalities for singular values $s_j(A-B) \leq s_j(A \oplus B)$, for $j \in \mathbb{N}$, where A and B are compact positive operators. It is clear that Zhan's inequalities for singular values fail for self-adjoint compact operators (e.g. take a compact positive operator A and set $B = -A$).

In this context, Theorem 3.7 provides with a submajorization inequality which is an extension for compact self-adjoint operators of Zhan's inequality, replacing the upper bound with $\mathrm{Spr}^+(A \oplus B)$. On the other hand, we point out that Theorem 3.7 cannot be improved to an entry-wise inequality between singular values. For example, take

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \oplus 0_{\ell^2(\mathbb{N})} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \oplus 0_{\ell^2(\mathbb{N})} \ .$$

Then $\lambda(A \oplus B) = \big((\ldots, 0, -1), (3, 1, 1, 0, \ldots) \big)$, so that

$$s(A - B) = (2, 2, 0 \ldots) \quad \text{and} \quad \mathrm{Spr}^+(A \oplus B) = (4, 1, 1, 0 \ldots) \ ,$$

which proves that $s_2(A - B) > \mathrm{Spr}^+_2(A \oplus B)$.

Similarly, the submajorization inequality obtained in Eq. (31) cannot be improved to an entry-wise inequality, as Tao's (7) for the positive case. For example take

$$A = \begin{pmatrix} 1/2 & 1/2 & 0 & 2 \\ 1/2 & 1/2 & 2 & 0 \\ 0 & 2 & 1/2 & 1/2 \\ 2 & 0 & 1/2 & 1/2 \end{pmatrix} \oplus 0_{\ell^2(\mathbb{N})} \ .$$

Then $\lambda(A) = \big((..., 0, -1, -2), (3, 2, 0, ...) \big)$ and $\mathrm{Spr}^+(A) = (5, 3, 0, ...)$. Taking its block $A_{1,2}$ as $B = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$ we have that $s(B) = (2, 2, 0, ..)$ but

$$4 = 2\, s_2(B) > \mathrm{Spr}^+_2(A) = 3 \, . \qquad \triangle$$

4 More inequalities for the spectral spread

In this section we apply Theorem 3.6 and obtain some submajorization inequalities (and then inequalities for unitarily invariant norms), involving the spectral spread of operators. We first deal with commutator inequalities and then obtain some Arithmetic-Geometric mean type inequalities.

4.1 Commutator inequalities

We begin with the following observation.

Lemma 4.1. *Let $A, B \in K(\mathcal{H})^{sa}$ be such that $\mathrm{rk}(A) = n \in \mathbb{N}$. Then*

$$\mathrm{tr}(AB) \le \mathrm{tr}\ \big(\lambda(A) \cdot \lambda(B) \big) \stackrel{\text{def}}{=} \sum_{k \in \mathbb{Z}_0} \lambda_i(A)\, \lambda_i(B) \ . \tag{35}$$

Proof. Since A has finite rank, there is a ONB for the range of A, denoted $R(A)$, say $\{x_i\}_{i=-m,\, i\neq 0}^r$ with $m + r = n$, $m, r \ge 0$, such that

$$A = \sum_{i=-m,\, i\neq 0}^{r} \lambda_i(A)\, x_i \otimes x_i \implies$$

$$\operatorname{tr}(A\,B) = \operatorname{tr}(B\,A) = \sum_{i=-m,\,i\neq 0}^{r} \lambda_i(A)\ \operatorname{tr}(B\,x_i\otimes x_i) = \sum_{i=-m,\,i\neq 0}^{r} \lambda_i(A)\,\langle B\,x_i\,,\,x_i\rangle\,.$$

Assume first that $B\in\mathcal{K}(\mathcal{H})^+$. Since $\lambda_{-i}(A)\le 0$ for every $i\in\mathbb{I}_m$, we get

$$\operatorname{tr}(A\,B) = \sum_{i=-m,\,i\neq 0}^{r} \lambda_i(A)\,\langle B\,x_i\,,\,x_i\rangle \le \sum_{i=1}^{r} \lambda_i(A)\,\langle B\,x_i\,,\,x_i\rangle\,. \tag{36}$$

We denote by $a=(\lambda_i(A))_{i=1}^r\in\mathbb{R}^r_{\ge 0}$, $b=(\langle B\,x_i\,,\,x_i\rangle)_{i=1}^r\in\mathbb{R}^r_{\ge 0}$, and Q the orthogonal projection onto $\operatorname{Span}\{x_i\}_{i=1}^r$. Using the Schur-Horn theorem (see [6] or [3]), the interlacing inequalities of Eq. (28) and item 6. of Lemma 5.7 we get that

$$\sum_{i=1}^{h} b_i \le \sum_{i=1}^{h}\lambda_i(B) \quad \text{for} \quad 1\le h\le r \implies \sum_{i=1}^{r} a_i\,b_i \le \sum_{i=1}^{r} a_i\,\lambda_i(B)\,. \tag{37}$$

Now we can get Eq. (35) for the positive case:

$$\operatorname{tr}(A\,B) \overset{(36)}{\le} \sum_{i=1}^{r} \lambda_i(A)\,\langle B\,x_i\,,\,x_i\rangle \overset{(37)}{\le} \sum_{i=1}^{r} \lambda_i(A)\,\lambda_i(B) = \operatorname{tr}(\lambda(A)\cdot\lambda(B)\,)\,, \tag{38}$$

since $\lambda_i(B)=0$ for $i<0$. Similarly, if $B\in -K(\mathcal{H})^+$ then

$$\operatorname{tr}(A\,B\,) \overset{(36)}{\le} \sum_{i=1}^{m} \lambda_{-i}(A)\,\langle B\,x_{-i}\,,\,x_{-i}\rangle \overset{(37)}{\le} \sum_{i=1}^{m}(-\lambda_{-i}(A)\,)\,\lambda_i(-B)\ . \tag{39}$$

