Hopf fibration in a C*-module

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Abstract

Let $\mathbf{X}$ be a right C^*-module over the unital C^*-algebra $\mathcal{A}$. As is usual, denote by $\mathcal{B}_{\mathcal{A}}(\mathbf{X})$ the C^*-algebra of bounded $\mathcal{A}$-linear adjointable operators acting in $\mathbf{X}$. We consider the elements $\mathbf{x} \in \mathbf{X}$ such that the submodule $[\mathbf{x}] = \{\mathbf{x}a : a \in \mathcal{A}\}$ generated by $\mathbf{x}$, is closed and complemented in $\mathbf{X}$. If this property holds, then the module $[\mathbf{x}]$ has a generator $\mathbf{x}_0$ such that $\langle \mathbf{x}_0, \mathbf{x}_0 \rangle$ is a projection in $\mathcal{A}$. Given $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, denote as usual $\theta_{\mathbf{x},\mathbf{y}} \in \mathcal{B}_{\mathcal{A}}(\mathbf{X})$ the rank one operator $\theta_{\mathbf{x},\mathbf{y}}(\mathbf{z}) = \mathbf{x}\langle \mathbf{y}, \mathbf{z} \rangle$ ($\mathbf{z} \in \mathbf{X}$). We denote by

$$\mathbf{X}_{\mathcal{P}} := \{\mathbf{x} \in \mathbf{X} : \langle \mathbf{x}, \mathbf{x} \rangle \text{ is a projection in } \mathcal{A}\},$$

which is shown here to be a C^∞ submanifold of $\mathbf{X}$. It is easy to see that $\theta_{\mathbf{x},\mathbf{x}}$ is a projection if and only if $\mathbf{x} \in \mathbf{X}_{\mathcal{P}}$. The *Hopf map* of $\mathbf{X}$ is the C^∞ map

$$\mathfrak{h} : \mathbf{X}_{\mathcal{P}} \to \{\text{projections in } \mathcal{B}_{\mathcal{A}}(\mathbf{X})\}, \ \mathfrak{h}(\mathbf{x}) = \theta_{\mathbf{x},\mathbf{x}}.$$

The set of projections in a C^*-algebra is known to be a differentiable submanifold of the algebra. We show that and that $\mathfrak{h}$ is a submersion onto de submanifold of rank one projections. We introduce a linear connection and a metric in $\mathbf{X}_{\mathcal{P}}$, and show that the tangent maps of $\mathfrak{h}$ are contractive between the corresponding tangent spaces. Also, for a fixed projection $p \in \mathcal{A}$, we study the submanifold $\mathbf{X}_p \subset \mathbf{X}_{\mathcal{P}}$, $\mathbf{X}_p := \{\mathbf{x} \in \mathbf{X} : \langle \mathbf{x}, \mathbf{x} \rangle = p\}$. We prove a minimality result for curves in $\mathbf{X}_p$ with given initial conditions. This result makes use of an generalizations of Krein's extension method (for symmetric transformations between Hilbert spaces) to the realm of Hilbert C^*-modules. We consider several examples of modules, and examine these geometric general properties in the examples.

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1 Introduction

Let $\mathbf{X}$ be a (right) C*-module over the unital C*-algebra $\mathcal{A}$. Denote by $\langle \mathbf{x}, \mathbf{y} \rangle \in \mathcal{A}$ the inner product (linear in the factor $\mathbf{y}$, skew-linear in $\mathbf{x}$), and by $\theta_{\mathbf{x},\mathbf{y}}$ the modular rank one operator $\theta_{\mathbf{x},\mathbf{y}}(\mathbf{z}) = \mathbf{x}\langle \mathbf{y}, \mathbf{z} \rangle$, for $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbf{X}$. In this paper we characterize the elements $\mathbf{x} \in \mathbf{X}$ such that the singly generated $\mathcal{A}$-linear manifold $[\mathbf{x}] = \{\mathbf{x}a : a \in \mathcal{A}\}$ is closed and ortho-complemented in $\mathbf{X}$. We shall call such submodules of $\mathbf{X}$ *singly generated submodules*. As we shall see below, $[\mathbf{x}]$ is a singly generated submodule if and only if $\langle \mathbf{x}, \mathbf{x} \rangle = a$ has closed range (i.e., is a positive element in $\mathcal{A}$ whose spectrum $\sigma(a)$ satisfies that $\inf(\sigma(a) \setminus \{0\}) > 0$). This condition is equivalent to the existence of a generator $\mathbf{x}'$ of $[\mathbf{x}]$ such that $\langle \mathbf{x}', \mathbf{x}' \rangle$ is a projection in $\mathcal{A}$. We denote

$$\mathbf{X}_{\mathcal{P}} := \{\mathbf{x} \in \mathbf{X} : \langle \mathbf{x}, \mathbf{x} \rangle \text{ is a projection in } \mathcal{A}\}.$$

The Hopf fibration $\mathfrak{h}$ of the title is the map

$$\mathfrak{h} : \mathbf{X}_{\mathcal{P}} \to \{\text{singly generated submodules of } \mathbf{X}\},\ \mathfrak{h}(\mathbf{x}) = [\mathbf{x}].$$

Alternatively, since there is a one to one correspondence between a complemented submodule $\mathbf{Y} \subset \mathbf{X}$ and its adjointable orthoprojection $P_{\mathbf{Y}}$, it will be useful for our computations to identify $[\mathbf{x}] \sim P_{[\mathbf{x}]}$. As we shall see, when $\mathbf{x} \in \mathbf{X}_{\mathcal{P}}$, one has $P_{[\mathbf{x}]} = \theta_{\mathbf{x},\mathbf{x}}$. We shall call these projections *rank one projections.* Thus we have the operator valued version of the map $\mathfrak{h}$:

$$\mathfrak{h} : \mathbf{X}_{\mathcal{P}} \to \{\text{rank one projections in } \mathcal{B}_{\mathcal{A}}(\mathbf{X})\},\ \mathfrak{h}(\mathbf{x}) = \theta_{\mathbf{x},\mathbf{x}},$$

where, as is usual notation, $\mathcal{B}_{\mathcal{A}}(\mathbf{X})$ is the C*-algebra of adjointable $\mathcal{A}$-linear bounded operators in $\mathbf{X}$. We abreviate

$$\mathbb{P}_1(\mathbf{X}) := \{\text{singly generated submodules of } \mathbf{X}\} \simeq \{\text{rank one projections in } \mathcal{B}_{\mathcal{A}}(\mathbf{X})\}.$$

As we shall see, the spaces involved are C^∞-manifolds, and the map $\mathfrak{h}$ is a C^∞ submersion.

If $\mathcal{B}$ is a C*-algebra, denote by $\mathcal{U}_{\mathcal{B}}$ the Banach-Lie group of unitary elements in $\mathcal{B}$. Its Banach-Lie algebra is the space $\mathcal{B}_{ah} = \{b \in \mathcal{B} : b^* = -b\}$ of anti-Hermitian elements of $\mathcal{B}$. Denote by $\mathcal{P}_{\mathcal{B}}$ the (Grassmann) manifold of projections in $\mathcal{B}$.

There is another fibration which is relevant in this discussion: the *product map*

$$\rho : \mathbf{X}_{\mathcal{P}} \to \mathcal{P}_{\mathcal{A}},\ \rho(\mathbf{x}) = \langle \mathbf{x}, \mathbf{x} \rangle.$$

In the special case when $\mathbf{X} = \mathcal{A}$, $\mathbf{X}_{\mathcal{P}}$ is the space of partial isometries in $\mathcal{A}$, and $\mathfrak{h}(\mathbf{x})$ and $\rho(\mathbf{x})$ are, respectively, the initial and final projections of the partial isometry $\mathbf{x}$.

We introduce a reductive structure in $\mathbf{X}_{\mathcal{P}}$, and a natural Finsler metric. We prove that with this metric, the map $\mathfrak{h}$ is contractive at the level of the tangent spaces (here the target space, the space of projections of the C*-algebra $\mathcal{B}_{\mathcal{A}}(\mathbf{X})$, is considered with its usual Grassmann geometry [5]). Moreover, $\mathfrak{h}$ maps geodesics of $\mathbf{X}_{\mathcal{P}}$ into geodesics of the Grassmann manifold of $\mathcal{B}_{\mathcal{A}}(\mathbf{X})$.

We characterize curves of minimal length in $\mathbf{X}_{\mathcal{P}}$, with given initial conditions. Namely, if $\mathbf{x} \in \mathbf{X}_{\mathcal{P}}$ and $\mathbf{v}$ is a tangent vector at $\mathbf{x}$, we prove the following facts:

1. there exists an anti-Hermitian operator $Z \in \mathcal{B}_{\mathcal{A}}(\mathbf{X})$ such that $Z\mathbf{x} = \mathbf{v}$, and Z has minimal norm among all anti-Hermitian operators with this property;

2. for any such Z (which may not be unique), the curve (in $\mathbf{X}_{\mathcal{P}}$)

$$\delta(t) = e^{tZ}\mathbf{x}$$

   has minimal length among all smooth curves in $\mathbf{X}_{\mathcal{P}}$ joining the same endpoints as δ, as long as $|t|\|Z\| \leq \pi/2$.

The first fact follows from adapting the so called *Krein's method* for finding extensions of a symmetric operator defined on a subspace, valid in the context of Hilbert spaces, to work for complemented submodules of a Hilbert C*-module. We do believe that this fact might be of interest, independently of its application in this context. The second fact follows from ideas first discussed in [8].

