

UNIFORM BOUNDS FOR THE NUMBER OF RATIONAL POINTS ON VARIETIES OVER GLOBAL FIELDS

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ABSTRACT. We extend the work of Salberger; Walsh; Castryck, Cluckers, Dittmann and Nguyen; and Vermeulen to prove the uniform dimension growth conjecture of Heath-Brown and Serre for varieties of degree at least 4 over global fields. As an intermediate step, we generalize the bounds of Bombieri and Pila to curves over global fields and in doing so we improve the B^ε factor by a $\log(B)$ factor.

1. INTRODUCTION

Let X be a projective variety defined over a global field K . A central problem in diophantine geometry is to find bounds for the number of K -rational points in X of bounded height, for some adequate height function. When $K = \mathbb{Q}$ and X is a hypersurface, perhaps the first account of such results with great generality is due to Cohen. Specifically, as a consequence of the results in [15] concerning Hilbert's irreducibility theorem, in the appendix of Heath-Brown's article [23] it is proved that for an absolutely irreducible form $G \in \mathbb{Z}[X_1, \dots, X_n]$ of degree $d \geq 2$, it holds

$$|\{\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{Z}^n : G(\mathbf{x}) = 0, \text{h.c.f.}(x_1, \dots, x_n) = 1, \max_i |x_i| \leq B\}| \lesssim_{\varepsilon, G} B^{n-\frac{3}{2}+\varepsilon},$$

for any $\varepsilon > 0$. Furthermore, in [23, Page 227] Heath-Brown posed the question:

Question 1.1. *Let G be an absolutely irreducible form with coefficients in \mathbb{Z} of degree $d \geq 2$ in n variables. Is it true that for every $\varepsilon > 0$ it holds*

$$|\{\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{Z}^n : G(\mathbf{x}) = 0, \max_i |x_i| \leq B\}| \lesssim_{\varepsilon, G} B^{n-2+\varepsilon}?$$

The results in [15] were later generalized by Serre in [40] to the context of projective varieties over number fields. Moreover, in [40], Serre proposed the following variation of Question 1.1 (see [41, Page 178]),

Question 1.2. *Let K be a number field of degree d_K , and let $X \subseteq \mathbb{P}_K^n$ be an integral projective variety which is not a linear variety. Let H be the absolute projective multiplicative height. Is there a constant c such that*

$$|\{\mathbf{x} \in X(K) : H(\mathbf{x}) \leq B\}| \lesssim_X B^{d_K \dim(X)} (\log(B))^c?$$

By considering the quadric $xy = zw$ in \mathbb{P}_K^3 , Serre remarked in [41, Page 178] that the logarithmic factor in Question 1.2 can not be dispensed. Moreover, in [42, Page 27], he formulated a variant to Question 1.2 where the logarithmic factor is replaced by a factor B^ε for all ε .

It turns out that for many applications one requires uniformity in the coefficients of the polynomial defining the variety X . The first breakthrough concerning bounds in terms of the degree and dimension of X is due to Bombieri and Pila in [3]. In that article they developed the now called determinant method of Bombieri-Pila and proved that the number of integral zeroes of size up to B of an absolutely irreducible polynomial $F(X, Y) \in \mathbb{Z}[X, Y]$ of degree $d \geq 2$ is $\lesssim_{\varepsilon, d} B^{\frac{1}{d}+\varepsilon}$ for all $\varepsilon > 0$. This result was used in [34] by Pila to prove uniform bounds for projective and affine

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varieties, for instance, he proved that the number of rational zeroes of height up to B of an absolutely irreducible form $G \in \mathbb{Z}[X_1, \dots, X_n]$ of degree d is $\lesssim_\varepsilon B^{n-2+\frac{1}{d}+\varepsilon}$.

The determinant method of Bombieri-Pila is of an affine nature. Subsequently, in [24] Heath-Brown developed a p -adic determinant method which allowed him to prove general uniform bounds for projective hypersurfaces. In particular, he gave a positive answer to Question 1.1 in the case $d = 2$ and also he proved the following projective version of the main result in [3].

Theorem 1.3 ([24]). *Let $F \in \mathbb{Z}[X_1, X_2, X_3]$ be an absolutely irreducible form of degree d . Then for all $\varepsilon > 0$ the number of rational zeroes of height up to B of F is at most $\lesssim_{\varepsilon, d} B^{\frac{2}{d}+\varepsilon}$.*

Moreover, in this article Heath-Brown stated a uniform version of Question 1.1 ([24, Conjecture 1.2]). A more general version of this conjecture for projective varieties over \mathbb{Q} appeared first in the literature in [7, Conjecture 3.3] where it is called the “dimension growth conjecture”.

Conjecture 1.4 (Dimension growth conjecture). *Let $X \subseteq \mathbb{P}_{\mathbb{Q}}^n$ be an integral projective variety of degree $d \geq 2$. Let H be the absolute projective multiplicative height. Then for any $\varepsilon > 0$ it holds:*

$$|\{\mathbf{x} \in X(\mathbb{Q}) : H(\mathbf{x}) \leq B\}| \lesssim_{\dim(X), d, \varepsilon} B^{\dim(X)+\varepsilon}.$$

Conjecture 1.4 was established in the case $d = 2$, and the cases $n = 3$ and $n = 4$ for any degree, by Heath-Brown in [24, Theorem 2] and [24, Theorems 3 and 9], respectively. For $n = 5$ and $d \geq 4$ the conjecture was proved by Broberg and Salberger in [6, Theorem 1], and for $n = 5$ and $d = 3$ it was proved by Browning and Heath-Brown in [8, Theorem 3]. Moreover, Conjecture 1.4 was proved for varieties of degree $d \geq 6$ for all n by Browning, Heath-Brown and Salberger in [9, Corollary 2].

In order to tackle Conjecture 1.4 in the cases of lower degrees, Salberger further developed the determinant method. In [37] he extended the p -adic determinant method devised in [24] to prove Conjecture 1.4 for $d \geq 4$ whenever X contains finitely many linear varieties defined over $\overline{\mathbb{Q}}$, of dimension $r - 1$. This result was superseded by those of the article [38], where Salberger introduced a global version of the method of Heath-Brown and proved the following theorem.

Theorem 1.5 ([9, 38]). *The dimension growth conjecture holds for every integral projective variety $X \subseteq \mathbb{P}_{\mathbb{Q}}^n$ of degree $d \geq 4$. In the case that X has degree $d = 3$, it holds*

$$|\{\mathbf{x} \in X(\mathbb{Q}) : H(\mathbf{x}) \leq B\}| \lesssim_{\dim(X), \varepsilon} B^{\dim(X)-1+\frac{2}{\sqrt{3}}+\varepsilon}.$$

Theorem 1.3 and Theorem 1.5 (the latter when $d \geq 4$) have exponents that are essentially optimal, up to the ε -factor. Then, it remained an open question if in general one could remove the factor B^ε . Building on the global determinant method of Salberger, and using the polynomial method in a clever way, in [46] Walsh proved that this is indeed the case for curves.

Theorem 1.6 ([46]). *Let $F \in \mathbb{Z}[X_1, X_2, X_3]$ be an absolutely irreducible form of degree d . Then the number of rational zeroes of height up to B of F is at most $\lesssim_d B^{\frac{2}{d}}$.*

This was further explored in [11] by Castryck, Cluckers, Dittmann, and Nguyen where they provided effective versions of several results in [17, 8, 38, 46], which allowed them to prove that the dimension growth conjecture holds without the ε factor when the degree is at least 5.

Theorem 1.7 ([11]). *Given $n > 1$, there exist constants $c = c(n)$ and $e = e(n)$, such that for all integral projective varieties $X \subseteq \mathbb{P}_{\mathbb{Q}}^n$ of degree $d \geq 5$ and all $B \geq 1$ one has*

$$|\{\mathbf{x} \in X(\mathbb{Q}) : H(\mathbf{x}) \leq B\}| \leq cd^e B^{\dim(X)}.$$

While there has been a lot of progress in giving uniform bounds for varieties defined over \mathbb{Q} , there are not as many results for global fields. In [5], Broberg generalized several of the estimates in [24] to number fields. Also, the determinant method was reinterpreted in the framework of Arakelov's theory in [12], [13] and [29]. For function fields, the Bombieri-Pila bound was proved for $\mathbb{F}_q(T)$ in [39] by Sedunova, adapting a reinterpretation given in [25] by Helfgott and Venkatesh of the determinant method of Bombieri-Pila. Also, by methods of model theory, in [14] Cluckers, Forey and Loeser proved a stronger bound than the one in [39] for $\mathbb{F}_q(T)$ with $\text{char}(\mathbb{F}_q)$ large enough. Quite recently, in [45] Vermeulen proved analogues of Theorem 1.6 and Theorem 1.7 for hypersurfaces over $\mathbb{F}_q(T)$ of degree $d \geq 64$.

In this article we extend the work of Salberger, Walsh, and Castryck, Cluckers, Dittmann and Nguyen on the determinant method of Heath-Brown and Salberger, to give uniform estimates for the number of rational points of bounded height on projective varieties defined over global fields. More precisely, we first prove the following extension of Theorem 1.6, [11, Theorem 2] and [45, Theorem 1.1] to global fields.

Theorem 1.8. *Let K be a global field of degree d_K . Let H be the absolute projective multiplicative height. For any integral projective curve $C \subseteq \mathbb{P}_K^n$ of degree d it holds*

$$|\{\mathbf{x} \in C(K) : H(\mathbf{x}) \leq B\}| \lesssim_{K,n} \begin{cases} d^4 B^{\frac{2d_K}{d}} & \text{if } K \text{ is a number field,} \\ d^8 B^{\frac{2d_K}{d}} & \text{if } K \text{ is a function field.} \end{cases}$$

Theorem 1.8 is new when K is a global field different from \mathbb{Q} and $\mathbb{F}_q(T)$. Previous to this result, in the number field case the only known bound was $O_{K,n,d}(B^{\frac{2d_K}{d}+\varepsilon})$ given in [5, Corollary 1]. For number fields different from \mathbb{Q} , our bound was simultaneously obtained by Liu in [30].

Adapting the strategy devised in [17, Remark 2.3] and developed in [11, Proposition 4.2.1], from Theorem 1.8 we deduce the following extension of [11, Theorem 3] and [45, Theorem 1.2] to global fields.

Theorem 1.9 (Bombieri-Pila type of bound). *Let K be a global field of degree d_K . For any integral curve $C \subseteq \mathbb{A}_K^n$ of degree d , it holds*

$$|\{\mathbf{x} \in C(K) \cap [B]_{\mathcal{O}_K}^n\}| \lesssim_{K,n} \begin{cases} d^3 B^{\frac{1}{d}} (\log(B) + d) & \text{if } K \text{ is a number field,} \\ d^7 B^{\frac{1}{d}} (\log(B) + d) & \text{if } K \text{ is a function field,} \end{cases}$$

where $[B]_{\mathcal{O}_K}^n$ is defined in Section 5.2.

We remark that in the function field case, we obtain more precise bounds in the exponent of d than the ones presented in Theorem 1.8 and Theorem 1.9 which depend on the characteristic of the field as in [45] (see Definition 3.22, Theorem 5.9 and Theorem 5.16 for the precise statements).

Next, we obtain the following estimate for the number of points of bounded height on affine hypersurfaces, extending [38, Theorem 0.4], [11, Theorem 4] and [45, Theorem 4.2] to global fields.

Theorem 1.10. *Let K be a global field of degree d_K . Given $n > 2$ there exist a constant $e = e(n)$ such that for all polynomials $f \in \mathcal{O}_K[Y_1, \dots, Y_n]$ of degree d , whose homogeneous part of degree d is absolutely irreducible, it holds*

$$|\{\mathbf{x} \in \mathcal{Z}(f) \cap [B]_{\mathcal{O}_K}^n\}| \lesssim_{K,n} d^e B^{n-2}, \text{ whenever } d \geq 5.$$

In the case when $d = 3, 4$, for any $\varepsilon > 0$ we have

$$|\{\mathbf{x} \in \mathcal{Z}(f) \cap [B]_{\mathcal{O}_K}^n\}| \lesssim_{K,n,\varepsilon} \begin{cases} B^{n-2+\varepsilon} & \text{if } d = 4, \\ B^{n-3+\frac{2}{\sqrt{3}}+\varepsilon} & \text{if } d = 3. \end{cases}$$

Theorem 1.10 is new when K is a number field different from \mathbb{Q} . When K is a function field, the result is new when K is a function field different from $\mathbb{F}_q(T)$ or $K = \mathbb{F}_q(T)$ and $3 \leq d < 64$.

By means of an effective projection argument as in [31] which relies on the Combinatorial Nullstellensatz [1] and on an argument of Mumford found in [32], from Theorem 1.10 we deduce the dimension growth conjecture for global fields. More precisely, we have

Theorem 1.11 (Dimension growth conjecture for global fields). *Let K be a global field of degree d_K . Given $n > 1$, there exists a constant $e = e(n)$ such that for all integral projective varieties $X \subseteq \mathbb{P}_K^n$ of degree d it holds*

$$|\{\mathbf{x} \in X(K) : H(\mathbf{x}) \leq B\}| \lesssim_{K,n} d^e B^{d_K \dim(X)} \text{ whenever } d \geq 5.$$

In the case $d = 3, 4$, for any $\varepsilon > 0$ we have

$$|\{\mathbf{x} \in X(K) : H(\mathbf{x}) \leq B\}| \lesssim_{K,n,\varepsilon} \begin{cases} B^{d_K(\dim(X)+\varepsilon)} & \text{if } d = 4, \\ B^{d_K(\dim(X)-1+\frac{2}{\sqrt{3}}+\varepsilon)} & \text{if } d = 3. \end{cases}$$

Theorem 1.11 is new for number fields different from \mathbb{Q} . When K is a function field, the result is new when K is different from $\mathbb{F}_q(T)$ or when $K = \mathbb{F}_q(T)$ and X is a hypersurface of degree $3 \leq d < 64$ or when $K = \mathbb{F}_q(T)$ and X is a projective variety of codimension at least 2 and degree $d \geq 3$. Arguably, these are the first bounds appearing in the literature estimating the number of points of bounded height for varieties over global fields different from \mathbb{Q} or $\mathbb{F}_q(T)$.

The proof of all these theorems are obtained by adapting and extending the strategies developed in [24, 37, 9, 38, 46, 11] to global fields. Roughly speaking, the proof are obtained by the polynomial method (as it was presented in [46]), namely by constructing a polynomial g of small degree vanishing on $\{\mathbf{x} \in X(K) : H(\mathbf{x}) \leq B\}$, which does not vanishes identically on X . Then we study the rational points lying in the irreducible components of $\mathcal{Z}(g) \cap X$. For those irreducible components of high degree we argue as in [11], while for those of small degree we rework and simplify the proof of [38, Main Lemma 3.2]. Carrying out this last step adds difficulties all along the article, which were not present in [11, 46]. Also new challenges appear from dealing with general global fields that will be explained as they arise. It is relevant to emphasize that this article presents a unified treatment that deals with number fields and function fields simultaneously.

When we were at the final stages of writing the first arXiv version of this manuscript, Liu uploaded to the arXiv the article [30] where he proved Theorem 1.8 for number fields by reinterpreting [38] in the framework of Arakelov theory and by using ideas of [46, 11].

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2. HEIGHTS AND PRIMES IN GLOBAL FIELDS

The purpose of this section is manifold. First we establish a normalization of the absolute values of a global field. We use this to define the height function that will be used in this article and recall some basic properties of it. Secondly, we prove a proposition that allows us to find affine coordinates of projective points with controlled affine height. Then we recall the theorems of Bombieri and Vaaler, and Thunder, that give solutions of controlled height of a system of linear equations. Next, we define and analyze two notions of heights of polynomials that will be used in this article. Finally we present some estimates regarding the distribution of primes in global fields and we comment on how to make all the bounds in all the statements in this manuscript effective on the dependence of the global field K .

Notation 2.1. We use the asymptotic notation $X = O(Y)$ or $X \lesssim Y$ to mean $|X| \leq C|Y|$ for some constant C . We also use $O_{K,n,d}(Y)$ or $\lesssim_{K,n,d} Y$ to mean that the implicit constants depend on K, n and d .

2.1. Absolute values and relative height. Throughout this paper, K denotes a global field, i.e. a finite separable extension of \mathbb{Q} or $\mathbb{F}_q(T)$, in which case we further assume that the field of constants is \mathbb{F}_q . We will denote by d_K the degree of the extension K/\mathbb{k} , where \mathbb{k} indistinctively denotes the base fields \mathbb{Q} or $\mathbb{F}_q(T)$.

Let K be a number field and let \mathcal{O}_K be its ring of integers. Then each embedding $\sigma : K \hookrightarrow \mathbb{C}$ induces a place v , by means of the equation

$$(2.1) \quad |x|_v := |\sigma(x)|_\infty^{\frac{n_v}{d_K}},$$

where $|\cdot|_\infty$ denotes the absolute value of \mathbb{R} or \mathbb{C} and $n_v = 1$ or 2 , respectively. Such places will be called the places at infinite, and denoted by $M_{K,\infty}$. Note that $\sum_{v \in M_{K,\infty}} n_v = d_K$. They are all the archimedean places of K . Since the complex embeddings come in pairs that differ by complex conjugation, we have $|M_{K,\infty}| \leq d_K$.

Now let \mathfrak{p} be a non-zero prime ideal of the number field K , and denote by $\text{ord}_{\mathfrak{p}}$ the usual \mathfrak{p} -adic valuation. Associated to \mathfrak{p} , we have the place v in K given by the equation

$$|x|_v := |x|_{\mathfrak{p}} := \mathcal{N}_K(\mathfrak{p})^{-\frac{\text{ord}_{\mathfrak{p}}(x)}{d_K}},$$

where $\mathcal{N}_K(\mathfrak{p})$ denotes the cardinal of the finite quotient $\mathcal{O}_K/\mathfrak{p}$. Similarly, the norm of a non-zero ideal $I \subseteq \mathcal{O}_K$, denoted by $\mathcal{N}_K(I)$, is just the cardinal of the finite quotient \mathcal{O}_K/I . Such places will be called the finite places, and will be denoted by $M_{K,\text{fin}}$. They are all the non-archimedean places of K . The set of places of K is then the union $M_{K,\infty} \cup M_{K,\text{fin}}$, and it will be denoted by M_K .

Now, let us suppose that K is a function field over \mathbb{F}_q , such that \mathbb{F}_q is algebraically closed in K (in other words, the constant field of K is \mathbb{F}_q). A prime in K is, by definition, a discrete valuation ring $\mathcal{O}_{(\mathfrak{p})}$ with maximal ideal \mathfrak{p} such that $\mathbb{F}_q \subseteq \mathfrak{p}$ and the quotient field of $\mathcal{O}_{(\mathfrak{p})}$ equals to K . By abuse of notation, when we refer to a prime in K , we will refer to the maximal ideal \mathfrak{p} . Associated to \mathfrak{p} , we have the usual \mathfrak{p} -adic valuation, that we will denote by $\text{ord}_{\mathfrak{p}}$. The degree of \mathfrak{p} , denoted by $\deg(\mathfrak{p})$ will be the dimension of $\mathcal{O}_{(\mathfrak{p})}/\mathfrak{p}$ as an \mathbb{F}_q -vector space, which is finite. Then the norm of \mathfrak{p} is defined as $\mathcal{N}_K(\mathfrak{p}) := q^{\deg(\mathfrak{p})}$. Any prime \mathfrak{p} of K induces a place v in K by the equation

$$|x|_v := |x|_{\mathfrak{p}} := \mathcal{N}_K(\mathfrak{p})^{-\frac{\text{ord}_{\mathfrak{p}}(x)}{d_K}}.$$

They are all the places in K . The set of all places in K is denoted by M_K . Now we fix an arbitrary place v_∞ in M_K above the place in $\mathbb{F}_q(T)$ defined by $\left|\frac{f}{g}\right|_\infty := q^{\deg(f)-\deg(g)}$. Its corresponding prime will be denoted \mathfrak{p}_∞ ; it has degree at most d_K . The ring of integers of K is the subset

$$\mathcal{O}_K := \{x \in K : |x|_v \leq 1 \text{ for all } v \in M_K, v \neq v_\infty\}.$$

Given $x \in \mathcal{O}_K \setminus \{0\}$ we define $\mathcal{N}_K(x) := \prod_{\mathfrak{p} \neq \mathfrak{p}_\infty} \mathcal{N}_K(\mathfrak{p})^{\text{ord}_{\mathfrak{p}}(x)}$. By definition, $\text{ord}_{\mathfrak{p}}(x) \geq 0$ for all $\mathfrak{p} \neq \mathfrak{p}_\infty$, so that $\mathcal{N}_K(x)$ is a positive integer. The primes in \mathcal{O}_K will be the primes $\mathfrak{p} \neq \mathfrak{p}_\infty$ of K . We will denote $M_{K,\infty} := \{v_\infty\}$ and $M_{K,\text{fin}} := M_K \setminus M_{K,\infty}$.

Now, given a global field K , we define the absolute multiplicative projective height of K of a point $\mathbf{x} = (x_0 : \dots : x_n) \in \mathbb{P}^n(K)$, to be the function

$$H(\mathbf{x}) := \prod_{v \in M_K} \max_i \{|x_i|_v\},$$

and the relative multiplicative projective height by

$$H_K(\mathbf{x}) := H(\mathbf{x})^{d_K}.$$

If $x \in K$, $H_K(x)$ will always denote the projective height $H_K(1 : x)$.

For our purposes, it will be necessary to understand how the affine height of a point behaves under the action of a polynomial. It is easy to show (see [26, Proposition B.2.5]) that if $f(T_1, \dots, T_n) = \sum_{(i_1, \dots, i_n)} c_{i_1, \dots, i_n} T_1^{i_1} \dots T_n^{i_n}$, $\mathbf{c} = (c_{i_1, \dots, i_n})_{i_1, \dots, i_n}$ and R is the number of (i_1, \dots, i_n) with $c_{i_1, \dots, i_n} \neq 0$, then

$$(2.2) \quad H_K(f(\mathbf{x})) \leq R^{d_K} H_K(1 : \mathbf{c}) H_K(1 : \mathbf{x})^{\deg(f)}.$$

Now, let ϕ_0, \dots, ϕ_r be homogeneous polynomials of degree m in X_0, \dots, X_n with coefficients in K . Let \mathbf{c} be the projective point consisting of the coefficients of all the polynomials ϕ_i . Let us suppose that the maximum number of monomials appearing in any one of the ϕ_i is R . Then for all $\mathbf{x} \in \mathbb{P}^n(K)$ such that $\phi_i(\mathbf{x})$ is not zero for all i , it holds

$$(2.3) \quad H_K(\phi_0(\mathbf{x}) : \dots : \phi_r(\mathbf{x})) \leq R^{d_K} H_K(\mathbf{c}) H_K(\mathbf{x})^m.$$

Also, for any $x \in \mathcal{O}_K \setminus \{0\}$, it holds

$$(2.4) \quad \mathcal{N}_K(x) \leq H_K(x).$$

We will require to lift a bounded subset in projective space to a subset in affine space. The next proposition, which is a generalization of [41, Section 13.4], states that this can be done in a controlled manner.

Proposition 2.2. *Let K be a global field, and let $d \geq 1$ be an integer. There exist effective computable constants $c_1 := c_1(K)$, $c_2 := c_2(K)$ and $c_3 := c_3(K)$ such that for every $\mathbf{x} \in \mathbb{P}^d(K)$ there exists $(y_0, \dots, y_d) \in \mathcal{O}_K^{d+1}$ a lift of \mathbf{x} verifying:*

- (1) *if K is a number field, for all embedding $\sigma : K \hookrightarrow \mathbb{C}$, $\max_i |\sigma(y_i)| \leq c_1 H(\mathbf{x})$, thus for all $v \in M_{K,\infty}$, $\max_i |y_i|_v \leq c_1 H_K(\mathbf{x})^{\frac{n_v}{d_K^2}}$ after relabeling c_1 . If K is a function field, for the place v_∞ it holds $\max_i |y_i|_{v_\infty} \leq c_1 H_K(\mathbf{x})^{\frac{1}{d_K}}$;*
- (2) *for any prime $\mathfrak{p} \notin M_{K,\infty}$ with $\mathcal{N}_K(\mathfrak{p}) > c_2$, it holds that $\mathfrak{p} \nmid \sum_{i=0}^d y_i \mathcal{O}_K$. Equivalently, for any place v corresponding to such primes it holds $\max_i |y_i|_v = 1$;*
- (3) *it holds*

$$\prod_{v \in M_{K,\text{fin}}} \max_i |y_i|_v \geq c_3.$$

Proof. Given \mathbf{x} as in the statement, we choose coordinates (x_0, \dots, x_d) with $x_i \in \mathcal{O}_K$ for all i and consider the ideal $\mathfrak{a}_{x_0, \dots, x_d} = \sum_{i=0}^d x_i \mathcal{O}_K$. Note that

$$H(\mathbf{x}) = \frac{1}{\mathcal{N}_K(\mathfrak{a}_{x_0, \dots, x_d})^{\frac{1}{d_K}}} \prod_{v \in M_{K,\infty}} \max_i |x_i|_v.$$

The ideal $\mathfrak{a}_{x_0, \dots, x_d}$ depends on the coordinates, but its ideal class depends only on \mathbf{x} . Hence, if we take integral ideals $\mathfrak{a}_1, \dots, \mathfrak{a}_r$, representing all the ideal classes of \mathcal{O}_K , satisfying Minkowski's bound $\mathcal{N}_K(\mathfrak{a}_l) \leq c_2$ for all l , it holds that $\mathfrak{a}_{x_0, \dots, x_d} \mathfrak{a}_l^{-1} = \alpha \mathcal{O}_K$ for some l and some $\alpha \in K^\times$. Thus, $\alpha^{-1} \mathcal{O}_K \mathfrak{a}_{x_0, \dots, x_d} = \mathfrak{a}_l$. We conclude that $(x'_0, \dots, x'_d) := (\alpha^{-1} x_0, \dots, \alpha^{-1} x_d)$ are coordinates of \mathbf{x} in \mathcal{O}_K^{d+1} and $\mathfrak{a}_{x'_0, \dots, x'_d} = \mathfrak{a}_l$. In particular, since $\mathcal{N}_K(\mathfrak{a}_l) \leq c_2$, all the prime ideals \mathfrak{p} dividing \mathfrak{a}_l have norm $\mathcal{N}_K(\mathfrak{p}) \leq c_2$. We conclude that

$$\text{for all prime } \mathfrak{p} \notin M_{K,\infty} \text{ with } \mathcal{N}_K(\mathfrak{p}) > c_2, \mathfrak{p} \nmid \sum_{i=0}^d x'_i \mathcal{O}_K.$$

Moreover, if $c_3 = c_2^{-\frac{1}{d_K}}$, we see that $\prod_{v \in M_{K,\text{fin}}} \max_i |x'_i|_v = \frac{1}{\mathcal{N}_K(\mathfrak{a}_l)^{\frac{1}{d_K}}} \geq c_3$.

If K is a function field, $M_{K,\infty}$ consists only of the place v_∞ , thus condition (1) of Proposition 2.2 is immediately verified with $c_1 := c_3^{-1}$. If K is a number field, the argument follows the same lines given in [41, Section 13.4], [31, Proposition 2.1], namely we find c_1 and $\varepsilon \in \mathcal{O}_K^\times$ such that the set of coordinates $(y_0, \dots, y_d) := (\varepsilon^{-1} x'_0, \dots, \varepsilon^{-1} x'_d)$ is a point in \mathcal{O}_K^{d+1} , representing \mathbf{x} , satisfying for all embedding $\sigma : K \hookrightarrow \mathbb{C}$ the bound $\max_i |\sigma(y_i)| \leq c_1 H_K(\mathbf{x})$. More precisely, let $\sigma_1, \dots, \sigma_r, \sigma_{r+1}, \dots, \sigma_{r+s}$ be all the non-equivalent embeddings of K , rearranged in such a way that σ_i is a real embedding for all $1 \leq i \leq r$, and σ_i is a complex embedding for all $i > r$. Since the \mathbb{Z} -span

$$W := \left\{ \left(\log(|\sigma_1(\varepsilon)|), \dots, \log(|\sigma_r(\varepsilon)|), \log(|\sigma_{r+1}(\varepsilon)|^2), \dots, \log(|\sigma_{r+s}(\varepsilon)|^2) \right) : \varepsilon \in \mathcal{O}_K^\times \right\}_{\mathbb{Z}}$$

has dimension $r + s - 1$ by Dirichlet's unit theorem, so the same holds for the \mathbb{Z} -span

$$W' := \left\{ \left(\log(|\sigma_1(\varepsilon)|), \dots, \log(|\sigma_r(\varepsilon)|), \log(|\sigma_{r+1}(\varepsilon)|), \dots, \log(|\sigma_{r+s}(\varepsilon)|) \right) : \varepsilon \in \mathcal{O}_K^* \right\}_{\mathbb{Z}}.$$

The vector $\mathbf{1} = (1, \dots, 1) \in \mathbb{R}^{r+s}$ does not lie in W' because e is transcendental. Then, we have that $W_1 := W \oplus \mathbb{Z}\mathbf{1}$ is a lattice in \mathbb{R}^{r+s} . Let $\|\cdot\|$ be the ℓ^∞ -norm in \mathbb{R}^{r+s} . Let $\Omega \subseteq \mathbb{R}^{r+s}$ be a bounded subset containing a representative of each class of \mathbb{R}^{r+s}/W_1 (this subset exists since \mathbb{R}^{r+s}/W_1 is compact) and set $C_{W_1} := \sup_{\mathbf{y} \in \Omega} \|\mathbf{y}\|$. Then it follows that

$$(2.5) \quad \text{for any } \mathbf{z} \in \mathbb{R}^{r+s} \text{ there exists } \mathbf{w} \in W_1 \text{ verifying } \|\mathbf{z} - \mathbf{w}\| \leq C_{W_1}.$$

Now, for any $\mathbf{x} \in \mathbb{P}^d(K)$ with coordinates (x_0, \dots, x_d) such that $\mathbf{a}_{x_0, \dots, x_d} = \mathbf{a}_l$ for some l , we have

$$H(\mathbf{x}) = \frac{1}{\mathcal{N}_K(\mathbf{a}_l)} \prod_j \max_i |\sigma_j(x_i)|^{\frac{n_j}{d_K}} \sim_K \prod_j \max_i |\sigma_j(x_i)|^{\frac{n_j}{d_K}},$$

where $n_j = 1$ if σ_j is real and $n_j = 2$ if it is complex. Let $h_\sigma := \max_i |\sigma(x_i)| > 0$. Let $\mathbf{z} := (\log(h_{\sigma_1}), \dots, \log(h_{\sigma_{r+s}}))$. By (2.5) there exists $\mathbf{w} \in W_1$ such that $\|\mathbf{z} - \mathbf{w}\| \leq C_{W_1}$. In particular, there exist $\varepsilon \in \mathcal{O}_K^*$ and $m \in \mathbb{Z}$ such that $\mathbf{w} = (\log|\sigma_1(\varepsilon)|, \dots, \log|\sigma_{r+s}(\varepsilon)|) + m\mathbf{1}$ and

$$-C_{W_1} \leq \log|\sigma_j(\varepsilon)| + m - \log(h_{\sigma_j}) \leq C_{W_1} \text{ for all } j.$$

Hence, if $C_K := e^{C_{W_1}}$, we have that there is some $\varepsilon \in \mathcal{O}_K^*$ and $t = e^m$ such that

$$C_K^{-1} \frac{h_{\sigma_j}}{t} \leq |\sigma_j(\varepsilon)| \leq C_K \frac{h_{\sigma_j}}{t} \text{ for all } j.$$

Equivalently,

$$C_K^{-1} \max_i |\sigma_j(x_i \varepsilon^{-1})| \leq t \leq C_K \max_i |\sigma_j(x_i \varepsilon^{-1})| \text{ for all } j.$$

Thus, since $\sum_i \frac{n_i}{d_K} = 1$, for any j_0 it follows that

$$\begin{aligned} H(\mathbf{x}) &= H(\varepsilon^{-1} \mathbf{x}) \sim_K \prod_j \max_i |\sigma_j(x_i \varepsilon^{-1})|^{\frac{n_j}{d_K}} = \prod_j \left(\max_i |\sigma_j(x_i \varepsilon^{-1})| C_K \right)^{\frac{n_j}{d_K}} C_K^{-\frac{n_j}{d_K}} \\ &\gtrsim_K \prod_j t^{\frac{n_j}{d_K}} \gtrsim_K \prod_j \left(\max_i |\sigma_{j_0}(x_i \varepsilon^{-1})| C_K^{-1} \right)^{\frac{n_j}{d_K}} \gtrsim_K \max_i |\sigma_{j_0}(x_i \varepsilon^{-1})|. \end{aligned}$$

We conclude that $H(\mathbf{x}) \gtrsim_K \max_{j,i} |\sigma_j(x_i \varepsilon^{-1})| \gtrsim_K \max_\sigma |\sigma(x_i \varepsilon^{-1})|$ for all i . Then $\mathbf{y} = (x_1 \varepsilon^{-1}, \dots, x_d \varepsilon^{-1})$ is a lift of \mathbf{x} with the required properties. \square

2.2. Small solutions to linear equations. For our purpose, we will need a strong version of Siegel's lemma, given by the well known theorem [4] of Bombieri and Vaaler, and its function field generalization [44] of Thunder. In order to state these results, we recall the notion of the height of a matrix. Let $A = (a_{ij})_{i,j}$ be an $m \times n$ matrix with entries in K and rank $m < n$. If $J \subseteq \{1, 2, \dots, n\}$ is a subset with $|J| = m$ elements, we write $A_J := (a_{i,j})_{1 \leq i \leq m, j \in J}$ for the corresponding submatrix. For each place $v \in M_K$ we define

$$H_v(A) := \begin{cases} \max_{|J|=m} |\det(A_J)|_v & \text{if } v \text{ is non-archimedean,} \\ |\det(AA^*)|_v^{\frac{1}{2}} & \text{if } v \text{ is archimedean,} \end{cases}$$

where A^* is the complex conjugate transpose of A . The Arakelov height of A is defined as the product

$$H_{\text{Ar}}(A) := \prod_{v \in M_K} H_v(A).$$

Let us state the theorems of Bombieri and Vaaler, and Thunder.