If $B\in K(\mathcal{H})^{sa}$, take $B^+\,,\,B^-\in K(\mathcal{H})^+$ the positive and negative parts of B. By (16),

$$B=B^+-B^- \quad , \quad \lambda_i(B)=\lambda_i(B^+) \quad \text{and} \quad \lambda_{-i}(B)=-\lambda_i(B^-) \tag{40}$$

for every $i\in\mathbb{N}$. Hence

$$\begin{aligned}
\operatorname{tr}(AB) &= \operatorname{tr}(AB^+) - \operatorname{tr}(AB^-)\\
&\overset{(38)}{\le} \sum_{i=1}^{r} \lambda_i(A)\,\lambda_i(B^+) + \operatorname{tr}(A(-B^-)\,)\\
&\overset{(39)}{\le} \sum_{i=1}^{r} \lambda_i(A)\,\lambda_i(B^+) + \sum_{i=1}^{m}(-\lambda_{-i}(A)\,)\,\lambda_i(B^-)\\
&\overset{(40)}{=} \sum_{i=1}^{r} \lambda_i(A)\,\lambda_i(B) + \sum_{i=1}^{m} \lambda_{-i}(A)\,\lambda_{-i}(B) = \operatorname{tr}(\lambda(A)\cdot\lambda(B)\,)\ .
\end{aligned}$$

This completes the proof. □

Remark 4.2. The inequality of Lemma 4.1 is also true for matrices, but in this case the finite sequence of entry-wise products $\lambda(A)\cdot\lambda(B)$ is not the same, since the vectors of eigenvalues (counting multiplicities) are considered with a different order. Indeed, for Hermitian matrices A and B, the vectors $\lambda(A)$ and $\lambda(B)$ are ordered decreasingly, without taking account the positivity of its entries (as opposed to the definition for compact operators, see Section 5.1 for details).

Nevertheless, the previous proof is still valid for matrices in the case $B\ge 0$. The general matrix case follows by adding multiples of the identity to reduce to the positive case. □

Theorem 4.3. *Let $A, X \in \mathcal{A}(\mathcal{H})^{sa}$. Then*

$$\lambda\big(\, i\,(A\,X - X\,A)\big) \prec_w \frac{1}{2}\,\mathrm{Spr}^+(A)\,\cdot\,\mathrm{Spr}^+(X)\,. \tag{41}$$

Proof. Let $A_0\,,\, X_0 \in K(\mathcal{H})^{sa}$ be associated to $A\,,\, X$ as in Remark 3.1. Since

$$i\,(AX - XA) = i\,(A_0X_0 - X_0A_0) \in K(\mathcal{H})^{sa}\ ,\ \mathrm{Spr}^+(A) = \mathrm{Spr}^+(A_0)\ ,\ \mathrm{Spr}^+(X) = \mathrm{Spr}^+(X_0)$$

we can assume, without loss of generality , that $A = A_0$ and $X = X_0$. If $k \in \mathbb{N}$, by Proposition 2.7, there exists an orthogonal projection $P \in B(\mathcal{H})$ with $k = \mathrm{tr}(P)$ such that

$$\sum_{j=1}^{k} \lambda_j\,\big(\, i\,(A\,X - X\,A)\big) = \mathrm{tr}\,\big(\, i\,(A\,X - X\,A)\,P\big)\,.$$

Moreover, since $XP - PX$ has finite rank then, by Lemma 4.1, we get that

$$\mathrm{tr}\,\big(\, i\,(AX - XA)\,P\big) = \mathrm{tr}\,\big(\, i\,(XP - PX)\,A\big) \le \mathrm{tr}\,\big(\,\lambda(i\,(XP - PX)\,)\cdot\lambda(A)\big)\,.$$

Now consider the block matrix representations induced by P:

$$X = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^* & X_{22} \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \implies i\,(XP - PX) = i\,\begin{bmatrix} 0 & -X_{12} \\ X_{12}^* & 0 \end{bmatrix}$$

Denote $X_{12} = B \in K(\mathcal{H})$; then by Proposition 5.8

$$\lambda(i\,(XP - PX)) = (s(B)\,,\, -s(B))^{\uparrow\downarrow}\,.$$

Now, Theorem 3.6 implies that $s(B) \prec_w \frac{1}{2}\,\mathrm{Spr}^+(X)$. The previous fact together with by item 6. in Lemma 5.7 show that

$$s(B)\cdot\mathrm{Spr}^+(A) \prec_w \frac{1}{2}\,\mathrm{Spr}^+(X)\cdot\mathrm{Spr}^+(A)\,. \tag{42}$$

Moreover, if we let $k' \le k$ be the number of non-zero singular values of B then

$$\begin{aligned} \mathrm{tr}\,\big(\,\lambda(i(XP - PX)\,)\cdot\lambda(A)\big) &= \sum_{j=1}^{k'} s_j(B)\,\lambda_j(A) - \sum_{j=1}^{k'} s_j(B)\,\lambda_{-j}(A) \\ &= \sum_{j=1}^{k'} s_j(B)\,\mathrm{Spr}^+_j(A) \overset{(42)}{\le} \frac{1}{2}\sum_{i=1}^{k'} \mathrm{Spr}^+_j(X)\,\mathrm{Spr}^+_j(A)\,. \end{aligned}$$

Combining the previous arguments it is clear that

$$\sum_{j=1}^{k} \lambda_j\,\big(\, i\,(AX - XA)\big) \le \frac{1}{2}\sum_{i=1}^{k'} \mathrm{Spr}^+_j(X)\,\mathrm{Spr}^+_j(A) \le \frac{1}{2}\sum_{i=1}^{k} \mathrm{Spr}^+_j(X)\,\mathrm{Spr}^+_j(A)\,,$$

which proves Eq. (41). □

Although Theorem 4.3 contains much information, its statement is rather technical. For example, the submajorization of Eq. (41) only gives information about the positive eigenvalues of $i\,(AX - XA)$. The next result, which is a consequence of Theorem 4.3, is more clear, and it has several direct implications (see Corollary 4.5 below). The idea is that, in order to include all the eigenvalues (positive and negative) of the commutator (to compute its singular values), we need to consider two times the entries of the spread sequences. This is the reason we use (and need) operators of the type $A \oplus A$ in the following statement.