We consider several examples of modules and algebras, and focus on their particular features. Among them, (i) $\mathbf{X} = \mathcal{A}^N$, (ii) $\mathbf{X}$ the standard module $H(\mathcal{A})$ of square summable sequences in $\mathcal{A}$, (iii) $\mathbf{X} = \mathcal{B}$ a finite von Neumann factor, such that $\mathcal{A} \subset \mathcal{B}$ has finite Jones index.

2 Singly generated submodules

Let $\mathbf{X}$ be a right C*-module over a unital C*-algebra $\mathcal{A}$. Let $\mathbf{x} \in \mathbf{X}$. Denote

$$[\mathbf{x}] := \{\mathbf{x}a : a \in \mathcal{A}\} \subset \mathbf{X}.$$

Our first goal will be to characterize the elements $\mathbf{x} \in \mathbf{X}$ such that $[\mathbf{x}]$ is a closed and complemented submodule of $\mathbf{X}$, i.e., those $\mathbf{x}$ such that $[\mathbf{x}] \oplus [\mathbf{x}]^{\perp} = \mathbf{X}$. Here, if $\mathbf{Y} \subset \mathbf{X}$, $\mathbf{Y}^{\perp} = \{\mathbf{z} \in \mathbf{X} : \langle \mathbf{y}, \mathbf{z} \rangle = 0 \text{ for all } \mathbf{y} \in \mathbf{Y}\}$.

Once we single out these elements, we shall study the structure of the set of all such singly generated complemented submodules. If $[\mathbf{x}]$ is complemented, it will be useful to compute the orthogonal projections $P_{[\mathbf{x}]}$ onto $[\mathbf{x}]$. As is usual notation, given $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, let $\theta_{\mathbf{x},\mathbf{y}} \in \mathcal{B}_{\mathcal{A}}(\mathbf{X})$ be the operator given by

$$\theta_{\mathbf{x},\mathbf{y}}(\mathbf{z}) = \mathbf{x}\langle \mathbf{y}, \mathbf{z} \rangle.$$

Remark 2.1. Note the following basic properties of the elementary operators $\theta_{\mathbf{x},\mathbf{y}}$:

1. $\theta^*_{\mathbf{x},\mathbf{y}} = \theta_{\mathbf{y},\mathbf{x}}$;
2. if $S, T \in \mathcal{B}_{\mathcal{A}}(\mathbf{X})$, $S\theta_{\mathbf{x},\mathbf{y}}T = \theta_{S\mathbf{x},T^*\mathbf{y}}$;
3. $\theta_{\mathbf{x}a,\mathbf{y}} = \theta_{\mathbf{x},\mathbf{y}a^*}$;
4. $\theta_{\mathbf{x},\mathbf{y}}\theta_{\mathbf{z},\mathbf{v}} = \theta_{\mathbf{x}\langle \mathbf{y},\mathbf{z}\rangle,\mathbf{v}} = \theta_{\mathbf{x},\mathbf{v}\langle \mathbf{z},\mathbf{y}\rangle}$.

A candidate for $P_{[\mathbf{x}]}$ could be $\theta_{\mathbf{x},\mathbf{x}}$. Our first result shows which elements $\mathbf{x}$ give rise to orthogonal projections:

Lemma 2.2. *Let $\mathbf{x} \in \mathbf{X}$. Then $\theta_{\mathbf{x},\mathbf{x}}$ is a projection in $\mathcal{B}_{\mathcal{A}}(\mathbf{X})$ if and only if $\langle \mathbf{x}, \mathbf{x} \rangle = p$ is a projection in $\mathcal{A}$.*

In this case, $\mathbf{x}p = \mathbf{x}$, and p will be denoted the support *of $\mathbf{x}$.*

Proof. Suppose that $\theta_{\mathbf{x},\mathbf{x}}$ is a projection in $\mathcal{B}_{\mathcal{A}}(\mathbf{X})$, and let $\langle \mathbf{x}, \mathbf{x} \rangle = a \geq 0$. The fact that $(\theta_{\mathbf{x},\mathbf{x}})^2 = \theta_{\mathbf{x},\mathbf{x}}$ means that

$$\theta_{\mathbf{x}a,\mathbf{x}} = \theta_{\mathbf{x},\mathbf{x}}.$$

Evaluating at $\mathbf{x}$, we get $\mathbf{x}a^2 = \mathbf{x}a$. If we compute the inner product of this identity against $\mathbf{x}$, we get $a^3 = a^2$. This implies that the spectrum $\sigma(a)$ of a is contained in the set of roots of the polynomial $t^3 - t^2$, namely $\{-1, 0, 1\}$. Since $a \geq 0$, $\sigma(a) \subset \{0, 1\}$, i.e., a is a projection in $\mathcal{A}$.

Conversely, suppose that $\langle \mathbf{x}, \mathbf{x} \rangle = p$ is a projection in $\mathcal{A}$. Then

$$\langle \mathbf{x}(1-p), \mathbf{x}(1-p) \rangle = (1-p)\langle \mathbf{x}, \mathbf{x} \rangle (1-p) = 0,$$

and, thus, $\mathbf{x} = \mathbf{x}p$. Then

$$(\theta_{\mathbf{x},\mathbf{x}})^2 = \theta_{\mathbf{x}p,\mathbf{x}} = \theta_{\mathbf{x},\mathbf{x}}.$$

□

An element $a \in \mathcal{A}$ has *closed range* if a^*a is invertible or otherwise 0 is an isolated point in $\sigma(a^*a)$. Clearly a has closed range if and only if a^* has closed range. Note that if a has closed range, then for any faithful $*$-representation π of $\mathcal{A}$ in a Hilbert space $\mathcal{H}$, then $\pi(a)\mathcal{H}$ is a closed subspace of $\mathcal{H}$. Also, in this case, the orthogonal projection $P_{\pi(a)\mathcal{H}}$ belongs to $\pi(\mathcal{A})$.

Thus there exists a projection p_a in $\mathcal{A}$ such that $p_a a = a$. Moreover, there exists $a^\dagger \in \mathcal{A}$, called the *Moore-Penrose pseudoinverse* of a, which satisfies (see, for instance [9])

$$aa^\dagger a = a, \ a^\dagger a a^\dagger = a^\dagger, \ aa^\dagger = p_a, \ a^\dagger a = p_{a^\dagger}.$$

Note that if $a \geq 0$ has closed range, $p_a = p_{a^\dagger}$. Clearly $a \geq 0$ has closed range if and only if $a^{1/2}$ has closed range.

Also note that a has closed range if and only if $a\mathcal{A}$ is closed.

Let us state our result characterizing $\mathbf{x} \in \mathbf{X}$ such that $[\mathbf{x}]$ is a complemented submodule:

Theorem 2.3. *Let* $\mathbf{x} \in \mathbf{X}$. *Then the following are equivalent:*

1. $[\mathbf{x}] \oplus [\mathbf{x}]^\perp = \mathbf{X}$.
2. $[\mathbf{x}]$ *is closed in* $\mathbf{X}$.
3. $\langle \mathbf{x}, \mathbf{x}\rangle^{1/2}$ (*or equivalently,* $\langle \mathbf{x}, \mathbf{x}\rangle$) *has closed range.*
4. *There exists a generator* $\mathbf{x}_0$ *of* $[\mathbf{x}]$ (*i.e.,* $[\mathbf{x}_0] = [\mathbf{x}]$) *such that* $\langle \mathbf{x}_0, \mathbf{x}_0\rangle = p$ *is a projection in* $\mathcal{A}$. *In this case.* $P_{[\mathbf{x}]} = \theta_{\mathbf{x}_0,\mathbf{x}_0}$.

Proof. 1. $\Rightarrow$ 2. is evident.

2. $\Rightarrow$ 3.: suppose that $\langle \mathbf{x}, \mathbf{x}\rangle a_n \to b$ in $\mathcal{A}$. Then $(\langle \mathbf{x}, \mathbf{x}\rangle a_n)$ is a Cauchy sequence. Note that

$$\|\mathbf{x}a_n - \mathbf{x}a_m\| = \|\langle \mathbf{x}(a_n - a_m), \mathbf{x}(a_n - a_m)\rangle\|^{1/2} = \|(a_n - a_m)^* \langle \mathbf{x}, \mathbf{x}\rangle (a_n - a_m)\|^{1/2}$$

$$= \|\langle \mathbf{x}, \mathbf{x}\rangle^{1/2}(a_n - a_m)\| = \|\langle \mathbf{x}, \mathbf{x}\rangle^{1/2}a_n - \langle \mathbf{x}, \mathbf{x}\rangle^{1/2}a_m\|,$$

and then $(\mathbf{x}a_n)$ is a Cauchy sequence in $[\mathbf{x}]$. Thus there exists $c \in \mathcal{A}$ such that $\mathbf{x}a_n \to \mathbf{x}c$. A similar computation as above shows that $\langle \mathbf{x}, \mathbf{x}\rangle^{1/2}a_n \to \langle \mathbf{x}, \mathbf{x}\rangle^{1/2}c$.