Theorem 2.3 ([4, Theorem 9], [44, Corollary 2]). *Let K be a global field. Suppose that $A = (a_{ij})$ is an $m \times n$ matrix of rank $m < n$ with entries in K . Then there exist an effective constant $C = C(K)$ and linearly independent $\mathbf{b}_1, \dots, \mathbf{b}_{n-m} \in K^n$ with $A\mathbf{b}_i^t = 0$ for all i , satisfying*

$$\prod_{i=1}^{n-m} H(\mathbf{b}_i) \leq C(K)^{n-m} H_{\text{Ar}}(A).$$

In this paper we will use a different formulation of Theorem 2.3, requiring a more explicit definition of the Arakelov height of a matrix. For that, let A be an $m \times n$ matrix of rank $m < n$ with entries in \mathcal{O}_K . Let Δ be the

greatest common divisor in \mathcal{O}_K of the determinants $\det(A_J)$ with $|J| = m$. Since for any non-archimedean place $v \notin M_{K,\infty}$ with corresponding prime \mathfrak{p} , it holds

$$H_v(A) = \max_{|J|=m} \mathcal{N}_K(\mathfrak{p})^{-\frac{\text{ord}_{\mathfrak{p}}(\det(A_J))}{d_K}} = \mathcal{N}_K(\mathfrak{p})^{-\min_{|J|=m} \frac{\text{ord}_{\mathfrak{p}}(\det(A_J))}{d_K}} = \mathcal{N}_K(\mathfrak{p})^{-\frac{\text{ord}_{\mathfrak{p}}(\Delta)}{d_K}},$$

we conclude that

$$\prod_{v \notin M_{K,\infty}} H_v(A) = \prod_{\mathfrak{p}|\Delta} \mathcal{N}_K(\mathfrak{p})^{-\frac{\text{ord}_{\mathfrak{p}}(\Delta)}{d_K}} = \mathcal{N}_K(\Delta)^{-\frac{1}{d_K}}.$$

Hence, we may express the Arakelov height of a matrix as the product:

$$H_{\text{Ar}}(A) = \begin{cases} \mathcal{N}_K(\Delta)^{-\frac{1}{d_K}} \prod_{v \in M_{K,\infty}} |\det(AA^*)|_v^{\frac{1}{2}} & \text{if } K \text{ is a number field,} \\ \mathcal{N}_K(\Delta)^{-\frac{1}{d_K}} q^{-\frac{\deg(\mathfrak{p}_{\infty})}{d_K} \min_{|J|=m} \text{ord}_{\mathfrak{p}_{\infty}}(\det(A_J))} & \text{if } K \text{ is a function field.} \end{cases}$$

Using this, we conclude the following implication of Theorem 2.3.

Theorem 2.4. *Let K be a global field of degree d_K . Suppose that $A = (a_{ij})$ is an $m \times n$ matrix of rank $m < n$ with entries in K . Then there exist an effective constant $C = C(K)$ and a non-zero solution $\mathbf{b} = (b_1, \dots, b_n) \in \mathcal{O}_K^n$ with $A\mathbf{b}^t = 0$, satisfying*

$$H_K(b_1 : \dots : b_n)^{n-m} \leq \begin{cases} C(K)^{(n-m)} \mathcal{N}_K(\Delta)^{-1} \prod_{v \in M_{K,\infty}} |\det(AA^*)|_v^{\frac{d_K}{2}} & \text{if } K \text{ is a number field,} \\ C(K)^{(n-m)} \mathcal{N}_K(\Delta)^{-1} q^{-\deg(\mathfrak{p}_{\infty}) \min_{|J|=m} \text{ord}_{\mathfrak{p}_{\infty}}(\det(A_J))} & \text{if } K \text{ is a function field.} \end{cases}$$

Another useful consequence of Theorem 2.3 (which is slightly stronger than Siegel's lemma) is the following result. Given an $m \times n$ matrix A with entries in K , we denote by $H_K(A)$ the K -relative height of the point in \mathbb{P}_K^{mn-1} corresponding to the matrix A .

Corollary 2.5. *Let K be a global field. Suppose that $A = (a_{i,j})$ is an $m \times n$ matrix of rank r with entries in K . Then there exists an effective constant $C = C(K)$ and a non-zero solution $\mathbf{x} \in \mathcal{O}_K^n$ of $A\mathbf{x} = 0$ with*

$$H_K(1 : \mathbf{x}) \leq C(K) \left(n^{\frac{d_K}{2}} H_K(A) \right)^{\frac{r}{n-r}}.$$

Proof. When K is a number field, in [2, Corollary 2.9.9] it is proved that Theorem 2.3 implies the existence of a non-zero $\mathbf{x} \in \mathcal{O}_K^n$ of $A\mathbf{x} = 0$ with

$$(2.6) \quad H_K(\mathbf{x}) \leq C(K) \left(n^{\frac{d_K}{2}} H_K(A) \right)^{\frac{r}{n-r}}.$$

By the same argument, if K is a global field one may find a non-zero solution $\mathbf{x} \in \mathcal{O}_K^n$ verifying (2.6). Then a solution verifying the statement of the lemma can be found by lifting \mathbf{x} with Proposition 2.2. \square

2.3. Polynomial heights. We will use a notion of height for a polynomial that is defined in [2, § 1.6]. Specifically, given a global field K of degree d_K , and $f = \sum_I a_I \mathbf{X}^I \in K[X_1, \dots, X_n]$ a non-zero polynomial, for any $v \in M_K$ we let

$$|f|_v := \max_I |a_I|_v.$$

Also, for $f = \sum_I a_I \mathbf{X}^I \in \mathbb{C}[X_1, \dots, X_n]$, we define $\ell_{\infty}(f) := \max_I |a_I|_{\infty}$ and $\ell_1(f) := \sum_I |a_I|_{\infty}$, where $|\cdot|_{\infty}$ is the usual absolute value of \mathbb{C} .

Following [2, § 1.6], the height of $f \in K[X_1, \dots, X_n]$ is defined as

$$H(f) := \prod_{v \in M_K} |f|_v,$$

thus, the height of f is just the projective height of the projective point defined by the coefficients of f . In particular, for any $\lambda \in K$ we have $H(\lambda f) = H(f)$.

Similarly, in this article we define the affine height of f by

$$(2.7) \quad H_{\text{aff}}(f) := \prod_{v \in M_K} \max\{1, |f|_v\}.$$

It holds that for any non-zero polynomial f , $H_{\text{aff}}(f) \geq H(f) \geq 1$.

We will also use the K -relative versions of the projective and affine heights of polynomials:

$$H_K(f) := H(f)^{d_K} = \left(\prod_{v \in M_K} |f|_v \right)^{d_K}, \quad H_{K,\text{aff}}(f) := H_{\text{aff}}(f)^{d_K} = \left(\prod_{v \in M_K} \max\{1, |f|_v\} \right)^{d_K}.$$

We will need the following easy variant of the inequality (2.2). First, let us suppose that K is a number field. Let $f = \sum_I a_I \mathbf{X}^I \in \mathbb{Z}[X_1, \dots, X_n]$ be a non-zero polynomial, and let $\mathbf{x} = (x_1, \dots, x_n) \in K^n$. For any place $v \in M_{K,\infty}$, it holds

$$|f(\mathbf{x})|_v \leq \sum_I |a_I|_v |\mathbf{x}^I|_v \leq \sum_I |a_I|_v \max_i \{1, |x_i|_v\}^{\deg(f)}.$$

Using that the coefficients a_I are integers, for any $v \notin M_{K,\infty}$, we have

$$|f(\mathbf{x})|_v \leq \max_I \{|a_I|_v |\mathbf{x}^I|_v\} \leq \max_i \{1, |x_i|_v\}^{\deg(f)}.$$

It holds that

$$(2.8) \quad H(f(\mathbf{x})) = \prod_{v \in M_K} \max\{1, |f(\mathbf{x})|_v\} \leq \left(\prod_{v \in M_{K,\infty}} \sum_I |a_I|^{\frac{n_v}{d_K}} \right) H(1 : \mathbf{x})^{\deg(f)} \leq \left[\prod_{v \in M_{K,\infty}} \left(\sum_I |a_I| \right)^{\frac{n_v}{d_K}} \right] H(1 : \mathbf{x})^{\deg(f)} \\ \leq \ell_1(f) H(1 : \mathbf{x})^{\deg(f)}.$$

Hence

$$(2.9) \quad H_K(f(\mathbf{x})) \leq \ell_1(f)^{d_K} H_K(1 : \mathbf{x})^{\deg(f)}, \quad \text{for all } f \in \mathbb{Z}[X_1, \dots, X_n].$$

Now, suppose that K is a global field over $\mathbb{F}_q(T)$. Let $f = \sum_I a_I \mathbf{X}^I \in \mathbb{F}_q[X_1, \dots, X_n]$ be a non-zero polynomial, and let $\mathbf{x} = (x_1, \dots, x_n) \in K^n$. Since all places in K are non-archimedean, and trivial on \mathbb{F}_q , for any $v \in M_K$ we have

$$|f(\mathbf{x})|_v \leq \max_I \{|a_I|_v |\mathbf{x}^I|_v\} \leq \max_i \{1, |x_i|_v\}^{\deg(f)},$$

from where it follows immediately the bound

$$(2.10) \quad H_K(f(\mathbf{x})) \leq H_K(1 : \mathbf{x})^{\deg(f)}, \quad \text{for } f \in \mathbb{F}_q[X_1, \dots, X_n].$$

2.4. Distribution of primes in global fields. Let K be a global field of degree d_K . For any $Q > 0$, let us denote

$$\mathcal{P} := \{\mathfrak{p} \text{ prime in } \mathcal{O}_K\}, \quad \mathcal{P}(Q) := \{\mathfrak{p} \in \mathcal{P} : \mathcal{N}_K(\mathfrak{p}) \leq Q\}.$$

There exist constants $c_{1,K}$, $c_{2,K}$ and $c_{3,K}$ such that

$$(2.11) \quad \sum_{\mathfrak{p} \in \mathcal{P}(Q)} \frac{\log(\mathcal{N}_K(\mathfrak{p}))}{\mathcal{N}_K(\mathfrak{p})} = \log(Q) + O_K(1),$$

$$(2.12) \quad c_{1,K} Q \leq \sum_{\mathfrak{p} \in \mathcal{P}(Q)} \log(\mathcal{N}_K(\mathfrak{p})) \leq c_{2,K} Q,$$

$$(2.13) \quad \frac{1}{2} Q \leq \sum_{\substack{\mathfrak{p} \\ Q \leq \mathcal{N}_K(\mathfrak{p}) \leq 2Q}} \log(\mathcal{N}_K(\mathfrak{p})) \quad \text{for all } Q \geq c_{3,K},$$

where the logarithms are in base q and Q is of the form q^h with $h \in \mathbb{N}$ when K is a function field. Indeed, if K is a number field, (2.11), (2.12) and (2.13) follow from Landau Prime Ideal Theorem (see [27, Theorem 5.33]) which

states that $|\mathcal{P}(Q)| \sim \frac{Q}{\log(Q)}$. Meanwhile, if K is a function field over \mathbb{F}_q of genus g , this follows from the Riemann Hypothesis over function fields (see [35, Theorem 5.12]), which states that the number of primes in K of degree N is $\frac{q^N}{N} + O_K(\frac{q^{\frac{N}{2}}}{N})$.

By summation by parts, there exists a constant $c_{4,K}$ such that for all $Q > 0$ it holds

$$(2.14) \quad \sum_{\mathfrak{p} \in \mathcal{P}(Q)} \frac{\log(\mathcal{N}_K(\mathfrak{p}))}{\mathcal{N}_K(\mathfrak{p})^{\frac{3}{2}}} \leq c_{4,K} Q^{-\frac{1}{2}}.$$

We will use the following lemma, which is similar to [38, Lemma 1.10]. We provide a simple proof of it, which holds for global fields.

Lemma 2.6. *Let $\mathfrak{J} \neq (1)$ be a non-zero ideal. Then*

$$\sum_{\mathfrak{p}|\mathfrak{J}} \frac{\log(\mathcal{N}_K(\mathfrak{p}))}{\mathcal{N}_K(\mathfrak{p})} \leq \max\{\log(\log(\mathcal{N}_K(\mathfrak{J}))), 0\} + O_K(1).$$

Proof. Let $c > 0$. By (2.11) we have

$$\sum_{\mathfrak{p}|\mathfrak{J}} \frac{\log(\mathcal{N}_K(\mathfrak{p}))}{\mathcal{N}_K(\mathfrak{p})} \leq \sum_{\substack{\mathfrak{p}|\mathfrak{J} \\ \mathcal{N}_K(\mathfrak{p}) \leq c}} \frac{\log(\mathcal{N}_K(\mathfrak{p}))}{\mathcal{N}_K(\mathfrak{p})} + \sum_{\substack{\mathfrak{p}|\mathfrak{J} \\ \mathcal{N}_K(\mathfrak{p}) > c}} \frac{\log(\mathcal{N}_K(\mathfrak{p}))}{c} \leq \max\{\log(c), 0\} + O_K(1) + \frac{\log(\mathcal{N}_K(\mathfrak{J}))}{c}.$$

The lemma follows by taking $c = \log(\mathcal{N}_K(\mathfrak{J}))$. \square

Remark 2.7. Let K be a function field of degree d_K . By inspecting the proof of [35, Theorem 5.12] one sees that $|\{\mathfrak{p} \text{ prime in } K : \deg(\mathfrak{p}) = N\}| = \frac{q^N}{N} + O_g(\frac{q^{\frac{N}{2}}}{N})$ is effective, with the implicit constant depending only on the genus of the function field K . Since the prime v_∞ has degree at most d_K , it follows that all the bounds in § 2.4 for function fields are effective, with constants depending only on q, d_K and the genus of K .

On the other hand, if K is a number field of degree d_K and discriminant Δ_K , by [27, Theorem 5.33] if one assumes the Generalized Riemann Hypothesis there exists an effective absolute constant $c > 0$ such that it holds $\sum_{\mathfrak{a}: \mathcal{N}_K(\mathfrak{a}) \leq Q} \Lambda_K(\mathfrak{a}) = Q + O(\sqrt{|\Delta_K|} Q \exp(-cd_K^2 \sqrt{\log(Q)}))$, where $\Lambda_K(\mathfrak{a})$ is the usual von Mangoldt function on ideals of \mathcal{O}_K . Then, all the bounds in § 2.4 for number fields would be effective, with constants depending only on d_K, Δ_K (see also [21] for explicit versions of Landau Prime Ideal Theorem under the Generalized Riemann Hypothesis).

3. THE LOCAL AND THE GLOBAL DETERMINANT METHOD OVER GLOBAL FIELDS

In this section we start by recalling some uniform estimates for Hilbert functions given by Salberger in [37]. Then, in Lemma 3.14 we generalize the local determinant method [37, Main Lemma 2.5] as it is presented in [11, Corollary 2.5] to varieties over global fields. Afterwards, our next main result is Theorem 3.20, which is a generalization of [11, Proposition 3.2.26] where the authors made explicit the dependence on the degree in the bounds of [46, Theorem 2.3] when $K = \mathbb{Q}$. Finally, in Lemma 3.23, we bound the contribution of the primes \mathfrak{p} for which the reduction modulo \mathfrak{p} of a homogeneous absolutely irreducible polynomial $f \in \mathcal{O}_K[X_0, \dots, X_{n+1}]$ is not absolutely irreducible. This generalizes [46, inequality (2.1)], [11, Corollary 3.2.3] and [45, Lemma 2.3] to global fields. In order to do that, we use the theory of effective Noether forms as in [11, 45], forcing us to distinguish the characteristic of the global field.

We remark that a difference between this article and [46, 11, 45] is the introduction of a finite subset of primes in the bounds. The reason to do so is rather technical and will be apparent much later (see the introduction to Section 6); in a few words it will be needed to prove the dimension growth conjecture for small values of d .

3.1. Uniform estimates for Hilbert functions. We start by recalling the definition of the Hilbert function of a projective variety.

Definition 3.1. Let K be a field and $I \subseteq K[X_0, \dots, X_n]$ be a homogeneous ideal. Given a non-negative integer k , we will denote the elements of degree k in $K[X_0, \dots, X_n]/I$ by $(K[X_0, \dots, X_n]/I)_k$. Then we define the Hilbert function as

$$h_I(k) := \dim_K(K[X_0, \dots, X_n]/I)_k.$$

If $X \subseteq \mathbb{P}_K^n = \text{Proj}(K[X_0, \dots, X_n])$ is a closed subscheme, then we define $h_X(k) := h_I(k)$ for the saturated homogeneous ideal I corresponding to $X = \text{Proj}(K[X_0, \dots, X_n]/I)$.

By a classic theorem [22, p. 51-52], there is a unique polynomial $p_I(t) \in \mathbb{Q}[t]$, the Hilbert polynomial, such that $h_I(k) = p_I(k)$ for all sufficiently large positive integers, verifying

$$p_I(t) = d \frac{t^r}{r!} + O(t^{r-1}),$$

where r is the dimension of $X = \text{Proj}(K[X_0, \dots, X_n]/I)$ and d is the degree of $X \subseteq \mathbb{P}^n$.

For our purposes, we need to have a uniform estimate in the implicit constant appearing in the error term of the Hilbert polynomial. For this, we require the following two lemmas of Salberger.

Lemma 3.2 ([37, Lemma 1.3]). *Let n be a positive integer and let K be a field. Let $X = \text{Proj}(K[X_0, \dots, X_n]/I) \subseteq \mathbb{P}_K^n$ be a closed subscheme, with I a homogeneous saturated ideal. Then X is defined by forms in $K[X_0, \dots, X_n]$ of degree bounded in terms of n and $p_I(t)$.*

Lemma 3.3 ([37, Lemma 1.4]). *Let d and n be positive integers and let K be a field. Then there are only finitely many possibilities for the Hilbert function $h_X : \mathbb{N} \rightarrow \mathbb{Z}$ of closed geometrically reduced equidimensional subschemes $X \subseteq \mathbb{P}_K^n$ of degree d .*

In order to obtain uniform results in the degree and the dimension, we use Gröbner basis as in Broberg's article [5]. We recall the following classic definition from [16, Chapter 2, § 2, Definition 1 and Chapter 8, § 4, p. 416] :

Definition 3.4 (Graded monomial ordering and leading monomial). A graded monomial ordering $<$ is a well-ordering on the set of monomials $\mathbf{X}^\alpha = X_0^{\alpha_0} \dots X_n^{\alpha_n}$ satisfying the following conditions for the corresponding order on the set of exponents $\alpha = (\alpha_0, \dots, \alpha_n) \in \mathbb{Z}_{\geq 0}^{n+1}$:

- (1) If $\alpha > \beta$ and $\gamma \in \mathbb{Z}_{\geq 0}^{n+1}$, then $\alpha + \gamma > \beta + \gamma$;
- (2) if $\alpha, \beta \in \mathbb{Z}_{\geq 0}^{n+1}$ and $\alpha_0 + \dots + \alpha_n \leq \beta_0 + \dots + \beta_n$, then $\alpha < \beta$.

Moreover, if $f = \sum_{\alpha} a_{\alpha} \mathbf{X}^{\alpha}$, the leading monomial of f , denoted by $LM(f)$, is the maximum of all \mathbf{X}^{α} with $a_{\alpha} \neq 0$.

Let $I \subseteq K[X_0, \dots, X_n]$ be a homogeneous monomial ideal, i.e. an ideal generated by monomials. By the same argument given in the proof of [16, Chapter 9, § 3, Proposition 3], $h_I(k)$ is the number of monomials not in I of degree k . Furthermore, given a homogeneous ideal $I \subseteq K[X_0, \dots, X_n]$, let $<$ be a graded monomial ordering, and consider the set $\{LM(f) : f \in I_k\}$, where I_k denotes the elements of degree k in I . Then $\{LM(f) : f \in I_k\} = \{LM(f_1), \dots, LM(f_m)\}$ for some polynomials f_1, \dots, f_m of degree k . It holds that $\{f_1, \dots, f_m\}$ is a basis for I_k and $\{LM(f_1), \dots, LM(f_m)\}$ is a basis of the k -degree graded part of the ideal $\langle LT(I) \rangle$, where

$$LT(I) := \{c \mathbf{X}^{\alpha} : \text{exists } f \in I \setminus \{0\} \text{ satisfying } LT(f) = c \mathbf{X}^{\alpha}\},$$

(see [16, Chapter 9, § 3, Proposition 9]). We conclude

$$(K[X_0, \dots, X_n]/I)_k \cong (K[X_0, \dots, X_n]/\langle LT(I) \rangle)_k,$$

and thus

$$h_I(k) = h_{\langle LT(I) \rangle}(k).$$

This is quite useful, since this reduces the problem of computing $h_I(k)$ to the problem of computing the Hilbert function of a monomial ideal, which is easy to do, and moreover, the monomial ideal $\langle LT(I) \rangle$ can be computed by means of a Gröbner basis of I . Furthermore, $(K[X_0, \dots, X_n]/\langle LT(I) \rangle)_k$ has as a K -basis those monomials which

are not the leading term of some form of degree k in I . From this, it follows easily that the same does hold for $(K[X_0, \dots, X_n]/I)_k$. This is because the monomials of degree k which are not the leading term of some form of degree k in I are linearly independent on $(K[X_0, \dots, X_n]/I)_k$, and have cardinal $h_I(k)$, thus they are a K -basis.

We will also require:

Definition 3.5. Let K be a field and $I \subseteq K[X_0, \dots, X_n]$ be a homogeneous ideal. Let $<$ be a graded monomial order on $\{\mathbf{X}^\alpha : \alpha \in \mathbb{Z}_{\geq 0}^{n+1}\}$ and $m \in \{0, \dots, n\}$. Define

$$\sigma_{I,m}(k) := \sum_{\alpha} \alpha_m,$$

where $\alpha = (\alpha_0, \dots, \alpha_n)$ runs over the exponent set of all monomials \mathbf{X}^α which are not leading monomials of any form of degree k in I .

Since the monomials of degree k which are not leading monomials of any form of degree k in I form a K -basis of $(K[X_0, \dots, X_n]/I)_k$, it holds

$$(3.1) \quad \sigma_{I,0}(k) + \dots + \sigma_{I,n}(k) = kh_I(k).$$

It turns out that the functions $\sigma_{I,m}(k)$ are numerical polynomials. More precisely:

Lemma 3.6 ([5, Lemma 1]). *Let $X = \text{Proj}(K[X_0, \dots, X_n]/I) \subseteq \mathbb{P}_K^n$ be a closed subscheme of dimension r and degree d , such that $I \subseteq K[X_0, \dots, X_n]$ is generated by forms of degrees at most δ . There is a positive integer $k_0 \lesssim_{n,\delta} 1$, such that for all $k \geq k_0$, $h_X(k)$ is equal to some polynomial of degree r , and $\sigma_{I,0}(k), \dots, \sigma_{I,n}(k)$ are equal to some polynomials of degree $r+1$. Furthermore, the coefficients of these polynomials can be bounded in terms of n and δ .*

As a consequence of Lemma 3.6, the limit $\lim_{k \rightarrow \infty} \frac{\sigma_{I,m}(k)}{kh_I(k)}$ exists. This, together with Lemma 3.2, Lemma 3.3, and equation (3.1) gives:

Lemma 3.7 ([37, Lemma 1.9]). *Let $K, I, <, \sigma_{I,m}$ be as above and suppose that $X = \text{Proj}(K[X_0, \dots, X_n]/I)$ is reduced and equidimensional. Denote $d = \deg(X)$. For all $m \in \{0, \dots, n\}$, there exists $0 \leq a_{I,m} \leq 1$ with $a_{I,0} + \dots + a_{I,n} = 1$ such that*

$$\frac{\sigma_{I,m}(k)}{kh_I(k)} = a_{I,m} + O_{d,n}\left(\frac{1}{k}\right), \quad m \in \{0, \dots, n\}.$$

3.2. Prime powers dividing certain determinants. We begin with the following lemma.

Lemma 3.8. *Let K be a global field, and let $X = \text{Proj}(K[X_0, \dots, X_n]/I)$ be a closed reduced subscheme of \mathbb{P}_K^n . The schematic closure of X in $\mathbb{P}_{\mathcal{O}_K}^n = \text{Proj}(\mathcal{O}_K[X_0, \dots, X_n])$ is equal to*

$$\Xi := \text{Proj}(\mathcal{O}_K[X_0, \dots, X_n]/(I \cap \mathcal{O}_K[X_0, \dots, X_n])).$$

Proof. Due to the reduced hypothesis, $I \cap \mathcal{O}_K[X_0, \dots, X_n]$ is reduced, thus Ξ is reduced. We have a ring morphism

$$\varphi : \mathcal{O}_K[X_0, \dots, X_n]/(I \cap \mathcal{O}_K[X_0, \dots, X_n]) \rightarrow K[X_0, \dots, X_n]/I.$$

Since, for any homogeneous prime ideal $\mathfrak{P} \subseteq K[X_0, \dots, X_n]/I$ which does not contain (X_0, \dots, X_n) , it holds that $\varphi^{-1}(\mathfrak{P})$ is an homogeneous prime ideal which does not contain (X_0, \dots, X_n) , we see that there exists a morphism $\text{Proj}(\varphi) : X \rightarrow \Xi$ given by $\mathfrak{P} \rightarrow \varphi^{-1}(\mathfrak{P})$. Now, note that the minimal primes of $K[X_0, \dots, X_n]/I$ correspond to the primes appearing in the primary decomposition of I . If $I = \mathfrak{P}_1 \cap \dots \cap \mathfrak{P}_r$, then $I \cap \mathcal{O}_K[X_0, \dots, X_n] = (\mathfrak{P}_1 \cap \mathcal{O}_K[X_0, \dots, X_n]) \cap \dots \cap (\mathfrak{P}_r \cap \mathcal{O}_K[X_0, \dots, X_n])$ and $\{\mathfrak{P}_1 \cap \mathcal{O}_K[X_0, \dots, X_n], \dots, \mathfrak{P}_r \cap \mathcal{O}_K[X_0, \dots, X_n]\}$ is the set of minimal primes of $I \cap \mathcal{O}_K[X_0, \dots, X_n]$, thus they are the set of generic points of Ξ . Since $\text{Proj}(\varphi)(\mathfrak{P}_i) = \mathfrak{P}_i \cap \mathcal{O}_K[X_0, \dots, X_n]$ for all i , the image of $\text{Proj}(\varphi)$ contains all the generic points of Ξ , thus its closure is Ξ . Since Ξ is reduced, by [20, (10.8)] we conclude that Ξ is the schematic closure of X . \square

From here on, K will denote a global field. We shall work with closed subschemes $X = \text{Proj}(K[X_0, \dots, X_n]/I)$ of \mathbb{P}_K^n , which are geometrically reduced, equidimensional of dimension r , and degree d . Also, we will denote

$$\Xi := \text{Proj}(\mathcal{O}_K[X_0, \dots, X_n]/(I \cap \mathcal{O}_K[X_0, \dots, X_n])).$$

Furthermore, given a prime $\mathfrak{p} \notin M_{K,\infty}$, if $\mathbb{F}_{\mathfrak{p}} := \mathcal{O}_K/\mathfrak{p}$, we let $I_{\mathfrak{p}} \subset \mathbb{F}_{\mathfrak{p}}[X_0, \dots, X_n]$ be the image of $I \cap \mathcal{O}_K[X_0, \dots, X_n]$ in $\mathbb{F}_{\mathfrak{p}}[X_0, \dots, X_n]$, and we set

$$X_{\mathfrak{p}} := \text{Proj}(\mathbb{F}_{\mathfrak{p}}[X_0, \dots, X_n]/I_{\mathfrak{p}}) = \Xi \times_{\mathbb{Z}} \mathbb{F}_{\mathfrak{p}}.$$

The following lemmas, that are global field variants of [38, Lemma 2.2, Lemma 2.3] show that the reductions modulo \mathfrak{p} of X , i.e. the $\mathbb{F}_{\mathfrak{p}}$ -schemes $X_{\mathfrak{p}}$, retain the same uniform bounds of Lemma 3.2 and Lemma 3.3.

Lemma 3.9. *Let d and n be given and $X, X_{\mathfrak{p}}$ as above. Then $X_{\mathfrak{p}}$ is defined by forms of degree $\lesssim_{d,n} 1$.*

Proof. In order to prove Lemma 3.9, we will use the key property that a flat morphism has all its fibres with the same Hilbert polynomial. Let X and Ξ be as above, and let $f : \Xi \rightarrow \text{Spec}(\mathcal{O}_K)$ be the structural morphism. If η is the generic point of $\text{Spec}(\mathcal{O}_K)$, by Lemma 3.8 the schematic closure of $f^{-1}(\eta) = X$ is Ξ . Then by [20, Proposition 14.14] f is flat.