Theorem 4.4. *Let $A, X \in \mathcal{A}(\mathcal{H})^{sa}$. Then*

$$s(AX - XA) \prec_w \frac{1}{2} \operatorname{Spr}^+(A \oplus A) \cdot \operatorname{Spr}^+(X \oplus X)\,. \tag{43}$$

Hence, for any unitarily invariant norm N we have that

$$N(AX - XA) \leq \frac{1}{2}\, g_N\big(\operatorname{Spr}^+(A \oplus A) \cdot \operatorname{Spr}^+(X \oplus X)\big)\,. \tag{44}$$

Proof. Using Eq. (41) applied to $A, -X \in \mathcal{A}(\mathcal{H})^{sa}$,

$$\lambda(-i\,(AX - XA)) \prec_w \frac{1}{2} \operatorname{Spr}^+(A) \cdot \operatorname{Spr}^+(-X) = \frac{1}{2} \operatorname{Spr}^+(A) \cdot \operatorname{Spr}^+(X)\,,$$

where we used that $\operatorname{Spr}^+(-X) = \operatorname{Spr}^+(X)$. By Eq. (18) in Remark 2.6 and item 4. in Lemma 5.7 we have that

$$\begin{aligned} s(AX - XA) &= \big(\,(\lambda_j(i\,(\,AX - XA)\,)\,)_{j\in\mathbb{N}}\,,\,(\lambda_j(-\,i\,(AX - XA)\,)_{j\in\mathbb{N}}\big)^{\downarrow} \\ &\prec_w \frac{1}{2}\,(\operatorname{Spr}^+(A) \cdot \operatorname{Spr}^+(X)\,,\, \operatorname{Spr}^+(A) \cdot \operatorname{Spr}^+(X)\,)^{\downarrow} \\ &= \frac{1}{2} \operatorname{Spr}^+(A \oplus A) \cdot \operatorname{Spr}^+(X \oplus X)\,, \end{aligned}$$

which proves Eq. (43). Finally, by Eq. (21) we know that (43) $\implies$ (44). □

The next result is formally analogous to [24, Theorem 3.1.] (which played a central role in [25]). We point out that the spectral spread of a self-adjoint matrix A differs from the spectral spread of the self-adjoint compact operator $\tilde{A}$ obtained from embedding A as a finite rank operator (see Section 5.1). Thus, we present a new approach to obtain this result, which is based on Theorem 4.4.

Corollary 4.5. *Let $A, B \in \mathcal{A}(\mathcal{H})^{sa}$ and $X \in K(\mathcal{H})$. Then*

$$s(AX - XB) \prec_w \operatorname{Spr}^+(A \oplus B) \cdot s(X)\,. \tag{45}$$

Proof. First take $A = B$ and assume that $X \in \mathcal{A}(\mathcal{H})^{sa}$. By Theorem 4.4, Proposition 3.4 (item 2) and Lemma 5.7 (item 6), we have that

$$s(AX - XA) \prec_w \frac{1}{2} \cdot \operatorname{Spr}^+(A \oplus A) \operatorname{Spr}^+(X \oplus X) \prec_w \operatorname{Spr}^+(A \oplus A) \cdot s(X)\,.$$

In the general case, let $C = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \in \mathcal{A}(\mathcal{H} \oplus \mathcal{H})^{sa}$ and $\hat{X} = \begin{bmatrix} 0 & X \\ X^* & 0 \end{bmatrix} \in K(\mathcal{H} \oplus \mathcal{H})^{sa}$. Then

$$C\hat{X} - \hat{X}C = \begin{bmatrix} 0 & AX - XB \\ (AX - XB)^* & 0 \end{bmatrix}\,,$$

and, by the first part of the proof

$$s(C\hat{X} - \hat{X}C) = (s(AX - XB)\,,\, s(AX - XB)\,)^{\downarrow} \prec_w \operatorname{Spr}^+(C \oplus C) \cdot s(\hat{X})\,.$$

Notice that Eq. (45) follows from the previous submajorization relation, since

$$s(\hat{X}) = (s(X)\,,\, s(X))^{\downarrow} \quad \text{and} \quad \operatorname{Spr}^+(C \oplus C) = (\operatorname{Spr}^+(A \oplus B)\,,\, \operatorname{Spr}^+(A \oplus B))^{\downarrow}\,.$$ □

Remark 4.6. Notice that inequality (45) cannot be improved to an entry-wise inequality, as Kittaneh's inequality in Eq. (6) for the positive case. For example, we can consider

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad , \quad B = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \quad \text{and} \quad X = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} ,$$

all of them embedded on $K(\mathcal{H})^{sa}$. Then

$$\lambda(A \oplus B) = \big((\ldots , 0 , -1) , (3 , 1 , 1 , 0 , \ldots) \big) \quad , \quad s(A\,X - X\,B) = (6 , 2 , 0 , \ldots)$$

and $s(X) = (3 , 1 , 0 , \ldots)$. Therefore $s_2(A\,X - X\,B) = 2 > \operatorname{Spr}_2^+(A \oplus B)\, s_2(X) = 1$. △

The next result gives an estimation for commutators, when $A, B \in \mathcal{A}(\mathcal{H}) = K(\mathcal{H}) + \mathbb{C}\,I$ are not necessarily self-adjoint (for related results see [18]).

Theorem 4.7. *Let $A, B \in \mathcal{A}(\mathcal{H})$ and $X \in K(\mathcal{H})$. Consider $A = A_1 + i\,A_2$, $B = B_1 + i\,B_2$ where $A_j, B_j \in \mathcal{A}(\mathcal{H})^{sa}$ for $j = 1, 2$. Then*

$$s(AX - XB) \prec_w \big(\operatorname{Spr}^+(A_1 \oplus B_1) + \operatorname{Spr}^+(A_2 \oplus B_2)\big) \cdot s(X) . \tag{46}$$

Proof. Notice that

$$AX - XB = (A_1 + i\,A_2)X - X(B_1 + i\,B_2) = A_1X - XB_1 + i\,(A_2X - XB_2) .$$

Then by Weyl's inequality for singular values (see Eq. (19)) and Corollary 4.5,

$$\begin{aligned} s(AX - XB) &\prec_w s(A_1X - XB_1) + s(A_2X - XB_2) \\ &\prec_w \operatorname{Spr}^+(A_1 \oplus B_1) \cdot s(X) + \operatorname{Spr}^+(A_2 \oplus B_2) \cdot s(X) \\ &= (\operatorname{Spr}^+(A_1 \oplus B_1) + \operatorname{Spr}^+(A_2 \oplus B_2)\,) \cdot s(X) . \end{aligned}$$

□

Corollary 4.8. *Let $A, B \in \mathcal{A}(\mathcal{H})$ and $X \in K(\mathcal{H})$. Consider $A = A_1 + i\,A_2$, $B = B_1 + i\,B_2$ where $A_j, B_j \in \mathcal{A}(\mathcal{H})^{sa}$ for $j = 1, 2$. Let $a_j, a'_j, b_j, b'_j \in \mathbb{R}$ be such that*

$$a_j\,I \le A_j \le a'_j\,I \quad \text{and} \quad b_j\,I \le B_j \le b'_j\,I \quad \text{for} \quad j = 1, 2 .$$