3. $\Rightarrow$ 4.: let $b = b^*$ be the Moore-Penrose pseudo-inverse of $a = \langle \mathbf{x}, \mathbf{x}\rangle^{1/2}$. Then put $\mathbf{x}_0 = \mathbf{x}b$. Note that

$$\langle \mathbf{x}_0, \mathbf{x}_0\rangle = \langle \mathbf{x}b, \mathbf{x}b\rangle = ba^2b = p_a.$$

As seen above, in this case $\mathbf{x}_0 p_a = \mathbf{x}_0$. Clearly $[\mathbf{x}_0] \subset [\mathbf{x}]$. Note also that $\mathbf{x} = \mathbf{x}p_a$:

$$\langle \mathbf{x}(1-p_a), \mathbf{x}(1-p_a)\rangle = \langle \mathbf{x}, \mathbf{x}\rangle - \langle \mathbf{x}p_a, \mathbf{x}\rangle - \langle \mathbf{x}, \mathbf{x}p_a\rangle + \langle \mathbf{x}p_a, \mathbf{x}p_a\rangle = a^2 - p_a a^2 - a^2 p_a + p_a a^2 p_a = 0.$$

Then $\mathbf{x} = \mathbf{x}p_a = \mathbf{x}ba = \mathbf{x}_0 a$, i.e., $[\mathbf{x}] \subset [\mathbf{x}_0]$.

4. $\Rightarrow$ 1.: $\theta_{\mathbf{x}_0,\mathbf{x}_0}$ is a selfadjoint projection, whose range is $[\mathbf{x}_0] = [\mathbf{x}]$. □

We shall call these (orthocomplemented) modules $[\mathbf{x}] \subset \mathbf{X}$ *singly generated modules.*

3 The space of generators (of singly generated submodules)

Let us denote by

$$\mathbf{X}_{\mathcal{P}} = \{\mathbf{x} \in \mathbf{X} : \langle \mathbf{x}, \mathbf{x}\rangle \text{ is a projection in } \mathcal{A}\}.$$

We shall study the structure of $\mathbf{X}_{\mathcal{P}}$. Recall that two projections p, q are said to be *equivalent* in $\mathcal{A}$, $p \sim q$, if there exists $v \in \mathcal{A}$ such that $v^*v = p$ and $vv^* = q$. In this case. v is called a *partial isometry*, with initial space p and final space q.

Proposition 3.1. *Suppose that* $\mathbf{x}_0$, $\mathbf{x}_1$ *generate the same module* $[\mathbf{x}_0] = [\mathbf{x}_1]$, *and* $\langle \mathbf{x}_0, \mathbf{x}_0 \rangle = p_0$ *and* $\langle \mathbf{x}_1, \mathbf{x}_1 \rangle = p_1$ *are projections in* $\mathcal{A}$. *Then* $p_0 \sim p_1$.

Proof. Since $[\mathbf{x}_0] = [\mathbf{x}_1]$ and $\langle \mathbf{x}_0, \mathbf{x}_0 \rangle = p_0$ and $\langle \mathbf{x}_1, \mathbf{x}_1 \rangle = p_1$ are projections, we have that $\theta_{\mathbf{x}_0,\mathbf{x}_0} = \theta_{\mathbf{x}_1,\mathbf{x}_1}$ is the orthogonal projection onto $[\mathbf{x}_0]$. Then, since $\mathbf{x}_0, \mathbf{x}_1 \in [\mathbf{x}_0]$, it holds

$$\mathbf{x}_1 = \theta_{\mathbf{x}_0,\mathbf{x}_0}(\mathbf{x}_1) = \mathbf{x}_0 \langle \mathbf{x}_0, \mathbf{x}_1 \rangle = \mathbf{x}_0 a \quad \text{and} \quad \mathbf{x}_0 = \theta_{\mathbf{x}_1,\mathbf{x}_1}(\mathbf{x}_0) = \mathbf{x}_1 \langle \mathbf{x}_1, \mathbf{x}_0 \rangle = \mathbf{x}_1 a^*.$$

Note that

$$p_1 a = \langle \mathbf{x}_0, \mathbf{x}_1 \rangle p_1 = \langle \mathbf{x}_0, \mathbf{x}_1 p_1 \rangle = \langle \mathbf{x}_0, \mathbf{x}_1 \rangle = a,$$

and, similarly,

$$a^* p_0 = \langle \mathbf{x}_1, \mathbf{x}_0 \rangle p_0 = \langle \mathbf{x}_1, \mathbf{x}_0 p_0 \rangle = \langle \mathbf{x}_1, \mathbf{x}_0 \rangle = a^*.$$

Then

$$p_1 = \langle \mathbf{x}_1, \mathbf{x}_1 \rangle = \langle \mathbf{x}_0 a, \mathbf{x}_0 a \rangle = a^* \langle \mathbf{x}_0, \mathbf{x}_0 \rangle a = a^* p_0 a = a^* a,$$

and

$$p_0 = \langle \mathbf{x}_0, \mathbf{x}_0 \rangle = \langle \mathbf{x}_1 a^*, \mathbf{x}_1 a^* \rangle = a \langle \mathbf{x}_1, \mathbf{x}_1 \rangle a = a p_1 a^* = a a^*.$$

□

We shall denote by $\sim_u$ the *unitary equivalence* of projections: $p_1 \sim_u p_2$ if there exists a unitary element $u \in \mathcal{A}$ such that $u p_1 u^* = p_2$.

Remark 3.2. The following simple example shows that if $[\mathbf{x}_1] = [\mathbf{x}_2]$, then $\langle \mathbf{x}_1, \mathbf{x}_1 \rangle$ and $\langle \mathbf{x}_2, \mathbf{x}_2 \rangle$ need not be unitarily equivalent. Let $\mathcal{A} = \mathbf{X} = \mathcal{B}(\ell^2)$, and consider $\mathbf{x}_1 = S^*$, $\mathbf{x}_2 = (S^*)^2$, where S is the shift operator. Both elements generate the same module:

$$\mathbf{x}_2 = (S^*)^2 = \mathbf{x}_1 S^* \ , \quad \text{and } \mathbf{x}_1 = (S^*)^2 S = \mathbf{x}_2 S,$$

but $\langle \mathbf{x}_1, \mathbf{x}_1 \rangle = S S^*$ has co-rank 1 and $\langle \mathbf{x}_2, \mathbf{x}_2 \rangle = S^2 (S^*)^2$ has co-rank 2, and thus are not unitarily equivalent.

Next, note that elements in $\mathbf{X}_{\mathcal{P}}$ that are close, give rise to unitarily equivalent projections in $\mathcal{A}$ and $\mathcal{B}_{\mathcal{A}}(\mathbf{X})$:

Proposition 3.3. *Let* $\mathbf{x}_0, \mathbf{x}_1 \in \mathbf{X}_{\mathcal{P}}$ *such that* $\|\mathbf{x}_0 - \mathbf{x}_1\| < \frac{1}{2}$, *and denote* $p_i = \langle \mathbf{x}_i, \mathbf{x}_i \rangle$ *and* $P_i = \theta_{\mathbf{x}_i,\mathbf{x}_i}$, $i = 1, 2$. *Then*

$$\|p_0 - p_1\| < 1 \quad \textit{and} \quad \|P_0 - P_1\| < 1.$$

In particular, $p_0 \sim_u p_1$ *in* $\mathcal{A}$ *and* $P_1 \sim_u P_2$ *in* $\mathcal{B}_{\mathcal{A}}(\mathbf{X})$

Proof. First,

$$\|p_0 - p_1\| = \|\langle \mathbf{x}_0, \mathbf{x}_0 \rangle - \langle \mathbf{x}_1, \mathbf{x}_1 \rangle\| \leq \|\langle \mathbf{x}_0, \mathbf{x}_0 \rangle - \langle \mathbf{x}_0, \mathbf{x}_1 \rangle\| + \|\langle \mathbf{x}_0, \mathbf{x}_1 \rangle - \langle \mathbf{x}_1, \mathbf{x}_1 \rangle\|$$

$$= \|\langle \mathbf{x}_0, \mathbf{x}_0 - \mathbf{x}_1 \rangle\| + \|\langle \mathbf{x}_0 - \mathbf{x}_1, \mathbf{x}_1 \rangle\| \leq \|\mathbf{x}_0\| \|\mathbf{x}_0 - \mathbf{x}_1\| + \|\|\mathbf{x}_0 - \mathbf{x}_1\| \|\mathbf{x}_1\| < 1,$$

because $\|\mathbf{x}_i\| = \|\langle \mathbf{x}_i, \mathbf{x}_1 \rangle\|^{1/2} = 1$, for $i = 1, 2$. Similarly,

$$||P_0 - P_1\| \leq \|\theta_{\mathbf{x}_0,\mathbf{x}_0} - \theta_{\mathbf{x}_0,\mathbf{x}_1}\| + \|\theta_{\mathbf{x}_0,\mathbf{x}_1} - \theta_{\mathbf{x}_1,\mathbf{x}_1}\| = \|\theta_{(\mathbf{x}_0 - \mathbf{x}_1),\mathbf{x}_0}\| + \|\theta_{\mathbf{x}_1,(\mathbf{x}_0 - \mathbf{x}_1)}\|.$$

Note that for $\mathbf{x},\mathbf{y},\mathbf{z}\in\mathbf{X}$, $\|\theta_{\mathbf{x},\mathbf{y}}(\mathbf{z})\|=\|\mathbf{x}\langle\mathbf{y},\mathbf{z}\rangle\|\le\|\mathbf{x}\|\|\mathbf{y}\|\|\mathbf{z}\|$. Then $\|\theta_{\mathbf{x},\mathbf{y}}\|\le\|\mathbf{x}\|\|\mathbf{y}\|$. Thus,

$$\|P_1-P_2\|\le\|\mathbf{x}_0-\mathbf{x}_1\|\|\mathbf{x}_0\|+\|\mathbf{x}_1\|\|\mathbf{x}_0-\mathbf{x}_1\|<1.$$

It is well known that in a C*-algebra, two projections at distance less than 1, are unitarily equivalent. □

Let us elaborate more the last assertion of this proof.