By hypothesis, X is geometrically reduced, so we may apply Lemma 3.3. There are thus $\lesssim_{d,n} 1$ possible Hilbert polynomials for the schemes $X \subseteq \mathbb{P}_K^n$. But the Hilbert polynomials of $X = \Xi \times_{\mathcal{O}_K} K$ and $X_{\mathfrak{p}} = \Xi \times_{\mathcal{O}_K} \mathbb{F}_{\mathfrak{p}}$ coincide since they are fibres of f , which is flat (this follows from [22, Theorem 9.9]). There are thus $\lesssim_{d,n} 1$ possible Hilbert polynomials also for the $\mathbb{F}_{\mathfrak{p}}$ -subschemes $X_{\mathfrak{p}}$. The result follows from Lemma 3.2. \square

Lemma 3.10. *Let d and n be given and $X, X_{\mathfrak{p}}$ be as above. Let A be the stalk of $X_{\mathfrak{p}}$ at some $\mathbb{F}_{\mathfrak{p}}$ -point P on $X_{\mathfrak{p}}$ of multiplicity μ . Let \mathfrak{m} be the maximal ideal of A . Let $g_{X,P} : \mathbb{Z}_{>0} \rightarrow \mathbb{Z}$ be the function where $g_{X,P}(k) = \dim_{A/\mathfrak{m}} \mathfrak{m}^k / \mathfrak{m}^{k+1}$. Then there are $\lesssim_{d,n} 1$ different functions $g_{X,P}$ among all pairs (X, P) as above, and we have*

$$g_{X,P}(k) \leq \mu \frac{k^{r-1}}{(r-1)!} + O_{d,n}(k^{r-2})$$

for all $k \geq 1$, where $r = \dim(X)$. Furthermore, If X is a hypersurface, the implicit constant in the second summand does not depend on d .

Proof. The function $g_{X,P}$ coincides with the Hilbert function of the projective tangent cone $\text{PTC}_P X \subseteq \mathbb{P}_{\mathbb{F}_{\mathfrak{p}}}^{n-1}$ of $X_{\mathfrak{p}}$ at P . Moreover, from Lemma 3.9 $X_{\mathfrak{p}}$ is defined by forms of degree $\lesssim_{d,n} 1$, thus this is also the case for $\text{PTC}_P X$ (this follows from the fact that the tangent cone can be effectively computed in terms of a Gröbner basis of I , and the fact that the degrees of the elements in a Gröbner basis of I are bounded in terms of n and the degrees of the elements in a basis of I , see [33, III.3] and [16, Chapter 9, § 7, Proposition 4]). By Lemma 3.6, there is a positive integer $k_0 \lesssim_{n,d} 1$ such that the Hilbert function $g_{X,P}(k)$ coincides with a polynomial $p_{X,P}(k)$ for all $k \geq k_0$, with coefficients bounded in terms of n and d . In particular, $g_{X,P}(k) \leq \mu_P \frac{k^{r-1}}{(r-1)!} + O_{d,n}(k^{r-2})$ for all $k \geq 1$.

If X is a hypersurface, the projective tangent cone $\text{PTC}_P X \subseteq \mathbb{P}_{\mathbb{F}_{\mathfrak{p}}}^{n-1}$ is a hypersurface of degree μ in \mathbb{P}^n , thus the function $g_{X,P}$ has the explicit formula

$$g_{X,P}(k) = \begin{cases} \binom{r+k}{r} & \text{for } k < \mu, \\ \binom{r+k}{r} - \binom{r+k-\mu}{r} & \text{for } k \geq \mu. \end{cases}$$

Now, the same argument in [11, Lemma 2.1] finishes the proof. \square

We will need the following lemmas.

Lemma 3.11 ([37, Proof of Main Lemma 2.5], [11, Lemma 2.2]). *Let $c, r, \mu > 0$ be integers. Let $g : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{Z}_{\geq 0}$ be a function with $g(0) = 1$ and satisfying $g(k) \leq \mu \frac{k^{r-1}}{(r-1)!} + ck^{r-2}$ for $k > 0$. Let $(n_i)_{i \geq 1}$ be the non-decreasing sequence of integers $m \geq 0$ where m occurs exactly $g(m)$ times. Then for any $s \geq 0$ we have*

$$n_1 + \dots + n_s \geq \left(\frac{r!}{\mu} \right)^{\frac{1}{r}} \frac{r}{r+1} s^{1+\frac{1}{r}} + O_{r,c}(s).$$

Lemma 3.12 ([11, Lemma 2.3], see also [30, Proposition A.1] for explicit bounds). *Consider A as in Lemma 3.10, and let $(n_i(A))_{i \geq 1}$ be the non-decreasing sequence of integers $k \geq 0$ where k occurs exactly $\dim_{A/\mathfrak{m}} \mathfrak{m}^k / \mathfrak{m}^{k+1}$ times. Write $A(s) = n_1(A) + \dots + n_s(A)$. Then*

$$A(s) \geq \left(\frac{r!}{\mu}\right)^{\frac{1}{r}} \frac{r}{r+1} s^{1+\frac{1}{r}} + O_{d,n}(s),$$

where $r = \dim(X)$. Furthermore, if X is a hypersurface, the implicit constant in the second summand does not depend on d .

In order to provide our generalization of [37, Lemma 2.4 and Main Lemma 2.5] to global fields, we follow the exposition in [11, §2]. We denote $\mathcal{O}_{\mathfrak{p}}$ for the localization of \mathcal{O}_K at the prime \mathfrak{p} .

Lemma 3.13. *Let R be a commutative noetherian local ring containing $\mathcal{O}_{\mathfrak{p}}$ as a subring and let $A = R/\mathfrak{p}R$. Let \mathfrak{m} be the maximal ideal of A and let $(n_i(A))_{i \in \mathbb{N}}$ be the non-decreasing sequence of integers $k \geq 0$ where k occurs exactly $\dim_{A/\mathfrak{m}} \mathfrak{m}^k / \mathfrak{m}^{k+1}$ times. Let r_1, \dots, r_s be elements of R and $\phi_1, \dots, \phi_s : R \rightarrow \mathcal{O}_{\mathfrak{p}}$ be ring homomorphisms such that $\phi_i|_{\mathcal{O}_{\mathfrak{p}}} = \text{Id}$. Then the determinant of the $s \times s$ matrix $(\phi_i(r_j))_{i,j}$ is divisible by $\mathfrak{p}^{A(s)}$ for $A(s) := n_1(A) + \dots + n_s(A)$.*

Proof. Choose one of the homomorphisms ϕ_i , say ϕ_1 . Let $I := \text{Ker}(\phi_1)$. Since the map $\mathcal{O}_{\mathfrak{p}} \hookrightarrow R \xrightarrow{\phi_1} \mathcal{O}_{\mathfrak{p}}$ is the identity, it holds $\phi_1(\mathcal{O}_{\mathfrak{p}}) = \mathcal{O}_{\mathfrak{p}}$, hence $R/I \cong \mathcal{O}_{\mathfrak{p}}$. If $\tilde{\phi}_1 : R/I \rightarrow \mathcal{O}_{\mathfrak{p}}$ is the induced homomorphism, then $\tilde{\phi}_1^{-1}(\mathfrak{p}\mathcal{O}_{\mathfrak{p}}) = (\mathfrak{p}R + I)/I$ is a maximal ideal, from where it follows that $\mathfrak{p}R + I$ is the maximal ideal of R . Furthermore, $\phi_i(I + \mathfrak{p}R)\mathcal{O}_{\mathfrak{p}} \neq \mathcal{O}_{\mathfrak{p}}$, otherwise we would find $x \in I + \mathfrak{p}R$ and $a \in \mathcal{O}_{\mathfrak{p}}$ with $a\phi_i(x) = 1$, hence $\phi_i(ax - 1) = 0$, with contradicts that $I + \mathfrak{p}R$ is the only maximal ideal of R . We conclude that $\phi_i(I)$ is a proper ideal of $\mathcal{O}_{\mathfrak{p}}$, and hence $\phi_i(I) \subseteq \mathfrak{p}\mathcal{O}_{\mathfrak{p}}$. In particular, for all i , $\phi_i(I^k) \subseteq \mathfrak{p}^k\mathcal{O}_{\mathfrak{p}}$. From Nakayama's lemma, it follows that for all $k \geq 0$, I^k/I^{k+1} is an $\mathcal{O}_{\mathfrak{p}}$ -module generated with at most $\dim_{A/\mathfrak{m}} \mathfrak{m}^k / \mathfrak{m}^{k+1}$ elements. Now, suppose that there are more than $g(k) := \dim_{A/\mathfrak{m}} \mathfrak{m}^k / \mathfrak{m}^{k+1}$ elements in $I^k - I^{k+1}$ among r_1, \dots, r_s , say r_1, \dots, r_q . There exists an $\mathcal{O}_{\mathfrak{p}}$ -module homomorphism $\lambda : \mathcal{O}_{\mathfrak{p}}^q \rightarrow I^k/I^{k+1}$ where $\lambda(\beta_1, \dots, \beta_q) = \beta_1 r_1 + \dots + \beta_q r_q \pmod{I^{k+1}}$. Since I^k/I^{k+1} is generated by at most $g(k) < q$ elements we conclude that there exists $(\beta_1, \dots, \beta_q) \in \text{Ker}(\lambda)$ with, say, $\beta_q = 1$ (this is because I^k/I^{k+1} is a finite $\mathcal{O}_{\mathfrak{p}}$ -module generated by at most $g(k)$ elements and $\mathcal{O}_{\mathfrak{p}}$ is local, then by Nakayama's lemma there exists $j \in \{1, \dots, q\}$, say $j = q$, such that $\text{span}_{\mathcal{O}_{\mathfrak{p}}} \{r_1, \dots, r_q\} = \text{span}_{\mathcal{O}_{\mathfrak{p}}} \{r_1, \dots, r_{q-1}\}$). Denoting $\rho_q = \beta_1 r_1 + \dots + \beta_{q-1} r_{q-1} + r_q$, we see that $\rho_q \in I^{k+1}$, furthermore, since the homomorphisms ϕ_1, \dots, ϕ_s restrict to the identity in $\mathcal{O}_{\mathfrak{p}}$, we see that the determinant of $(\phi_i(r_j))_{i,j}$ will not change if we replace r_q with ρ_q . If we continue making such elementary transformations, we arrive to a situation where there are at most $\dim_{A/\mathfrak{m}} \mathfrak{m}^k / \mathfrak{m}^{k+1}$ elements in $I^k - I^{k+1}$ among r_1, \dots, r_s , for each $k \geq 0$.

After rearranging, we have that r_j is in $I^{n_j(A)}$ for all $j = 1, \dots, s$. This implies that all the elements $\phi_1(r_j), \dots, \phi_s(r_j)$ are divisible by $\mathfrak{p}^{n_j(A)}$, hence $\mathfrak{p}^{A(s)}$ divides the determinant of $(\phi_i(r_j))_{i,j}$. \square

Lemma 3.14. *Let \mathfrak{p} be a prime and let $X, r, d, n, \Xi, X_{\mathfrak{p}}$ be as above. Let P be an $\mathbb{F}_{\mathfrak{p}}$ -point of multiplicity μ_P on $X_{\mathfrak{p}}$ and let ξ_1, \dots, ξ_s be points in \mathcal{O}_K^{n+1} , such that, for all i , ξ_i has its coordinates not all divisible by \mathfrak{p} and it represents a K -rational point of X which reduces modulo \mathfrak{p} to P . Let F_1, \dots, F_s be forms in (X_0, \dots, X_n) with coefficients in \mathcal{O}_K , and let $\det(F_j(\xi_l))_{j,l}$ be the determinant of the $s \times s$ matrix $(F_j(\xi_l))_{j,l}$. Then there exists a non-negative integer $N = \left(\frac{r!}{\mu_P}\right)^{\frac{1}{r}} \frac{r}{r+1} s^{1+\frac{1}{r}} + O_{d,n}(s)$ such that $\mathfrak{p}^N \mid \det(F_j(\xi_l))_{j,l}$. Furthermore, if X is a hypersurface, the implicit constant in the second summand does not depend on d .*

Proof. Let $P = (z_0 : z_1 : \dots : z_n)$ with $z_i \in \mathbb{F}_{\mathfrak{p}}$ for all i . It holds that there is some i with $z_i \neq 0$, which we may suppose that it is $i = 0$. By hypothesis, it also follows that the x_0 -coordinates of the points ξ_1, \dots, ξ_s are not divisible by \mathfrak{p} and we replace the forms $F_j, j = 1, \dots, s$ by the rational functions $f_j = F_j(1, \frac{x_1}{x_0}, \dots, \frac{x_n}{x_0})$ without changing the \mathfrak{p} -adic valuation of the determinant.

Now, we define a prime ideal in $\mathcal{O}_K[X_0, \dots, X_n]$ by $\mathfrak{P} := (\mathfrak{p}, X_0 - z_0, \dots, X_n - z_n)$. Let $R := \mathcal{O}_K[X_0, \dots, X_n]_{\mathfrak{P}}$. It is clear that R is noetherian local ring containing $\mathcal{O}_{\mathfrak{p}}$ as a subring, and the rational functions $f_j, j = 1, \dots, s$ are elements in R . Moreover, the evaluations $\text{ev}_{\xi_1}, \dots, \text{ev}_{\xi_s} : \mathcal{O}_K[X_0, \dots, X_n] \rightarrow \mathcal{O}_K$ are ring homomorphisms, which restrict to the identity on \mathcal{O}_K , and can be extended to ring homomorphisms $\overline{\text{ev}_{\xi_1}}, \dots, \overline{\text{ev}_{\xi_s}} : R \rightarrow \mathcal{O}_{\mathfrak{p}}$ which restrict

to the identity on $\mathcal{O}_{\mathfrak{p}}$. Taking $\phi_i := \overline{\text{ev}_{\xi_i}}$ for all i , we see that R and ϕ_1, \dots, ϕ_s are as in Lemma 3.13, hence $\mathfrak{p}^{A(s)} | \det(\phi_i(f_j))_{j,l} = \det(F_j(\xi_i))_{j,l}$. The proof finishes using Lemma 3.12. \square

3.3. From local estimates to global estimates. In this section, $X \subseteq \mathbb{P}_K^{n+1}$ will denote a geometrically integral hypersurface of degree d . Hence, X is defined by an absolutely irreducible homogeneous polynomial $f = \sum_I a_I \mathbf{X}^I \in \mathcal{O}_K[X_0, \dots, X_{n+1}]$ of degree d . We shall begin with an important remark concerning the polynomial f defining X .

Remark 3.15 (Primitive polynomials in global fields). We recall that f is said to be primitive if $\mathfrak{a} := \sum_I a_I \mathcal{O}_K = (1)$. Also, given a prime \mathfrak{p} of \mathcal{O}_K , we will say that f is \mathfrak{p} -primitive if \mathfrak{p} does not divide \mathfrak{a} . Note that in general we can not take the polynomial f defining X to be primitive as in the case $\mathbb{k} = \mathbb{Q}$ or $\mathbb{F}_q(T)$. Indeed, by the proof of Proposition 2.2, the ideal class of $\mathfrak{a} = \sum_I a_I \mathcal{O}_K$ is determined by the projective point with coordinates $(a_I)_I$. Hence, if \mathcal{O}_K is not a principal domain, it may occur that the ideal class of \mathfrak{a} is not principal, thus there is no non-zero scalar $\lambda \in \mathcal{O}_K$ such that λf is primitive. Nevertheless, by Proposition 2.2 we may find a non-zero $\lambda \in \mathcal{O}_K$ such that λf is \mathfrak{p} -primitive for all primes $\mathfrak{p} \subseteq \mathcal{O}_K$ with $\mathcal{N}_K(\mathfrak{p}) > c_2$, where c_2 is the constant in Proposition 2.2. Also, by Proposition 2.2, for all $v \in M_{K,\infty}$, it holds $\max_I |\lambda a_I|_v \leq c_1 H(f)^{\frac{n_v}{d_K}}$ if K is a number field, and $\max_I |\lambda a_I|_{v_\infty} \leq c_1 H(f)$ if K is a function field, for some effective computable constants c_1 , and c_3 which depend on K . By definition 2.7, it holds

$$H_{K,\text{aff}}(\lambda f) = \left(\prod_{v \in M_K} \max_I \{1, |\lambda a_I|_v\} \right)^{d_K} = \left(\prod_{v \in M_{K,\infty}} \max_I \{1, |\lambda a_I|_v\} \right)^{d_K} \leq c_1^{d_K} H_K(f).$$

Furthermore, since the height is invariant under multiplication by non-zero scalars, it holds $H_K(\lambda f) = H_K(f)$. For this reason we may assume that X is defined by an absolutely irreducible homogeneous polynomial $f = \sum_I a_I \mathbf{X}^I \in \mathcal{O}_K[X_0, \dots, X_{n+1}]$ of degree d , which is \mathfrak{p} -primitive for all primes \mathfrak{p} with $\mathcal{N}_K(\mathfrak{p}) > c_2$, and

$$H_{K,\text{aff}}(f) \leq c_1^{d_K} H_K(f).$$

We proceed to generalize [46, Theorem 2.2], [11, Corollary 2.9] and [45, Lemma 2.1] to global fields.

Theorem 3.16. *Let \mathfrak{p} be a prime for which the reduction $X_{\mathfrak{p}}$ is absolutely irreducible, and for which either $\text{char}(K) = 0$ and $\mathcal{N}_K(\mathfrak{p}) > \max\{27d^4, c_2\}$, or $0 < \text{char}(K) \leq \max\{27d^4, c_2\}$ and $\mathcal{N}_K(\mathfrak{p}) \geq d^{\frac{14}{3}}$, or $\text{char}(K) > \max\{27d^4, c_2\}$. Let (ξ_1, \dots, ξ_s) be a tuple of points in \mathcal{O}_K^{n+1} , such that, for all i , ξ_i has its coordinates not all divisible by \mathfrak{p} and it represents a K -rational point of X . Let $F_1, \dots, F_s \in \mathcal{O}_K[X_0, \dots, X_{n+1}]$ be forms with \mathcal{O}_K -coefficients, and write Δ for the determinant of $(F_i(\xi_j))_{i,j}$. Then, there exists some*

$$e_{\mathfrak{p}} \geq n!^{\frac{1}{n}} \frac{n}{n+1} \frac{s^{1+\frac{1}{n}}}{\mathcal{N}_K(\mathfrak{p}) + O_n(d^2 \mathcal{N}_K(\mathfrak{p})^{\frac{1}{2}})} + O_n(s)$$

such that $\mathfrak{p}^{e_{\mathfrak{p}}} | \Delta$.

Proof. Let P be an $\mathbb{F}_{\mathfrak{p}}$ -point on $X_{\mathfrak{p}}$ and let us write s_P for the cardinal of the subset $I_P \subseteq \{1, \dots, s\}$ of indices l such that ξ_l reduces to P modulo \mathfrak{p} . Let

$$N_P = \left(\frac{n!}{\mu_P} \right)^{\frac{1}{n}} \frac{n}{n+1} s_P^{1+\frac{1}{n}} + O_n(s_P)$$

be the integer in Lemma 3.14. Then by Lemma 3.14, for each $s_P \times s_P$ minor $A = (a_{k,l})_{k,l}$ of $(F_i(\xi_j))_{i,j}$ with $l \in I_P$, it holds that $\mathfrak{p}^{N_P} | \det(A)$. If we apply this to all the $\mathbb{F}_{\mathfrak{p}}$ -points on $X_{\mathfrak{p}}$ and use the Laplace expansion, we see that $\mathfrak{p}^{e_{\mathfrak{p}}} | \Delta$, where

$$(3.2) \quad e_{\mathfrak{p}} \geq \sum_{P \in X(\mathbb{F}_{\mathfrak{p}})} N_P = n!^{\frac{1}{n}} \frac{n}{n+1} \left(\sum_{P \in X(\mathbb{F}_{\mathfrak{p}})} \frac{s_P^{1+\frac{1}{n}}}{\mu_P^{\frac{1}{n}}} \right) + O_n \left(\sum_{P \in X(\mathbb{F}_{\mathfrak{p}})} s_P \right) = n!^{\frac{1}{n}} \frac{n}{n+1} \left(\sum_{P \in X(\mathbb{F}_{\mathfrak{p}})} \frac{s_P^{1+\frac{1}{n}}}{\mu_P^{\frac{1}{n}}} \right) + O_n(s).$$

To conclude the proof, we need to estimate $\sum_{P \in X(\mathbb{F}_p)} \frac{s_P^{1+\frac{1}{n}}}{\mu_P^{\frac{1}{n}}}$. By Hölder's inequality,

$$s = \sum_{P \in X(\mathbb{F}_p)} s_P \leq \left(\sum_{P \in X(\mathbb{F}_p)} \mu_P \right)^{\frac{1}{n+1}} \left(\sum_{P \in X(\mathbb{F}_p)} \frac{s_P^{1+\frac{1}{n}}}{\mu_P^{\frac{1}{n}}} \right)^{\frac{n}{n+1}},$$

hence

$$(3.3) \quad \sum_{P \in X(\mathbb{F}_p)} \frac{s_P^{1+\frac{1}{n}}}{\mu_P^{\frac{1}{n}}} \geq \frac{s^{1+\frac{1}{n}}}{\left(\sum_{P \in X(\mathbb{F}_p)} \mu_P \right)^{\frac{1}{n}}}.$$

If we denote $n_p := \sum_{P \in X(\mathbb{F}_p)} \mu_P$, from (3.2) and (3.3) we see that

$$(3.4) \quad e_p \geq n!^{\frac{1}{n}} \frac{n}{n+1} \frac{s^{1+\frac{1}{n}}}{n_p^{\frac{1}{n}}} + O_n(s).$$

In order to bound n_p , namely the number of \mathbb{F}_p -points in X_p counted with multiplicity, we observe that the singular points of X_p , say $X_{p,\text{sing}}$, consist of the intersection of f and the partial derivatives $\frac{\partial f}{\partial x_i}$. Hence, we may suppose that $X_{p,\text{sing}}$ is contained in the intersection of f and $\frac{\partial f}{\partial x_0}$, which is the intersection of two hypersurfaces of degree at most d . By Bézout's Theorem [18, Example 8.4.6], the number of irreducible components of $X_{p,\text{sing}}$ is bounded by d^2 , and their degree is also bounded by d^2 . Then the standard Schwarz-Zippel estimate for $X_{p,\text{sing}}$ yields $|X_{p,\text{sing}}(\mathbb{F}_p)| \lesssim_n d^2 \mathcal{N}_K(\mathfrak{p})^{n-1}$. Hence

$$(3.5) \quad n_p - |X(\mathbb{F}_p)| = \sum_{P \in X(\mathbb{F}_p)} (\mu_P - 1) \leq (d-1)|X_{p,\text{sing}}| \lesssim_n d^3 \mathcal{N}_K(\mathfrak{p})^{n-1} \lesssim_n d \mathcal{N}_K(\mathfrak{p})^{n-\frac{1}{2}}.$$

Claim 3.17. $|X(\mathbb{F}_p)| \leq \mathcal{N}_K(\mathfrak{p})^n + O_n(d^2 \mathcal{N}_K(\mathfrak{p})^{n-\frac{1}{2}})$.

Proof of Claim 3.17. This follows from an effective version of the Lang-Weil estimate. If K is a number field, by hypothesis, we have $\mathcal{N}_K(\mathfrak{p}) > \max\{27d^4, c_2\}$. On the other hand, if K is a function field over $\mathbb{F}_q(T)$ with $\text{char}(K) > \max\{27d^4, c_2\}$, it holds $\mathcal{N}_K(\mathfrak{p}) \geq q \geq \text{char}(K) > \max\{27d^4, c_2\}$. In either case, since X_p is absolutely irreducible and $\mathcal{N}_K(\mathfrak{p}) > 27d^4$, from [10, Corollary 5.6] it follows that the number of \mathbb{F}_p -points of X_p counted without multiplicity is bounded by

$$\frac{\mathcal{N}_K(\mathfrak{p})^{n+1} + (d-1)(d-2)\mathcal{N}_K(\mathfrak{p})^{n+\frac{1}{2}} + (5d^2 + d + 1)\mathcal{N}_K(\mathfrak{p})^n - 1}{\mathcal{N}_K(\mathfrak{p}) - 1} \leq \mathcal{N}_K(\mathfrak{p})^n + O_n(d^2 \mathcal{N}_K(\mathfrak{p})^{n-\frac{1}{2}}).$$

Similarly, if K is a function field over $\mathbb{F}_q(T)$ with $\text{char}(K) < \max\{27d^4, c_2\}$, by hypothesis $\mathcal{N}_K(\mathfrak{p}) > d^{\frac{14}{3}}$, thus [10, Theorem 5.2] gives that the number of \mathbb{F}_p -points of X_p counted without multiplicity is bounded by

$$\frac{\mathcal{N}_K(\mathfrak{p})^{n+1} + (d-1)(d-2)\mathcal{N}_K(\mathfrak{p})^{n+\frac{1}{2}} + 5d^{\frac{13}{3}}\mathcal{N}_K(\mathfrak{p})^n - 1}{\mathcal{N}_K(\mathfrak{p}) - 1} \leq \mathcal{N}_K(\mathfrak{p})^n + O_n(d^2 \mathcal{N}_K(\mathfrak{p})^{n-\frac{1}{2}}).$$

□

Combining (3.5) and Claim (3.17) we obtain $n_p \leq \mathcal{N}_K(\mathfrak{p})^n + O_n(d^2 \mathcal{N}_K(\mathfrak{p})^{n-\frac{1}{2}})$. Applying the bound $|x^{\frac{1}{n}} - 1| \leq |x - 1|$ to $x = \frac{n_p}{\mathcal{N}_K(\mathfrak{p})}$, we conclude that $n_p^{\frac{1}{n}} \leq \mathcal{N}_K(\mathfrak{p}) + O_n(d^2 \mathcal{N}_K(\mathfrak{p})^{n-\frac{1}{2}})$. Replacing this in (3.4) finishes the proof. □

Our next objective is to obtain a global bound for the determinant Δ in Theorem 3.16, without making any assumptions over its reductions X_p . We follow the presentation given in [11, Proposition 3.2.26] where the authors made explicit the dependence of the degree in the bounds of [46, Theorem 2.3].

Definition 3.18. Let c_2 be the constant in Proposition 2.2. We let

$$\beta := \begin{cases} 27d^4 & \text{if } \text{char}(K) = 0, \\ d^{\frac{14}{3}} & \text{if } 0 < \text{char}(K) \leq \max\{27d^4, c_2\}, \\ 1 & \text{if } \text{char}(K) > \max\{27d^4, c_2\}. \end{cases}$$

Given $f \in \mathcal{O}_K[X_0, \dots, X_{n+1}]$ we define $b(f) := 0$ if f is not absolutely irreducible, otherwise we set

$$b(f) := \prod_{\mathfrak{p} \in \mathcal{P}_f} \exp\left(\frac{\log(\mathcal{N}_K(\mathfrak{p}))}{\mathcal{N}_K(\mathfrak{p})}\right),$$

where

$$(3.6) \quad \mathcal{P}_f := \{\mathfrak{p} \notin M_{K,\infty} : \mathcal{N}_K(\mathfrak{p}) > \max\{\beta, c_2\} \text{ and } f \bmod \mathfrak{p} \text{ is not absolutely irreducible}\}.$$

Remark 3.19. Observe that if $X \subseteq \mathbb{P}_K^{n+1}$ is a geometrically integral hypersurface defined by an absolutely irreducible homogeneous polynomial $f \in \mathcal{O}_K[X_0, \dots, X_{n+1}]$ which is \mathfrak{p} -primitive, then $X_{\mathfrak{p}}$ is geometrically irreducible if and only if $f \bmod \mathfrak{p}$ is absolutely irreducible.

Theorem 3.20. Let (ξ_1, \dots, ξ_s) be a tuple of K -rational points in X , let $F_{li} \in \mathcal{O}_K[X_0, \dots, X_{n+1}]$, $1 \leq l \leq L$, $1 \leq i \leq s$, be homogeneous polynomials with integer coefficients, and write Δ_l for the determinant of $(F_{li}(\xi_j))_{ij}$. Let Δ be the greatest common divisor of the Δ_l , and assume that $\Delta \neq 0$. Let $\{\mathfrak{p}_1, \dots, \mathfrak{p}_u\}$ be (a possibly empty) subset of primes and set $\mathfrak{q} := \prod_{i=1}^u \mathfrak{p}_i$ if $u \geq 1$ and $\mathfrak{q} := (1)$ if $u = 0$. Let \mathcal{P}_X be the collection of primes $\mathfrak{p} \notin M_{K,\infty}$ such that either $\mathcal{N}_K(\mathfrak{p}) \leq \max\{\beta, c_2\}$, $X_{\mathfrak{p}}$ is not geometrically irreducible, or $\mathfrak{p} \in \{\mathfrak{p}_1, \dots, \mathfrak{p}_u\}$, namely $\mathcal{P}_X = \mathcal{P}_f \cup \{\mathfrak{p} : \mathcal{N}_K(\mathfrak{p}) \leq \max\{\beta, c_2\}\} \cup \{\mathfrak{p}_1, \dots, \mathfrak{p}_u\}$. Then there is some non-zero ideal \mathfrak{J} , relative prime with all the primes lying in \mathcal{P}_X , such that $\mathfrak{J}|\Delta$, and

$$\log(\mathcal{N}_K(\mathfrak{J})) \geq \frac{n!^{\frac{1}{n}}}{n+1} s^{1+\frac{1}{n}} (\log(s) - O_{n,K}(1) - n(\log(\beta) + \max\{\log(\log(\mathcal{N}_K(\mathfrak{q}))), 0\} + \log b(f))).$$

Proof. We fix \mathcal{O}_K^{n+1} -coordinates for the tuple (ξ_1, \dots, ξ_s) satisfying the conditions of Proposition 2.2. Let us suppose that all prime \mathfrak{p} with $\mathcal{N}_K(\mathfrak{p}) \leq s^{\frac{1}{n}}$ is contained in \mathcal{P}_X . Then by (2.11) and Lemma 2.6 we have

$$\begin{aligned} \frac{1}{n} \log(s) + O_K(1) &= \sum_{\substack{\mathfrak{p} \\ \mathcal{N}_K(\mathfrak{p}) \leq s^{\frac{1}{n}}}} \frac{\log(\mathcal{N}_K(\mathfrak{p}))}{\mathcal{N}_K(\mathfrak{p})} \leq \sum_{\substack{\mathfrak{p} \\ \mathcal{N}_K(\mathfrak{p}) \leq \max\{\beta, c_2\}}} \frac{\log(\mathcal{N}_K(\mathfrak{p}))}{\mathcal{N}_K(\mathfrak{p})} + \sum_{\substack{\mathfrak{p} \\ \mathfrak{p} \in \mathcal{P}_f}} \frac{\log(\mathcal{N}_K(\mathfrak{p}))}{\mathcal{N}_K(\mathfrak{p})} + \sum_{\substack{\mathfrak{p} \\ \mathfrak{p}|\mathfrak{q}}} \frac{\log(\mathcal{N}_K(\mathfrak{p}))}{\mathcal{N}_K(\mathfrak{p})} \\ &\leq \log(\beta) + \log(b(f)) + \max\{\log(\log(\mathcal{N}_K(\mathfrak{q}))), 0\} + O_K(1). \end{aligned}$$

Thus, if the term $O_{n,K}(1)$ is large enough, we may take $\mathfrak{J} = (1)$, since the right hand side of the bound in Theorem 3.20 is negative.