Then, for every unitarily invariant norm $N(\cdot)$ we have that

$$N(AX - XB) \le \big(\, \max\{a'_1, b'_1\} - \min\{a_1, b_1\} + \max\{a'_2, b'_2\} - \min\{a_2, b_2\}\,\big)\, N(X) .$$

Proof. Notice that since $a_j\,I \le A_j \le a'_j\,I$ and $b_j\,I \le B_j \le b'_j\,I$ for $j = 1, 2$, then

$$\operatorname{Spr}^+(A_j \oplus B_j) \le \big(\, \max\{a'_j, b'_j\} - \min\{a_j, b_j\}\,\big)\, \mathbb{1} \ \text{ for } j = 1, 2 .$$

From Theorem 4.7 we now see that

$$s(AX - XB) \prec_w \big(\, \max\{a'_1, b'_1\} - \min\{a_1, b_1\} + \max\{a'_2, b'_2\} - \min\{a_2, b_2\}\,\big)\, s(X) . \tag{47}$$

Now the result follows from Eq. (47) and Remark 2.8. □

Theorem 4.9. Let $A,\, X \in \mathcal{A}(\mathcal{H})^{sa}$ and $U = e^{i\,X} \in \mathcal{U}(\mathcal{H})$. Then

$$s(A - U^*AU) \prec_w \frac{1}{2}\,\mathrm{Spr}^+(X \oplus X) \cdot \mathrm{Spr}^+(A \oplus A)\,.$$

Proof. Take $X = X_0 + \lambda\, I$, then $U = e^{i\,\lambda}\, e^{i\,X_0} = e^{i\,\lambda}\, U_0$ where $U_0 = e^{i\,X_0} \in \mathcal{U}(\mathcal{H})$. Since $s(A - U^*AU) = s(A_0 - U^*A_0\, U)$ we can assume, without loss of generality, that $X = X_0$, $U = U_0$ and $A = A_0$, since Spr^+ is invariant under translations.

Let $A(\cdot) : [0,1] \to K(\mathcal{H})^{sa}$ be the smooth function given by $A(t) = e^{-i\,t\,X}\, A\, e^{i\,tX}$, for $t \in [0,1]$. Notice that $A(0) = A$ and $A(1) = U^*A\,U$; using Weyl's inequality in Eq. (19),

$$s(A - U^*AU) \prec_w \sum_{j=0}^{m-1} s\big(\, A(\tfrac{j}{m}) - A(\tfrac{j+1}{m})\,\big) \quad \text{for every} \quad m \in \mathbb{N}\,. \tag{48}$$

Notice that $A(t+h) = e^{-i\,t\,X}\, A(h)\, e^{i\,t\,X}$ with $e^{i\,t\,X} \in \mathcal{U}(\mathcal{H})$, for $t,\ h,\ t+h \in [0,1]$. Thus

$$s(A(\tfrac{j}{m}) - A(\tfrac{j+1}{m})) \ = s(A - A(\tfrac{1}{m})) \quad \text{for} \quad j \in \mathbb{I}_{m-1}\,. \tag{49}$$

Since $A - A(0) = 0$ and $\frac{d}{dt}A(t)|_{t=0} = i\,(AX - XA)$ we get that

$$s(A - A(\tfrac{1}{m})) \ = \tfrac{1}{m}\, s(AX - XA) + O(m) \quad \text{with} \quad \lim_{m\to\infty} m\, O(m) = 0\,. \tag{50}$$

Hence, by Theorem 4.4 we have that

$$s(A - A(\tfrac{1}{m})) \ \prec_w \tfrac{1}{2\,m}\,\mathrm{Spr}^+(X \oplus X)\, \cdot\, \mathrm{Spr}^+(A \oplus A) + O(m)\,. \tag{51}$$

Therefore, by Eqs. (48) and (49) we have that, for sufficiently large m,

$$s(A - U^*AU) \prec_w \frac{1}{2}\,\mathrm{Spr}^+(X \oplus X)\, \cdot\, \mathrm{Spr}^+(A \oplus A) + m\, O(m)\,.$$

The statement now follows by taking the limit $m \to \infty$ in the expression above. □

4.2 AG-type inequalities and submajorization

Assume that $C,\, S \in B(\mathcal{H})$ are such that $C^*C + S^*S \le I$ and let $E \in K(\mathcal{H})^+$. Then, we can apply the operator Arithmetic Geometric mean (AG) inequality in Eq. (1) (see [8]) and conclude that for $j \in \mathbb{I}_k$ we have that

$$s_j(SEC^*) = s_j(\,(SE^{1/2})(CE^{1/2})^*) \le \frac{1}{2}s_j\,\big(\,E^{1/2}(S^*S + C^*C)E^{1/2}\,\big) \le \frac{1}{2}s_j(E)\,. \tag{52}$$

Eq. (52) was derived in [4], where it was also shown to be stronger than the AG-type inequalities obtained in [1] (see [15, 17] for related singular values inequalities). In particular (see Remark 2.8) for any unitarily invariant norm N we have that

$$N(SEC^*) \le \frac{1}{2}N(E)\,. \tag{53}$$

From Eqs. (52) and (53) it is possible to derive more general AG-type inequalities. Nevertheless, Eqs. (52) and (53) fail for arbitrary self-adjoint $E \in K(\mathcal{H})^{sa}$:

Example 4.10. As in previous examples, we shall use that a matrix $A \in \mathcal{M}_n(\mathbb{C})$ can be embedded as a finite rank operator, which allow us to build counterexamples using matrices. Now we see that Eq. (53) is false if $E \not\geq 0$. Consider $N(X) = \|X\|_2 = (\operatorname{tr}(X^*X))^{1/2}$ the Frobenius norm (which is unitarily invariant). Let

$$S = \begin{bmatrix} \sin(\pi/3) & 0 \\ 0 & \sin(\pi/5) \end{bmatrix}, \ E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} \cos(\pi/3) & 0 \\ 0 & \cos(\pi/5) \end{bmatrix}.$$

Then $C^*C + S^*S = I$ and $\|S\,E\,C^*\|_2 \approx 0.7598$ meanwhile $\frac{1}{2}\,\|E\|_2 = \frac{1}{2}$. △

In what follows we obtain a generalization of Eq. (53) for arbitrary self-adjoint $E \in K(\mathcal{H})^{sa}$; these are new inequalities in the sense that they involve upper bounds in terms of the spectral spread of self-adjoint operators in $\mathcal{A}(\mathcal{H})$. We point out that our results are based on the (weaker) submajorization relations; nevertheless, these results imply inequalities with respect to unitarily invariant norms as in Eq. (53).