Remark 3.4. (see [17]) If P,Q are projections in a C*-algebra $\mathcal{B}$, such that $\|P-Q\|<1$, then there exists a unitary $\mathbf{U}_P(Q)$ such that

- $\mathbf{U}_P(P)=1$.
- $\mathbf{U}_P(Q)P\mathbf{U}_P(Q)^*=Q$.
- The map $\mathcal{P}_\mathcal{B}\times\mathcal{P}_\mathcal{B}\to\mathcal{U}_\mathcal{B}$, $(P,Q)\mapsto\mathbf{U}_P(Q)$ is a C^∞ map.

First we shall use this fact for projections p in $\mathcal{A}$, with the action of the unitary group $\mathcal{U}_\mathcal{A}$ of $\mathcal{A}$. There are many ways to choose the unitaries $\mathbf{U}_{p_0}(p)$ (which provide cross sections for the action of $\mathcal{U}_\mathcal{A}$ on the unitary orbit of p_0). We shall consider the reductive cross section (see [17], [5]), based on the exponential map of the manifold $\mathcal{P}_\mathcal{A}$: for all $p\in\mathcal{P}_\mathcal{A}$ such that $\|p-p_0\|<1$, there exists a unique $z=z_{p_0}(p)$, which is a C^∞ map on p, with $z^*=-z$, $p_0zp_0=(1-p_0)z(1-p_0)=0$ (z is p_0-co-diagonal), $\|z\|<\pi/2$, such that $e^zp_0e^{-z}=p$. Then we choose

$$\mathbf{U}_{p_0}(p)=e^z=e^{z_{p_0}(p)}.$$

We have that (see [17])

$$\mathbf{U}_{p_0}:\{p\in\mathcal{P}_\mathcal{A}:\|p-p_0\|<1\}\xrightarrow{\simeq}\{u\in\mathcal{U}_\mathcal{A}:u(2p_0-1)=(2p_0-1)u^*\text{ and }\|u-1\|<\sqrt{2}\},$$

is a C^∞-diffeomorphism.

Lemma 3.5. *The set $\{u\in\mathcal{U}_\mathcal{A}:u(2p_0-1)=(2p_0-1)u^*$ and $\|u-1\|<\sqrt{2}\}$ is a C^∞ complemented submanifold of $\mathcal{U}_\mathcal{A}$*

Proof. If $u\in\mathcal{U}_\mathcal{A}$ satisfies that $\|u-1\|<2$, then there exists a unique z such that $z^*=-z$, $\|z\|<\pi$ and e^z. If, additionally, $\|u-1\|<\sqrt{2}$, an elementary computation shows that $\|z\|<\pi/2$. Moreover, the exponential map

$$\{z\in\mathcal{A}:z^*=-z,\|z\|<\pi/2\}\xrightarrow{\exp}\{u\in\mathcal{U}_\mathcal{A}:\|u-1\|<1\}\ ,\quad\exp(z)=e^z$$

is a C^∞ diffeomorphism. Note that $u=e^z$ satisfies $u(2p_0-1)=(2p_0-1)u^*$ if and only if $z(2p_0-1)=-(2p_0-1)z$. Indeed, if $z(2p_0-1)=-(2p_0-1)z$ then $z^n(2p_0-1)=(2p_0-1)(-z)^n$, and thus $e^z(2p_0-1)=(2p_0-1)e^{-z}$. Conversely, if $u(2p_0-1)=(2p_0-1)u^*$, then $u^n(2p_0-1)=(2p_0-1)(u^*)^n$, and therefore for any continuous function f on $\sigma(u)\subset\{e^{it}:t\in(-\pi/2,\pi/2)\}$, one has $f(u)(2p_0-1)=(2p_0-1)f(u^*)$. In particular, for $f(e^{it})=it$, $f(u)=z$, for the unique $z^*=-z$ with $\|z\|<\pi/2$, we have $z(2p_0-1)=(2p_0-1)z^*=-(2p_0-1)z$. Therefore, the above diffeomorphism exp maps

$$\{z\in\mathcal{A}:z^*=-z,\|z\|<\pi/2,z(2p_0-1)=-(2p_0-1)z\}$$

onto

$$\{u \in \mathcal{U}_{\mathcal{A}} : \|u - 1\| < 1, u(2p_0 - 1) = (2p_0 - 1)u^{-1}\}.$$

The former set is an open subset of the linear manifold

$$\{z \in \mathcal{A} : z^* = -z, z \text{ is } p_0 - \text{co-diagonal}\}.$$

Recall that z is p-co-diagonal if $pzp = p^{\perp}zp^{\perp} = 0$. It is easy to see that z anti-commutes with $2p_0 - 1$ if and only if it is p_0-co-diagonal. Clearly this linear manifold is complemented in $\mathcal{A}_{ah}$, by the set of elements in $\mathcal{A}_{ah}$ which commute with p_0 (are p_0-diagonal). Therefore, $\{u \in \mathcal{U}_{\mathcal{A}} : \|u - 1\| < 1, u(2p_0 - 1) = (2p_0 - 1)u^{-1}\}$ is a complemented C$^\infty$-submanifold of $\mathcal{U}_{\mathcal{A}}$. □

Given a fixed $p_0 \in \mathcal{P}_{\mathcal{A}}$, consider the set $\mathbf{X}_{p_0} \subset \mathbf{X}_{\mathcal{P}}$ given by

$$\mathbf{X}_{p_0} = \{\mathbf{x} \in \mathbf{X} : \langle \mathbf{x}, \mathbf{x} \rangle = p_0\}.$$

Remark 3.6. As we shall see in the examples, $\mathbf{X}_{p_0}$ may be empty. However, if it is non empty, in [2] it was shown that $\mathbf{X}_{p_0}$ is a C$^\infty$ submanifold of $\mathbf{X}$. Let us sketch the proof of this fact. Recall that if $\langle \mathbf{x}, \mathbf{x} \rangle = p_0$, then $\mathbf{x} = \mathbf{x}p_0$, and thus $\mathbf{X}_{p_0} \subset \mathbf{X}p_0$. Moreover, if $\mathbf{x}, \mathbf{y} \in \mathbf{X}_{p_0}$, then $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{x}p_0, \mathbf{y}p_0 \rangle = p_0 \langle \mathbf{x}, \mathbf{y} \rangle p_0 \in p_0\mathcal{A}p_0$. Note that $\mathbf{X}p_0 \subset \mathbf{X}$ is a complemented analytic submanifold, that $p_0\mathcal{A}p_0 \subset \mathcal{A}$ is a C*-subalgebra with unit p_0, and that $\mathbf{X}p_0$ is a (complete) C*-right-module over $p_0\mathcal{A}p_0$. Thus, in order to prove that $\mathbf{X}_{p_0}$ is a submanifold of $\mathbf{X}$, it suffices to show that $\mathbf{X}_{p_0}$ is a submanifold of $\mathbf{X}p_0$. This means that we can start over again, supposing that $p_0 = 1$.

In this context, denote by $\mathcal{A}^+_{\bullet} = \{g \in \mathcal{A} : g \geq 0 \text{ and } g \text{ is invertible }\}$, and note that $\{\mathbf{x} \in \mathbf{X} : \langle \mathbf{x}, \mathbf{x} \rangle \in \mathcal{A}^+_{\bullet}\}$ is open in $\mathbf{X}$. The map

$$\rho : \{\mathbf{x} \in \mathbf{X} : \langle \mathbf{x}, \mathbf{x} \rangle \in \mathcal{A}^+_{\bullet}\} \to \mathcal{A}^+_{\bullet} \ , \quad \rho(\mathbf{x}) = \langle \mathbf{x}, \mathbf{x} \rangle$$

is a C$^\infty$ submersion, with a global smooth section (the set $\mathcal{A}^+_{\bullet}$ is open in $\mathcal{A}_h = \{x \in \mathcal{A} : x^* = x\}$, and therefore a submanifold of $\mathcal{A}$). Indeed, pick $\mathbf{x}_0$ such that $\langle \mathbf{x}_0, \mathbf{x}_0 \rangle = 1$ and put $\mathbf{s} : \mathcal{A}^+_{\bullet} \to \{\mathbf{x} \in \mathbf{X} : \langle \mathbf{x}, \mathbf{x} \rangle \in \mathcal{A}^+_{\bullet}\}$, $\mathbf{s}(g) = \mathbf{x}_0 g^{1/2}$. It is easy to see that $\mathbf{s}$ is a C$^\infty$ global section for ρ. Thus ρ is a C$^\infty$ submersion, and then

$$\mathbf{X}_1 = \{\mathbf{x} \in \mathbf{X} : \langle \mathbf{x}, \mathbf{x} \rangle = 1\} = \rho^{-1}(\{1\})$$

is a C$^\infty$ submanifold of $\mathbf{X}$.

In [1] (Prop. 2.4), it is also shown that the action of $\mathcal{U}_{\mathcal{B}_{\mathcal{A}}(\mathbf{X})}$ in $\mathbf{X}_{p_0}$, given by $U \cdot \mathbf{x} = U\mathbf{x}$ is locally transitive: if $\mathbf{x}_1, \mathbf{x}_2 \in \mathbf{X}_{p_0}$ satisfy that $\|\mathbf{x}_1 - \mathbf{x}_2\| < \frac{1}{2}$, then there exists $U = U_{\mathbf{x}_1, \mathbf{x}_2} \in \mathcal{U}_{\mathcal{B}_{\mathcal{A}}(\mathbf{X})}$, which is a C^∞ map in the parameters $\mathbf{x}_1, \mathbf{x}_2$, such that $U\mathbf{x}_1 = \mathbf{x}_2$.