Hence we may suppose that there is some prime $\mathfrak{p} \notin \mathcal{P}_X$ with $\mathcal{N}_K(\mathfrak{p}) \leq s^{\frac{1}{n}}$. Applying Theorem 3.16 for each prime $\mathfrak{p} \notin \mathcal{P}_X$ with $\mathcal{N}_K(\mathfrak{p}) \leq s^{\frac{1}{n}}$, we obtain $\prod_{\mathfrak{p} \notin \mathcal{P}_X : \mathcal{N}_K(\mathfrak{p}) \leq s^{\frac{1}{n}}} \mathfrak{p}^{e_{\mathfrak{p}}} | \Delta$. Let $\mathfrak{J} := \prod_{\mathfrak{p} \notin \mathcal{P}_X : \mathcal{N}_K(\mathfrak{p}) \leq s^{\frac{1}{n}}} \mathfrak{p}^{e_{\mathfrak{p}}}$. Then the Landau estimate (2.12) yields

$$\begin{aligned} \log(\mathcal{N}_K(\mathfrak{J})) &= \sum_{\substack{\mathfrak{p} \notin \mathcal{P}_X \\ \mathcal{N}_K(\mathfrak{p}) \leq s^{\frac{1}{n}}}} e_{\mathfrak{p}} \log(\mathcal{N}_K(\mathfrak{p})) \geq \frac{n!^{\frac{1}{n}} n}{n+1} s^{1+\frac{1}{n}} \sum_{\substack{\mathfrak{p} \notin \mathcal{P}_X \\ \mathcal{N}_K(\mathfrak{p}) \leq s^{\frac{1}{n}}}} \frac{\log(\mathcal{N}_K(\mathfrak{p}))}{\mathcal{N}_K(\mathfrak{p}) + O_n(d^2 \mathcal{N}_K(\mathfrak{p})^{\frac{1}{2}})} - O_n(s) \sum_{\substack{\mathfrak{p} \\ \mathcal{N}_K(\mathfrak{p}) \leq s^{\frac{1}{n}}}} \log(\mathcal{N}_K(\mathfrak{p})) \\ &\geq \frac{n!^{\frac{1}{n}} n}{n+1} s^{1+\frac{1}{n}} \sum_{\substack{\mathfrak{p} \notin \mathcal{P}_X \\ \mathcal{N}_K(\mathfrak{p}) \leq s^{\frac{1}{n}}}} \frac{\log(\mathcal{N}_K(\mathfrak{p}))}{\mathcal{N}_K(\mathfrak{p}) + O_n(d^2 \mathcal{N}_K(\mathfrak{p})^{\frac{1}{2}})} - O_{n,K}(s^{1+\frac{1}{n}}). \end{aligned}$$

The inequality $\frac{1}{x+O_n(d^2x^{\frac{1}{2}})} \geq \frac{1}{x} - O_n(d^2)\frac{1}{x^{\frac{3}{2}}}$ for all $x \geq 0$, the Landau estimates (2.11), (2.12), and (2.14), and Lemma 2.6, allow us to deduce

$$\begin{aligned}
\sum_{\substack{\mathfrak{p} \notin \mathcal{P}_X \\ \mathcal{N}_K(\mathfrak{p}) \leq s^{\frac{1}{n}}}} \frac{\log(\mathcal{N}_K(\mathfrak{p}))}{\mathcal{N}_K(\mathfrak{p}) + O_n(d^2\mathcal{N}_K(\mathfrak{p})^{\frac{1}{2}})} &\geq \sum_{\substack{\mathfrak{p} \\ \mathcal{N}_K(\mathfrak{p}) \leq s^{\frac{1}{n}}}} \frac{\log(\mathcal{N}_K(\mathfrak{p}))}{\mathcal{N}_K(\mathfrak{p})} - \sum_{\mathfrak{p} \in \mathcal{P}_X} \frac{\log(\mathcal{N}_K(\mathfrak{p}))}{\mathcal{N}_K(\mathfrak{p})} - O_n(d^2) \sum_{\substack{\mathfrak{p} \notin \mathcal{P}_X \\ \mathcal{N}_K(\mathfrak{p}) \leq s^{\frac{1}{n}}}} \frac{\log(\mathcal{N}_K(\mathfrak{p}))}{\mathcal{N}_K(\mathfrak{p})^{\frac{3}{2}}} \\
&\geq \frac{\log(s)}{n} - O_K(1) - \sum_{\substack{\mathfrak{p} \\ \mathcal{N}_K(\mathfrak{p}) \leq \max\{\beta, c_2\}}} \frac{\log(\mathcal{N}_K(\mathfrak{p}))}{\mathcal{N}_K(\mathfrak{p})} - \sum_{\mathfrak{p}|q} \frac{\log(\mathcal{N}_K(\mathfrak{p}))}{\mathcal{N}_K(\mathfrak{p})} - \log b(f) - O_n(d^2) \sum_{\substack{\mathfrak{p} \notin \mathcal{P}_X \\ \mathcal{N}_K(\mathfrak{p}) \leq s^{\frac{1}{n}}}} \frac{\log(\mathcal{N}_K(\mathfrak{p}))}{\mathcal{N}_K(\mathfrak{p})^{\frac{3}{2}}} \\
&\geq \frac{\log(s)}{n} - \log(\beta) - \max\{\log \log(\mathcal{N}_K(q)), 0\} - \log b(f) - O_K(1) - O_n(d^2) \sum_{\substack{\mathfrak{p} \\ \mathcal{N}_K(\mathfrak{p}) > \beta}} \frac{\log(\mathcal{N}_K(\mathfrak{p}))}{\mathcal{N}_K(\mathfrak{p})^{\frac{3}{2}}} \\
&\geq \frac{\log(s)}{n} - \log(\beta) - \max\{\log(\log(\mathcal{N}_K(q))), 0\} - \log b(f) - O_K(1) + O_n(d^2\beta^{-\frac{1}{2}}) \\
&\geq \frac{\log(s)}{n} - \log(\beta) - \max\{\log(\log(\mathcal{N}_K(q))), 0\} - \log b(f) - O_{n,K}(1).
\end{aligned}$$

The bound in Theorem 3.20 follows. \square

Remark 3.21. When $\mathfrak{q} = (1)$, Theorem 3.20 gives the bounds in [11, Proposition 3.2.26] and [45, Proposition 2.4] in the cases $K = \mathbb{Q}$ and $K = \mathbb{F}_q(T)$, respectively.

In [46] Walsh proved the bound $b(f) \lesssim_d \max\{\log(H_{K,\text{aff}}(f)), 1\}$ for $K = \mathbb{Q}$. In order to keep track of the dependence on the degree of f , in [11, Corollary 3.2.3] and [45, Lemma 2.3] the authors gave an effective bound of $b(f)$ for $K = \mathbb{Q}$ and $K = \mathbb{F}_q(T)$, respectively. We follow their strategy to give an effective bound for global fields. We will use the following notation

Definition 3.22. Let $d \in \mathbb{N}$, and let K be a global field. Then for any $a_1, a_2, a_3, a_4 \in \mathbb{R}$ we denote

$$[a_1, a_2, a_3, a_4] := \begin{cases} a_1 & \text{if } \text{char}(K) = 0, \\ a_2 & \text{if } 0 < \text{char}(K) \leq d(d-1), \\ a_3 & \text{if } d(d-1) < \text{char}(K) \leq \max\{27d^4, c_2\}, \\ a_4 & \text{if } \text{char}(K) > \max\{27d^4, c_2\}. \end{cases}$$

Lemma 3.23. Let $f = \sum_I a_I \mathbf{X}^I \in \mathcal{O}_K[X_0, \dots, X_{n+1}]$ be an absolutely irreducible homogeneous polynomial of degree $d \geq 2$. We have

$$b(f) \lesssim_{K,n} \max\{d^{[-\frac{2}{3}, \frac{4}{3}, -\frac{4}{3}, 2]} \log(H_{K,\text{aff}}(f)), 1\}.$$

Proof. We will use Noether forms as they are presented in [36, 28, 19]. Let \mathcal{P}_f be as in (3.6). First, let us suppose that K is a number field. By [36, Satz 4], there is a homogeneous form Φ with coefficients in \mathbb{Z} , of degree $d^2 - 1$ and

$$(3.7) \quad \ell_1(\Phi) \leq d^{3(d^2-1)} \left[\binom{n+d}{n} 3^d \right]^{d^2-1},$$

such that Φ applied to the coefficients of f is non-zero, but is divisible by any prime in \mathcal{P}_f . In particular, by inequalities (2.4), (2.9) we have

$$(3.8) \quad \prod_{\mathfrak{p} \in \mathcal{P}_f} \mathcal{N}_K(\mathfrak{p}) \leq \mathcal{N}_K(\Phi(a_I)_I) \leq H_K(\Phi(a_I)_I) \leq \ell_1(\Phi)^{d_K} H_K(1 : (a_I)_I)^{\deg(\Phi)} = \ell_1(\Phi)^{d_K} H_{K,\text{aff}}(f)^{\deg(\Phi)}.$$

Let $c := \ell_1(\Phi)^{d_K} H_{K,\text{aff}}(f)^{\deg(\Phi)}$. Using the above inequality and (2.11), we have

$$\begin{aligned} \log(b(f)) &= \sum_{\mathfrak{p} \in \mathcal{P}_f} \frac{\log(\mathcal{N}_K(\mathfrak{p}))}{\mathcal{N}_K(\mathfrak{p})} \leq \sum_{\substack{\mathfrak{p} \\ \max\{27d^4, c_2\} < \mathcal{N}_K(\mathfrak{p}) \leq \log c}} \frac{\log(\mathcal{N}_K(\mathfrak{p}))}{\mathcal{N}_K(\mathfrak{p})} + \sum_{\substack{\mathfrak{p} \in \mathcal{P}_f \\ \log c < \mathcal{N}_K(\mathfrak{p})}} \frac{\log(\mathcal{N}_K(\mathfrak{p}))}{\log c} \\ &\leq \max\{\log(\log c) - 4\log d, 0\} + O_K(1) + \frac{\log c}{\log c} \\ &\leq \max\{\log(\deg(\Phi) \log(H_{K,\text{aff}}(f))) - 4\log d, \log(d_K \log \ell_1(\Phi)) - 4\log d\} + O_K(1). \end{aligned}$$

Using that $\deg(\Phi) = d^2 - 1$ and (3.7), the result follows in the number field case. Indeed, note that

$$(3.9) \quad \begin{aligned} \log(d_K \deg(\Phi) \log(H_{K,\text{aff}}(f))) - 4\log d &\leq \log(d_K) + \log(d^2 \log(H_{K,\text{aff}}(f))) - 4\log d \\ &\leq \log(d_K) + \log(\log(H_{K,\text{aff}}(f))) - 2\log d. \end{aligned}$$

Meanwhile,

$$(3.10) \quad \begin{aligned} \log(d_K \log \ell_1(\Phi)) &\leq \log \left(d_K \log \left(d^{3(d^2-1)} \left[\binom{n+d}{n} 3^d \right]^{d^2-1} \right) \right) \\ &\leq \log(d_K) + 2\log(d) + \log(3\log(d) + d\log(3) + d\log(n+d) - \log(d!)). \end{aligned}$$

Now we bound $\log(d_K \log \ell_1(\Phi)) - 4\log d$. Since

$$\frac{3\log(d) + d\log(3) + d\log(n+d) - \log(d!)}{d^2} \lesssim_n 1,$$

from (3.10) we obtain $\log(d_K \log \ell_1(\Phi)) - 4\log(d) \lesssim_{n,K} 1$. This together with (3.9) gives the bound for $b(f)$ in the number field case.

Now let us suppose that K is a function field over $\mathbb{F}_q(T)$. We will follow [45], and consider separately the cases of small and large characteristic.

Let K be with $0 < \text{char}(K) \leq d(d-1)$. By [28, Theorem 7], there is a homogeneous form Φ with coefficients in \mathbb{Z} , of degree $12d^6$ such that Φ applied to the coefficients of f is non-zero, but divisible by any prime in \mathcal{P}_f . On the other hand, if $\text{char}(K) > d(d-1)$, by [19, Lemma 2.4], the result in [36, Satz 4] still holds in the function field case and then we may take Φ of degree $d^2 - 1$. In any case, since the coefficients of Φ are integers, their height in K is 1, then $\ell_1(\Phi) = 1$. Then by inequalities (2.2), (2.4), (2.10) we have

$$(3.11) \quad \prod_{\mathfrak{p} \in \mathcal{P}_f} \mathcal{N}_K(\mathfrak{p}) \leq \mathcal{N}_K(\Phi(a_I)_I) \leq H_K(\Phi(a_I)_I) \leq \begin{cases} H_{K,\text{aff}}(f)^{12d^6} & \text{if } 0 < \text{char}(K) \leq d(d-1), \\ H_{K,\text{aff}}(f)^{d^2-1} & \text{if } \text{char}(K) > d(d-1). \end{cases}$$

Now, denoting $c := H_{K,\text{aff}}(f)^{\deg(\Phi)}$, by (2.11) and (2.12) it follows that

$$\begin{aligned} \log(b(f)) &= \sum_{\mathfrak{p} \in \mathcal{P}_f} \frac{\log(\mathcal{N}_K(\mathfrak{p}))}{\mathcal{N}_K(\mathfrak{p})} \leq \sum_{\substack{\mathfrak{p} \\ \max\{\beta, c_2\} < \mathcal{N}_K(\mathfrak{p}) \leq \log c}} \frac{\log(\mathcal{N}_K(\mathfrak{p}))}{\mathcal{N}_K(\mathfrak{p})} + \sum_{\substack{\mathfrak{p} \in \mathcal{P}_f \\ \log c < \mathcal{N}_K(\mathfrak{p})}} \frac{\log(\mathcal{N}_K(\mathfrak{p}))}{\log c} \\ &\leq \max\{\log(\deg(\Phi) \log(H_{K,\text{aff}}(f))) - \log(\beta), 0\} + O_K(1) \\ &\leq \max\{\log(\deg(\Phi)) + \log(\log(H_{K,\text{aff}}(f))) - \log(\beta), 0\} + O_K(1). \end{aligned}$$

From this last estimate it follows the bounds for function fields. \square

4. AN ADEQUATE CHANGE OF COORDINATES

For technical reasons that will appear in the next section, given two non-zero polynomials $f, h \in \mathcal{O}_K[X_0, \dots, X_{n+1}]$, we will need to obtain a lower bound for the quantity $H_K(fh)$ in terms of $H_K(f)$ and $H_K(h)$. This turns out to be rather easy in the function field case. Indeed, suppose that K is a function field. Then every place $v \in M_K$ is non-archimedean. Thus, by Gauss' lemma, (e.g. see [2, Lemma 1.6.3]), it holds

$$(4.1) \quad H_K(fh) = H_K(f)H_K(h).$$

In the number field case, while it is possible to bound $H_K(fh)$ in terms of $H_K(f)$ and $H_K(h)$, this bound will necessarily depend on the degrees of f and g , which is not good enough for our purposes (see Remark 5.6). In Claim 5.7 we will prove that, if f is of an adequate form, it is possible to bound from below $H_K(fh)$ in terms of $H_K(f)$ and of the degree of f . By making an adequate change of variables as in [46, §3], in this section we will show that any polynomial can be taken into a polynomial that verifies the hypothesis of Claim 5.7.

Let us suppose that K is a number field. Let $f = \sum_I c_I \mathbf{X}^I \in \mathcal{O}_K[X_0, \dots, X_{n+1}]$ be a homogeneous polynomial of degree d . Let $a_0, \dots, a_n \in \mathcal{O}_K$ and consider the new variables $X'_i := X_i - a_i X_{n+1}$ for all $i = 0, \dots, n+1$. Then the linear change of variables gives

$$\begin{aligned} \tilde{f}(X_0, \dots, X_{n+1}) &:= f(X'_0, \dots, X'_n, X_{n+1}) = \sum_{i_0 + \dots + i_{n+1} = d} c_{i_0, \dots, i_{n+1}} (X_0 - a_0 X_{n+1})^{i_0} \dots (X_n - a_n X_{n+1})^{i_n} X_{n+1}^{i_{n+1}} \\ &= \pm \left(\sum_{i_0 + \dots + i_{n+1} = d} c_{i_0, \dots, i_{n+1}} a_0^{i_0} \dots a_n^{i_n} \right) X_{n+1}^d + \dots = \pm f(a_0, \dots, a_n, 1) X_{n+1}^d + \dots. \end{aligned}$$

So, the coefficient of X_{n+1}^d in \tilde{f} is $\pm f(a_0, \dots, a_n, 1)$. Next, we will require that this coefficient is large enough. More precisely, we shall prove the following generalization of [46, § 3] and [11, Lemma 3.4.2].

Lemma 4.1. *Let K be a number field and let $f = \sum_I c_I \mathbf{X}^I \in \mathcal{O}_K[X_0, \dots, X_{n+1}]$ be a homogeneous polynomial of degree d . There exist integers a_0, \dots, a_n with $0 \leq a_i \leq d$ for all i such that*

$$\prod_{v \in M_{K, \infty}} |f(a_0, \dots, a_n, 1)|_v \geq 3^{-\frac{(n+1)d}{d_K}} 2^{-d} \prod_{v \in M_{K, \infty}} |f|_v.$$

Proof. Given an embedding $\sigma : K \rightarrow \mathbb{C}$, we define $\sigma(f) := \sum_I \sigma(c_I) \mathbf{X}^I$ and $f^G = \prod_{\sigma} \sigma(f)$ where the product runs over all the embeddings of K in \mathbb{C} . By (2.1) it holds

$$(4.2) \quad \prod_{v \in M_{K, \infty}} |f(a_0, \dots, a_n, 1)|_v = |f^G(a_0, \dots, a_n, 1)|^{\frac{1}{d_K}}.$$

By [11, Lemma 3.4.2], there exist integers a_0, \dots, a_n with $0 \leq a_i \leq d$ for all i such that

$$|f^G(a_0, \dots, a_n, 1)| \geq 3^{-(n+1)d} \ell_{\infty}(f^G).$$

Since f^G is a polynomial of degree $d_K d$, by [2, Lemma 1.6.11] we have

$$|f^G(a_0, \dots, a_n, 1)| \geq 3^{-(n+1)d} 2^{-dd_K} \prod_{\sigma} \ell_{\infty}(\sigma(f)).$$

If v is the place corresponding to σ , we have $\ell_{\infty}(\sigma(f)) = |f|_v^{\frac{d_K}{n_v}}$. Then, since the complex embeddings come in conjugate pairs, both corresponding to the same place, it follows that

$$(4.3) \quad |f^G(a_0, \dots, a_n, 1)| \geq 3^{-(n+1)d} 2^{-dd_K} \left(\prod_{v \in M_{K, \infty}} |f|_v \right)^{d_K}.$$

Combining (4.2) and (4.3) yields the desired result. \square

From Lemma 4.1 we find integers a_0, \dots, a_n with $0 \leq a_i \leq d$ for all i , such that $\tilde{f}(X_0, \dots, X_n, X_{n+1}) = f(X'_0, \dots, X'_n, X_{n+1}) = c_{\tilde{f}} X_{n+1}^d + \dots$, with $\prod_{v \in M_{K, \infty}} |c_{\tilde{f}}|_v \geq 3^{-\frac{(n+1)d}{d_K}} 2^{-d} \prod_{v \in M_{K, \infty}} |f|_v$. Since for all $v \in M_{K, \infty}$, it holds

$$(4.4) \quad |\tilde{f}|_v \leq \left| \binom{n+d+1}{n+1} \right|_v |d|_v^{n+1} |f|_v \leq |2(n+1)|_v^d |d|_v^{n+1} |f|_v,$$

we conclude that

$$(4.5) \quad \prod_{v \in M_{K, \infty}} |c_{\tilde{f}}|_v \geq 3^{-\frac{(n+1)d}{d_K}} 2^{-(d+1)} (n+1)^{-d} d^{-(n+1)} \prod_{v \in M_{K, \infty}} |\tilde{f}|_v \geq C^{-nd^{1+\frac{1}{n}}} \prod_{v \in M_{K, \infty}} |\tilde{f}|_v,$$

where C is a positive constant depending on K and n . The bound is not sharp, but having it in this form will be convenient for our purposes. Now, if \mathfrak{p} is a prime such that f is \mathfrak{p} -primitive, it also follows that \tilde{f} is \mathfrak{p} -primitive. Furthermore, if $\tilde{f} = \sum_J b_J \mathbf{X}^J$, the proof of Proposition 2.2 shows that there exists a constant $c_1 = c_1(K)$ and a unit $\varepsilon \in \mathcal{O}_K^\times$ such that $\varepsilon \tilde{f}$ satisfies

$$H_K(1 : (\varepsilon b_J)_J) \leq c_1^{d_K} H_K(\varepsilon \tilde{f}).$$

Since the \mathfrak{p} -adic valuation of any unit is 0, and by the product formula it holds $\prod_{v \in M_{K,\infty}} |\varepsilon|_v = 1$, we see that the coefficient of X_{n+1}^d in $f_1 := \varepsilon \tilde{f}$ also satisfies (4.5). Moreover, from the identity $f(X_0, \dots, X_{n+1}) = \varepsilon^{-1} f_1(X_0 + a_0 X_{n+1}, \dots, X_n + a_n X_{n+1}, X_{n+1})$, proceeding as in (4.4), we have that for all $v \in M_{K,\infty}$

$$(4.6) \quad |f|_v \leq |\varepsilon|_v^{-1} 2(n+1)^d d^{m+1} |v| f_1|_v.$$

Also, for all $v \in M_{K,\text{fin}}$, it holds

$$(4.7) \quad |f|_v = |f_1|_v,$$

hence (4.6) and (4.7) imply

$$H_K(f) \leq \prod_{v \in M_{K,\infty}} |\varepsilon|_v^{-1} 2(n+1)^d d^{m+1} |v| f_1|_v \prod_{v \in M_{K,\text{fin}}} |f_1|_v \leq C'^{d^{1+\frac{1}{n}}} H_K(f_1),$$

for some constant C' depending only on n . We summarize all this in the next proposition.

Proposition 4.2. *Let K be a number field, and let $f = \sum_I a_I \mathbf{X}^I \in \mathcal{O}_K[X_0, \dots, X_{n+1}]$ be an absolutely irreducible homogeneous polynomial of degree $d \geq 2$, which is \mathfrak{p} -primitive for all primes \mathfrak{p} with $\mathcal{N}_K(\mathfrak{p}) > c_2$ and*

$$(4.8) \quad H_{K,\text{aff}}(f) \leq c_1^{d_K} H_K(f).$$

Then there are integers a_0, \dots, a_n with $0 \leq a_i \leq d$ and a unit $\varepsilon \in \mathcal{O}_K^\times$ such that $f_1(X_0, \dots, X_n, X_{n+1}) := \varepsilon f(X_0 - a_0 X_{n+1}, \dots, X_n - a_n X_{n+1}, X_{n+1})$ is a polynomial with coefficient c_{f_1} in X_{n+1}^d verifying

$$\prod_{v \in M_{K,\infty}} |c_{f_1}|_v \geq C^{-nd^{1+\frac{1}{n}}} \prod_{v \in M_{K,\infty}} |f_1|_v \quad \text{and} \quad H_K(f) \leq C'^{d^{1+\frac{1}{n}}} H_K(f_1)$$

for some positive constants $C = C(n, K)$, $C' = C(n)$. Furthermore, f_1 verifies (4.8) and it holds $b(f) = b(f_1)$.

5. CONSTRUCTION OF THE AUXILIARY HYPERSURFACE AND BOUNDS FOR THE NUMBER OF RATIONAL POINTS OF CURVES OVER GLOBAL FIELDS

This section is divided in two parts, the first one dealing with the proof of Theorem 1.8 extending and improving the bound of Heath-Brown [24] and the second one dealing with Theorem 1.9 extending and improving the bounds of Bombieri and Pila.

The main technical tool in the first part is Theorem 5.2 (which is a generalization of [46, Theorem 1.3], [11, Theorem 3.1.1] and [45, Theorem 3.1] to global fields) where we construct an auxiliary hypersurface of bounded degree that vanishes on the rational points of the hypersurface X of prescribed height. From Theorem 5.2 and Bézout's theorem, Theorem 1.8 follows for plane projective curves. By means of a projection argument used in [31] we deduce the general form of Theorem 1.8. In the proof of Theorem 5.2, the change of variables done in Section 4 will allow us to assume that in the number field case the polynomial defining X has its height concentrated on a prescribed coefficient. Unlike [45], in the function field case the proof presented here does not need a change of variables. As in [46, 11, 45], the bound on the degree of the auxiliary hypersurface improves as the height of the polynomial defining X gets larger. A difference with [46, 11, 45] is that the hypersurface we construct vanishes on the rational points of X of bounded height and prescribed reduction modulo \mathfrak{p} for many primes \mathfrak{p} , and this produces a saving in the degree of the hypersurface, which will be needed in Section 6.

The proof of Theorem 1.9 follows the same strategy of the proof of Theorem 1.8, the main technical tool being Theorem 5.14, which is an adaptation of Theorem 5.2 to affine hypersurfaces. In the proof of Theorem 5.14 we follow the strategy devised in [17, Remark 2.3] and developed in [11, Proposition 4.2.1].

5.1. The projective case. We proceed to generalize [46, Theorem 1.3], [11, Theorem 3.1.1] and [45, Theorem 3.1] to global fields. Moreover, this generalization also improves [38, Theorem 2.2] when $(B_0, \dots, B_{n+1}) = (B, \dots, B)$ by removing a logarithmic factor. We will require the following notation.

Notation 5.1. Let $X \subseteq \mathbb{P}_K^{n+1}$ be a projective variety. Given $B \in \mathbb{R}_{>0}$, we will denote

$$X(K, B) := \{x \in X(K) : H_K(x) \leq B\},$$

Moreover, if $\{\mathfrak{p}_1, \dots, \mathfrak{p}_u\}$ is a subset of primes such that for all $1 \leq i \leq u$ we let P_i be a non-singular $\mathbb{F}_{\mathfrak{p}_i}$ -point on $X_{\mathfrak{p}_i}$, we denote

$$X(K, B; P_1, \dots, P_u) := \{x \in X(K, B) : x \text{ specialises to } P_i \text{ in } X_{\mathfrak{p}_i} \text{ for all } i\}.$$

We denote

$$N(X, K, B) := |X(K, B)|.$$

Theorem 5.2. *Let K be a global field of degree d_K . Let $X \subseteq \mathbb{P}_K^{n+1}$ be an integral hypersurface of degree $d \geq 2$, defined by an irreducible homogeneous polynomial $f \in \mathcal{O}_K[X_0, \dots, X_{n+1}]$ of degree d . Let $\{\mathfrak{p}_1, \dots, \mathfrak{p}_u\}$ be (a possibly empty) subset of primes and let P_i be a non-singular $\mathbb{F}_{\mathfrak{p}_i}$ -point on $X_{\mathfrak{p}_i}$ for each $i \in \{1, \dots, u\}$. Set $\mathfrak{q} := \prod_{i=1}^u \mathfrak{p}_i$ if $u \geq 1$ and $\mathfrak{q} := (1)$ if $u = 0$. Then for any $B \geq 1$, there exists a homogeneous $g \in \mathcal{O}_K[X_0, \dots, X_{n+1}]$ of degree*

$$\begin{aligned} M \lesssim_{K,n} B^{\frac{n+1}{nd} - \frac{1}{n}} d^{[4, \frac{14}{3}, \frac{14}{3}, 0] - \frac{1}{n} b(f) \max\{\log(\mathcal{N}_K(\mathfrak{q}), 1)\}} & \frac{H_K(f)^{\frac{1}{n} - \frac{1}{d+1}} \mathcal{N}_K(\mathfrak{q})}{H_K(f)^{\frac{1}{n} - \frac{1}{d+1}} \mathcal{N}_K(\mathfrak{q})} + d^{1 - \frac{1}{n}} \log(B \mathcal{N}_K(\mathfrak{q})) \\ & + d^{[4 - \frac{1}{n}, 7, \frac{14}{3} - \frac{1}{n}, 3]} \left(\frac{\max\{\log(\mathcal{N}_K(\mathfrak{q})), 1\}}{\mathcal{N}_K(\mathfrak{q})} + 1 \right), \end{aligned}$$

not divisible by f and vanishing on all $X(K, B; P_1, \dots, P_u)$.

Remark 5.3. Observe that if a polynomial f is irreducible but not absolutely irreducible over K , then by the argument in [24, Corollary 1], there exists a homogeneous polynomial $g \in \mathcal{O}_K[X_0, \dots, X_{n+1}]$ of degree d , not divisible by f and vanishing on all K -rational points of X . Thus, in the proof of Theorem 5.2 we may assume that f is absolutely irreducible.

In order to prove Theorem 5.2, first we will prove a seemingly weaker version of it where the polynomial f defining the hypersurface X satisfies the following assumption, afterwards the work done in Section 4 will be used to deduce the general case.

Assumption 5.4. $f \in \mathcal{O}_K[X_0, \dots, X_{n+1}]$ is an absolutely irreducible homogeneous polynomial of degree d that verifies:

- (1) f is \mathfrak{p} -primitive for any prime \mathfrak{p} of \mathcal{O}_K with $\mathcal{N}_K(\mathfrak{p}) > c_2$;
- (2) $H_{K, \text{aff}}(f) \leq c_1^{d_K} H_K(f)$.
- (3) If K is a number field, the coefficient c_f of X_{n+1}^d in f verifies:

$$\prod_{v \in M_{K, \infty}} |c_f|_v \geq C^{-nd^{1+\frac{1}{n}}} \prod_{v \in M_{K, \infty}} |f|_v,$$

where C is a constant that depends only on K and n .

Proof of Theorem 5.2 assuming f verifies Assumption 5.4. The proof follows the usual strategy of the polynomial method: first we find a small characteristic subset $\mathcal{C} \subseteq X(K, B; P_1, \dots, P_u)$, which detects the algebraic structure of $X(K, B; P_1, \dots, P_u)$. Then, we use a dimensional argument (which usually is a variation of the Siegel's lemma) to construct an adequate polynomial g of small complexity that vanishes on \mathcal{C} . From the nature of the subset \mathcal{C} , it will follow that g vanishes on $X(K, B; P_1, \dots, P_u)$ and has the desired property.

In order to accomplish the above strategy, we let M be a positive integer of size

$$(5.1) \quad M \sim_{K,n} B^{\frac{n+1}{nd^{\frac{1}{n}}}} \frac{d^{[4, \frac{14}{3}, \frac{14}{3}, 0] - \frac{1}{n}} b(f) \max\{\log(\mathcal{N}_K(\mathfrak{q})), 1\}}{H_K(f)^{\frac{1}{n}} \frac{1}{d^{1+\frac{1}{n}}} \mathcal{N}_K(\mathfrak{q})} + d^{1-\frac{1}{n}} \log(B\mathcal{N}_K(\mathfrak{q})) \\ + d^{[4-\frac{1}{n}, 7, \frac{14}{3}-\frac{1}{n}, 3]} \left(\frac{\max\{\log(\mathcal{N}_K(\mathfrak{q})), 1\}}{\mathcal{N}_K(\mathfrak{q})} + 1 \right).$$

This integer will be the degree of the polynomial g to be constructed. Now we construct the characteristic subset, which should be a subset such that any polynomial of degree M , vanishing on it also vanishes on $X(K, B; P_1, \dots, P_u)$, without vanishing completely on X . The construction is rather natural. Using Proposition 2.2 we represent points in $X(K, B; P_1, \dots, P_u)$ with coordinates in \mathcal{O}_K^{n+2} . Afterwards, we take $\mathcal{C} := \{\xi_1, \dots, \xi_s\}$ to be a maximal subset of $X(K, B; P_1, \dots, P_u)$ with the property that if A is the matrix whose i^{th} row is the evaluation of the different monomials of degree M at ξ_i , then A has rank s .