Proposition 4.11. *Let $C,\, S \in B(\mathcal{H})$ be such that $C^*C + S^*S = P$, where $P = P^2$, and let $E \in K(\mathcal{H})^{sa}$. Then*

$$2\, s(SEC^*) \prec_w \operatorname{Spr}^+(E)\,. \tag{54}$$

Proof. By considering the polar decomposition of S and C and Eq. (20), we can assume that $S,\, C \in B(\mathcal{H})^+$. In this case, since $C = CP = PCP$ and $S = SP = PSP$ then $S(PEP)C = SEC$. Also, by Eq. (28), $\operatorname{Spr}^+(PEP) \prec_w \operatorname{Spr}^+(E)$. Hence, working in $B(R(P)\,)$, we can assume further that $P = I$. Therefore $C^2 + S^2 = I$, so that C and $S = (I - C^2)^{1/2}$ commute. This fact implies that $U = \begin{bmatrix} C & -S \\ S & C \end{bmatrix} \in \mathcal{U}(\mathcal{H} \oplus \mathcal{H})$. Moreover,

$$U(E \oplus 0_{\mathcal{H}})U^* = \begin{bmatrix} CEC & CES \\ SEC & SES \end{bmatrix} \in K(\mathcal{H} \oplus \mathcal{H})^{sa}$$

$$\overset{(31)}{\Longrightarrow}\ 2\, s(SEC) \prec_w \operatorname{Spr}^+(U(E \oplus 0_{\mathcal{H}})U^*) = \operatorname{Spr}^+(E \oplus 0_{\mathcal{H}}) \overset{(29)}{=} \operatorname{Spr}^+(E)\,,$$

where we have used Theorem 3.6 and Eq. (29) in Proposition 3.4. □

Remark 4.12. With the notation of Proposition 4.11, if $E \in K(\mathcal{H})^+$ then $\operatorname{Spr}^+(E) \overset{(26)}{=} s(E)$, Hence, Proposition 4.11 implies that $N(SEC) \leq \frac{1}{2}\, N(E)$ and we recover Eq. (53).

In case $E \in K(\mathcal{H})^{sa} \setminus K(\mathcal{H})^+$ then it turns out that $s(E) \prec \operatorname{Spr}^+(E)$, with strict majorization i.e., if N is a strictly convex unitarily invariant norm then

$$N(E) < N(\operatorname{Spr}^+(E))\,.$$

For example, if we consider $N(X) = \|X\|_2 = (\operatorname{tr}(X^*X))^{1/2}$ then we have that

$$\|E\|_2^2 = \sum_{j \in \mathbb{Z}_0} \lambda_j(E)^2 < \sum_{j \in \mathbb{N}} (\lambda_j(E) - \lambda_{-j}(E))^2 = \|\operatorname{Spr}^+(E)\|_2^2\,,$$

for every Hilbert-Schmidt self-adjoint operator $E \notin K(\mathcal{H})^+$. △

Next we state a well known result of R. Douglas [13] which contains criteria for the factorization of operators, that we will need in the sequel.

Theorem 4.13. Let A, $B \in B(\mathcal{H})$. Then the following conditions are equivalent:

1. $R(A) \subseteq R(B)$.
2. There exists $\lambda \in \mathbb{R}_+$ such that $AA^* \leq \lambda BB^*$.
3. There exists $C \in B(\mathcal{H})$ such that $A = BC$.

In this case, there exists an unique

$$C \in B(\mathcal{H}) \quad \text{such that} \quad A = BC \quad \text{and} \quad R(C) \subseteq \overline{R(B^*)} = \ker B^{\perp} \ . \tag{55}$$

□

Now we can state our main result on AG-type inequalities for unitarily invariant norms.

Theorem 4.14. *Let A, $B \in B(\mathcal{H})$ and let $E \in K(\mathcal{H})^{sa}$. Then*

$$s(A\,E\,B^*) \prec_w \frac{1}{2}\ \text{Spr}^+\left((A^*A + B^*B)^{1/2}\, E\, (A^*A + B^*B)^{1/2} \right). \tag{56}$$

Proof. By Theorem 4.13, since $A^*A \leq A^*A + B^*B$, the (linear) operator equations

$$A = S\,(A^*A + B^*B)^{1/2} \quad \text{and} \quad B = C\,(A^*A + B^*B)^{1/2} \tag{57}$$

admit unique solutions S, $C \in B(\mathcal{H})$ also verifying that

$$\ker(A^*A + B^*B)^{1/2} = R(\,(A^*A + B^*B)^{1/2})^{\perp} \overset{(55)}{\subseteq} R(S^*)^{\perp} \cap R(C^*)^{\perp} = \ker S \cap \ker C \ .$$

Hence, if $z \in R(\,(A^*A + B^*B)^{1/2})$ and $x \in \mathcal{H}$ is such that $z = (A^*A + B^*B)^{1/2}x$ then

$$\begin{aligned} \langle (S^*S + C^*C)z, z\rangle \ &= \|S(A^*A + B^*B)^{1/2}x\|^2 + \|C(A^*A + B^*B)^{1/2}x\|^2 \\ &\overset{(57)}{=} \|A\,x\|^2 + \|B\,x\|^2 = \langle (A^*A + B^*B)\,x\,,\,x\rangle = \|z\|^2 \ . \end{aligned}$$

On the other hand, if $z \in \ker(AA^* + BB^*)^{1/2}$ then $(S^*S + C^*C)z = 0$. Hence, if we let P denote the orthogonal projection onto the closure of $R((A^*A + B^*B)^{1/2})$ then the previous facts show that $\langle (S^*S + C^*C)z, z\rangle = \langle Pz, z\rangle$, for $z \in \mathcal{H}$; thus, $S^*S + C^*C = P$. Moreover

$$AEB^* = S\,(A^*A + B^*B)^{1/2}\, E\,(A^*A + B^*B)^{1/2}\, C^* \ .$$

We now apply Proposition 4.11 to the expression in the left-hand side of the previous identity and get Eq. (56). □

Example 4.15. Let A, B and E be as in Theorem 4.14, but assume that $E \geq 0$. Since $s\left((A^*A + B^*B)^{1/2}\, E\,(A^*A + B^*B)^{1/2} \right) = s\left(E^{1/2}\,(A^*A + B^*B)\, E^{1/2} \right)$ and $\text{Spr}^+(C) = s(C)$ for $C \in K(\mathcal{H})^+$, then Eq. (1) and its consequence Eq. (52), being entry-wise inequalities, are stronger that Eq. (56) in the positive case. As in the previous inequalities of this paper, Eq. (56) is a substitute of Eq. (52) in the self-adjoint non positive case.