Theorem 3.7. *The set* $\mathbf{X}_{\mathcal{P}}$ *is a* C^∞ *submanifold of* $\mathbf{X}$.

Proof. Pick $\mathbf{x}_0 \in \mathbf{X}_{\mathcal{P}}$ with $\langle \mathbf{x}_0, \mathbf{x}_0 \rangle = p_0$. Denote by $\mathcal{D}_{\mathbf{x}_0}$ the set

$$\mathcal{D}_{\mathbf{x}_0} = \{\mathbf{x} \in \mathbf{X}_{\mathcal{P}} : \|\langle \mathbf{x}, \mathbf{x} \rangle - p_0\| < 1\}.$$

Note that since the map $\mathbf{X} \ni \mathbf{x} \mapsto \langle \mathbf{x}, \mathbf{x} \rangle \in \mathcal{A}$ is continuous, the set $\mathcal{D}_{\mathbf{x}_0} \subset \mathbf{X}_{\mathcal{P}}$ is relatively open in $\mathbf{X}_{\mathcal{P}}$. Consider the map

$$\Phi : \mathcal{D}_{\mathbf{x}_0} \to \mathbf{X}_{p_0} \times \mathcal{U}_{\mathcal{A}}, \ \Phi(\mathbf{x}) = (\mathbf{x}\mathbf{U}_{p_0}(\langle \mathbf{x}, \mathbf{x} \rangle), \mathbf{U}_{p_0}(\langle \mathbf{x}, \mathbf{x} \rangle)) \,.$$

First note that Φ is well defined: since $\mathbf{U}_{p_0}(\langle \mathbf{x}, \mathbf{x}\rangle) p_0 \mathbf{U}_{p_0}(\langle \mathbf{x}, \mathbf{x}\rangle)^* = \langle \mathbf{x}, \mathbf{x}\rangle$, we have that

$$p_0 = \mathbf{U}^*_{p_0}(\langle \mathbf{x}, \mathbf{x}\rangle)\langle \mathbf{x}, \mathbf{x}\rangle \mathbf{U}_{p_0}(\langle \mathbf{x}, \mathbf{x}\rangle) = \langle \mathbf{x}\mathbf{U}_{p_0}(\langle \mathbf{x}, \mathbf{x}\rangle), \mathbf{x}\mathbf{U}_{p_0}(\langle \mathbf{x}, \mathbf{x}\rangle)\rangle,$$

i.e., $\mathbf{x}\mathbf{U}_{p_0}(\langle \mathbf{x}, \mathbf{x}\rangle) \in \mathbf{X}_{p_0}$. Next, let us compute the range of Φ: we claim that it is an open set of $\mathbf{X}_{p_0} \times \mathcal{U}_\mathcal{A}$, and therefore it provides a local chart for $\mathbf{X}_\mathcal{P}$ on a neighbourhood of $\mathbf{x}_0$.

We claim that

$$\Phi : \mathcal{D}_{\mathbf{x}_0} \to \mathbf{X}_{p_0} \times \{u \in \mathcal{U}_\mathcal{A} : u(2p_0 - 1) = (2p_0 - 1)u^* \text{ and } \|u - 1\| < \sqrt{2}\}$$

is a C^∞-diffeomorphism. Indeed, its inverse is

$$\Psi : \mathbf{X}_{p_0} \times \{u \in \mathcal{U}_\mathcal{A} : u(2p_0 - 1) = (2p_0 - 1)u^*, \text{ and } \|u - 1\| < \sqrt{2}\} \to \mathcal{D}_{\mathbf{x}_0} \ , \quad \Psi(\mathbf{y}, u) = \mathbf{y}u^*.$$

Note first that $\Psi(\mathbf{y}, u) \in \mathcal{D}_{\mathbf{x}_0}$:

$$\|\langle \Psi(\mathbf{y}, u), \Psi(\mathbf{y}, u)\rangle - p_0\| = \|\langle \mathbf{y}u^*, \mathbf{y}u^*\rangle - p_0\| = \|u\langle \mathbf{y}, \mathbf{y}\rangle u^* - p_0\| = \|up_0u^* - p_0\| < 1,$$

by Remark 3.4. Moreover, if $\mathbf{x} \in \mathcal{D}_{\mathbf{x}_0}$,

$$\Psi \circ \Phi(\mathbf{x}) = \Psi(\mathbf{x}\mathbf{U}_{p_0}(\langle \mathbf{x}, \mathbf{x}\rangle), \mathbf{U}_{p_0}(\langle \mathbf{x}, \mathbf{x}\rangle)) = \mathbf{x}\mathbf{U}_{p_0}(\langle \mathbf{x}, \mathbf{x}\rangle)(\mathbf{U}_{p_0}(\langle \mathbf{x}, \mathbf{x}\rangle))^* = \mathbf{x}.$$

If $(\mathbf{y}, u) \in \mathbf{X}_{p_0} \times \{u \in \mathcal{U}_\mathcal{A} : u(2p_0 - 1) = (2p_0 - 1)u^*, \text{ and } \|u - 1\| < \sqrt{2}\}$, then $\langle \mathbf{y}u^*, \mathbf{y}u^*\rangle = up_0u^*$, and, by the characteristic property of u, one has $\mathbf{U}_{p_0}(up_0u^*) = u$. Then

$$\Phi \circ \Psi(\mathbf{y}, u) = \Phi(\mathbf{y}u^*) = (\mathbf{y}u^*\mathbf{U}_{p_0}(\langle \mathbf{y}u^*, \mathbf{y}u^*\rangle), \mathbf{U}_{p_0}(\langle \mathbf{y}u^*, \mathbf{y}u^*\rangle)) = (\mathbf{y}u^*u, u) = (\mathbf{y}, u).$$

Note that $\{u \in \mathcal{U}_\mathcal{A} : u(2p_0 - 1) = (2p_0 - 1)u^*, \text{ and } \|u - 1\| < \sqrt{2}\}$ is open in $\{u \in \mathcal{U}_\mathcal{A} : u(2p_0 - 1) = (2p_0 - 1)u^*\}$, which is a complemented submanifold of $\mathcal{U}_\mathcal{A}$.

Thus we have found a local chart for any $\mathbf{x}_0 \in \mathbf{X}_\mathcal{P}$. It is fairly straightforward to verify that the transition maps between different charts are C^∞ . □

4 The product and Hopf maps

Consider the following two projection-valued maps: first, the product map ρ

$$\rho : \mathbf{X}_\mathcal{P} \to \mathcal{P}_\mathcal{A} \ , \quad \rho(\mathbf{x}) = \langle \mathbf{x}, \mathbf{x}\rangle;$$

next, the Hopf map

$$\mathfrak{h} : \mathbf{X}_\mathcal{P} \to \mathcal{P}_{\mathcal{B}_\mathcal{A}(\mathbf{X})} \ , \quad \mathfrak{h}(\mathbf{x}) = \theta_{\mathbf{x},\mathbf{x}}.$$

The manifold $\mathbf{X}_\mathcal{P}$ admits the left action of the unitary group $\mathcal{U}_{\mathcal{B}_\mathcal{A}(\mathbf{X})}$ of the C*-algebra $\mathcal{B}_\mathcal{A}(\mathbf{X})$:

$$U \cdot \mathbf{x} = U\mathbf{x}.$$

Note that $\langle U\mathbf{x}, U\mathbf{x}\rangle = \langle \mathbf{x}, \mathbf{x}\rangle \in \mathcal{P}_\mathcal{A}$, thus the action is well defined, and preserves the fibers of ρ: $\rho(U(\mathbf{x})) = \rho(\mathbf{x})$. In particular, this means that the action is not transitive.

The map $\mathfrak{h}$, on the other hand, is equivariant under this action. Recall the natural action of $\mathcal{U}_{\mathcal{B}_\mathcal{A}(\mathbf{X})}$ on $\mathcal{P}_{\mathcal{B}_\mathcal{A}(\mathbf{X})}$: $U \bullet P = UPU^*$. In the special case of projections of the form $P = \theta_{\mathbf{x},\mathbf{x}}$ it takes the form

$$U \bullet \theta_{\mathbf{x},\mathbf{x}} = U\theta_{\mathbf{x},\mathbf{x}}U^* = \theta_{U\mathbf{x},U\mathbf{x}}.$$

Thus

$$\mathfrak{h}(U\mathbf{x}) = \theta_{U(\mathbf{x}),U(\mathbf{x})} = U \bullet \mathfrak{h}(\mathbf{x}).$$

The manifold $\mathbf{X}_{\mathcal{P}}$ admits also the right action of the unitary group $\mathcal{U}_{\mathcal{A}}$ of $\mathcal{A}$:

$$\mathbf{x} \cdot u = \mathbf{x}u.$$

Note that $\langle \mathbf{x}u, \mathbf{x}u\rangle = u^*\langle \mathbf{x}, \mathbf{x}\rangle u$, i.e., ρ is equivariant for the right action of $\mathcal{U}_{\mathcal{A}}$ (the right action of $\mathcal{U}_{\mathcal{A}}$ on $\mathcal{P}_{\mathcal{A}}$ is $p \cdot u = u^*pu$):

$$\rho(\mathbf{x} \cdot u) = u^*\langle \mathbf{x}, \mathbf{x}\rangle u = \rho(\mathbf{x}) \cdot u.$$

On the other hand, $\mathfrak{h}$ is invariant for the right action (the right action preserves the fibers of $\mathfrak{h}$):

$$\mathfrak{h}(\mathbf{x} \cdot u) = \theta_{(\mathbf{x}u),(\mathbf{x}u)} = \theta_{\mathbf{x},\mathbf{x}} = \mathfrak{h}(\mathbf{x}),$$

because $\mathbf{x}u\langle \mathbf{x}u, \mathbf{y}\rangle = (\mathbf{x}u)u^*\langle \mathbf{x}, \mathbf{y}\rangle = \mathbf{x}\langle \mathbf{x}, \mathbf{y}\rangle$.