The next step is to construct an adequate polynomial by means of the Bombieri-Vaaler theorem. Given an integer D , we write $\mathcal{B}[D]$ for the set of monomials of degree D ; it follows that $|\mathcal{B}[D]| = \binom{D+n+1}{n+1}$. Since the elements of $f\mathcal{B}[M-d]$ provide linearly independent polynomials vanishing on $X(K, B; P_1, \dots, P_u)$, it follows that

$$(5.2) \quad s = \text{rank}(A) \leq |\mathcal{B}[M]| - |\mathcal{B}[M-d]|.$$

Now, let us note that since A has coefficients in \mathcal{O}_K , if the inequality (5.2) is strict we can find a polynomial in the \mathcal{O}_K -span of $\mathcal{B}[M]$ which vanishes on \mathcal{C} (so it also vanishes on $X(K, B; P_1, \dots, P_u)$, by the maximality of \mathcal{C}), and it is not divisible by f . This would conclude the proof. Then we may suppose that

$$(5.3) \quad s = |\mathcal{B}[M]| - |\mathcal{B}[M-d]|.$$

The remaining of the proof is to show that (5.3) does not hold if the implicit constant in (5.1) is large enough.

Let us note that (5.3) implies

$$(5.4) \quad s = \frac{dM^n}{n!} + O_n(d^2 M^{n-1}).$$

From (5.4) and the fact that $M \gtrsim d^2$ we obtain

$$(5.5) \quad \log(s) = \log(d) + n \log(M) - O_n(1),$$

and

$$(5.6) \quad \frac{s^{\frac{1}{n}}}{M} = \frac{d^{\frac{1}{n}}}{n!^{\frac{1}{n}}} + O_n\left(\frac{d^2}{M}\right).$$

Now, let Δ be the greatest common divisor in \mathcal{O}_K of the determinants of the $s \times s$ -minors of A and let us denote

$$r := |\mathcal{B}[M]|.$$

To find an \mathcal{O}_K -homogeneous polynomial of degree M that vanishes on \mathcal{C} amounts to find a non-trivial solution in \mathcal{O}_K^r of the system of linear equations given by A . For that, using Theorem 2.4, there exists a polynomial $g = \sum_I a_I \mathbf{X}^I \in \mathcal{O}_K[X_0, \dots, X_{n+1}]$ of degree M , vanishing on \mathcal{C} and satisfying the inequality

$$(5.7) \quad H_K(g)^{r-s} \leq \begin{cases} C(K)^{(r-s)d_K} \mathcal{N}_K(\Delta)^{-1} \prod_{v \in M_{K,\infty}} |\det(AA^*)|_v^{\frac{d_K}{2}} & \text{if } K \text{ is a number field,} \\ C(K)^{(r-s)d_K} \mathcal{N}_K(\Delta)^{-1} q^{-\deg(\mathfrak{p}_\infty) \min_{|J|=s} \text{ord}_{\mathfrak{p}_\infty}(\det(A_J))} & \text{if } K \text{ is a function field.} \end{cases}$$

Claim 5.5. *It holds that*

$$(5.8) \quad H_K(g)^{r-s} \leq C(K)^{(r-s)} \mathcal{N}_K(\Delta)^{-1} \tilde{c}_1^{Ms} s!^{\frac{d_K}{2}} r^{\frac{d_K s}{2}} B^{Ms}.$$

Proof of Claim 5.5. The proof amounts to bound the contribution of the places $v \in M_{K,\infty}$ in the right-hand side of (5.7). Since the coordinates of the points ξ_i were chosen according to Proposition 2.2, given any $1 \leq i \leq s$ and any monomial of degree M corresponding to an index set $\mathbf{J} \subseteq \mathbb{N}^{n+2}$, it holds that for any place $v \in M_{K,\infty}$,

$$(5.9) \quad |\xi_i^{\mathbf{J}}|_v = |\xi_{i,0}^{j_0} \dots \xi_{i,n+1}^{j_{n+1}}|_v \leq \begin{cases} c_1^M (B^{j_0} \dots B^{j_{n+1}})^{\frac{n_v}{d_K^2}} = c_1^M B^{\frac{M n_v}{d_K^2}} & \text{if } K \text{ is a number field,} \\ c_1^M (B^{j_0} \dots B^{j_{n+1}})^{\frac{1}{d_K}} = c_1^M B^{\frac{M}{d_K}} & \text{if } K \text{ is a function field.} \end{cases}$$

Let us suppose that K is a number field. Let $v \in M_{K,\infty}$. If $\sigma \in S_s$ then

$$|(AA^*)_{i,\sigma(i)}|_v = \left| \sum_{k=1}^r a_{i,k} \overline{a_{\sigma(i),k}} \right|_v \leq c_1^{2M} |r|_v B^{\frac{2M n_v}{d_K^2}}.$$

Hence

$$|\det(AA^*)|_v = \left| \sum_{\sigma \in S_s} (-1)^\sigma \prod_{i=1}^s (AA^*)_{i,\sigma(i)} \right|_v \leq |s!|_v \left(c_1^{2M} |r|_v B^{\frac{2M n_v}{d_K^2}} \right)^s.$$

The above inequality allows us to conclude that

$$(5.10) \quad \prod_{v \in M_{K,\infty}} |\det(AA^*)|_v^{\frac{d_K}{2}} \leq \prod_{v \in M_{K,\infty}} c_1^{d_K M s} |s!|_v^{\frac{d_K}{2}} B^{\frac{M s n_v}{d_K}} |r|_v^{\frac{d_K s}{2}} \leq c_1^{d_K^2 M s} s!^{\frac{d_K}{2}} r^{\frac{d_K s}{2}} B^{M s}.$$

On the other hand, if K is a function field over $\mathbb{F}_q(T)$, recalling that for all $x \in K$, $|x|_{v_\infty} = \mathcal{N}_K(\mathfrak{p}_\infty)^{-\frac{\text{ord}_{\mathfrak{p}_\infty}(x)}{d_K}}$, (5.9) implies that every entry a_{ij} of A verifies $\text{ord}_{\mathfrak{p}_\infty}(a_{ij}) \geq -\frac{M}{\deg(\mathfrak{p}_\infty)}(d_K \log_q(c_1) + \log_q(B))$. Then, for any minor A_J it holds

$$\begin{aligned} \text{ord}_{\mathfrak{p}_\infty}(\det(A_J)) &= \text{ord}_{\mathfrak{p}_\infty} \left(\sum_{\sigma \in S_s} (-1)^\sigma \prod_{i=1}^s (A_J)_{i,\sigma(i)} \right) \geq \min_{\sigma \in S_s} \left\{ \text{ord}_{\mathfrak{p}_\infty} \left((-1)^\sigma \prod_{i=1}^s (A_J)_{i,\sigma(i)} \right) \right\} \\ &\geq \min_{\sigma \in S_s} \left\{ \sum_{i=1}^s \text{ord}_{\mathfrak{p}_\infty}((A_J)_{i,\sigma(i)}) \right\} \geq -\frac{sM}{\deg(\mathfrak{p}_\infty)}(d_K \log_q(c_1) + \log_q(B)), \end{aligned}$$

and from this, we get the bound

$$(5.11) \quad q^{-\deg(\mathfrak{p}_\infty) \min_{|J|=s} \text{ord}_{\mathfrak{p}_\infty}(\det(A_J))} \leq q^{sM(d_K \log_q(c_1) + \log_q(B))} = c_1^{d_K s M} B^{sM}.$$

Denoting $\tilde{c}_1 := c_1^{\frac{d_K^2}{2}}$ or $\tilde{c}_1 := c_1^{d_K}$ according to K being a number field or a function field, respectively, the inequalities (5.10) and (5.11) applied in (5.7) yields the desired result. \square

Now, the standing hypothesis (5.3) implies that g must be of the form $g = fh$ with h homogeneous of degree $M - d$. We shall bound $H_K(g) = H_K(fh)$ in terms of the height $H_K(f)$.

Remark 5.6. If K is a number field, for $f, h \in K[X_1, \dots, X_n]$, it holds (e.g. see [2, Theorem 1.6.13]) that $H_K(fh) \gtrsim_{\deg(f), \deg(h)} H_K(f)H_K(h)$. Since in our case the polynomial h has degree $M - d$, such a bound would depend on the parameter M . The next claim shows that under the Assumption 5.4(3) it is possible to obtain a bound independent of the degree of h .

Claim 5.7. *There exists some positive constant C , depending only on n, K , such that $H_K(g) \geq C^{-nd^{1+\frac{1}{n}}} H_K(f)$.*

Proof of Claim 5.7. If K is a function field, then by identity (4.1) and the fact that $H_K(h) \geq 1$ the claim follows by taking $C = 1$.

Now, let us suppose that K is a number field. If $h = \sum_I c_I \mathbf{X}^I$, let $\mathbf{c}_h = (c_I)_I$ be the projective point defined by the coordinates of h . Following the proof of Proposition 2.2, we find a non-zero scalar $\lambda \in \mathcal{O}_K$ such that $(\lambda c_I)_I$ satisfies the conditions of Proposition 2.2. Thus, we get $\prod_{v \in M_{K,\text{fin}}} |\lambda h|_v \geq c_3$. Since the polynomial height is invariant

under multiplication by scalars, we get $H(g) = H(\lambda g) = H(f(\lambda h))$. Hence, after multiplication by a scalar, we may suppose that $g = fh$ with $h \in \mathcal{O}_K[X_0, \dots, X_{n+1}]$ satisfying

$$(5.12) \quad \prod_{v \in M_{K, \text{fin}}} |h|_v \geq c_3.$$

Let W be the greatest monomial (in right to left lexicographical order) appearing in h with non-zero coefficient and let w its coefficient. Then the monomial WX_{n+1}^d appears in g with $c_f w$ as its coefficient, where c_f is the coefficient of X_{n+1}^d in f . By Gauss' lemma (see [2, Lemma 1.6.3]), it holds

$$(5.13) \quad H(g) = H(fh) = \prod_{v \in M_K} |fh|_v = \prod_{v \in M_{K, \text{fin}}} |f|_v \prod_{v \in M_{K, \text{fin}}} |h|_v \prod_{v \in M_{K, \infty}} |fh|_v.$$

Now we bound $|fh|_v$ for any infinite place v . Since WX_{n+1}^d appears in g with coefficient $c_f w$, from Assumption 5.4 we have

$$(5.14) \quad \prod_{v \in M_{K, \infty}} |fh|_v \geq \prod_{v \in M_{K, \infty}} |c_f w|_v \geq C^{-nd^{1+\frac{1}{n}}} \prod_{v \in M_{K, \infty}} |f|_v \prod_{v \in M_{K, \infty}} |w|_v.$$

Moreover, if given $v \in M_{K, \infty}$, we let σ_v be the embedding corresponding to v , then

$$\prod_{v \in M_{K, \infty}} |w|_v = \prod_{v \text{ real}} |\sigma_v(x)|^{\frac{1}{d_K}} \prod_{v \text{ complex}} |\sigma_v(x)|^{\frac{2}{d_K}} = \left(\prod_{\sigma} |\sigma(w)| \right)^{\frac{1}{d_K}} \geq 1$$

where the last inequality is because w is an algebraic integer and $\prod_{\sigma} \sigma(w)$ is the constant term of the minimal polynomial of w . Thus from (5.14) it follows

$$(5.15) \quad \prod_{v \in M_{K, \infty}} |fh|_v \geq C^{-nd^{1+\frac{1}{n}}} \prod_{v \in M_{K, \infty}} |f|_v.$$

Using inequalities (5.12) and (5.15) in (5.13) we deduce

$$(5.16) \quad H(g) \geq c_3 C^{-nd^{1+\frac{1}{n}}} \prod_{v \in M_K} |f|_v = c_3 C^{-nd^{1+\frac{1}{n}}} H(f).$$

Thus, from (5.16), after relabelling C , the claim follows. \square

Multiplying (5.8) by $\mathcal{N}_K(\Delta)$ and using Claim 5.7 it follows that

$$(5.17) \quad \left(C^{-nd^{1+\frac{1}{n}}} H_K(f) \right)^{r-s} \mathcal{N}_K(\Delta) \leq C(K)^{(r-s)} \tilde{c}_1^{Ms} s!^{\frac{d_K}{2}} r^{\frac{d_K s}{2}} B^{Ms}.$$

Now we take logarithm in (5.17). By Theorem 3.20, there is some non-zero ideal \mathfrak{J} , relative prime with all the primes lying in \mathcal{P}_X , such that $\mathfrak{J}|\Delta$, and

$$(5.18) \quad \log(\mathcal{N}_K(\mathfrak{J})) \geq \frac{n!^{\frac{1}{n}}}{n+1} s^{1+\frac{1}{n}} (\log(s) - O_{n,K}(1) - n(\log(\beta) + \max\{\log(\log(\mathcal{N}_K(\mathfrak{q}))), 0\} + \log b(f))).$$

Recall that given $i \in \{1, \dots, u\}$, P_i is a non-singular $\mathbb{F}_{\mathfrak{p}_i}$ -point on $X_{\mathfrak{p}_i}$. So, by Lemma 3.14, there exists a non-negative $N_i = n!^{\frac{1}{n}} \frac{n}{n+1} s^{1+\frac{1}{n}} + O_n(s)$ such that $\mathfrak{p}_i^{N_i} |\Delta$. Hence, if we set $\mathfrak{J}' := \prod_{i=1}^u \mathfrak{p}_i^{N_i}$, then $\mathfrak{J}'|\Delta$, it is relative prime with \mathfrak{J} , and

$$(5.19) \quad \log(\mathcal{N}_K(\mathfrak{J}')) = n!^{\frac{1}{n}} \frac{n}{n+1} s^{1+\frac{1}{n}} \log(\mathcal{N}_K(\mathfrak{q})) + O_n(s \log(\mathcal{N}_K(\mathfrak{q}))).$$

On the one hand, bounding $\log(\mathcal{N}_K(\Delta))$ by means of (5.18) and (5.19), the logarithm of the left hand side of (5.17) has as lower bound

$$(5.20) \quad \begin{aligned} & \frac{n!^{\frac{1}{n}}}{n+1} s^{1+\frac{1}{n}} (\log(s) + n \log(\mathcal{N}_K(\mathfrak{q})) + O_{n,K}(1) - n(\log(\beta) + \max\{\log(\log(\mathcal{N}_K(\mathfrak{q}))), 0\} + \log b(f))) \\ & + O_n(s \log(\mathcal{N}_K(\mathfrak{q}))) + (r-s) \log(H_K(f)) - (r-s) nd^{1+\frac{1}{n}} \log(C). \end{aligned}$$

On the other hand, the logarithm of the right hand side of (5.17) is

$$(5.21) \quad (r-s) \log(C(K)) + Ms \log(\tilde{c}_1) + \frac{d_K}{2} \log(s!) + Ms \log(B) + \frac{sd_K}{2} \log(r).$$

Since $\log(s!) \leq s \log(s)$, $r = |\mathcal{B}[M]| = \binom{M+n+1}{n+1} \leq (M+1)^{n+1}$ and $M \gtrsim d$, by (5.5) it holds $\log(r) \lesssim_n \log(M) \lesssim_n \log(s)$, and so the terms $\frac{d_K}{2} \log(s!)$ and $\frac{sd_K}{2} \log(r)$ are majorized by $s^{1+\frac{1}{n}}$. Replacing (5.20) and (5.21) in (5.17) gives

$$(5.22) \quad \begin{aligned} & \frac{n!^{\frac{1}{n}}}{n+1} s^{1+\frac{1}{n}} (\log(s) + n \log(\mathcal{N}_K(\mathbf{q})) + O_{n,K}(1) - n(\log(\beta) + \max\{\log(\log(\mathcal{N}_K(\mathbf{q}))), 0\} + \log(b(f)))) \\ & \leq Ms \log(\tilde{c}_1) + Ms \log(B) - (r-s) \log(H_K(f)) + (r-s) \log(C_K) + (r-s)nd^{1+\frac{1}{n}} \log(C) + O_n(s \log(\mathcal{N}_K(\mathbf{q}))). \end{aligned}$$

We will now estimate the right hand side of (5.22). By (5.1), $M \gtrsim d^2$ and this implies

$$r-s = |\mathcal{B}[M-d]| = \binom{M-d+n+1}{n+1} = \frac{M^{n+1}}{(n+1)!} + O_n(dM^n),$$

hence from (5.4) we obtain

$$(5.23) \quad \frac{r-s}{Ms} = \frac{\frac{1}{(n+1)!} + O_n(dM^{-1})}{\frac{d}{n!} + O_n(d^2M^{-1})} = \frac{1}{d(n+1)} \frac{1 + O_n(dM^{-1})}{1 + O_n(dM^{-1})} = \frac{1}{d(n+1)} + O_n\left(\frac{1}{M}\right).$$

Since $\log(\tilde{c}_1)$ and $\frac{d^{\frac{1}{n}}n}{n+1} \log(C)$ are majorized by $\frac{d^{\frac{1}{n}}n}{n+1} O_{n,K}(1)$, and since by (5.1), $M \gtrsim \log(\mathcal{N}_K(\mathbf{q}))$, using (5.23) and that $M \gtrsim d^2 \gtrsim d^{1+\frac{1}{n}}$, the quotient by Ms of the right hand side of (5.22) is equal to

$$(5.24) \quad \begin{aligned} & \log(\tilde{c}_1) + \log(B) - \left(\frac{1}{d(n+1)} + O_n\left(\frac{1}{M}\right) \right) \log(H_K(f)) + \left(\frac{1}{d(n+1)} + O_n\left(\frac{1}{M}\right) \right) nd^{1+\frac{1}{n}} \log(C) \\ & + O_n\left(\frac{1}{M} \log(\mathcal{N}_K(\mathbf{q}))\right) = \log(B) - \left(\frac{1}{d(n+1)} + O_n\left(\frac{1}{M}\right) \right) \log(H_K(f)) + \frac{d^{\frac{1}{n}}n}{n+1} O_{n,K}(1). \end{aligned}$$

We will now estimate the left hand side of (5.22). Using the inequalities (5.5) and (5.6) the quotient by Ms in the left hand side of (5.22) is equal to

$$(5.25) \quad \begin{aligned} & \frac{n!^{\frac{1}{n}}}{n+1} \left(\frac{d^{\frac{1}{n}}}{n!^{\frac{1}{n}}} + O_n\left(\frac{d^2}{M}\right) \right) \left(\log(s) + n \log(\mathcal{N}_K(\mathbf{q})) + O_{n,K}(1) - n(\log(\beta) + \max\{\log(\log(\mathcal{N}_K(\mathbf{q}))), 0\} + \log(b(f))) \right) \\ & = \frac{d^{\frac{1}{n}}n}{n+1} \left(1 + O_n\left(\frac{d^{2-\frac{1}{n}}}{M}\right) \right) \left(\log(M) + \frac{1}{n} \log(d) + \log(\mathcal{N}_K(\mathbf{q})) + O_{n,K}(1) - \log(\beta) - \max\{\log(\log(\mathcal{N}_K(\mathbf{q}))), 0\} - \log(b(f)) \right) \\ & = \frac{d^{\frac{1}{n}}n}{n+1} \left(\log(M) + \log(\mathcal{N}_K(\mathbf{q})) - \log(\beta d^{-\frac{1}{n}}) + O_{n,K}(1) - \left(1 + O_n\left(\frac{d^{2-\frac{1}{n}}}{M}\right) \right) \log(b(f)) - \max\{\log(\log(\mathcal{N}_K(\mathbf{q}))), 0\} \right), \end{aligned}$$

where the terms $O_n(\frac{d^{2-\frac{1}{n}}}{M} \log(M))$, $O_n(\frac{d^{2-\frac{1}{n}}}{M} \log(\mathcal{N}_K(\mathbf{q})))$, $O_n(\frac{d^{2-\frac{1}{n}}}{M} \max\{\log(\log(\mathcal{N}_K(\mathbf{q}))), 0\})$, and $O_n(\frac{d^{2-\frac{1}{n}}}{M} \log(d))$ were merged in the term $O_{n,K}(1)$.

Replacing the inequalities (5.24) and (5.25) in (5.22) gives

$$(5.26) \quad \begin{aligned} & \frac{d^{\frac{1}{n}} n}{n+1} \left(\log(M) + \log(\mathcal{N}_K(\mathbf{q})) - \log(\beta d^{-\frac{1}{n}}) + O_{n,K}(1) - \left(O_n \left(\frac{d^{2-\frac{1}{n}}}{M} \right) + 1 \right) \log(b(f)) - \max\{\log(\log(\mathcal{N}_K(\mathbf{q}))), 0\} \right) \\ & \leq \log(B) - \left(\frac{1}{d(n+1)} + O_n \left(\frac{1}{M} \right) \right) \log(H_K(f)). \end{aligned}$$

The leading term in the right hand side (5.26) depends on the size of $H_K(f)$. In the case when $H_K(f)$ is small, namely $H_K(f) \leq B^{2d(n+1)}$, we use (5.1) to bound $M \geq d^{1-\frac{1}{n}} \log(B)$. From this it follows

$$\frac{\log(H_K(f))}{M} \leq \frac{2d(n+1) \log(B)}{M} \lesssim_n d^{\frac{1}{n}}.$$

Assumption 5.4(2) and Lemma 3.23 yield

$$(5.27) \quad b(f) \lesssim_{K,n} \max\{d^{[-2, \frac{4}{3}, -\frac{4}{3}, 2]} \log(H_K(f)), 1\}.$$

Then it holds that $d^{2-\frac{1}{n}} \frac{\log(b(f))}{M} \lesssim_{K,n} 1$, hence in (5.26) the terms $O_{K,n} \left(\frac{\log(H_K(f))}{M} \right)$ and $O_n \left(d^{2-\frac{1}{n}} \frac{\log(b(f))}{M} \right)$ can be merged in the term $O_{n,K}(1)$. Then, rearranging (5.26), we obtain

$$\log(M) \leq \frac{(n+1)}{nd^{\frac{1}{n}}} \log(B) - \log(\mathcal{N}_K(\mathbf{q})) - \frac{\log(H_K(f))}{d^{1+\frac{1}{n}} n} + \log(\beta d^{-\frac{1}{n}}) + O_{n,K}(1) + \log(b(f)) + \max\{\log(\log(\mathcal{N}_K(\mathbf{q}))), 0\}.$$

Hence we reach a contradiction if the implicit constant in (5.1) is large enough.

In the case when $H_K(f)$ is large, namely $H_K(f) \geq B^{2d(n+1)}$, let us take the implicit constant in (5.1) large enough such that $O_n \left(\frac{\log(H_K(f))}{M} \right)$ is bounded by $\frac{\log(H_K(f))}{4d(n+1)}$. Then the right hand side of (5.26) is

$$\log(B) - \frac{\log(H_K(f))}{d(n+1)} - O_n \left(\frac{\log(H_K(f))}{M} \right) \leq -\frac{\log(H_K(f))}{2d(n+1)} + \frac{\log(H_K(f))}{4d(n+1)} \leq -\frac{\log(H_K(f))}{4d(n+1)}.$$

Recalling that by (5.1), $M \gtrsim d^2$, we conclude that $O \left(\frac{d^{2-\frac{1}{n}}}{M} \right) = O(d^{-\frac{1}{n}})$. Then, rearranging (5.26) we obtain

$$(5.28) \quad \begin{aligned} \log(M) & \leq O_{n,K}(1) + \log(\beta d^{-\frac{1}{n}}) - \log(\mathcal{N}_K(\mathbf{q})) + \left(1 + O_n(d^{-\frac{1}{n}}) \right) \log(b(f)) \\ & \quad + \max\{\log(\log(\mathcal{N}_K(\mathbf{q}))), 0\} - \frac{\log(H_K(f))}{4nd^{1+\frac{1}{n}}}. \end{aligned}$$

By (5.27),

$$(5.29) \quad \begin{aligned} O_n(1) \log(b(f)) - \frac{H_K(f)}{4nd^{1+\frac{1}{n}}} & \leq O_{n,K}(1) \log(\max\{d^{[-2, \frac{4}{3}, -\frac{4}{3}, 2]} \log(H_K(f)), 1\}) - \frac{\log(H_K(f))}{4nd^{1+\frac{1}{n}}} \\ & \leq O_{n,K}(1) \max\{\log \log(H_K(f)) + \left[-2, \frac{4}{3}, -\frac{4}{3}, 2 \right] \log(d), 0\} - \frac{\log(H_K(f))}{4nd^{1+\frac{1}{n}}} \\ & \leq \max \left\{ 0, \left[-2, \frac{4}{3}, -\frac{4}{3}, 2 \right] \log(d) + O_{n,K}(1) \log \log(H_K(f)) - \frac{\log(H_K(f))}{4nd^{1+\frac{1}{n}}} \right\} \\ & \leq \max \left\{ 0, \left[-2, \frac{4}{3}, -\frac{4}{3}, 2 \right] \log(d) + \left(1 + \frac{1}{n} \right) \log(d) + O_{n,K}(1) \right\}. \end{aligned}$$

In the last line, we used that given $c > 0$, for any $x > 1$ we have $\log \log(x) - \frac{\log(x)}{c} \leq \log(c) + O(1)$ (see [11, Lemma 3.3.5]). Using (5.29) in (5.28) we get

$$\begin{aligned} \log(M) &\leq O_{n,K}(1) + \log(\beta d^{-\frac{1}{n}}) - \log(\mathcal{N}_K(\mathfrak{q})) + \max\{\log(\log(\mathcal{N}_K(\mathfrak{q}))), 0\} \\ &\quad + \max\left\{0, \left(\left[-1, \frac{7}{3}, -\frac{1}{3}, 3\right] + \frac{1}{n}\right) \log(d) + O_{n,K}(1)\right\} \\ &\leq O_{n,K}(1) + \max\left\{\log\left(\beta d^{-\frac{1}{n}}\right), \log(\beta d^{[-1, \frac{7}{3}, -\frac{1}{3}, 3]}) + O_{n,K}(1)\right\} - \log(\mathcal{N}_K(\mathfrak{q})) + \max\{\log(\log(\mathcal{N}_K(\mathfrak{q}))), 0\}. \end{aligned}$$

Hence if the implicit constant in (5.1) is large enough, we arrive at a contradiction. \square

Proof of Theorem 5.2. By Remark 5.3, we may assume that f is absolutely irreducible. Then, by Remark 3.15, we may assume, after multiplication by a non-zero scalar in \mathcal{O}_K that f satisfies conditions (1) and (2) in Assumption 5.4. This is enough to conclude the proof in the function field case.

For number fields, we use Proposition 4.2 to find integers a_0, \dots, a_n with $0 \leq a_i \leq d$ and an unit $\varepsilon \in \mathcal{O}_K^\times$ such that $f_1(X_0, \dots, X_n, X_{n+1}) := \varepsilon f(X_0 - a_0 X_{n+1}, \dots, X_n - a_n X_{n+1}, X_{n+1})$ satisfies all the conditions in Assumption 5.4. For each $i \in \{1, \dots, u\}$, let P'_i be the image of P_i under the linear transformation $(x_0, \dots, x_{n+1}) \mapsto (x_0 + a_0 x_{n+1}, \dots, x_n + a_n x_{n+1}, x_{n+1})$; it is a non-singular \mathbb{F}_p -rational point on $\mathcal{Z}(f_1)_p$. Then we can apply Theorem 5.2 to f_1 and P'_1, \dots, P'_u to conclude that for any $B' \geq 1$ there exists a homogeneous polynomial $g_1 \in \mathcal{O}_K[X_0, \dots, X_{n+1}]$ of degree

$$(5.30) \quad \begin{aligned} M \lesssim_{K,n} B' \frac{d^{[4, \frac{14}{3}, \frac{14}{3}, 0] - \frac{1}{n}} b(f_1) \max\{\log(\mathcal{N}_K(\mathfrak{q})), 1\}}{\mathcal{N}_K(\mathfrak{q}) H_K(f_1)^{\frac{1}{n} \frac{1}{d^{1+\frac{1}{n}}}}} + d^{1-\frac{1}{n}} \log(B' \mathcal{N}_K(\mathfrak{q})) \\ + d^{[4-\frac{1}{n}, 7, \frac{14}{3}-\frac{1}{n}, 3]} \left(\frac{\max\{\log(\mathcal{N}_K(\mathfrak{q})), 1\}}{\mathcal{N}_K(\mathfrak{q})} + 1 \right), \end{aligned}$$

not divisible by f_1 , which vanishes on $\mathcal{Z}(f_1)(K, B'; P'_1, \dots, P'_u)$. Thus, $g := g_1(X_0 + a_0 X_{n+1}, \dots, X_n + a_n X_{n+1}, X_{n+1})$ is a polynomial of degree at most M , vanishing on $X(K, \frac{B'}{(2d)^{d_K}}; P_1, \dots, P_u)$. Indeed, if $L(X_0, \dots, X_{n+1}) := (X_0 - a_0 X_{n+1}, \dots, X_n - a_n X_{n+1}, X_{n+1})$, by (2.3), it holds

$$H_K(L(x_0 : \dots : x_{n+1})) \leq (2d)^{d_K} H_K(x_0 : \dots : x_{n+1}).$$

Take $B' = (2d)^{d_K} B$. By Proposition 4.2, f_1 verifies $b(f) = b(f_1)$ and $H_K(f) \leq C'^{d^{1+\frac{1}{n}}} H_K(f_1)$. Replacing this in (5.30), it follows that M verifies the bound on the statement of the theorem. \square

As a consequence of Theorem 5.2, Lemma 3.23, Remark 3.15, and Bézout's theorem, we generalize [46, Theorem 1.2], [11, Corollary 3.1.2] and [45, Theorem 1.1]:

Corollary 5.8. *Let K be a global field of degree d_K . For any irreducible homogeneous polynomial $f \in \mathcal{O}_K[X_0, X_1, X_2]$ of degree d and for any $B \geq 1$, if $\mathcal{Z}(f) \subseteq \mathbb{P}_K^2$ is the corresponding curve, it holds*

$$N(\mathcal{Z}(f), K, B) \lesssim_K B^{\frac{2}{d}} \frac{d^{[4, \frac{14}{3}, \frac{14}{3}, 0]} b(f)}{H_K(f)^{\frac{1}{d^2}}} + d \log(B) + d^{[4, 8, \frac{14}{3}, 4]} \lesssim_K d^{[4, 8, \frac{14}{3}, 4]} B^{\frac{2}{d}}.$$

Now we are in position to prove Theorem 1.8 of the introduction by extending Corollary 5.8 to general curves using a projection argument. This is accomplished by means of an effective change of variables, as in [31]. The approach presented here differs from the one given in [11, §5].