Nevertheless, as it happens with Eq. (4), for the general self-adjoint case the submajorization relation (56) cannot be improved to an entry-wise inequality as (52). Consider

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1 \end{pmatrix} \quad \text{and} \quad E = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}.$$

In this case $F = A^*A + B^*B = \begin{pmatrix} \frac{13}{4} & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \frac{13}{4} \end{pmatrix}$ is such that

$$\lambda(F^{\frac{1}{2}} E\, F^{\frac{1}{2}}) \ \ = \big((\dots, 0\,, \tfrac{-13}{4})\,, (\tfrac{39}{4}\,, 2\,, 0\,, \dots) \big) \ \text{ and } \ \mathrm{Spr}^+(F^{\frac{1}{2}} E\, F^{\frac{1}{2}}) = (13\,, 2\,, 0\,, \dots).$$

Moreover $s(A\,E\,B^*) \approx (4,74\,,\, 1,58\,,\, 1\,,\, 0....)$, which shows that

$$3,16 \approx 2\, s_2(A\,E\,B^*) > \mathrm{Spr}_2^+(F^{\frac{1}{2}} E\, F^{\frac{1}{2}}) = 2\ . \qquad \triangle$$

The following results complements Proposition 4.11.

Proposition 4.16. *Let $C,\, S \in B(\mathcal{H})^+$ be such that $C^2 + S^2 = P$, with $P = P^2$, and let $E_1,\, E_2 \in K(\mathcal{H})^{sa}$. Then*

$$s(SE_1C + CE_2S) \prec_w \frac{1}{2}\ \mathrm{Spr}^+(E_1 \oplus -E_2)\,. \tag{58}$$

Proof. As in Proposition 4.11, we can assume that $P = I$ and construct a 2×2 unitary matrix $U = \begin{bmatrix} C & -S \\ S & C \end{bmatrix} \in \mathcal{U}(\mathcal{H} \oplus \mathcal{H})$. Then

$$U(E_1 \oplus -E_2)U^* = \begin{bmatrix} CE_1C - SE_2S & CE_1S + SE_2C \\ SE_1C + CE_2S & SE_1S - CE_2C \end{bmatrix} \in K(\mathcal{H} \oplus \mathcal{H})^{sa}$$

$$\stackrel{(31)}{\Longrightarrow}\ 2\, s(SE_1C + CE_2S) \prec_w \mathrm{Spr}^+(U(E_1 \oplus -E_2)U^*) = \mathrm{Spr}^+(E_1 \oplus -E_2)\,,$$

where we have used Theorem 3.6. $\square$

It is easy to see that, if $E \in K(\mathcal{H})^{sa}$ then

$$\mathrm{Spr}^+(E \oplus -E) = \big(\, 2\, |\lambda(E)|\, \big)^{\downarrow} = 2\ s(E)\ .$$

We use this in the following:

Corollary 4.17. *Let $C,\, S \in B(\mathcal{H})^+$ be such that $C^2 + S^2 = P$, with $P = P^2$, and let $E \in K(\mathcal{H})^{sa}$. Then*

$$s(\,\mathrm{Re}(SEC)\,) \prec_w \frac{1}{4}\ \mathrm{Spr}^+(E \oplus -E) = \frac{1}{2}\ s(E) \tag{59}$$

Proof. Notice that $\mathrm{Re}(SEC) = \frac{SEC + CES}{2}$ and apply Proposition 4.16 with $E_1 = E_2 = E$. $\square$

We end this last section by showing that several of the main results in this work are equivalent. It is worth pointing out that each reformulation has a quite different appeal. Indeed, notice that the statements involve the key result on the spectral spread (Theorem 3.6), AG-type inequalities, commutator inequalities and Zhan's type inequalities, all with respect to submajorization. The reader can write down the corresponding norm inequalities for arbitrary unitarily invariant norms.

Theorem 4.18. *The following inequalities are equivalent: for arbitrary $E,\, F \in K(\mathcal{H})^{sa}$, $X \in K(\mathcal{H})$ and $A,\, B,\, P \in B(\mathcal{H})$, with $P = P^* = P^2$.*

1. $2\, s(P\,E\,(I-P)) \prec_w \mathrm{Spr}^+(E)$;
2. $s(A\,E\,B^*) \prec_w \frac{1}{2}\,\mathrm{Spr}^+((A^*A+B^*B)^{1/2}\,E\,(A^*A+B^*B)^{1/2})$;
3. $s(EF-FE) \prec_w \frac{1}{2}\,\mathrm{Spr}^+(E\oplus E)\,\cdot\,\mathrm{Spr}^+(F\oplus F)$;
4. $s(EX-XF) \prec_w \mathrm{Spr}^+(E\oplus F)\,\cdot\,s(X)$;
5. $s(E-F) \prec_w \mathrm{Spr}^+(E\oplus F)$.

Proof. By inspection of the proof of Theorem 4.14 we see that 1. → 2 (vía Proposition 4.11). On the other hand, if we take $A=P$ and $B=I-P$ in 2. we see that $A^*A+B^*B=I$ and we recover item 1. These facts show the equivalence of items 1 and 2.

By inspection of the proofs of Theorem 4.4 and its Corollary 4.5 we see that 1. → 3. → 4.

On the other hand, if we let $X=I$ in item 4. we get item 5. Thus, 4 → 5.

To prove 5. → 1. we can assume that $\mathcal{H}=\mathcal{K}\oplus\mathcal{K}$ and that $P=P_{\mathcal{K}\oplus\{0\}}$. Then

$$E=\begin{bmatrix} E_1 & B \\ B^* & E_2 \end{bmatrix}\begin{matrix}\mathcal{K}\\ \mathcal{K}\end{matrix} \quad \text{with} \quad s(PE(I-P))=s(B)\in c_0(\mathbb{N})^{\downarrow}\,.$$

Let $B=U|B|$ be the polar decomposition of B. Then $U\in B(\mathcal{K})$ is a partial isometry. If we construct the partial isometry $W\in B(\mathcal{H})=B(\mathcal{K}\oplus\mathcal{K})$ given by

$$W=\begin{bmatrix} U & 0 \\ 0 & I_{\mathcal{K}} \end{bmatrix} \quad \text{then} \quad W^*EW=\begin{bmatrix} U^*E_1U & U^*B \\ B^*U & E_2 \end{bmatrix}=\begin{bmatrix} U^*E_1U & |B| \\ |B| & E_2 \end{bmatrix}.$$