Let us show first that ρ is a locally trivial fiber bundle. Note that the fiber $\rho^{-1}(p_0)$ over $p_0 \in \mathcal{P}_{\mathcal{A}}$ is

$$\rho^{-1}(p_0) = \{\mathbf{x} \in \mathbf{X}_{\mathcal{P}} : \langle \mathbf{x}, \mathbf{x}\rangle = p_0\} = \mathbf{X}_{p_0}.$$

Proposition 4.1. *The map $\rho : \mathbf{X}_{\mathcal{P}} \to \mathcal{P}_{\mathcal{A}}$, $\rho(\mathbf{x}) = \langle \mathbf{x}, \mathbf{x}\rangle$ is a C^∞-locally trivial fiber bundle.*

Proof. Clearly, ρ is C^∞: it is the restriction of the inner product map $(\mathbf{x}, \mathbf{y}) \mapsto \langle \mathbf{x}, \mathbf{y}\rangle$ to the diagonal submanifold $\{(\mathbf{x}, \mathbf{x}) : x \in \mathbf{X}_{\mathcal{P}}\}$ of $\mathbf{X} \times \mathbf{X}$. Fix $\mathbf{x} \in \mathbf{X}_{\mathcal{P}}$ with $\rho(\mathbf{x}_0) = p_0$. Recall from the proof of Theorem 3.7 the diffeomorphism

$$\Phi : \mathcal{D}_{\mathbf{x}_0} \to \mathbf{X}_{p_0} \times \{u \in \mathcal{U}_{\mathcal{A}} : u(2p_0 - 1) = (2p_0 - 1)u^* \text{ and } \|u - 1\| < \sqrt{2}\}$$

From the previous remarks, it follows that the right hand factor $\{u \in \mathcal{U}_{\mathcal{A}} : u(2p_0 - 1) = (2p_0 - 1)u^* \text{ and } \|u - 1\| < \sqrt{2}\}$ is diffeomorphic to

$$\{p \in \mathcal{P}_{\mathcal{A}} : \|p - p_0\| < 1\}.$$

Thus, composing with this diffeomorphism, we obtain a slight modification of the local chart Φ, namely $\hat{\Phi}$

$$\hat{\Phi} : \mathcal{D}_{\mathbf{x}_0} \to \mathbf{X}_{p_0} \times \{p \in \mathcal{P}_{\mathcal{A}} : \|p - p_0\| < 1\},$$

which provides a local trivialization of ρ near $\mathbf{x}_0$. □

We consider now the map $\mathfrak{h}$. Fix $\mathbf{x}_0 \in \mathbf{X}_{\mathcal{P}}$, with $\langle \mathbf{x}_0, \mathbf{x}_0\rangle = p_0$. The fiber $\mathfrak{h}^{-1}(\theta_{\mathbf{x}_0,\mathbf{x}_0})$ over $\theta_{\mathbf{x}_0,\mathbf{x}_0}$ is naturally isomorphic to the space

$$\mathbf{F}_{p_0} = \{\text{partial isometries of } \mathcal{A} \text{ with final space } p_0\}$$

Indeed, note that we have seen that $\mathbf{x}, \mathbf{x}_0 \in \mathbf{X}_{\mathcal{P}}$ satisfy $\theta_{\mathbf{x},\mathbf{x}} = \theta_{\mathbf{x}_0,\mathbf{x}_0}$ if and only if $\mathbf{x} = \mathbf{x}_0 v$, where $v : p \sim p_0$ (namely: $v = \langle \mathbf{x}_0, \mathbf{x}\rangle$).

The range of the map $\mathfrak{h}$ consists of *rank one* modular projections (i.e., projections of the form $\theta_{\mathbf{x},\mathbf{x}}$). Recall that we denote this set by

$$\mathbb{P}_1(\mathbf{X}) = \{\theta_{\mathbf{x},\mathbf{x}} : x \in \mathbf{X}_{\mathcal{P}}\} = \{P \in \mathcal{P}_{\mathcal{B}_{\mathcal{A}}(\mathbf{X})} : R(P) \text{ is a singly generated submodule of } \mathbf{X}\}$$

Proposition 4.2. *The map* $\mathfrak{h} : \mathbf{X}_{\mathcal{P}} \to \mathbb{P}_1(\mathbf{X})$*,* $\mathfrak{h}(\mathbf{x}) = \theta_{\mathbf{x},\mathbf{x}}$ *is a* C^∞ *locally trivial fiber bundle.*

Proof. Fix $\mathbf{x}_0 \in \mathbf{X}_{\mathcal{P}}$, and consider the open neighbourhood of $\mathbf{x}_0$ given by

$$\mathcal{E}_{\mathbf{x}_0} = \{\mathbf{x} \in \mathbf{X}_{\mathcal{P}} : \|\theta_{\mathbf{x},\mathbf{x}} - \theta_{\mathbf{x}_0,\mathbf{x}_0}\| < 1\}.$$

The fact that $\|\theta_{\mathbf{x},\mathbf{x}} - \theta_{\mathbf{x}_0,\mathbf{x}_0}\| < 1$ implies, as remarked above, the existence of a unitary operator $U = \mathbf{U}_{\theta_{\mathbf{x}_0,\mathbf{x}_0}}(\theta_{\mathbf{x},\mathbf{x}}) \in \mathcal{U}_{\mathcal{B}_{\mathcal{A}}(\mathbf{X})}$, which is a C$^\infty$-map of $\mathbf{x}$, such that $U\theta_{\mathbf{x}_0,\mathbf{x}_0}U^* = \theta_{\mathbf{x},\mathbf{x}}$. Consider the map

$$\Upsilon : \mathcal{E}_{\mathbf{x}_0} \to \{P \in \mathcal{P}_{\mathcal{B}_{\mathcal{A}}(\mathbf{X})} : \|P - \theta_{\mathbf{x}_0,\mathbf{x}_0}\| < 1\} \times \{z \in \mathbf{X}_{\mathcal{P}} : \theta_{\mathbf{z},\mathbf{z}} = \theta_{\mathbf{x}_0,\mathbf{x}_0}\},$$

given by

$$\Upsilon(\mathbf{x}) = (\theta_{\mathbf{x},\mathbf{x}}, \mathbf{U}_{\theta_{\mathbf{x}_0,\mathbf{x}_0}}(\theta_{\mathbf{x},\mathbf{x}})\mathbf{x})^*.$$

Note that it is well defined: $U = \mathbf{U}_{\theta_{\mathbf{x}_0,\mathbf{x}_0}}(\theta_{\mathbf{x},\mathbf{x}})$ satisfies $\theta_{U^*(\mathbf{x}),U^*(\mathbf{x})} = U^*(\theta_{\mathbf{x},\mathbf{x}})U = \theta_{\mathbf{x}_0,\mathbf{x}_0}$. Also, it is C^∞. Moreover, it is a diffeomorphism with inverse

$$\Xi : \{P \in \mathcal{P}_{\mathcal{B}_{\mathcal{A}}(\mathbf{X})} : \|P - \theta_{\mathbf{x}_0,\mathbf{x}_0}\| < 1\} \times \{\mathbf{z} \in \mathbf{X}_{\mathcal{P}} : \theta_{\mathbf{z},\mathbf{z}} = \theta_{\mathbf{x}_0,\mathbf{x}_0}\} \to \mathcal{E}_{\mathbf{x}_0},$$

$$\Xi(P, \mathbf{z}) = \mathbf{U}_{\theta_{\mathbf{x}_0,\mathbf{x}_0}}(P)\mathbf{z}.$$

It is well defined:

$$\mathbf{U}_{\theta_{\mathbf{x}_0,\mathbf{x}_0}}(P)\theta_{\mathbf{z},\mathbf{z}}\mathbf{U}_{\theta_{\mathbf{x}_0,\mathbf{x}_0}}(P)^* = \mathbf{U}_{\theta_{\mathbf{x}_0,\mathbf{x}_0}}(P)\theta_{\mathbf{x}_0,\mathbf{x}_0}\mathbf{U}_{\theta_{\mathbf{x}_0,\mathbf{x}_0}}(P)^* = P,$$

by Remark 3.4. It is straightforward to verify that Ξ is the inverse of Υ.