Theorem 5.9. *Let K be a global field of degree d_K . Given $n > 1$, for any integral projective curve $C \subseteq \mathbb{P}_K^n$ of degree d it holds*

$$N(C, K, B) \lesssim_{K,n} d^{[4, 8, \frac{14}{3}, 4]} B^{\frac{2}{d}}.$$

Proof. We are going to reduce Theorem 5.9 to the case of a planar curve, by means of an adequate change of variables, as in [31]. More specifically, by the method of the proof of [31, Theorem 4.3] there exists linear forms $L_0, L_1, L_2 \in \mathcal{O}_K[X_0, \dots, X_n]$ of height bounded by $\lesssim_{k,n} d^{n-2}$ such that $\varphi : C \rightarrow \mathbb{P}^2$ given by $\varphi(\mathbf{x}) := (L_0(\mathbf{x}) : L_1(\mathbf{x}) : L_2(\mathbf{x}))$ is a finite morphism with fibres of size at most d and $\varphi(C)$ is a geometrically integral projective curve defined over K of degree $\deg(\varphi(C)) \leq d$. Furthermore, by (2.3), any point $\mathbf{x} \in C(K)$ of K -relative height at most B verifies $H_K(\varphi(\mathbf{x})) \lesssim_{k,n} d^{n-2} H_K(\mathbf{x}) \lesssim_{k,n} d^{n-2} B$. Hence, there is some constant $c = c(K, n)$ such that it holds

$$(5.31) \quad N(C, K, B) \leq dN(\varphi(C), K, cd^{n-2}B).$$

By Corollary 5.8, we have

$$(5.32) \quad N(\varphi(C), K, cd^{n-2}B) \lesssim_{K,n} d^{[4,8, \frac{14}{3}, 4]} (cd^{n-2}B)^{\frac{2}{d}} \lesssim_{K,n} d^{[4,8, \frac{14}{3}, 4]} B^{\frac{2}{d}}.$$

The conclusion of Theorem 5.9 follows from (5.31) and (5.32). \square

Remark 5.10. By Remark 5.3 and Bézout's theorem, if $f \in \mathcal{O}_K[X_0, X_1, X_2]$ is irreducible but not absolutely irreducible, we have $N(\mathcal{Z}(f), K, B) \leq d^2$. In consequence, the argument of the proof of Theorem 5.9 shows that for any integral projective curve $C \subseteq \mathbb{P}_K^n$ of degree d , which is not geometrically irreducible, it holds $N(C, K, B) \leq d^2$.

5.2. The affine case. In Theorem 5.2 we found a hypersurface of small degree vanishing on the rational points up to height B of a given projective hypersurface X . Now we will obtain a similar result in the affine case. For this purpose, given a number field K , for any $x \in \mathcal{O}_K$ we define

$$|\overline{x}| := \max_{\sigma: K \hookrightarrow \mathbb{C}} |\sigma(x)|,$$

and

$$[B]_{\mathcal{O}_K} := \{x \in \mathcal{O}_K : |\overline{x}| \leq B^{\frac{1}{d_K}}\}.$$

When K is a function field, we define

$$[B]_{\mathcal{O}_K} := \{x \in \mathcal{O}_K : |x|_{v_\infty} \leq B^{\frac{1}{d_K}}\}.$$

Remark 5.11. The previous definition is to ensure that given $\mathbf{x} \in \mathbb{P}^{n+1}(K)$ with $H_K(\mathbf{x}) \leq B$, by Proposition 2.2 it follows that \mathbf{x} has a lift $(y_1, \dots, y_{n+2}) \in \mathbb{A}^{n+2}(K)$ which lies in the box $[c_1 B]_{\mathcal{O}_K}^{n+2}$.

Similarly to the projective case, for any $X \subseteq \mathbb{A}_K^n$ affine variety, given $B \in \mathbb{R}_{>0}$ we will use the notation

$$X_{\text{aff}}(\mathcal{O}_K, B) := X(K) \cap [B]_{\mathcal{O}_K}^n,$$

Moreover, if $\{\mathfrak{p}_1, \dots, \mathfrak{p}_u\}$ is a subset of primes such that for all $1 \leq i \leq u$ we let P_i be a non-singular $\mathbb{F}_{\mathfrak{p}_i}$ -point on $X_{\mathfrak{p}_i}$, we denote

$$X_{\text{aff}}(\mathcal{O}_K, B; P_1, \dots, P_u) := \{\mathbf{x} \in X_{\text{aff}}(\mathcal{O}_K, B) : \mathbf{x} \text{ specialises to } P_i \text{ in } X_{\mathfrak{p}_i} \text{ for all } i\}.$$

We denote

$$N_{\text{aff}}(X, \mathcal{O}_K, B) := |X_{\text{aff}}(\mathcal{O}_K, B)|.$$

We need the next application of the bound of Bombieri and Vaaler presented in Corollary 2.5, which will be useful also in Lemma 6.1 and Lemma 6.2. A variant of this for $K = \mathbb{Q}$ and $K = \mathbb{F}_q(T)$ can be found in [24, Theorem 4] and [45, Lemma 3.5] with different proofs.

Lemma 5.12. *Let $f \in \mathcal{O}_K[X_1, \dots, X_{n+2}]$ be a homogeneous polynomial of degree d .*

- (1) *Let $B \geq 1$. Then, either $H_{K,\text{aff}}(f) \lesssim_K \binom{d+n+1}{n+1}^{d_K \binom{d+n+1}{n+1}} B^{d \binom{d+n+1}{n+1}}$ or there exists a homogeneous polynomial g of degree d , vanishing on all $\mathcal{Z}(f)_{\text{aff}}(\mathcal{O}_K, B)$, and not divisible by f ;*
- (2) *Let $B \geq 1$. Then, either $H_{K,\text{aff}}(f) \lesssim_K \binom{d+n+1}{n+1}^{d_K \binom{d+n+1}{n+1}} B^{d \binom{d+n+1}{n+1}}$, or there exists a homogeneous polynomial g of degree d , vanishing on all $\{\mathbf{x} = (x_1, \dots, x_{n+2}) \in [B]_{\mathcal{O}_K}^{n+2} : x_1 = 1 \text{ and } f(1, x_2, \dots, x_{n+2}) = 0\}$, and not divisible by f .*

Proof. Let $R := \binom{d+n+1}{n+1}$ and let $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(N)} \in \mathcal{Z}(f)_{\text{aff}}(\mathcal{O}_K, B)$. Consider the $N \times R$ matrix M whose i^{th} row consists of the R possible monomials of degree d in the variables X_1, \dots, X_{n+2} evaluated in $\mathbf{x}^{(i)}$. We have

$$H_K(M) = \left(\prod_{v \in M_{K,\infty}} \max_{i,j} \{1, |M_{ij}|_v\} \right)^{d_K} \leq B^d.$$

Denoting by \mathbf{f} the vector in \mathcal{O}_K^{n+2} consisting of the coefficients of f , it holds $M\mathbf{f} = 0$. Since $\mathbf{f} \neq 0$, the matrix M must have rank at most $R - 1$. Hence, by Corollary 2.5 the linear system $M\mathbf{g}$ has a non-zero $\mathbf{g} \in \mathcal{O}_K^R$ verifying

$$H_{K,\text{aff}}(\mathbf{g}) \leq C(K) \left(R^{\frac{d_K}{2}} H_K(M) \right)^{\frac{\text{rank}(M)}{R - \text{rank}(M)}} \leq C(K) \binom{d+n+1}{n+1}^{d_K \binom{d+n+1}{n+1}} B^{d \binom{d+n+1}{n+1}}.$$

Let $g \in \mathcal{O}_K[X_1, \dots, X_{n+2}]$ be the polynomial of degree d corresponding to \mathbf{g} . Then, either f divides g , in which case $H_K(f) = H_K(g) = H_K(\mathbf{g}) \leq H_{K,\text{aff}}(\mathbf{g}) \leq \binom{d+n+1}{n+1}^{d_K \binom{d+n+1}{n+1}} B^{d \binom{d+n+1}{n+1}}$, or g is a non-zero homogeneous polynomial of degree d , not divisible by f and vanishing on all $\mathcal{Z}(f)_{\text{aff}}(\mathcal{O}_K, B)$. This proves Lemma 5.12(1). The proof of Lemma 5.12(2) is analogous. \square

Lemma 5.13. *Let K be a global field of degree d_K . Let $f = \sum_I a_I \mathbf{X}^I \in \mathcal{O}_K[X_1, \dots, X_{n+2}]$ be an absolutely irreducible homogeneous polynomial of degree d . If we let $1 \leq y \leq H_{K,\text{aff}}(f)$ then*

$$d^{[4, \frac{14}{3}, \frac{14}{3}, 0] - \frac{1}{n}} \frac{b(f)}{H_{K,\text{aff}}(f)^{\frac{1}{n} \frac{1}{d^{1+\frac{1}{n}}}}} \lesssim_{n,K} d^{[2, 6, 2, 2] - \frac{1}{n}} \frac{\log(y) + d^{[2, 1+\frac{1}{n}, \frac{8}{3}, 1+\frac{1}{n}]} y^{\frac{1}{n} \frac{1}{d^{1+\frac{1}{n}}}}}{y^{\frac{1}{n} \frac{1}{d^{1+\frac{1}{n}}}}}.$$

Proof. The proof is similar to the one of [11, Lemma 4.2.3] for number fields, and to the one of [45, Lemma 3.4] for function fields, but instead of using [11, Corollary 3.2.3] or [45, Lemma 2.3] one uses Lemma 3.23. \square

Now we use Lemma 5.12(1) and Lemma 5.13 to prove the following generalization of [11, Proposition 4.2.1] and [45, Lemma 3.6]:

Theorem 5.14. *Let K be a global field of degree d_K . Let $n > 0$ be an integer. Let $X \subseteq \mathbb{A}_K^{n+1}$ be an integral hypersurface of degree $d > 0$, defined by an irreducible polynomial $f \in \mathcal{O}_K[X_1, \dots, X_{n+1}]$ of degree d . For each i write f_i for the degree i homogeneous part of f . Let $\{\mathfrak{p}_1, \dots, \mathfrak{p}_u\}$ be (a possibly empty) subset of primes and P_i be a non-singular $\mathbb{F}_{\mathfrak{p}_i}$ -point on $X_{\mathfrak{p}_i}$ for each $i \in \{1, \dots, u\}$. Set $\mathfrak{q} := \prod_{i=1}^u \mathfrak{p}_i$ if $u \geq 1$ and $\mathfrak{q} = (1)$ if $u = 0$. Fix $B \geq 1$. Then there is a polynomial $g \in \mathcal{O}_K[X_1, \dots, X_{n+1}]$ of degree*

$$M \lesssim_{K,n} B^{\frac{1}{d^{1+\frac{1}{n}}}} d^{[2, 6, 2, 2] - \frac{1}{n}} \frac{\min\{\log(H_K(f_d)) + d \log(B) + d^{[2, 1+\frac{1}{n}, \frac{8}{3}, 1+\frac{1}{n}]} d^{[2, -\frac{4}{3}, \frac{8}{3}, -2]} b(f)\} \max\{\log(\mathcal{N}_K(\mathfrak{q})), 1\}}{H_K(f_d)^{\frac{1}{n} \frac{1}{d^{1+\frac{1}{n}}}} \mathcal{N}_K(\mathfrak{q})} + d^{1-\frac{1}{n}} \log(B \mathcal{N}_K(\mathfrak{q})) + d^{[4-\frac{1}{n}, 7, \frac{14}{3}-\frac{1}{n}, 3]} \left(\frac{\max\{\log(\mathcal{N}_K(\mathfrak{q})), 1\}}{\mathcal{N}_K(\mathfrak{q})} + 1 \right),$$

not divisible by f , and vanishing on all $X_{\text{aff}}(\mathcal{O}_K, B; P_1, \dots, P_u)$.

Proof. The proof follows the strategy devised in [17, Remark 2.3] and developed in [11, Proposition 4.2.1]. Recall that \mathbb{k} denotes the field \mathbb{Q} if K is a number field, or $\mathbb{F}_q(T)$ if K is a function field. By Remark 5.3 and Remark 3.15, we may assume, after multiplication by a non-zero scalar in \mathcal{O}_K that $f = \sum_I a_I \mathbf{X}^I$ is absolutely irreducible and it is \mathfrak{p} -primitive for any prime \mathfrak{p} of \mathcal{O}_K with $\mathcal{N}_K(\mathfrak{p}) > c_2$, and $H_{K,\text{aff}}(f) \leq c_1^{d_K} H_K(f)$.

For each $L \in \mathcal{O}_K$ we consider the polynomial $F_L \in \mathcal{O}_K[X_1, \dots, X_{n+2}]$ given by $F_L(X_1, \dots, X_{n+2}) := \sum_{i=0}^d L^i f_i X_{n+2}^{d-i}$. Then F_L is an absolutely irreducible homogeneous polynomial of degree d . Furthermore, if we denote by X_L the projective hypersurface in \mathbb{P}_K^{n+1} of degree d defined by F_L , we see that each point $(x_1, \dots, x_{n+1}) \in X(\mathcal{O}_K)$ gives a K -rational point (x_1, \dots, x_{n+1}, L) in X_L . Given $1 \leq i \leq u$, we denote $\tilde{P}_i := (P_i, L) \in (X_L)_{\mathfrak{p}_i}(\mathbb{F}_{\mathfrak{p}_i})$. If $\mathfrak{p}_i \nmid L$, then we see that \tilde{P}_i is a non-singular $\mathbb{F}_{\mathfrak{p}_i}$ -point on $(X_L)_{\mathfrak{p}_i}$.

Observe that given a subset of \mathcal{O}_K^{n+1} of size $\lesssim_K 1$, there exists a $g \in \mathcal{O}_K[X_1, \dots, X_{n+1}]$ of degree $\lesssim_{K,n} 1$ vanishing on it which is not divisible by f . Applying this to the subset $[B]_{\mathcal{O}_K}^{n+1}$ with $B \lesssim_K 1$, we see that it is enough to prove Theorem 5.14 when $B \gtrsim_K 1$.

Let \mathcal{P}' be the subset of primes $L \in \mathcal{O}_k$ with $H_k(L) \in [(\frac{B}{2})^{\frac{1}{d_K}}, B^{\frac{1}{d_K}}]$ such that $\mathfrak{q} = \prod_{i=1}^u \mathfrak{p}_i \nmid L$. First let us suppose that $|\mathcal{P}'| \leq \frac{1}{2} |\{L \in \mathcal{O}_k \text{ prime} : H_k(L) \in [(\frac{B}{2})^{\frac{1}{d_K}}, B^{\frac{1}{d_K}}]\}|$. It follows that

$$\sum_{\mathfrak{p}|\mathfrak{q}} \frac{\log(\mathcal{N}_K(\mathfrak{p}))}{\mathcal{N}_K(\mathfrak{p})} \gtrsim_K \log(B) \gtrsim_K \sum_{\substack{\mathfrak{p} \\ \mathcal{N}_K(\mathfrak{p}) \leq B}} \frac{\log(\mathcal{N}_K(\mathfrak{p}))}{\mathcal{N}_K(\mathfrak{p})},$$

where the second inequality is due to (2.11). Moreover, for any prime $\mathfrak{p}_i | \mathfrak{q}$ by definition any element of the subset $X_{\text{aff}}(\mathcal{O}_K, B; P_1, \dots, P_u)$ reduces exactly to one element modulo \mathfrak{p}_i , namely P_i . Then, by the larger sieve of Gallagher as in [31, Lemma 3.2], it holds that $|X_{\text{aff}}(\mathcal{O}_K, B; P_1, \dots, P_u)| \lesssim_K 1$. Hence, once again, there exists $g \in \mathcal{O}_K[X_1, \dots, X_{n+1}]$ of degree $\lesssim_K 1$ vanishing on $X_{\text{aff}}(\mathcal{O}_K, B; P_1, \dots, P_u)$ which is not divisible by f .

Then, we may suppose that

$$(5.33) \quad |\mathcal{P}'| \geq \frac{1}{2} |\{L \in \mathcal{O}_k \text{ prime} : H_k(L) \in [(\frac{B}{2})^{\frac{1}{d_K}}, B^{\frac{1}{d_K}}]\}|.$$

We will split the rest of the proof in two cases: when there exists $L \in \mathcal{P}'$ for which $L \nmid f_0$ and when for all $L \in \mathcal{P}'$, $L | f_0$.

In the first case, F_L is \mathfrak{p} -primitive for all primes of norm at least c_2 . Moreover, for all i , \tilde{P}_i is a non-singular $\mathbb{F}_{\mathfrak{p}_i}$ -point on $(X_L)_{\mathfrak{p}_i}$. We apply Theorem 5.2 to F_L to find a homogeneous polynomial $G_L \in \mathcal{O}_K[X_1, \dots, X_{n+2}]$ of degree

$$(5.34) \quad M \lesssim_{K,n} B^{\frac{n+1}{nd}} \frac{d^{[4, \frac{14}{3}, \frac{14}{3}, 0] - \frac{1}{n}} b(F_L) \max\{\log(\mathcal{N}_K(\mathfrak{q})), 1\}}{H_K(F_L)^{\frac{1}{n} \frac{1}{d^{1+\frac{1}{n}}}} \mathcal{N}_K(\mathfrak{q})} + d^{1-\frac{1}{n}} \log(B \mathcal{N}_K(\mathfrak{q})) \\ + d^{[4-\frac{1}{n}, 7, \frac{14}{3}-\frac{1}{n}, 3]} \left(\frac{\max\{\log(\mathcal{N}_K(\mathfrak{q})), 1\}}{\mathcal{N}_K(\mathfrak{q})} + 1 \right),$$

not divisible by F_L , and vanishing on all $X_L(K, B; \tilde{P}_1, \dots, \tilde{P}_u)$. By Remark 3.15, there exists $\lambda \in \mathcal{O}_K^\times$ such that $H_{K,\text{aff}}(\lambda F_L) \leq c_1^{d_K} H_K(F_L)$. Since $b(F_L) = b(\lambda F_L)$, inequality (5.34) gives

$$(5.35) \quad M \lesssim_{K,n} B^{\frac{n+1}{nd}} \frac{d^{[4, \frac{14}{3}, \frac{14}{3}, 0] - \frac{1}{n}} b(\lambda F_L) \max\{\log(\mathcal{N}_K(\mathfrak{q})), 1\}}{H_{K,\text{aff}}(\lambda F_L)^{\frac{1}{n} \frac{1}{d^{1+\frac{1}{n}}}} \mathcal{N}_K(\mathfrak{q})} + d^{1-\frac{1}{n}} \log(B \mathcal{N}_K(\mathfrak{q})) \\ + d^{[4-\frac{1}{n}, 7, \frac{14}{3}-\frac{1}{n}, 3]} \left(\frac{\max\{\log(\mathcal{N}_K(\mathfrak{q})), 1\}}{\mathcal{N}_K(\mathfrak{q})} + 1 \right).$$

From the definition of F_L we see that for all $v \in M_K$ it holds

$$|\lambda F_L|_v = \max_i |\lambda L^i f_i X_{n+2}^{d-i}|_v \geq |\lambda L^d f_d|_v = |\lambda|_v |L^d|_v |f_d|_v.$$

Then

$$(5.36) \quad H_{K,\text{aff}}(\lambda F_L) = \left(\prod_{v \in M_{K,\infty}} \max\{1, |\lambda F_L|_v\} \right)^{d_K} \geq \left(\prod_{v \in M_{K,\infty}} |\lambda|_v |L^d|_v |f_d|_v \right)^{d_K} \geq H_K(L^d) H_K(f_d) \geq \frac{B^d}{2^d} H_K(f_d).$$

Hence, from Lemma 5.13 we have the bound

$$d^{[4, \frac{14}{3}, \frac{14}{3}, 0] - \frac{1}{n}} \frac{b(\lambda F_L)}{H_{K,\text{aff}}(\lambda F_L)^{\frac{1}{n} \frac{1}{d^{1+\frac{1}{n}}}}} \lesssim_{n,K} \left(\frac{B}{2} \right)^{-\frac{1}{nd}} \frac{d^{[2, 6, 2, 2] - \frac{1}{n}} \log(H_K(f_d)) + d \log(B) + d^{[2, 1+\frac{1}{n}, \frac{8}{3}, 1+\frac{1}{n}]} }{H_K(f_d)^{\frac{1}{n} \frac{1}{d^{1+\frac{1}{n}}}}}$$

Replacing this on the first summand on the right hand side of (5.35), it follows that this summand is at most

$$(5.37) \quad \lesssim_{n,K} B^{\frac{1}{d^n}} d^{[2,6,2,2]-\frac{1}{n}} \frac{(\log(H_K(f_d)) + d \log(B) + d^{[2,1+\frac{1}{n}, \frac{8}{3}, 1+\frac{1}{n}]}) \max\{\log(\mathcal{N}_K(\mathfrak{q})), 1\}}{H_K(f_d)^{\frac{1}{n}} d^{\frac{1}{d^{1+\frac{1}{n}}}} \mathcal{N}_K(\mathfrak{q})}.$$

On the other hand, since the reduction modulo \mathfrak{p} of F_L for all primes \mathfrak{p} not dividing L is absolutely irreducible whenever f is so, and the number of primes in \mathcal{O}_K dividing L is at most d_K , it follows from Definition 3.18 that $b(F_L)$ coincides with $b(f)$ up to a factor $O_K(1)$.

This and (5.36) give that the first summand on the right hand side (5.35) is bounded by

$$(5.38) \quad B^{\frac{1}{d^n}} d^{[4, \frac{14}{3}, \frac{14}{3}, 0]-\frac{1}{n}} \frac{b(f) \max\{\log(\mathcal{N}_K(\mathfrak{q})), 1\}}{H_K(f_d)^{\frac{1}{n}} d^{\frac{1}{d^{1+\frac{1}{n}}}} \mathcal{N}_K(\mathfrak{q})}.$$

By inequalities (5.37) and (5.38), it follows that the polynomial G_L has degree at most

$$(5.39) \quad M \lesssim_{K,n} B^{\frac{1}{d^n}} d^{[2,6,2,2]-\frac{1}{n}} \frac{\min\{\log(H_K(f_d)) + d \log(B) + d^{[2,1+\frac{1}{n}, \frac{8}{3}, 1+\frac{1}{n}]} d^{[2, -\frac{4}{3}, \frac{8}{3}, -2]} b(f)\} \max\{\log(\mathcal{N}_K(\mathfrak{q})), 1\}}{H_K(f_d)^{\frac{1}{n}} d^{\frac{1}{d^{1+\frac{1}{n}}}} \mathcal{N}_K(\mathfrak{q})} \\ + d^{1-\frac{1}{n}} \log(B \mathcal{N}_K(\mathfrak{q})) + d^{[4-\frac{1}{n}, 7, \frac{14}{3}-\frac{1}{n}, 3]} \left(\frac{\max\{\log(\mathcal{N}_K(\mathfrak{q})), 1\}}{\mathcal{N}_K(\mathfrak{q})} + 1 \right).$$

Thus the polynomial $g(X_1, \dots, X_{n+1}) := G_L(X_1, \dots, X_{n+1}, L)$ has degree bounded by the right hand side of (5.39), it is not divisible by f and it vanishes on the subset

$$\mathcal{A} := \{\mathbf{x} \in \mathcal{Z}(f) \cap \mathcal{O}_K^{n+1} : \mathbf{x} \text{ specialises to } P_i \text{ for all } i \text{ and } H_K(\mathbf{x} : L) \leq B\}.$$

Now we will see that $\mathcal{Z}(f)_{\text{aff}}(\mathcal{O}_K, B; P_1, \dots, P_u)$ is contained in \mathcal{A} . For that, it is enough to see that for $\mathbf{x} \in [B]_{\mathcal{O}_K}^{n+1}$, it holds $H_K(\mathbf{x} : L) \leq B$. In the case when K is a number field, for any $v \in M_{K,\infty}$, $|L|_v = |L|_{\infty}^{\frac{n_v}{d_K}} \leq B^{\frac{n_v}{d_K^2}}$. Then, for all $(x_1, \dots, x_{n+1}) \in [B]_{\mathcal{O}_K}^{n+1}$, we have

$$(5.40) \quad H_K(x_1 : \dots : x_{n+1} : L) = \left(\prod_{v \in M_K} \max_i \{|x_i|_v, |L|_v\} \right)^{d_K} \leq \left(\prod_{v \in M_{K,\infty}} \max \left\{ |\sigma_v(x_i)|^{\frac{n_v}{d_K}}, B^{\frac{n_v}{d_K^2}} \right\} \right)^{d_K} \leq \left(\prod_{v \in M_{K,\infty}} B^{\frac{n_v}{d_K^2}} \right)^{d_K} = B.$$

In the case when K is a function field, since v_{∞} was chosen above the place $|\cdot|_{\infty}$, it holds $|L|_{v_{\infty}} = |L|_{\infty}^{\frac{n_{v_{\infty}}}{d_K}} \leq B^{\frac{n_{v_{\infty}}}{d_K^2}} \leq B^{\frac{1}{d_K}}$, where $n_{v_{\infty}}$ is the degree of the finite extension $K_{v_{\infty}}/\mathbb{k}_{\infty}$, with $K_{v_{\infty}}$ and \mathbb{k}_{∞} the completions of K and \mathbb{k} with respect to v_{∞} and $|\cdot|_{\infty}$, respectively. Then, for all $(x_1, \dots, x_{n+1}) \in [B]_{\mathcal{O}_K}^{n+1}$, we have

$$(5.41) \quad H_K(x_1 : \dots : x_{n+1} : L) = \left(\prod_{v \in M_K} \max_i \{|x_i|_v, |L|_v\} \right)^{d_K} \leq \left(\max \left\{ |x_i|_{v_{\infty}}, B^{\frac{1}{d_K}} \right\} \right)^{d_K} \leq B.$$

This ends the proof in the case when there exists a prime element $L \in \mathcal{P}'$ verifying that $L \nmid f_0$.

Now we must deal with the other case, namely, when $\prod_{L \in \mathcal{P}'} L \mid f_0$. If $f_0 \neq 0$, using (2.4) we deduce

$$\sum_{L \in \mathcal{P}'} \log(\mathcal{N}_K(L)) = d_K \sum_{L \in \mathcal{P}'} \log(L) \leq \log(\mathcal{N}_K(f_0)) \leq \log(H_K(f_0)).$$

Since

$$H_K(f_0) \leq H_{K,\text{aff}}(f_0) = \left(\prod_v \max\{1, |f_0|_v\} \right)^{d_K} \leq \left(\prod_v \max\{1, |f_i|_v\} \right)^{d_K} = H_{K,\text{aff}}(f),$$

we deduce

$$(5.42) \quad d_K \sum_{L \in \mathcal{P}'} \log(L) \leq \log(H_{K,\text{aff}}(f)).$$

Now, if $H_{K,\text{aff}}(f) \gtrsim_K \binom{d+n+1}{n+1}^{d_K \binom{d+n+1}{n+1}} B^{d \binom{d+n+1}{n+1}}$, we are done by Lemma 5.12(1). Otherwise, by (5.42) we get

$$d_K \sum_{L \in \mathcal{P}'} \log(L) \leq d_K \binom{d+n+1}{n+1} \log \left(\binom{d+n+1}{n+1} \right) + d \binom{d+n+1}{n+1} \log(B) + O_K(1).$$

By (5.33) and inequality (2.12) for $K = \mathbb{k}$, there exists a constant c depending only on \mathbb{k} , such that

$$(5.43) \quad d_K c B^{\frac{1}{d_K}} \leq d \binom{d+n+1}{n+1} \log(B) + d_K \binom{d+n+1}{n+1} \log \left(\binom{d+n+1}{n+1} \right) + O_K(1).$$

From (5.43) it follows that B is bounded by a polynomial in d , and hence $B^{\frac{1}{ndn}} \lesssim_{n,K} 1$.

Let us take $L = 1$ and consider the polynomial F_1 . Then Theorem 5.2 for F_1 gives a homogeneous polynomial $G_1 \in \mathcal{O}_K[X_1, \dots, X_{n+2}]$ of degree

$$(5.44) \quad M \lesssim_{K,n} B^{\frac{1}{dn}} d^{[4, \frac{14}{3}, \frac{14}{3}, 0] - \frac{1}{n}} \frac{b(F_1) \max\{\log(\mathcal{N}_K(\mathbf{q})), 1\}}{H_K(F_1)^{\frac{1}{n} \frac{1}{d^{1+\frac{1}{n}}}} \mathcal{N}_K(\mathbf{q})} + d^{1-\frac{1}{n}} \log(B \mathcal{N}_K(\mathbf{q})) \\ + d^{[4-\frac{1}{n}, 7, \frac{14}{3}-\frac{1}{n}, 3]} \left(\frac{\max\{\log(\mathcal{N}_K(\mathbf{q})), 1\}}{\mathcal{N}_K(\mathbf{q})} + 1 \right),$$

not divisible by F_1 , and vanishing on all $X_1(K, B; \tilde{P}_1, \dots, \tilde{P}_u)$. Now we use Remark 3.15 to find $\lambda \in \mathcal{O}_K^\times$ such that $H_{K,\text{aff}}(\lambda F_1) \leq c_1^{d_K} H_K(F_1)$. The same argument used to prove inequality (5.36) gives $H_{K,\text{aff}}(\lambda F_1) \geq H_K(f_d)$. This, together with the fact that $b(F_1) = b(f)$, Lemma 3.23 and Lemma 5.13 allows us to conclude the bound

$$(5.45) \quad d^{[4, \frac{14}{3}, \frac{14}{3}, 0] - \frac{1}{n}} \frac{b(\lambda F_1)}{H_{K,\text{aff}}(\lambda F_1)^{\frac{1}{n} \frac{1}{d^{1+\frac{1}{n}}}}} \lesssim_{n,K} d^{[2, 6, 2, 2] - \frac{1}{n}} \frac{\min\{\log H_K(f_d) + d^{[2, 1+\frac{1}{n}, \frac{8}{3}, 1+\frac{1}{n}]} b(f), d^{[2, -\frac{4}{3}, \frac{8}{3}, -2]} b(f)\}}{H_K(f_d)^{\frac{1}{n} \frac{1}{d^{1+\frac{1}{n}}}}}.$$

From (5.40) and (5.41) with $L = 1$, and from (5.44) and (5.45), it follows that $g(X_1, \dots, X_{n+1}) := G_1(X_1, \dots, X_{n+1}, 1)$ verifies the conclusion of Theorem 5.14.

In order to finish the proof of Theorem 5.14, it remains to cover the case when $f_0 = 0$, namely $f(0) = 0$. By the Combinatorial nullstellensatz (see [1, Theorem 1.2]), we may find $\mathbf{A} = (a_1, \dots, a_{n+1}) \in \mathcal{O}_{\mathbb{k}}^{n+1}$ with $f(a_1, \dots, a_{n+1}) \neq 0$ and $H_{\mathbb{k}}(a_i) \leq d$ for all $1 \leq i \leq n+1$. Let us consider the polynomial $\tilde{f}(\mathbf{X}) := f(\mathbf{X} + \mathbf{A})$; we see that $\tilde{f}(0) \neq 0$, $H_K(f_d) = H_K(\tilde{f}_d)$, and $b(\tilde{f}) = b(f)$. Thus, reasoning as in the two previous cases with \tilde{f} in place of f and $\tilde{B} = B + d$ in place of B we obtain a polynomial $\tilde{g} \in \mathcal{O}_K[X_1, \dots, X_{n+1}]$ of degree

$$M \sim_{K,n} \tilde{B}^{\frac{1}{dn}} d^{[2, 6, 2, 2] - \frac{1}{n}} \frac{\min\{\log(H_K(f_d)) + d \log(\tilde{B}) + d^{[2, 1+\frac{1}{n}, \frac{8}{3}, 1+\frac{1}{n}]} b(f), d^{[2, -\frac{4}{3}, \frac{8}{3}, -2]} b(f)\} \max\{\log(\mathcal{N}_K(\mathbf{q})), 1\}}{H_K(f_d)^{\frac{1}{n} \frac{1}{d^{1+\frac{1}{n}}}} \mathcal{N}_K(\mathbf{q})} \\ + d^{1-\frac{1}{n}} \log(\tilde{B} \mathcal{N}_K(\mathbf{q})) + d^{[4-\frac{1}{n}, 7, \frac{14}{3}-\frac{1}{n}, 3]} \left(\frac{\max\{\log(\mathcal{N}_K(\mathbf{q})), 1\}}{\mathcal{N}_K(\mathbf{q})} + 1 \right),$$

not divisible by \tilde{f} , and vanishing on $\mathcal{Z}(\tilde{f})_{\text{aff}}(\mathcal{O}_K, \tilde{B}; P_1, \dots, P_u)$. Then $g(\mathbf{X}) = \tilde{g}(\mathbf{X} - \mathbf{A})$ verifies the conclusion of Theorem 5.14. \square

As a consequence of Theorem 5.14, Bézout's theorem, and the estimate $\frac{\log(H_K(f_d))}{H_K(f_d)^{\frac{1}{d^2}}} \lesssim d^2$, we get the next improvement of the bounds of Bombieri and Pila.