Then, by item 5. in Proposition 3.4 we get that $\mathrm{Spr}^+(W^*EW)\prec_w \mathrm{Spr}^+(E)$ and $s(B)=s(|B|)$. Hence, in order to show item 1. we can assume that $B\in K(\mathcal{K})^{sa}$. In this case, taking the self-adjoint unitary operator $R=\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}\in B(\mathcal{K}\oplus\mathcal{K})$, we have that

$$RER=\begin{bmatrix} E_2 & B \\ B & E_1 \end{bmatrix} \implies \frac{E+RER}{2}=\begin{bmatrix} \frac{E_1+E_2}{2} & B \\ B & \frac{E_1+E_2}{2} \end{bmatrix}.$$

By item 3. in Theorem 2.7 (Weyl inequality) and item 4. in Proposition 3.4

$$\lambda\left(\frac{E+RER}{2}\right)\prec \frac{\lambda(E)+\lambda(RER)}{2}=\lambda(E)\implies \mathrm{Spr}^+\left(\frac{E+RER}{2}\right)\prec_w \mathrm{Spr}^+(E)\,.$$

Therefore, in order to show item 1. we can assume that $B\in K(\mathcal{K})^{sa}$ and $E_1=E_2$.

Take now $Z=\frac{1}{\sqrt{2}}\begin{bmatrix} I & I \\ -I & I \end{bmatrix}\in B(\mathcal{K}\oplus\mathcal{K})$ that is a unitary operator. Since now

$$E=\begin{bmatrix} E_1 & B \\ B & E_1 \end{bmatrix} \implies Z^*EZ=\begin{bmatrix} E_1-B & 0 \\ 0 & E_1+B \end{bmatrix}\,,\ \mathrm{Spr}^+(Z^*EZ)=\mathrm{Spr}^+(E)\,. \tag{60}$$

Hence, using item 5 we get that

$$2\,s(B)=s\big([E_1-B]-[E_1+B]\big)\prec_w \mathrm{Spr}^+\big([E_1-B]\oplus[E_1+B]\big)\overset{(60)}{=}\mathrm{Spr}^+(E)\,.$$

This shows that 5 → 1. and we are done. □

5 Appendix

In this section we first compare the results obtained herein with previous results involving the related notion of spectral spread of Hermitian matrices. Then, we show how the arguments and techniques used in this work can be adapted to obtain new results in the matrix context. Finally, we include some facts related with submajorization of real sequences in c_0.

5.1 Matrix versions

In this section $\mathcal{M}_d(\mathbb{C})$ denotes the algebra of $d \times d$ complex matrices, $\mathcal{H}(d) \subset \mathcal{M}_d(\mathbb{C})$ denotes the real subspace of Hermitian (self-adjoint) matrices and we write $\mathcal{M}_d(\mathbb{C})^+$ for the cone of positive semi-definite matrices. For $A \in \mathcal{H}(d)$ we let $\lambda(A) = (\lambda_i(A))_{i=1}^d$ denote the eigenvalues of A counting multiplicities and arranged in non-increasing order. For arbitrary $C \in \mathcal{M}_d(\mathbb{C})$ we let $s(C) = (s_i(C))_{i=1}^d = \lambda((C^*C)^{1/2})$ denote the singular values of C, counting multiplicities and arranged in non-increasing order.

Next, we recall the spectral spread for self-adjoint matrices considered in [24].

Definition 5.1. Given $A \in \mathcal{H}(d)$ then the spectral spread of A, denoted $\operatorname{Spr}^+(A)$, is

$$\operatorname{Spr}^+(A) = (\lambda_1(A) - \lambda_d(A), \lambda_2(A) - \lambda_{d-1}(A), \ldots, \lambda_k(A) - \lambda_{n-k+1}(A)) \in \mathbb{R}^k_{\geq 0}\,,$$

where $k = [d/2]$ (integer part). △

As already mentioned in Remark 3.5, the definition of spread for compact self-adjoint operators is not exactly a generalization of the spread for matrices, as in Definition 5.1 (see [24]). Indeed, if we let $A \in \mathcal{M}_d(\mathbb{C})^+$ be invertible and we let $\operatorname{Spr}^+(A) = (\operatorname{Spr}^+_i(A))_{i=1}^d$ then

$$0 \leq \operatorname{Spr}^+_i(A) < \lambda_i(A) \quad \text{for} \quad 1 \leq i \leq k\,,$$

where $k = [d/2]$. On the other hand, if we embed A as a finite rank (and therefore compact) operator $\tilde{A} \in K(\mathcal{H})^+$ then $\operatorname{Spr}^+_i(\tilde{A}) = \lambda_i(A) > 0$, for $1 \leq i \leq d$, and $\operatorname{Spr}^+_i(\tilde{A}) = 0$, for $i \geq d+1$. That is, the sequence $\operatorname{Spr}^+(\tilde{A})$ is not obtained by appending zeros to the vector $\operatorname{Spr}^+(A)$.

Definition 5.2. Given two real vectors $a = (a_i)_{i=1}^d$, $b = (b_i)_{i=1}^d \in \mathbb{R}^d$ then recall that a is submajorized by b (in $\mathbb{R}^d$) if

$$\sum_{i=1}^k a_i^\downarrow \leq \sum_{i=1}^k b_i^\downarrow \quad \text{for} \quad 1 \leq k \leq d\,, \tag{61}$$

where $a^\downarrow = (a_i^\downarrow)_{i=1}^d \in \mathbb{R}^d$ is the vector obtained by rearranging the entries of $a \in \mathbb{R}^d$ in non-increasing order (see [6]). In case $a = (a_i)_{i=1}^r \in \mathbb{R}^r_{\geq 0}$ and $b = (b_i)_{i=1}^s \in \mathbb{R}^s_{\geq 0}$ have non-negative entries (but possibly different sizes) we say that $a \prec_w b$ if $\tilde{a} = (a, 0_p) \prec_w \tilde{b} = (b, 0_q)$ where $\tilde{a}$ and $\tilde{b}$ are obtained from a, b by appending zeros, in order to equate sizes (for convenient $p, q \geq 0$). △

We notice that submajorization between real vectors a and b differs from the notion of submajorization between the real sequences obtained by appending zero entries to a and b.