This diffeomorphism provides a local trivialization of the map $\mathfrak{h}$. □

5 $\mathbf{X}_{\mathcal{P}}$ as an homogeneous reductive metric space

We saw above that $\mathbf{X}_{\mathcal{P}}$ carries the left action of $\mathcal{U}_{\mathcal{B}_{\mathcal{A}}(\mathbf{X})}$ and the right action of $\mathcal{U}_{\mathcal{A}}$. We can condense both actions in the left action of the group $\mathcal{U}_{\mathcal{B}_{\mathcal{A}}(\mathbf{X})} \times \mathcal{U}_{\mathcal{A}}$: if $(U, u) \in \mathcal{U}_{\mathcal{B}_{\mathcal{A}}(\mathbf{X})} \times \mathcal{U}_{\mathcal{A}}$ and $\mathbf{x} \in \mathbf{X}_{\mathcal{P}}$,

$$(U, u) \bullet \mathbf{x} = (U\mathbf{x})u^* = U\mathbf{x}u^*.$$

Clearly $(U, u)(V, v) \bullet \mathbf{x} = (UV, uv) \bullet \mathbf{x} = U(V(\mathbf{x}v^*))u^* = (U, u) \bullet ((V, v) \bullet \mathbf{x})$. First note that the action is locally transitive:

Proposition 5.1. *If* $\mathbf{x}_1, \mathbf{x}_2 \in \mathbf{X}_{\mathcal{P}}$ *satisfy that* $\|\mathbf{x}_1 - \mathbf{x}_2\| < \frac{1}{4}$*, then there exists* $(U, u) \in \mathcal{U}_{\mathcal{B}_{\mathcal{A}}(\mathbf{X})} \times \mathcal{U}_{\mathcal{A}}$ *such that* $(U, u) \bullet \mathbf{x}_1 = \mathbf{x}_2$*.*

Proof. Recall that $\|\rho(\mathbf{x}_1) - \rho(\mathbf{x}_2)\| = \|\langle \mathbf{x}_1, \mathbf{x}_1\rangle - \langle \mathbf{x}_2, \mathbf{x}_2\rangle\| \le 2\|\mathbf{x}_1 - \mathbf{x}_2\| < \frac{1}{2} < 1$. Then there exists $u = u_{\mathbf{x}_1,\mathbf{x}_2} \in \mathcal{U}_{\mathcal{A}}$, the cross section induced by the exponential mapping in the manifold of projections in $\mathcal{A}$, such that $u\rho(\mathbf{x}_1)u^* = \rho(\mathbf{x}_2)$, i.e., $\mathbf{x}_2 u \in \mathbf{X}_{\rho(\mathbf{x}_1)}$. Note also that (see Lemma 3.5)

$$\|\mathbf{x}_2 u - \mathbf{x}_1\| \le \|\mathbf{x}_2 u - \mathbf{x}_2\| + \|\mathbf{x}_2 - \mathbf{x}_1\| \le \|1 - u\| + \|\mathbf{x}_2 - \mathbf{x}_1\| < \frac{1}{2}.$$

Here we used the fact that $\|u - 1\| = \frac{1}{2}\|u\rho(\mathbf{x}_1)u^* - \rho(\mathbf{x}_1)\| = \frac{1}{2}\|\rho(\mathbf{x}_2) - \rho(\mathbf{x}_1)\| < \frac{1}{4}$ (see [17]). Then, as recalled in last paragraph of Remark 3.6, there exists $U = U_{\mathbf{x}_2 u} \in \mathcal{U}_{\mathcal{B}_{\mathcal{A}}(\mathbf{X})}$ such that $U(\mathbf{x}_1) = \mathbf{x}_2 u$, i.e., $U(\mathbf{x}_1 u^*) = \mathbf{x}_2$. □

Remark 5.2. Note that both U and u above are C^∞-maps in terms of $\mathbf{x}_1, \mathbf{x}_2 \in \mathbf{X}_\mathcal{P}$.

Let us characterize the *isotropy subgroups* of this action: given $\mathbf{x}_0 \in \mathbf{X}_\mathcal{P}$, with $\rho(\mathbf{x}_0) = \langle \mathbf{x}_0, \mathbf{x}_0 \rangle = p_0$ and $\mathfrak{h}(\mathbf{x}_0) = \theta_{\mathbf{x}_0,\mathbf{x}_0} = P_0$, the isotropy subgroup $\mathcal{I}_{\mathbf{x}_0}$ at $\mathbf{x}_0$ is

$$\mathcal{I}_{\mathbf{x}_0} = \{(V,v) \in \mathcal{U}_{\mathcal{B}_\mathcal{A}(\mathbf{X})} \times \mathcal{U}_\mathcal{A} : (V,v) \bullet \mathbf{x}_0 = \mathbf{x}_0\} = \{(V,v) \in \mathcal{U}_{\mathcal{B}_\mathcal{A}(\mathbf{X})} \times \mathcal{U}_\mathcal{A} : V(\mathbf{x}_0) = \mathbf{x}_0 v\}.$$

Lemma 5.3. *If $(V,v) \in \mathcal{I}_{\mathbf{x}_0}$, then V commutes with P_0 and v commutes with p_0.*

Proof. $V\mathbf{x}_0 = \mathbf{x}_0 v$ implies that

$$VP_0V^* = V\theta_{\mathbf{x}_0,\mathbf{x}_0}V^* = \theta_{V\mathbf{x}_0,V\mathbf{x}_0} = \theta_{\mathbf{x}_0 v,\mathbf{x}_0 v} = \theta_{\mathbf{x}_0,\mathbf{x}_0} = P_0,$$

and

$$v^* p_0 v = v^* \langle \mathbf{x}_0, \mathbf{x}_0 \rangle v = \langle \mathbf{x}_0 v, \mathbf{x}_0 v \rangle = \langle V\mathbf{x}_0, V\mathbf{x}_0 \rangle = \langle \mathbf{x}_0, \mathbf{x}_0 \rangle = p_0.$$

□

It follows that the isotropy algebra $\mathfrak{i}_{\mathbf{x}_0}$ at $\mathbf{x}_0$, i.e., the Banach-Lie algebra of $\mathcal{I}_{\mathbf{x}_0}$, is

$$\mathfrak{i}_{\mathbf{x}_0} = \{((X,x) \in \mathcal{B}_\mathcal{A}(\mathbf{X}) \times \mathcal{A} : X^* = -X, x^* = -x \text{ and } X\mathbf{x}_0 = \mathbf{x}_0 x\}.$$

Let us characterize the elements $(X,x) \in \mathfrak{i}_{\mathbf{x}_0}$ as pairs of 2×2 matrices in terms of the projections $P_0 \in \mathcal{B}_\mathcal{A}(\mathbf{X})$ and $p_0 \in \mathcal{A}$:

Proposition 5.4. *With the current notations,*

$$\mathfrak{i}_{\mathbf{x}_0} = \{(\begin{pmatrix} \theta_{\mathbf{x}_0 z_0,\mathbf{x}_0} & 0 \\ 0 & Z' \end{pmatrix}, \begin{pmatrix} z_0 & 0 \\ 0 & z' \end{pmatrix}) : Z'^* = -Z', z'^* = -z', z_0^* = -z_0 \text{ and } z_0 = p_0 z_0 p_0\}.$$

Proof. Any pair of the form $(\begin{pmatrix} \theta_{\mathbf{x}_0 z_0,\mathbf{x}_0} & 0 \\ 0 & Z' \end{pmatrix}, \begin{pmatrix} z_0 & 0 \\ 0 & z' \end{pmatrix})$ is clearly a pair of anti-hermitian elements: $\theta^*_{\mathbf{x}_0 z_0,\mathbf{x}_0} = \theta_{\mathbf{x}_0,\mathbf{x}_0 z_0} = \theta_{\mathbf{x}_0 z_0^*,\mathbf{x}_0} = -\theta_{\mathbf{x}_0 z_0,\mathbf{x}_0}$ (here we use that $\theta_{\mathbf{x},\mathbf{y}a} = \theta_{\mathbf{x}a^*,\mathbf{y}}$). Also,

$$\begin{pmatrix} \theta_{\mathbf{x}_0 z_0,\mathbf{x}_0} & 0 \\ 0 & Z' \end{pmatrix} \begin{pmatrix} \mathbf{x}_0 \\ 0 \end{pmatrix} = \begin{pmatrix} \theta_{\mathbf{x}_0 z_0,\mathbf{x}_0}(\mathbf{x}_0) \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbf{x}_0 z_0 \langle \mathbf{x}_0, \mathbf{x}_0 \rangle \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbf{x}_0 z_0 p_0 \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbf{x}_0 \\ 0 \end{pmatrix} z_0,$$

because $z_0 p_0 = z_0$.