Corollary 5.15. *Let K be a global field of degree d_K . For any irreducible polynomial $f \in \mathcal{O}_K[X_1, X_2]$ of degree d and any $B \geq 1$, if $Z(f) \subseteq \mathbb{A}_K^2$ denotes the corresponding curve, it holds*

$$N_{\text{aff}}(Z(f), \mathcal{O}_K, B) \lesssim_K B^{\frac{1}{d}} \frac{\min\{d^{[2,6,2,2]} \log(H_K(f_d)) + d^{[3,7,3,3]} \log(B) + d^{[4,8,\frac{14}{3},4]}, d^{[4,\frac{14}{3},\frac{14}{3},0]} b(f)\}}{H_K(f_d)^{\frac{1}{d^2}}} \\ + d \log(B) + d^{[4,8,\frac{14}{3},4]} \lesssim_K d^{[3,7,3,3]} B^{\frac{1}{d}} (\log(B) + d^{[1,1,\frac{5}{3},1]}).$$

Now we are in position to prove Theorem 1.9 of the introduction by extending Corollary 5.15 to general curves using a projection argument. This is accomplished by means of an effective change of variables, as in [31].

We also have:

Theorem 5.16. *Let K be a global field of degree d_K . Given $n > 1$, for any integral curve $C \subseteq \mathbb{A}_K^n$ of degree d , it holds*

$$N_{\text{aff}}(C, \mathcal{O}_K, B) \lesssim_{K,n} d^{[3,7,3,3]} B^{\frac{1}{d}} (\log(B) + d^{[1,1,\frac{5}{3},1]}).$$

Proof. We argue as in the proof of Theorem 5.9, first reducing Theorem 5.16 to the case of a planar curve and then we apply Corollary 5.15. We remark that the change of variables in [31, Theorem 4.3] works in the affine setting, namely one takes the projective closure of \overline{C} of C in \mathbb{P}_K^n and then one constructs an adequate finite morphism $\varphi: \overline{C} \rightarrow \mathbb{P}^2$, which maps C to the affine part $\mathbb{A}^2 = \mathbb{P}^2 \cap \varphi(\mathbb{A}^n)$. \square

Remark 5.17. As in Remark 5.10, if $C \subseteq \mathbb{A}_K^n$ is an integral curve which is not geometrically irreducible, then $N_{\text{aff}}(C, \mathcal{O}_K, B) \leq d^2$.

6. PARTITIONING A HYPERSURFACE IN \mathbb{P}^3 BY CURVES OF SMALL DEGREE

The purpose of this section is to prove Theorem 6.9, which is an affine variant of [38, Theorem 3.16] for varieties defined over global fields. This will be used in Section 7 to prove the dimension growth conjecture in the cases when the variety has degree $4 \leq d \leq 15$ if K is a number field, and in the cases when the variety has degree $4 \leq d \leq 65$ if K is a function field.

In [38, §3] Salberger developed a technique that allowed him to partition the \mathbb{Q} -rational points of bounded height of a projective variety $X \subseteq \mathbb{P}_{\mathbb{Q}}^n$ in a small number of subvarieties of codimension at most 2. We follow the same strategy of [38], that is, we construct a family of hypersurfaces which vanish on the rational points of X of bounded height and prescribed reduction modulo \mathfrak{p} for many primes \mathfrak{p} . The degree of these hypersurfaces is bounded also in terms of the density of this subset of primes, and this is reflected in the parameter \mathfrak{q} in the statements of the theorems of the previous sections. This will allow us to control the number and the degree of the subvarieties of the partition. The main technical tool is Proposition 6.3 which is a variant of [38, Main Lemma 3.2]. For its proof, instead of using [38, Theorem 2.2] and [38, Lemma 2.8] we rely on the results of Section 5.2 and we give a streamlined construction of a suitable large subset of primes with small norm. This allows us to simplify the presentation of the proof given in [38].

Before ending this brief introduction we mention that all bounds in this section can be made effective on the dependence on the degree d . Unlike the previous sections, we choose not to make this dependence explicit because this would further complicate the proof and the final bound obtained by the methods of this section is double exponential on d .

For any $X \subseteq \mathbb{P}_K^n$ projective variety, given $B \in \mathbb{R}_{>0}$ we will denote

$$X_{\text{aff}}(\mathcal{O}_K, B) := \{(x_1, \dots, x_n) \in [B]_{\mathcal{O}_K}^n : (1 : x_1 : \dots : x_n) \in X(K)\},$$

$$N_{\text{aff}}(X, \mathcal{O}_K, B) := |X_{\text{aff}}(\mathcal{O}_K, B)|.$$

Moreover, if $\{\mathfrak{p}_1, \dots, \mathfrak{p}_u\}$ is a subset of primes such that for all $1 \leq i \leq u$ we let P_i be a non-singular $\mathbb{F}_{\mathfrak{p}_i}$ -point on $X_{\mathfrak{p}_i}$, we denote

$$X_{\text{aff}}(\mathcal{O}_K, B; P_1, \dots, P_u) := \{\mathbf{x} \in X_{\text{aff}}(\mathcal{O}_K, B) : \mathbf{x} \text{ specialises to } P_i \text{ in } X_{\mathfrak{p}_i} \text{ for all } i\},$$

We will use the following notation. Given a hypersurface $X \subseteq \mathbb{P}_K^{n+1}$ and $\mathbf{x} \in X$, we will denote

$$\pi_{\mathbf{x}} := \prod_{\substack{\mathfrak{p} \in M_{K, \text{fin}} \\ \mathbf{x} \text{ specialises to a} \\ \text{singular } \mathbb{F}_{\mathfrak{p}}\text{-point in } X_{\mathfrak{p}}}} \mathfrak{p} \quad \text{and} \quad \pi_X := \prod_{\substack{\mathfrak{p} \in M_{K, \text{fin}} \\ X_{\mathfrak{p}} \text{ is not geometrically integral}}} \mathfrak{p}.$$

The following lemmas, which generalize Lemma [38, Lemma 3.1] and [38, Lemma 3.2] give estimates for $\pi_{\mathbf{x}}$ and π_X .

Lemma 6.1. *Let $X \subseteq \mathbb{P}_K^{n+1}$ be a geometrically integral hypersurface defined over K by a homogeneous polynomial $f \in \mathcal{O}_K[X_0, \dots, X_{n+1}]$ of degree $d \geq 2$. Let us suppose that X is the only hypersurface of degree d containing $X_{\text{aff}}(\mathcal{O}_K, B)$. Then for any non-singular point \mathbf{x} in $X_{\text{aff}}(\mathcal{O}_K, B)$, it holds $\log(\mathcal{N}_K(\pi_{\mathbf{x}})) \lesssim_{n, K, d} \log(B)$.*

Proof. By Lemma 5.12(2) and the assumption on X , we have that $H_{K, \text{aff}}(f) \lesssim_K \binom{d+n+1}{n+1}^{d_K} B^{d \binom{d+n+1}{n+1}}$. Now, given a non-singular point \mathbf{x} , there exists $1 \leq j \leq n+1$ such that $\frac{\partial f}{\partial X_j}(\mathbf{x}) \neq 0$. Then for all prime \mathfrak{p} appearing in the product $\pi_{\mathbf{x}}$, we have $\mathfrak{p} \mid \frac{\partial f}{\partial X_j}(\mathbf{x})$. By (2.2) and (2.4), this implies:

$$\mathcal{N}_K(\pi_{\mathbf{x}}) \leq H_K \left(\frac{\partial f}{\partial X_j}(\mathbf{x}) \right) \leq \binom{d+n}{n+1}^{d_K} H_{K, \text{aff}} \left(\frac{\partial f}{\partial X_j} \right) H_K(1 : \mathbf{x})^{d-1} \lesssim_{n, K} d^{(n+2)d_K} \binom{d+n+1}{n+1}^{d_K} B^{d \binom{d+n+1}{n+1} + d-1}$$

By taking logarithms, we conclude $\log(\mathcal{N}_K(\pi_{\mathbf{x}})) \lesssim_{n, K, d} \log(B)$. \square

Lemma 6.2. *Let $X \subseteq \mathbb{P}_K^{n+1}$ be a geometrically integral hypersurface defined by a homogeneous polynomial $f \in \mathcal{O}_K[X_0, \dots, X_{n+1}]$ of degree $d \geq 2$. Then, either there is a polynomial $g \in \mathcal{O}_K[X_1, \dots, X_{n+1}]$ of degree d not divisible by f which contains $X_{\text{aff}}(\mathcal{O}_K, B)$, or $\log(\mathcal{N}_K(\pi_X)) \lesssim_{n, K, d} \log(B)$.*

Proof. If there is no polynomial $g \in \mathcal{O}_K[X_0, \dots, X_{n+1}]$ of degree d , not divisible by f , which contains $X_{\text{aff}}(\mathcal{O}_K, B)$, by Lemma 5.12(2) it holds $H_{K, \text{aff}}(f) \lesssim_K \binom{d+n+1}{n+1}^{d_K} B^{d \binom{d+n+1}{n+1}}$. Now, note that any prime $\mathfrak{p} \mid \pi_X$ is also a prime for which f modulo \mathfrak{p} is not absolutely irreducible. Then, following the proof of inequalities (3.8) and (3.11) in Lemma 3.23, we have

$$\prod_{\mathfrak{p} \in \pi_X} \mathcal{N}_K(\mathfrak{p}) \leq \begin{cases} d^{3(d^2-1)} \left[\binom{d+n}{n} 3^d \right]^{d^2-1} H_{K, \text{aff}}(f)^{d^2-1} & \text{if } \text{char}(K) = 0, \\ H_{K, \text{aff}}(f)^{12d^6} & \text{if } 0 < \text{char}(K) \leq d(d-1), \\ H_{K, \text{aff}}(f)^{d^2-1} & \text{if } \text{char}(K) > d(d-1). \end{cases}$$

The proof finishes by taking logarithms. \square

Now we are in condition to prove the main technical lemma of this section, which is a variant of [38, Main Lemma 3.2]. In what follows, by a prime divisor on X we shall mean a closed integral subscheme of codimension one.

Proposition 6.3. *Let $n \geq 2$ and let $X \subseteq \mathbb{P}_K^{n+1}$ be a geometrically integral hypersurface defined by a polynomial $f \in \mathcal{O}_K[X_0, \dots, X_{n+1}]$ of degree $d \geq 2$. Then there exists a family $\{D_{\gamma}\}_{\gamma \in \Gamma}$ of prime divisors on X , and a (possibly empty) subset $\{\mathcal{Z}(\mathfrak{q})\}_{\mathfrak{q} \in \mathcal{Q}}$ of effective cycles of codimension 2 on X with \mathcal{Q} a subset of ideals, such that the following conditions hold:*

- (1) *The index subset Γ has size $|\Gamma| \lesssim_{n, K, d} B^{\frac{1}{d \cdot n}}$ and for each $\gamma \in \Gamma$, it holds $\deg(D_{\gamma}) \lesssim_{n, K, d} \log(B)$.*
- (2) *There is some positive constant $c \lesssim_{n, K, d} 1$ such that*

$$\sum_{\mathfrak{q} \in \mathcal{Q}} \deg(\mathcal{Z}(\mathfrak{q})) \lesssim_{n, K, d} B^{\frac{n}{d \cdot n}} \exp\left(c(\log(B))^{\frac{1}{2}}\right).$$

- (3) For each non-singular point $\mathbf{x} \in X_{\text{aff}}(\mathcal{O}_K, B)$ which does not lie in $\bigcup_{\gamma \in \Gamma} D_\gamma$ there exists $\mathbf{q} \in \mathcal{Q}$ such that $\mathbf{x} \in \text{Supp}(\mathcal{Z}(\mathbf{q}))$ and such \mathbf{x} specialises to a non-singular $\mathbb{F}_{\mathbf{p}_i}$ -point for each prime \mathbf{p} dividing \mathbf{q} .

When $n = 2$, Proposition 6.3 says that we may partition a hypersurface X in such a way that those points $\mathbf{x} \in X_{\text{aff}}(\mathcal{O}_K, B)$ either lie in a small family of curves of low degree, or lie in an exceptional subset of controlled size.

Roughly speaking, the proof of Proposition 6.3 is as follows. First, we localize a big subset of primes \mathcal{P} such that for all $\mathbf{p} \in \mathcal{P}$, the reduction $X_{\mathbf{p}}$ is geometrically irreducible, and all non-singular $\mathbf{x} \in X_{\text{aff}}(\mathcal{O}_K, B)$ specialises to a non-singular point in $X_{\mathbf{p}}$ for many primes $\mathbf{p} \in \mathcal{P}$. Then we construct a hypersurface Y that covers all points of $X_{\text{aff}}(\mathcal{O}_K, B)$. By the second assumption on the subset \mathcal{P} , a large family $\{D_\gamma\}_{\gamma \in \Gamma}$ of irreducible components of $\mathbf{x} \in X \cap Y$ will have points which specialise to non-singular points for many primes in \mathcal{P} . Then by the first assumption on \mathcal{P} and the Lang-Weil estimate, it will turn out that those non-singular points missed by $\{D_\gamma\}_{\gamma \in \Gamma}$ lie in a subvariety of codimension 2 with controlled degree.

Proof. First let us suppose that $B \lesssim_K 1$. Then cover $X_{\text{aff}}(\mathcal{O}_K, B)$ by $\lesssim_{n,K} 1$ hyperplanes and let $(D_\gamma)_{\gamma \in \Gamma}$ be this family of hyperplane sections and let $\mathcal{Q} = \emptyset$.

Now, let us suppose that $X_{\text{aff}}(\mathcal{O}_K, B)$ is contained in another hypersurface Y of degree d . In this case let $(D_\gamma)_{\gamma \in \Gamma}$ be the components of $X \cap Y$. By Bezout's Theorem [18, Example 8.4.6], $|\Gamma| \leq d^2$ and $\deg(D_\gamma) \leq d^2$ for each $\gamma \in \Gamma$. Let $\mathcal{Q} = \emptyset$.

In the previous two cases, all the assertions of Proposition 6.3 are verified. Thus, we may suppose that $X \subseteq \mathbb{P}_K^{n+1}$ is the only hypersurface of degree d containing $X_{\text{aff}}(\mathcal{O}_K, B)$ and that $B \gtrsim_K 1$. The next step will be to construct a family of auxiliary hypersurfaces.

By Lemma 6.1 and Lemma 6.2, there are positive constants $k_1 \lesssim_{n,K,d} 1$, $\delta \lesssim_{n,K,d} 1$ such that

$$(6.1) \quad \text{for all non-singular } \mathbf{x} \in X_{\text{aff}}(\mathcal{O}_K, B), \mathcal{N}_K(\pi_{\mathbf{x}}) \leq B^{k_1} \text{ and } \sum_{\mathbf{p} | \pi_{\mathbf{x}}} \log(\mathcal{N}_K(\mathbf{p})) \leq \delta \log(B).$$

We begin by localizing an adequate large subset of primes.

Claim 6.4. *Given $c' \geq 1$, for all $c > 2(\delta + c')$, if $I := [c \log(B), 2c \log(B)]$ we have $\sum_{\substack{\mathbf{p} \nmid \pi_{\mathbf{x}} \\ \mathcal{N}_K(\mathbf{p}) \in I}} \log(\mathcal{N}_K(\mathbf{p})) \geq c' \log(B)$ for $B \gtrsim_K 1$.*

Proof of Claim 6.4. Let $c \geq 1$. Let us suppose that $\sum_{\substack{\mathbf{p} \nmid \pi_{\mathbf{x}} \\ \mathcal{N}_K(\mathbf{p}) \in I}} \log(\mathcal{N}_K(\mathbf{p})) \leq c' \log(B)$. Then by (2.13) for $B \gtrsim_K 1$ it holds:

$$\frac{1}{2} c \log(B) \leq \sum_{\substack{\mathbf{p} \\ \mathcal{N}_K(\mathbf{p}) \in I}} \log(\mathcal{N}_K(\mathbf{p})) = \sum_{\substack{\mathbf{p} | \pi_{\mathbf{x}} \\ \mathcal{N}_K(\mathbf{p}) \in I}} \log(\mathcal{N}_K(\mathbf{p})) + \sum_{\substack{\mathbf{p} \nmid \pi_{\mathbf{x}} \\ \mathcal{N}_K(\mathbf{p}) \in I}} \log(\mathcal{N}_K(\mathbf{p})) \leq (\delta + c') \log(B).$$

Thus we arrive at a contradiction taking $c > 2(\delta + c')$. \square

Let $c' := k_1 + 1$, and let us take $c := 2(\delta + k_1 + 1) + 1$, $I := [c \log(B), 2c \log(B)]$. Let $\mathbf{q}^* := \prod_{\substack{\mathbf{p} \nmid \pi_{\mathbf{x}} \\ \mathcal{N}_K(\mathbf{p}) \in I}} \mathbf{p}$. By (2.12) and Claim 6.4,

$$(6.2) \quad (k_1 + 1) \log(B) \leq \log(\mathcal{N}_K(\mathbf{q}^*)) \leq 2c_{2,K} c \log(B).$$

Let $\mathbf{q} = \prod_{i=1}^u \mathbf{p}_i \mid \mathbf{q}^*$. For each i denote $(X_{\mathbf{p}_i})_{\text{ns}}$ for the non-singular locus of $X_{\mathbf{p}_i}$, and let $(P_1, \dots, P_u) \in \prod_{i=1}^u (X_{\mathbf{p}_i})_{\text{ns}}(\mathbb{F}_{\mathbf{p}_i})$. We choose a hypersurface $Y(P_1, \dots, P_u)$ as the homogenization of the one in Theorem 5.14. By Remark 3.15, we may assume that $H_{K,\text{aff}}(f) \leq c_1^{d_K} H_K(f)$. By Lemma 3.23 and (6.2),

$$(6.3) \quad \deg(Y(P_1, \dots, P_u)) \lesssim_{K,n,d} B^{\frac{1}{d^n}} \mathcal{N}_K(\mathbf{q})^{-1} \log(\mathcal{N}_K(\mathbf{q})) + \log(B \mathcal{N}_K(\mathbf{q})) + 1 \lesssim_{K,n,d} B^{\frac{1}{d^n}} \mathcal{N}_K(\mathbf{q})^{-1} \log(\mathcal{N}_K(\mathbf{q})) + \log(B) + 1.$$

When $\mathbf{q} = (1)$ (in which case we use the convention $u = 0$), there is a projective hypersurface Y vanishing on $X_{\text{aff}}(\mathcal{O}_K, B)$ and not identically zero on X , with

$$(6.4) \quad \deg(Y) \lesssim_{K,n,d} B^{\frac{1}{d^n}} + \log(B) + 1 \lesssim_{K,n,d} B^{\frac{1}{d^n}}.$$

Now we will define the subset Γ and the subset of prime divisors $D_\gamma \subseteq X$. This divisors will be the irreducible components of $X \cap Y$ which are contained in $Y(P_1, \dots, P_u)$ for some sequence $(P_1, \dots, P_u) \in \prod_{i=1}^u (X_{\mathfrak{p}_i})_{\text{ns}}(\mathbb{F}_{\mathfrak{p}_i})$ with $\mathfrak{q} = \prod_{i=1}^u \mathfrak{p}_i$ verifying $\mathcal{N}_K(\mathfrak{q}) \geq B^{\frac{1}{d^n}}$. Then by (6.3), (6.4) and Bézout's theorem [18, Example 8.4.6] it holds

$$\deg(D_\gamma) \leq \deg(X) \deg(Y(P_1, \dots, P_u)) \lesssim_{n,K,d} \log(B),$$

$$|\Gamma| \leq \deg(X \cap Y) \leq \deg(X) \deg(Y) \lesssim_{n,K,d} B^{\frac{1}{d^n}},$$

thus $\{D_\gamma\}_\gamma$ verifies the conditions in Proposition 6.3(1).

Claim 6.5. *For all $\mathbf{x} \in X_{\text{aff}}(\mathcal{O}_K, B)$ non-singular not lying in $\bigcup_{\gamma \in \Gamma} D_\gamma$ there is some $\mathfrak{q}|\mathfrak{q}^*$, relatively prime to $\pi_{\mathbf{x}}\pi_X$ and such that $B^{\frac{1}{d^n}} \leq \mathcal{N}_K(\mathfrak{q}) \leq 2cB^{\frac{1}{d^n}} \log(B)$.*

Proof. Let $\mathbf{x} \in X_{\text{aff}}(\mathcal{O}_K, B)$ be a non-singular point which does not lie in $\bigcup_{\gamma \in \Gamma} D_\gamma$. From (6.1) and (6.2) it follows that there is some ideal factor \mathfrak{q} of \mathfrak{q}^* , relatively prime to $\pi_{\mathbf{x}}\pi_X$, that verifies $\mathcal{N}_K(\mathfrak{q}) \geq B^{\frac{1}{d^n}}$. Let $\mathfrak{q}|\mathfrak{q}^*$ of minimal norm such that it is relatively prime with $\pi_{\mathbf{x}}\pi_X$ and $\mathcal{N}_K(\mathfrak{q}) \geq B^{\frac{1}{d^n}}$. Let us suppose that $\mathcal{N}_K(\mathfrak{q}) > 2cB^{\frac{1}{d^n}} \log(B)$. Since any prime $\mathfrak{p}|\mathfrak{q}^*$ has norm in I , there is some prime $\mathfrak{p}|\mathfrak{q}$ with $\mathcal{N}_K(\mathfrak{p}) \leq 2c \log(B)$. Thus $\mathcal{N}_K(\mathfrak{q}\mathfrak{p}^{-1}) \geq B^{\frac{1}{d^n}}$, which contradicts the minimality of \mathfrak{q} . \square

If \mathbf{x} and $\mathfrak{q} = \prod_{i=1}^u \mathfrak{p}_i$ are as in Claim 6.5, then \mathbf{x} specialises to a non-singular $\mathbb{F}_{\mathfrak{p}_i}$ -rational point P_i on $X_{\mathfrak{p}_i}$ for all i , thus $\mathbf{x} \in Y(P_1, \dots, P_u)$. Let $D_{\mathbf{x}}$ be an irreducible component of $X \cap Y$ containing \mathbf{x} . It can not hold that $D_{\mathbf{x}} \subseteq Y(P_1, \dots, P_u)$, since otherwise $\mathbf{x} \in \bigcup_{\gamma \in \Gamma} D_\gamma$. Hence, there is some $t \in \{0, \dots, u-1\}$ with $D_{\mathbf{x}} \subseteq Y(P_1, \dots, P_t)$ but $D_{\mathbf{x}} \not\subseteq Y(P_1, \dots, P_{t+1})$. If $D'_{\mathbf{x}}$ is the irreducible component of $X \cap Y(P_1, \dots, P_{t+1})$ containing \mathbf{x} , then $\mathbf{x} \in D_{\mathbf{x}} \cap D'_{\mathbf{x}}$. This will motivate the construction of the cycles.

Given $\mathfrak{q} = \mathfrak{q}_{t+1} = \prod_{i=1}^{t+1} \mathfrak{p}_i$ with $t \geq 0$, let $Z(P_1, \dots, P_{t+1})$ be the formal sum of all the irreducible components of $D \cap D'$ where D is an irreducible component of $X \cap Y(P_1, \dots, P_t)$ and $D' \neq D$ is an irreducible component of $X \cap Y(P_1, \dots, P_{t+1})$. By Bézout's theorem [18, Example 8.4.6], inequalities (6.3) and (6.4), if $\mathfrak{q}_t := \mathfrak{q}_{t+1}\mathfrak{p}_{t+1}^{-1}$, then it holds

(6.5)

$$\begin{aligned} \deg(Z(P_1, \dots, P_{t+1})) &\leq \deg(X) \deg(Y(P_1, \dots, P_t)) \deg(Y(P_1, \dots, P_{t+1})) \\ &\lesssim_{K,n,d} \left(B^{\frac{1}{d^n}} \mathcal{N}_K(\mathfrak{q}_{t+1})^{-1} \mathcal{N}_K(\mathfrak{p}_{t+1}) \log(\mathcal{N}_K(\mathfrak{q}_t)) + \log(B) + 1 \right) \left(B^{\frac{1}{d^n}} \mathcal{N}_K(\mathfrak{q}_{t+1})^{-1} \log(\mathcal{N}_K(\mathfrak{q}_{t+1})) + \log(B) + 1 \right). \\ &\lesssim_{K,n,d} \left(B^{\frac{1}{d^n}} \mathcal{N}_K(\mathfrak{q}_{t+1})^{-1} (\log(B))^2 + \log(B) + 1 \right) \left(B^{\frac{1}{d^n}} \mathcal{N}_K(\mathfrak{q}_{t+1})^{-1} \log(B) + \log(B) + 1 \right). \end{aligned}$$

We define

$$Z(\mathfrak{q}) = Z(\mathfrak{q}_{t+1}) := \sum_{(P_1, \dots, P_{t+1}) \in \prod_{i=1}^{t+1} (X_{\mathfrak{p}_i})_{\text{ns}}(\mathbb{F}_{\mathfrak{p}_i})} Z(P_1, \dots, P_{t+1}).$$

In order to bound the degree of $Z(\mathfrak{q}_{t+1})$ we need to estimate the cardinal of $\prod_{i=1}^{t+1} (X_{\mathfrak{p}_i})_{\text{ns}}(\mathbb{F}_{\mathfrak{p}_i})$.

Claim 6.6. *It holds $|\prod_{i=1}^{t+1} (X_{\mathfrak{p}_i})_{\text{ns}}(\mathbb{F}_{\mathfrak{p}_i})| \leq \mathcal{N}_K(\mathfrak{q}_{t+1})^n \exp\left(C(\log(B))^{\frac{1}{2}}\right)$ for some $C \lesssim_{K,n,d} 1$.*

Proof of Claim 6.6. By definition, for all i , $X_{\mathfrak{p}_i}$ is geometrically irreducible. Then the Lang-Weil estimate gives $|X_{\mathfrak{p}_i}(\mathbb{F}_{\mathfrak{p}_i})| \leq \mathcal{N}_K(\mathfrak{p}_i)^n + A\mathcal{N}_K(\mathfrak{p}_i)^{n-\frac{1}{2}}$ for some positive constant $A \lesssim_{d,n} 1$. This, together with the facts that $t \lesssim_{K,n,d} \log(B)$ and that for all i , $\mathfrak{p}_i \in I$, thus $\mathcal{N}_K(\mathfrak{p}_i) \geq \frac{1}{2}c \log(B)$, yield

$$\left| \prod_{i=1}^{t+1} (X_{\mathfrak{p}_i})_{\text{ns}}(\mathbb{F}_{\mathfrak{p}_i}) \right| \leq \mathcal{N}_K(\mathfrak{q}_{t+1})^n \prod_{i=1}^{t+1} \left(1 + A\mathcal{N}_K(\mathfrak{p}_i)^{-\frac{1}{2}} \right) \leq \mathcal{N}_K(\mathfrak{q}_{t+1})^n \left(1 + \frac{2^{\frac{1}{2}}A}{c^{\frac{1}{2}}(\log(B))^{\frac{1}{2}}} \right)^{t+1} \leq \mathcal{N}_K(\mathfrak{q}_{t+1})^n \exp\left(C(\log(B))^{\frac{1}{2}}\right),$$

for some $C \lesssim_{K,n,d} 1$. \square

By (6.5) and Claim 6.6, it follows that

$$\begin{aligned} \deg(Z(\mathbf{q}_{t+1})) &\lesssim_{K,n,d} \left(B^{\frac{1}{d^n}} \mathcal{N}_K(\mathbf{q}_{t+1})^{-1} (\log(B))^2 + \log(B) + 1 \right) \left(B^{\frac{1}{d^n}} \mathcal{N}_K(\mathbf{q}_{t+1})^{-1} \log(B) + \log(B) + 1 \right) \\ &\quad \mathcal{N}_K(\mathbf{q}_{t+1})^n \exp \left(C(\log(B))^{\frac{1}{2}} \right) \\ &\lesssim_{K,n,d} \left(B^{\frac{2}{d^n}} \mathcal{N}_K(\mathbf{q}_{t+1})^{n-2} (\log(B))^3 + B^{\frac{1}{d^n}} \mathcal{N}_K(\mathbf{q}_{t+1})^{n-1} (\log(B))^3 + B^{\frac{1}{d^n}} \mathcal{N}_K(\mathbf{q}_{t+1})^{n-1} (\log(B))^2 \right. \\ &\quad \left. + B^{\frac{1}{d^n}} \mathcal{N}_K(\mathbf{q}_{t+1})^{n-1} (\log(B))^2 + B^{\frac{1}{d^n}} \mathcal{N}_K(\mathbf{q}_{t+1})^{n-1} \log(B) + (\log(B))^2 \mathcal{N}_K(\mathbf{q}_{t+1})^n \right) \exp \left(C(\log(B))^{\frac{1}{2}} \right). \end{aligned}$$

If $\mathcal{N}_K(\mathbf{q}) \lesssim_{K,n,d} B^{\frac{1}{d^n}} \log(B)$, it follows that

$$(6.6) \quad \deg(Z(\mathbf{q}_{t+1})) \lesssim_{K,n,d} B^{\frac{n}{d^n}} (\log(B))^{n+2} \exp \left(C(\log(B))^{\frac{1}{2}} \right) \lesssim_{K,n,d} B^{\frac{n}{d^n}} \exp \left(C'(\log(B))^{\frac{1}{2}} \right)$$

for some positive constant $C' \lesssim_{K,n,d,\varepsilon} 1$.