Therefore, the statements of this paper relative to compact operators, which are formally analogous to those of [24], such as Proposition 3.4, Theorem 3.6, Theorem 3.7 and Corollary

4.5, do need new proofs in this context. On the other hand, a matrix version of Theorem 4.4 would be

Theorem 5.3. *Let $A, X \in \mathcal{H}(d)$. Then*

$$s(AX - XA) \prec_w \frac{1}{2}\,\mathrm{Spr}^+(A \oplus A)\,\cdot\,\mathrm{Spr}^+(X \oplus X)\,, \tag{62}$$

where the product is performed entry-wise and we use Definition 5.2. □

Proof. We obtain a proof of this result in the matrix context following the argument described in the proof of Theorem 4.4; indeed, a detailed look at that proof shows that it is formally a consequence of the inequality

$$2\,s(B) \prec_w \mathrm{Spr}^+(A) \quad \text{where} \quad A = \begin{bmatrix} A_1 & B \\ B^* & A_2 \end{bmatrix} \begin{matrix} \mathbb{C}^k \\ \mathbb{C}^{d-k} \end{matrix} \in \mathcal{H}(d) \tag{63}$$

(which is a formal analogue Theorem 3.6, and that was already proved for matrices in [24]), the analogue Lemma 4.1 i.e., that for $C,\, D \in \mathcal{H}(d)$ then

$$\mathrm{tr}(CD) \leq \mathrm{tr}(\lambda(C) \cdot \lambda(D))\,, \tag{64}$$

(which is trivial in the finite dimensional case, see Remark 4.2) and the inequality

$$\lambda(i\,(C\,D - D\,E)) \prec_w \frac{1}{2}\,\mathrm{Spr}^+(C)\,\cdot\,\mathrm{Spr}^+(D) \quad \text{for} \quad C,\, D \in \mathcal{H}(d)\,, \tag{65}$$

whose proof follows the lines of the proof of Theorem 4.3, now using Eqs. (63) and (64). □

Theorem 5.3 is a new result, and it implies the matrix version of Corollary 4.5 (that was proved in [24]). Similarly, the matrix reformulation of Theorems 4.7 and 4.9 are still true, since the proofs of those results given in this paper can be easily adapted to the finite dimensional case.

With respect to section 4.2 about AG-type inequalities, the matrix versions of all the results of this section are new, but we have to make small modifications of the statements in this context and obtain new proofs. This is because (in general) for a matrix $E \in \mathcal{H}(d)$,

$$\mathrm{Spr}^+(E \oplus 0_d) \neq \mathrm{Spr}^+(E)$$

(we used that the previous sequences do agree in the compact case in the proof of Proposition 4.11), not only because these vectors have different sizes, but because the initial entries of these vectors may be different. On the other hand, it is straightforward to show that

Proposition 5.4. Let $E \in \mathcal{H}(n)$. Then

1. In general, as in the compact case, we have that

$$s(E) \prec \mathrm{Spr}^+(E \oplus 0_d)\ .$$

2. If $E \in \mathcal{M}_d(\mathbb{C})^+$ or $E \in -\mathcal{M}_d(\mathbb{C})^+$ then

$$\mathrm{Spr}^+(E \oplus 0_d) = s(E)$$

and, if $k = [d/2]$,

$$\mathrm{Spr}^+_i(E) \leq \mathrm{Spr}^+_i(E \oplus 0_d) \quad \text{for} \quad i \in \mathbb{I}_k\ .$$ □

It is easy to see that the proof of Proposition 4.11 can be adapted to show that

Proposition 5.5. *Let $C,\, S\in \mathcal{M}_d(\mathbb{C})$ be such that $C^*C+S^*S=P$, where $P=P^2$, and let $E\in\mathcal{H}(d)$. Then*

$$2\, s(SEC^*) \prec_w \operatorname{Spr}^+(E\oplus 0_d)\ . \qquad \square$$

Using Proposition 5.5 we can easily see that

Theorem 5.6. *Let $A,\, B\in\mathcal{M}_d(\mathbb{C})$ and let $E\in\mathcal{H}(d)$. Then, if $F=(A^*A+B^*B)^{1/2}$,*

$$s(A\,E\,B^*) \prec_w \frac{1}{2}\operatorname{Spr}^+(F\,E\,F\oplus 0_d)\,. \qquad \square$$

We remark that both statements of Proposition 5.5 and Theorem 5.6 fail if one replaces $\operatorname{Spr}^+(E\oplus 0_d)$ by $\operatorname{Spr}^+(E)$ (or $\operatorname{Spr}^+(F\,E\,F\oplus 0_d)$ by $\operatorname{Spr}^+(F\,E\,F)$), as in the infinite dimensional case. Indeed, it suffices to take $E=I$ and choose S and C in such a way that $S^*S+C^*C=I$ but $SC^*\neq 0$. Finally, the matrix version of the statements of Proposition 4.16 and Corollary 4.17 are still true with the same proofs as for compact operators.

5.2 Submajorization for real sequences in c_0

Here we collect several well known results about majorization, used throughout our work. For detailed proofs of these results and general references in submajorization theory see [14].

In the next result we describe several elementary but useful properties of (sub)majorization between real vectors. The details are left to the reader.

Lemma 5.7. Let $x\,,\,y\,,\,z\,,\,w\in c_0(\mathbb{M})\cap\mathbb{R}^{\mathbb{M}}$ be real sequences. Then,

1. $x+y\prec x^{\uparrow\downarrow}+y^{\uparrow\downarrow}$;
2. If $x\prec y$ then $|x|\prec_w |y|$;
3. If $x\prec z$ and $y\prec w$ then $x+y\prec z+w$.

If we assume that $\mathbb{M}=\mathbb{N}$, then

4. If $x\prec_w y$ and $z\prec_w z$ then $(x,z)\prec_w (y,w)$.

If we assume further that $x\,,\,y\,,\,z\in c_0(\mathbb{N})$ are non-negative sequences then,

5. $x\cdot y\prec_w x^{\downarrow}\cdot y^{\downarrow}$;
6. If $x\prec_w y$ and $y,\,z\in c_0(\mathbb{N})^{\downarrow}$ then $x\cdot z\prec_w y\cdot z$. $\square$

Proposition 5.8. Let $E\in K(\mathcal{H})^{sa}$ and set $\hat{E}=\begin{bmatrix}0 & E\\ E^* & 0\end{bmatrix}\in K(\mathcal{H}\oplus\mathcal{H})$. Then

$$\lambda(\hat{E})=\big(\,s(E)\,,\,-s(E^*)\,\big)^{\uparrow\downarrow}=\big(\,s(E)\,,\,-s(E)\,\big)^{\uparrow\downarrow}. \qquad \square$$

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