Then we define

$$\mathcal{Q} := \left\{ \mathbf{q} | \mathbf{q}^* : \mathbf{q} \neq (1) \text{ and } \mathcal{N}_K(\mathbf{q}) \leq 2cB^{\frac{1}{d^n}} \log(B) \right\}.$$

Claim 6.7. *There is some $C'' \lesssim_{K,n,d} 1$ such that it holds $|\mathcal{Q}| \leq \exp \left(C'' \frac{\log(B)}{\log(\log(B))} \right)$.*

Proof of Claim 6.7. It will suffice to bound the number of divisors of \mathbf{q}^* . By Landau Prime Ideal Theorem or the Riemann Hypothesis over function fields, there is a constant $c_K \lesssim_{K,n,d} 1$ such that $|\{\mathbf{p} : \mathcal{N}_K(\mathbf{p}) \in I\}| \leq c_K \frac{c \log(B)}{\log(c \log(B))}$. Then

$$|\mathcal{Q}| \leq |\{\mathbf{q} : \mathbf{q} | \mathbf{q}^*\}| \leq 2^{|\{\mathbf{p} : \mathcal{N}_K(\mathbf{p}) \in I\}|} \leq \exp \left(c_K \frac{c \log(B)}{\log(c \log(B))} \right) \leq \exp \left(C'' \frac{\log(B)}{\log(\log(B))} \right),$$

for some $C'' \lesssim_{K,n,d} 1$. □

Thus, (6.6) and Claim (6.7) imply that

$$\sum_{\mathbf{q} \in \mathcal{Q}} \deg(Z(\mathbf{q})) \lesssim_{K,n,d} B^{\frac{n}{d^n}} \exp \left(C'' \frac{\log(B)}{\log(\log(B))} \right) \exp \left(C'(\log(B))^{\frac{1}{2}} \right) \lesssim_{K,n,d} B^{\frac{n}{d^n}} \exp \left(C'''(\log(B))^{\frac{1}{2}} \right),$$

for some $C''' \lesssim_{K,n,d} 1$. This proves Proposition 6.3(2). Moreover, by Claim 6.5 and the construction of $\{Z(\mathbf{q})\}_{\mathbf{q} \in \mathcal{Q}}$, it follows that $\{D_\gamma\}_{\gamma \in \Gamma}$ and $\{Z(\mathbf{q})\}_{\mathbf{q} \in \mathcal{Q}}$ verify the statement of Proposition 6.3(3). □

Remark 6.8. In Proposition 6.3(2) one could get the better bound $\lesssim_{K,n,d} B^{\frac{n}{d^n}} \exp(c \frac{(\log(B))^{\frac{1}{2}}}{\log(\log(B))})$ if in the proof of Claim 6.6 one bounds the quantity $\prod_{i=1}^{t+1} (1 + \mathcal{N}_K(\mathbf{p}_i)^{-\frac{1}{2}})$ as in [43, §1.5.5]. We choose not to do so because when we use it in the proof of an affine variant of the dimension growth conjecture (Proposition 7.3) this saving will be absorbed in a factor B^ε . This remark also applies to the bound in the first summand of the next theorem.

Theorem 6.9. *Let $X \subseteq \mathbb{P}_K^3$ be a geometrically integral hypersurface of degree d , defined over K , and let X_{ns} be the non-singular locus of X . Then for all $\nu > 0$ there exists a subset of $\lesssim_{K,d} B^{\frac{1}{\sqrt{d}}}$ geometrically integral curves $D_\lambda \subseteq X$, $\lambda \in \Lambda = \Lambda_\nu$, of degree at most $\frac{1}{\nu}$, such that it holds*

$$N_{\text{aff}} \left(X_{ns} - \bigcup_{\lambda \in \Lambda} D_\lambda, \mathcal{O}_K, B \right) \lesssim_{K,d} \begin{cases} B^{\frac{2}{\sqrt{d}}} \exp \left(c(\log(B))^{\frac{1}{2}} \right) + \frac{1}{\nu^4} B^{\frac{1}{\sqrt{d}} + \nu} \log(B) & \text{if } K \text{ is a number field,} \\ B^{\frac{2}{\sqrt{d}}} \exp \left(c(\log(B))^{\frac{1}{2}} \right) + \frac{1}{\nu^8} B^{\frac{1}{\sqrt{d}} + \nu} \log(B) & \text{if } K \text{ is a function field,} \end{cases}$$

for some positive constant $c \lesssim_{K,d} 1$.

Proof. Let $\{D_\gamma\}_{\gamma \in \Gamma}$, and $\{Z(\mathfrak{q})\}_{\mathfrak{q} \in \mathcal{Q}}$ as in Proposition 6.3 for $n = 2$. By Proposition 6.3(3), it holds

$$N_{\text{aff}}\left(X_{\text{ns}} - \bigcup_{\gamma \in \Gamma} D_\gamma, \mathcal{O}_K, B\right) \leq \sum_{\mathfrak{q} \in \mathcal{Q}} \deg(Z(\mathfrak{q})).$$

Hence, by Proposition 6.3(2), there is a positive constant $c \lesssim_{K,d} 1$ such that it holds

$$(6.7) \quad N_{\text{aff}}\left(X_{\text{ns}} - \bigcup_{\gamma \in \Gamma} D_\gamma, \mathcal{O}_K, B\right) \lesssim_{K,d} B^{\frac{2}{\sqrt{d}}} \exp\left(c(\log(B))^{\frac{1}{2}}\right).$$

Now, let $\Lambda \subseteq \Gamma$ be the subset of all indexes $\gamma \in \Gamma$ for which $\deg(D_\gamma) \leq \frac{1}{\nu}$. Then

$$(6.8) \quad N_{\text{aff}}\left(X_{\text{ns}} - \bigcup_{\lambda \in \Lambda} D_\lambda, \mathcal{O}_K, B\right) \leq N_{\text{aff}}\left(X_{\text{ns}} - \bigcup_{\gamma \in \Gamma} D_\gamma, \mathcal{O}_K, B\right) + |\Gamma| \max_{\gamma \in \Gamma \setminus \Lambda} N_{\text{aff}}(D_\gamma, \mathcal{O}_K, B).$$

It remains to bound $N_{\text{aff}}(D_\gamma, \mathcal{O}_K, B)$ for any $\gamma \in \Gamma \setminus \Lambda$. Now, we apply Theorem 5.16 to each D_γ with $\gamma \in \Gamma \setminus \Lambda$ to obtain

$$N_{\text{aff}}(D_\gamma, \mathcal{O}_K, B) \lesssim_K \begin{cases} \frac{1}{\nu^4} B^\nu \log(B) & \text{if } K \text{ is a number field} \\ \frac{1}{\nu^8} B^\nu \log(B) & \text{if } K \text{ is a function field.} \end{cases}$$

By Proposition 6.3(1), $|\Gamma| \lesssim_{K,d} B^{\frac{1}{\sqrt{d}}}$, from which we deduce

$$(6.9) \quad |\Gamma| \max_{\gamma \in \Gamma \setminus \Lambda} N_{\text{aff}}(D_\gamma, \mathcal{O}_K, B) \lesssim_{K,d} \begin{cases} \frac{1}{\nu^4} B^{\frac{1}{\sqrt{d}} + \nu} \log(B) & \text{if } K \text{ is a number field,} \\ \frac{1}{\nu^8} B^{\frac{1}{\sqrt{d}} + \nu} \log(B) & \text{if } K \text{ is a function field.} \end{cases}$$

The theorem follows from (6.7), (6.8), and (6.9). \square

7. UNIFORM DIMENSION GROWTH CONJECTURE

In this section we prove Theorem 1.10 and Theorem 1.11 of the introduction, which generalize [38, Theorem 0.3 and Theorem 0.4], [11, Theorem 1 and Theorem 4], and [45, Theorem 1.3 and Theorem 4.2] to varieties over global fields. The proof follows the strategy given in [9]. For this, we use an inductive argument: we establish Theorem 1.10 in the case $X \subseteq \mathbb{A}_K^n$ with $n = 3$ (this is the content of Proposition 7.3), and afterwards we intersect X with an adequate hyperplane H so that $X \cap H$ is of smaller dimension and suitable to apply the inductive hypothesis. As in [9] we deduce Theorem 1.11 by using a projection argument to reduce it to the case of a hypersurface, and then by taking affine cones and applying Theorem 1.10.

In order to prove Proposition 7.3 we proceed as in [11, Proposition 4.3.4], namely, first we establish that for large degree, the counting function $N_{\text{aff}}(\mathcal{Z}(f), \mathcal{O}_K, B)$ is at most $\lesssim_K d^e B$. This is done by using Theorem 5.14 to cover $X_{\text{aff}}(\mathcal{O}_K, B)$ with a hypersurface of small degree and then by bounding the contribution of the rational points coming from the irreducible components of $X \cap Y$ using Theorem 5.16 for the ones of degree at least 2 and Proposition 7.1 for the ones of degree 1. Having proved Proposition 7.3 for large degree, the remaining cases are dealt with Theorem 6.9.

Now we begin to carry out this program. First we establish the following generalization of [11, Proposition 4.3.3], where the authors give an effective estimate of [9, Proposition 1, case $D = 1$], with explicit dependence on the degree of the hypersurface.

Proposition 7.1. *Let K be a global field. There exists a constant $c = c(K, n)$ such that for all $f \in \mathcal{O}_K[X_1, X_2, X_3]$ of degree $d \geq 3$ satisfying that the homogeneous part of higher degree f_d is irreducible, and for all finite sets I of curves $C \subseteq \mathbb{A}_K^3$ of degree 1 lying on the hypersurface $\mathcal{Z}(f)$ defined by f , and all $B \geq 1$, it holds*

$$N_{\text{aff}}\left(\mathcal{Z}(f) \cap \left(\bigcup_{C \in I} C\right), \mathcal{O}_K, B\right) \leq cd^6 B + |I|.$$

Proof. Let us write $I = I_1 \cup I_2$ where

$$I_1 := \{C \in I : N_{\text{aff}}(C, \mathcal{O}_K, B) \leq 1\}, \quad I_2 := \{C \in I : N_{\text{aff}}(C, \mathcal{O}_K, B) > 1\}.$$

It is clear that

$$N_{\text{aff}}\left(\mathcal{Z}(f) \cap \left(\bigcup_{C \in I_1} C\right), \mathcal{O}_K, B\right) \leq |I_1|.$$

Now, for any $C \in I_2$ there exist $\mathbf{a} = (a_1, a_2, a_3), \mathbf{w} = (w_1, w_2, w_3) \in \mathcal{O}_K^3$ with $\mathbf{a} \in [B]_{\mathcal{O}_K}^3$, and $C(K) = \{\mathbf{a} + \lambda \mathbf{w} : \lambda \in K\}$.

Claim 7.2. *Let C be the line defined by $\mathbf{a} + \lambda \mathbf{w}$ with $\mathbf{a} \in [B]_{\mathcal{O}_K}^3$ and $\mathbf{w} \in \mathcal{O}_K^3$. Then*

$$|C(K) \cap [B]_{\mathcal{O}_K}^3| \lesssim_K \frac{B}{H_K(\mathbf{w})}.$$

Proof of Claim 7.2. While in the case $K = \mathbb{Q}$ or $K = \mathbb{F}_q(T)$ the claim is trivial, in general the proof is more involved due to the fact \mathcal{O}_K may not be a principal ideal domain. This is why we use the geometry of numbers and the theory of divisors.

By Proposition 2.2, we may suppose that \mathbf{w} is \mathfrak{p} -primitive for all \mathfrak{p} of \mathcal{O}_K with $\mathcal{N}_K(\mathfrak{p}) > c_2$, and $\prod_{v \in M_{K, \text{fin}}} \max_i |w_i|_v \geq c_3$. In particular, for all $v \in M_{K, \text{fin}}$, $\max_i \text{ord}_{\mathfrak{p}_v}(w_i) \lesssim_K 1$. Let $S = M_{K, \infty} \cup \{v : \mathcal{N}_K(\mathfrak{p}_v) \leq c_2\}$. We let

$$\mathcal{O}_{K, S} := \{x \in \mathcal{O}_K : |x|_v \leq 1 \text{ for all } v \notin S\}.$$

If K is a number field, then

$$\begin{aligned} C(K) \cap [B]_{\mathcal{O}_K}^3 &= \{\mathbf{a} + \lambda \mathbf{w} \in \mathcal{O}_K^3 : |a_i + \lambda w_i| \leq B^{\frac{1}{d_K}} \text{ for all } i\} \\ &\subseteq \left\{ \mathbf{a} + \lambda \mathbf{w} \in \mathcal{O}_K^3 : \lambda \in \mathcal{O}_{K, S}, |\sigma(\lambda)| \leq \frac{2B^{\frac{1}{d_K}}}{\max_i |\sigma(w_i)|} \forall \sigma : K \hookrightarrow \mathbb{C}, |\lambda|_v \leq c_4 \forall v \in S \setminus M_{K, \infty} \right\}, \end{aligned}$$

for some $c_4 := c_4(K)$. Now, if K_v denotes the completion of K with respect to v , we have that $\mathcal{O}_{K, S}$ is a lattice under the usual embedding $\mathcal{O}_{K, S} \hookrightarrow \prod_{v \in S} K_v$. Denoting $B_v(0, r_v)$ for the usual disk in K_v of center 0 and radius r_v , we have

$$\begin{aligned} |C(K) \cap [B]_{\mathcal{O}_K}^3| &\leq \left| \mathcal{O}_{K, S} \cap \prod_{v \in M_{K, \infty}} B_v\left(0, \frac{2B^{\frac{1}{d_K}}}{\max_i |\sigma(w_i)|}\right) \times \prod_{v \in S \setminus M_{K, \infty}} B_v(0, c_4) \right| \\ &\lesssim_K \prod_{\sigma \text{ real}} \frac{2B^{\frac{1}{d_K}}}{\max_i |\sigma(w_i)|} \prod_{\sigma \text{ complex}} \frac{4B^{\frac{2}{d_K}}}{\max_i |\sigma(w_i)|^2} \lesssim_K \frac{B}{H_K(\mathbf{w})}. \end{aligned}$$

If K is a function field over $\mathbb{F}_q(T)$, then

$$\begin{aligned} C(K) \cap [B]_{\mathcal{O}_K}^3 &= \{\mathbf{a} + \lambda \mathbf{w} \in \mathcal{O}_K^3 : |a_i + \lambda w_i|_{v_\infty} \leq B^{\frac{1}{d_K}} \text{ for all } i\} \\ &\subseteq \left\{ \mathbf{a} + \lambda \mathbf{w} \in \mathcal{O}_K^3 : \lambda \in \mathcal{O}_{K, S}, |\lambda|_{v_\infty} \leq \frac{2B^{\frac{1}{d_K}}}{\max_i |w_i|_{v_\infty}}, |\lambda|_v \leq c_4 \forall v \in S \setminus \{v_\infty\} \right\} := A \end{aligned}$$

for some constant $c_4 := c_4(K)$. Now we argue as in [31, Proposition 2.2]. Any $\lambda \in A$ satisfies $H_K(\lambda) \lesssim_K \frac{B}{H_K(\mathbf{w})}$. Since $H_K(\lambda) = H_K(\lambda^{-1})$, the positive divisor $\sum_{v: \text{ord}_v(\lambda) \geq 0} \text{ord}_v(\lambda) \cdot v$ has degree $\lesssim_K \log_q\left(\frac{B}{H_K(\mathbf{w})}\right)$; by [31, Proposition 2.2], there are at most $\lesssim_K \frac{B}{H_K(\mathbf{w})}$ such divisors. Since $\text{ord}_v(\lambda) \gtrsim_K 1$ for all $v \in S \setminus \{v_\infty\}$ and $\sum_{v \in M_K} \text{ord}_v(\lambda) \deg(v) = 0$, we conclude that the subset A has at most $\lesssim_K \frac{B}{H_K(\mathbf{w})}$ elements. \square

Now, for any $\mathbf{w} \in K^3$ there are at most $d(d-1)$ lines $C \in I_2$ in the direction of \mathbf{w} (this is because each such line intersects a generic hyperplane in \mathbb{A}^3 in a common point of the hypersurfaces defined by f and its directional derivate in the direction \mathbf{w}).

Since $C \in I_2$, it verifies the hypothesis of Claim 7.2, then we have $H_K(\mathbf{w}) \leq cB$. Moreover, since $C \subseteq \mathcal{Z}(f)$, $f_d(\mathbf{w}) = 0$. Consider the set $A_i := \{\mathbf{w} \in \mathbb{P}^2(K) : f_d(\mathbf{w}) = 0, H_K(\mathbf{w}) = i\}$. Then by Claim 7.2,

$$(7.1) \quad N_{\text{aff}}\left(\mathcal{Z}(f) \cap \bigcup_{C \in I} C, \mathcal{O}_K, B\right) \leq |I_1| + (d-1)d \sum_{i=1}^{\lfloor cB \rfloor} |A_i| \frac{cB}{i}.$$

Now Corollary 5.8 implies that $\sum_{i=1}^k |A_i| \lesssim_K d^4 k^{\frac{2}{d}}$. This, and summation by parts give

$$(7.2) \quad \sum_{i=1}^{\lfloor cB \rfloor} |A_i| \frac{cB}{i} = \sum_{i=1}^{\lfloor cB \rfloor} |A_i| \frac{cB}{\lfloor cB \rfloor} + \sum_{i=1}^{\lfloor cB \rfloor} \left(\sum_{j=i}^{\lfloor cB \rfloor} |A_j| \right) \left(\frac{cB}{i} - \frac{cB}{i+1} \right) \lesssim_K d^4 B^{\frac{2}{d}} + \sum_{i=1}^{\lfloor cB \rfloor} d^4 k^{\frac{2}{d}} \frac{cB}{k(k+1)} \lesssim_K d^4 B,$$

where the last bound is because $d \geq 3$. Replacing (7.2) in (7.1) finishes the proof. \square

Proposition 7.3. *Let K be a global field of degree d_K , and let $e = 18$ if K is a number field and $e = 64$ if K is a function field. Then for all polynomial $f \in \mathcal{O}_K[X_1, X_2, X_3]$ of degree d whose homogeneous part of highest degree is absolutely irreducible, it holds*

$$N_{\text{aff}}(\mathcal{Z}(f), \mathcal{O}_K, B) \lesssim_K d^e B \text{ whenever } d \geq 5.$$

In the case $d = 3, 4$, for all $\varepsilon > 0$ it holds

$$N_{\text{aff}}(\mathcal{Z}(f), \mathcal{O}_K, B) \lesssim_{K, \varepsilon} \begin{cases} B^{1+\varepsilon} & \text{if } d = 4, \\ B^{\frac{2}{\sqrt{3}}+\varepsilon} & \text{if } d = 3. \end{cases}$$

Proof. The strategy of the proof follows [11, Proposition 4.3.4], namely, first we establish that for large degree, the counting function $N_{\text{aff}}(\mathcal{Z}(f), \mathcal{O}_K, B)$ is at most $\lesssim_K d^e B$. This is done by using Theorem 5.14 to cover $X_{\text{aff}}(\mathcal{O}_K, B)$ with a hypersurface of small degree. Then one bounds the contribution of the rational points coming from the irreducible components of $X \cap Y$. Now, the contribution of those irreducible components of degree 1 is bounded with Proposition 7.1. By Bézout's theorem, the degree of these curves can be large as $\lesssim_{K, d} B^{\frac{1}{\sqrt{d}}}$, hence one can not apply directly Theorem 5.16 to deal with the components of $X \cap Y$ of degree $\gtrsim_K \log(B)$. This technical obstruction will be dealt by assuming that the degree of X is large enough.

Having proved Proposition 7.3 for large degree, the remaining cases are dealt with Theorem 6.9. Then, after enlarging the implicit constant, one concludes $N_{\text{aff}}(\mathcal{Z}(f), \mathcal{O}_K, B) \lesssim_K d^e B$. We remark that, while Theorem 6.9 gives the bound $N_{\text{aff}}(\mathcal{Z}(f), \mathcal{O}_K, B) \lesssim_{K, d} B$ for $d \geq 5$, it gives a double exponential dependence on d ; it is for this reason that we do not use it to prove Proposition 7.3 for all $d \geq 5$.

For any prime for which $f_d \bmod \mathfrak{p}$ is absolutely irreducible the reduction of f modulo \mathfrak{p} is absolutely irreducible. Hence $b(f) \leq b(f_d)$. Using Lemma 3.23 and Theorem 5.14, for any $B \geq 1$ we find a polynomial $g \in \mathcal{O}_K[X_1, X_2, X_3]$ of degree

$$(7.3) \quad \lesssim_K \begin{cases} d^{\frac{7}{2}} B^{\frac{1}{\sqrt{d}}} & \text{if } K \text{ is a number field,} \\ d^7 B^{\frac{1}{\sqrt{d}}} & \text{if } K \text{ is a function field,} \end{cases}$$

not divisible by f and vanishing on all $\mathcal{Z}(f)_{\text{aff}}(\mathcal{O}_K, B)$. Let $\mathcal{Z}(f, g)$ be the intersection of $f = 0$ and $g = 0$ (here the intersection is being considered with its reduced structure). Let \mathcal{C} be the subset of irreducible components of $\mathcal{Z}(f, g)$. By Bézout's theorem [18, Example 8.4.6],

$$(7.4) \quad |\mathcal{C}| \leq \sum_{C \in \mathcal{C}} \deg(C) \lesssim_K \begin{cases} d^{\frac{9}{2}} B^{\frac{1}{\sqrt{d}}} & \text{if } K \text{ is a number field,} \\ d^8 B^{\frac{1}{\sqrt{d}}} & \text{if } K \text{ is a function field.} \end{cases}$$

By Proposition 7.1 and (7.4), the contribution of the irreducible components of $\mathcal{Z}(f, g)$ of degree 1 is

$$(7.5) \quad \lesssim_K d^6 B + |\mathcal{C}| \lesssim_K \begin{cases} d^6 B & \text{if } K \text{ is a number field,} \\ d^8 B & \text{if } K \text{ is a function field.} \end{cases}$$

Now, let us suppose that C_1, \dots, C_k are the irreducible components of $\mathcal{Z}(f, g)$ of degree greater than 1, arranged in such a way that $\deg(C_i) \leq \log(B)$ for all $1 \leq i \leq m$, and $\deg(C_i) > \log(B)$ for all $i > m$. Let us denote $\delta_i := \deg(C_i)$. By Theorem 5.16, for all i it holds

$$(7.6) \quad N_{\text{aff}}(C_i, \mathcal{O}_K, B) \lesssim_K \begin{cases} \delta_i^3 B^{\frac{1}{\delta_i}} (\log(B) + \delta_i) & \text{if } K \text{ is a number field,} \\ \delta_i^7 B^{\frac{1}{\delta_i}} (\log(B) + \delta_i) & \text{if } K \text{ is a function field.} \end{cases}$$

Claim 7.4. *For all $1 \leq i \leq m$ it holds $N_{\text{aff}}(C_i, \mathcal{O}_K, B) \lesssim_K B^{\frac{1}{2}} (\log(B) + 1)$.*

Proof of Claim 7.4. Let us suppose that K is a number field. Let us set $\psi(\delta) := \delta^4 B^{\frac{1}{\delta}}$. Then $\log(\psi(\delta)) = 4 \log(\delta) + \frac{\log(B)}{\delta} = \log(B)(4 \log_B(\delta) + \frac{1}{\delta})$. Since the function $\delta_i \mapsto 4 \log_B(\delta_i) + \frac{1}{\delta_i}$ is decreasing in $(0, \frac{\log(B)}{4})$ and increasing in $(\frac{\log(B)}{4}, +\infty)$, the maximum value of $\psi(\delta)$ in $[2, \log(B)]$ is

$$(7.7) \quad \max\{\psi(2), \psi(\log(B))\} = \max\{2^4 B^{\frac{1}{2}}, (\log(B))^4 B^{\frac{1}{\log(B)}}\} \lesssim B^{\frac{1}{2}}.$$

The inequalities (7.6), (7.7), and the trivial bound $\delta \geq 1$ give Claim 7.4 for number fields. The case of function fields is analogous. \square

By Claim 7.4 and (7.4) we have

$$(7.8) \quad \sum_{i=1}^m N_{\text{aff}}(C_i, \mathcal{O}_K, B) \lesssim_K B^{\frac{1}{2}} (\log(B) + 1) m \lesssim_K \begin{cases} d^{\frac{9}{2}} B & \text{if } K \text{ is a number field,} \\ d^8 B & \text{if } K \text{ is a function field.} \end{cases}$$

On the other hand, if $\delta > \log(B)$ then $B^{\frac{1}{\delta}}$ is bounded, thus (7.4) and (7.6) imply

$$(7.9) \quad \sum_{i=m+1}^k N_{\text{aff}}(C_i, \mathcal{O}_K, B) \lesssim_K \begin{cases} \sum_{i=m+1}^k \delta_i^4 \lesssim_K \left(\sum_{i=m+1}^k \delta_i\right)^4 \lesssim_K d^{18} B^{\frac{4}{\sqrt{d}}} & \text{if } K \text{ is a number field,} \\ \sum_{i=m+1}^k \delta_i^8 \lesssim_K \left(\sum_{i=m+1}^k \delta_i\right)^8 \lesssim_K d^{64} B^{\frac{8}{\sqrt{d}}} & \text{if } K \text{ is a function field.} \end{cases}$$

Combining (7.3), Claim 7.5, (7.4), and (7.9), Proposition 7.3 follows for $d \geq 16$ if K is a number field and for $d \geq 64$ if K is a function field.

For the remaining values of d , by Theorem 6.9 with $\nu = \frac{1}{2\sqrt{d}}$, there is a constant $c \lesssim_{K,d} 1$ such that for all $B \geq 1$ there exists a subset of $\lesssim_{K,d} B^{\frac{1}{\sqrt{d}}}$ geometrically integral curves $D_\lambda \subseteq X$, $\lambda \in \Lambda$, of degree at most $\lesssim_K \sqrt{d}$ such that for all $\varepsilon > 0$ it verifies

$$N_{\text{aff}}\left(X_{\text{ns}} - \bigcup_{\lambda \in \Lambda} D_\lambda, \mathcal{O}_K, B\right) \lesssim_{K,d,\varepsilon} B^{\frac{2}{\sqrt{d}} + \varepsilon} + B^{\frac{2}{\sqrt{d}}},$$

with $c \lesssim_{K,d} 1$ (we used that $\exp(c(\log(B))^{\frac{1}{2}}) \lesssim_{d,K,\varepsilon} B^\varepsilon$). Hence, for all $\varepsilon > 0$ we have

$$N_{\text{aff}}\left(X_{\text{ns}} - \bigcup_{\lambda \in \Lambda} D_\lambda, \mathcal{O}_K, B\right) \lesssim_{K,d,\varepsilon} \begin{cases} B & \text{if } d \geq 5, \\ B^{1+\varepsilon} & \text{if } d = 4, \\ B^{\frac{2}{\sqrt{3}} + \varepsilon} & \text{if } d = 3. \end{cases}$$

Thus, it remains to bound the contribution of the points lying in any of the curves D_λ , and the points lying in the complement of $X \setminus X_{\text{ns}}$. By Proposition 7.1, those curves $D_\lambda \subseteq \Lambda$ of degree 1 contribute at most $\lesssim_{K,d} B$ points, while by Theorem 5.16 those curves $D_\lambda \subseteq \Lambda$ of degree at least 2 contribute at most $\lesssim_{K,d} B^{\frac{1}{2} + \varepsilon}$ points. On the other hand, by Bézout's theorem, $X \setminus X_{\text{ns}}$ is a union of irreducible curves the sum of whose degrees is bounded by a constant. Applying Theorem 5.8 to those curves of degree at least 2, and Proposition 7.1 to those of degree 1 yields that the rational points coming from $X \setminus X_{\text{ns}}$ is $\lesssim_{K,d} B$. \square

Lemma 7.5. *Let K be a global field. Let $n \geq 3$ and $X \subseteq \mathbb{P}_K^n$ be a geometrically integral hypersurface of degree d . Let $\kappa = (n+1)(d^2-1)$ if K is a number field, and $\kappa = 12(n+1)d^7$ if K is a function field. Then there exists a non-zero form $F \in \mathcal{O}_K[Y_0, \dots, Y_n]$ of degree at most κ such that $F(A) = 0$ whenever the hyperplane section $H_A \cap X$ is not geometrically integral, where $A \in (\mathbb{P}^n)^*$ and $H_A \subseteq \mathbb{P}^n$ denotes the hyperplane corresponding to the linear form A .*

Proof. The proof is similar to the one given in [11, Proposition 4.3.7] for $K = \mathbb{Q}$ and in [45, Lemma 4.6] for $K = \mathbb{F}_q(T)$. \square

Now we are in condition to prove the dimension growth conjecture for varieties over global fields, namely Theorem 1.10 and Theorem 1.11 for the introduction.

Proof of Theorem 1.10. Let $n \geq 3$ and $X \subseteq \mathbb{A}_K^n$ be a geometrically integral hypersurface of degree $d \geq 5$ defined by a polynomial $f \in \mathcal{O}_K[X_1, \dots, X_n]$ with absolutely irreducible highest degree part. We proceed by induction on n , where the base case $n = 3$ is Proposition 7.3.

Now, let us assume that $n > 3$ and that the theorem holds for all lower n . Let f_d be the homogeneous part of highest degree of f ; since it is absolutely irreducible, it defines a geometrically integral hypersurface in \mathbb{P}_K^{n-1} . Applying the Combinatorial Nullstellensatz (see [1, Theorem 1.2]) to F of Lemma 7.5 there exists $A = (a_1, \dots, a_n)$ such that the hyperplane section $\{f_d = 0\} \cap \{\sum_i a_i X_i = 0\}$ is geometrically integral of degree d , with all $a_i \in \mathcal{O}_K$ having $H_K(a_i) \leq n(d^2 - 1)$ or $H_K(a_i) \leq 12nd^7$ if K is a number field or a function field, respectively. Let $\gamma = 2$ or $\gamma = 7$ if K is a number field or a function field, respectively. Since there exists a constant $c(n) \lesssim_n 1$ such that for any $(x_1, \dots, x_n) \in [B]_{\mathcal{O}_K}^n$ we have $a_1 x_1 + \dots + a_n x_n \in [c(n)d^\gamma B]_{\mathcal{O}_K}$, it follows that

$$N_{\text{aff}}(\mathcal{Z}(f), \mathcal{O}_K, B) \leq \sum_{k \in [c(n)d^\gamma B]_{\mathcal{O}_K}} N_{\text{aff}}\left(\{f = 0\} \cap \left\{\sum_i a_i X_i = k\right\}, \mathcal{O}_K, B\right).$$

For each k , the variety $\{f = 0\} \cap \{\sum_i a_i X_i = k\}$ is a hypersurface in the affine plane $\{\sum_i a_i X_i = k\}$, thus after a change of variables it is described by a polynomial $g \in \mathcal{O}_K[X_1, \dots, X_{n-1}]$ whose homogeneous part of highest degree is absolutely irreducible by the construction of A . The proof follows from the induction hypothesis and the fact that $|[B]_{\mathcal{O}_K}| \sim_K B$. \square

Proof of Theorem 1.11. We make a change of variables as in the proof of Theorem 5.9 to reduce Theorem 1.11 to the case of a hypersurface. Hence let $n \geq 3$ and consider an irreducible polynomial $f \in \mathcal{O}_K[X_0, \dots, X_n]$ of degree d . Then we take the affine cone $C(f)$ defined by $\mathcal{Z}(f)$; it is an affine hypersurface in \mathbb{A}_K^{n+1} . By Proposition 2.2 or by Remark 5.11, for any point in $\mathcal{Z}(f)(K, B)$, there is a lift in \mathbb{A}_K^{n+1} which lies in $\mathcal{Z}(C(f))_{\text{aff}}(\mathcal{O}_K, [c_1 B]_{\mathcal{O}_K}^{n+1})$. Thus

$$N(\mathcal{Z}(f), K, B) \leq N_{\text{aff}}(C(f), \mathcal{O}_K, c_1 B).$$

If f is absolutely irreducible, applying Theorem 1.10 finishes the proof. Otherwise, by Remark 5.3, there exists a homogeneous polynomial $g \in \mathcal{O}_K[X_0, \dots, X_{n+1}]$ of degree d not divisible by f and vanishing on all K -rational point of $C(f)$. Then

$$(7.10) \quad N_{\text{aff}}(C(f), \mathcal{O}_K, c_1 B) \leq N_{\text{aff}}(C(f) \cap \mathcal{Z}(g), \mathcal{O}_K, c_1 B).$$

Now, it holds that $C(f) \cap \mathcal{Z}(g)$ is a variety of dimension $n - 2$. By a hyperplane section argument as in [8, Page 91], it holds

$$(7.11) \quad N_{\text{aff}}(C(f) \cap \mathcal{Z}(g), \mathcal{O}_K, B) \lesssim_{K,n} dB^{n-2}.$$

Then inequalities (7.10) and (7.11) yield the conclusion of Theorem 1.11 for integral projective varieties which are not geometrically irreducible. \square

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