Conversely, let $(X,x) \in \mathfrak{i}_{\mathbf{x}_0}$. By Lemma 5.3, the matrices of X and x in terms of P_0 and p_0, respectively, are diagonal. Thus $X = \begin{pmatrix} T & 0 \\ 0 & X' \end{pmatrix}$ where T is an anti-hermitian (adjointable) operator acting in the module $[\mathbf{x}_0]$. Thus T is an operator of the form $T = \theta_{\mathbf{x}_0 z_0,\mathbf{x}_0}$, with $z_0 \in \mathcal{A}_{ah}$. Note that, since $\mathbf{x}_0 = \mathbf{x}_0 p_0$, we have that $\mathbf{x}_0 z_0 = T\mathbf{x}_0$ satisfies

$$\mathbf{x}_0 z_0 = T\mathbf{x}_0 = T\mathbf{x}_0 p_0 = (T\mathbf{x}_0)p_0 = \mathbf{x}_0 p_0 z_0 p_0,$$

i.e., we can choose the coefficient z_0 to satisfy that $z_0 = p_0 z_0 p_0$. Finally, the fact that $X\mathbf{x}_0 = \mathbf{x}_0 x$ implies that the $1,1$-entry of x is z_0, i.e.,

$$x = \begin{pmatrix} z_0 & 0 \\ 0 & x' \end{pmatrix},$$

for $x' \in \mathcal{A}_{ah}$. □

We shall now introduce a *reductive structure* in the homogeneous space $\mathbf{X}_{\mathcal{P}}$. That is, a distribution

$$\mathbf{X}_{\mathcal{P}} \ni \mathbf{x}_0 \longmapsto \mathbb{H}_{\mathbf{x}_0} \subset \mathcal{B}_{\mathcal{A}}(\mathbf{X})_{ah} \times \mathcal{A}_{ah}$$

of closed (real) linear subspaces of $\mathcal{B}_{\mathcal{A}}(\mathbf{X})_{ah} \times \mathcal{A}_{ah}$ satisfying the following conditions:

1. $\mathbb{H}_{\mathbf{x}_0} \oplus \mathfrak{i}_{\mathbf{x}_0} = \mathcal{B}_{\mathcal{A}}(\mathbf{x})_{ah} \times \mathcal{A}_{ah}$.
2. $\mathbb{H}_{\mathbf{x}_0}$ is $\mathcal{I}_{\mathbf{x}_0}$ invariant: if $(V, v) \in \mathcal{I}_{\mathbf{x}_0}$, then $(V, v) \cdot \mathbb{H}_{\mathbf{x}_0} \cdot (V^*, v^*) = \mathbb{H}_{\mathbf{x}_0}$.
3. The map $\mathbf{x}_0 \mapsto \mathbb{H}_{\mathbf{x}_0}$ is smooth, i.e., if $P_{\mathbb{H}_{\mathbf{x}_0}}$ denotes the projection onto $\mathbb{H}_{\mathbf{x}_0}$ given by the decomposition $\mathbb{H}_{\mathbf{x}_0} \oplus \mathfrak{i}_{\mathbf{x}_0} = \mathcal{B}_{\mathcal{A}}(\mathbf{X})_{ah} \times \mathcal{A}_{ah}$, then the map
$$\mathbf{X}_{\mathcal{P}} \ni \mathbf{x}_0 \longmapsto \mathbf{P}_{\mathbb{H}_{\mathbf{x}_0}} \in \mathcal{B}(\mathcal{B}_{\mathcal{A}}(\mathbf{X})_{ah} \times \mathcal{A}_{ah})$$
is C^∞.

Definition 5.5. *For $\mathbf{x}_0 \in \mathbf{X}_{\mathcal{P}}$, let*

$$\mathbb{H}_{\mathbf{x}_0} := \{(\begin{pmatrix} 0 & Y \\ -Y^* & 0 \end{pmatrix}, \begin{pmatrix} a & y \\ -y^* & 0 \end{pmatrix}) : a^* = -a\}$$

$$= \{(Z, z) \in \mathcal{B}_{\mathcal{A}}(\mathbf{X})_{ah} \times \mathcal{A}_{ah} : P_0 Z P_0 = P_0^\perp Z P_0^\perp = 0, p_0^\perp z p_0^\perp = 0\}.$$

Proposition 5.6. *The distribution $\{\mathbb{H}_{\mathbf{x}_0} : \mathbf{x}_0 \in \mathbf{X}_{\mathcal{P}}\}$ is a reductive structure for $\mathbf{X}_{\mathcal{P}}$.*

Proof. We have to check the three conditions above. Clearly $\mathbb{H}_{\mathbf{x}_0} \cap \mathfrak{i}_{\mathbf{x}_0} = \{0\}$. Also it is apparent that $\mathbb{H}_{\mathbf{x}_0} + \mathfrak{i}_{\mathbf{x}_0} = \mathcal{B}_{\mathcal{A}}(\mathbf{X})_{ah} \times \mathcal{A}_{ah}$. The invariance under conjugation by elements in $\mathcal{I}_{\mathbf{x}_0}$ is a straightforward matrix computation. In order to see that the distribution is smooth, note that the projection onto $\mathbb{H}_{\mathbf{x}_0}$ given by this decomposition is

$$\mathbf{P}_{\mathbb{H}_{\mathbf{x}_0}}(X, x) = (P_0 X P_0 + P_0^\perp X P_0^\perp, p_0^\perp x p_0^\perp).$$

Since the maps $\mathbf{x}_0 \mapsto P_0 = \mathfrak{h}(\mathbf{x}_0)$ and $\mathbf{x}_0 \mapsto p_0 = \rho(\mathbf{x}_0)$ are C^∞, it is clear that $\mathbf{x}_0 \mapsto \mathbf{P}_{\mathbb{H}_{\mathbf{x}_0}}$ is C^∞. ☐

For a fixed $\mathbf{x}_0 \in \mathbf{X}_{\mathcal{P}}$, denote

$$\pi_{\mathbf{x}_0} : \mathcal{U}_{\mathcal{B}_{\mathcal{A}}(\mathbf{X})} \times \mathcal{U}_{\mathcal{A}} \to \mathbf{X}_{\mathcal{P}} \ , \quad \pi_{\mathbf{x}_0}(U, u) = (U, u) \bullet \mathbf{x}_0 = U(\mathbf{x}_0 u^*).$$

Remark 5.7. As in classical differential geometry, a reductive structure induces a linear connection in $\mathbf{X}_{\mathcal{P}}$. The procedure is formally identical as in the classic setting ([11], or [15] where the C*-algebraic context is considered). For instance, for any tangent vector $\mathbf{v} \in (T\mathbf{X}_{\mathcal{P}})_{\mathbf{x}_0}$ at $\mathbf{x}_0 \in \mathbf{X}_{\mathcal{P}}$, there exists a unique $(Z_{\mathbf{v}}, z_{\mathbf{v}}) = (Z, z) \in \mathbb{H}_{\mathbf{x}_0}$ such that $d(\pi_{\mathbf{x}_0})_{(1,1)}(Z, z) = \mathbf{v}$, and the unique geodesic δ of the linear connection such that $\delta(0) = \mathbf{x}_0$ and $\dot{\delta}(0) = \mathbf{v}$ is

$$\delta(t) = e^{t(Z,z)} \bullet \mathbf{x}_0 = e^{tZ}(\mathbf{x}_0 e^{-tz}).$$

Next, we prove that these geodesics are pushed to geodesics of $\mathcal{P}_{\mathcal{B}_{\mathcal{A}}(\mathbf{X})}$ by the map $\mathfrak{h}$.

Corollary 5.8. *If δ is a geodesic in $\mathbf{X}_{\mathcal{P}}$, then $\mathfrak{h}(\delta)$ is a geodesic in $\mathcal{P}_{\mathcal{B}_{\mathcal{A}}(\mathbf{X})}$, which starts at $P_0 = \mathfrak{h}(\mathbf{x}_0)$ with initial velocity $[Z, P_0]$.*

Proof. Note that if $\delta(t) = e^{tZ}\mathbf{x}_0 e^{-tz}$, for $(Z, z) \in \mathbb{H}_{\mathbf{x}_0}$, then

$$\mathfrak{h}(\delta(t)) = \mathfrak{h}(e^{tZ}\mathbf{x}_0 e^{-tz}) = \theta_{e^{tZ}\mathbf{x}_0 e^{-tz}, e^{tZ}\mathbf{x}_0 e^{-tz}} = \theta_{e^{tZ}\mathbf{x}_0, e^{tZ}\mathbf{x}_0} = e^{tZ}\theta_{\mathbf{x}_0,\mathbf{x}_0} e^{-tZ} = e^{tZ} P_0 e^{-tZ}.$$

Then it is clear that $\dot{\mathfrak{h}(\delta)}(0) = [Z, P_0]$. Recall from Section 1 the form of the geodesics of $\mathcal{P}_{\mathcal{B}_{\mathcal{A}}(\mathbf{X})}$: the curve $\mathfrak{h}(\delta(t))$ is a geodesic because $Z \in \mathcal{B}_{\mathcal{A}}(\mathbf{X})_{ah}$ is co-diagonal with respect to P_0. □

The manifold $\mathbf{X}_{\mathcal{P}}$ becomes a metric space, if we consider the Finsler metric given by the norm of $\mathbf{X}$ at every tangent space $(T\mathbf{X})_{\mathbf{x}}$, for every $\mathbf{x} \in \mathbf{X}_{\mathcal{P}}$ (recall that $(T\mathbf{X})_{\mathbf{x}}$ is a complemented subspace of $\mathbf{X}$). As remarked in the introduction, $\mathcal{P}_{\mathcal{B}(\mathbf{X})}$, being the Grassmann manifold of a C*-algebra, is a well studied metric space, with the Finsler metric given by the norm of $\mathcal{B}_{\mathcal{A}}(\mathbf{X})$ at every tangent space (see [17], [5]). Then, for these metrics, we have

Theorem 5.9. *The tangent maps of* $\mathfrak{h} : \mathbf{X}_{\mathcal{P}} \to \mathcal{P}_{\mathcal{B}_{\mathcal{A}}(\mathbf{X})}$, $\mathfrak{h}(\mathbf{x}) = \theta_{\mathbf{x},\mathbf{x}}$ *are contractive at every point* $\mathbf{x}_0$*, i.e.,*

$$\|d\mathfrak{h}_{\mathbf{x}_0}(\mathbf{v})\|_{\mathcal{B}_{\mathcal{A}}(\mathbf{X})} \leq \|\mathbf{v}\|_{\mathbf{X}},$$

for every $\mathbf{v} \in (T\mathbf{X}_{\mathcal{P}})_{\mathbf{x}_0}$ *and every* $\mathbf{x}_0 \in \mathbf{X}_{\mathcal{P}}$*.*

Proof. Denote $\theta_{\mathbf{x}_0,\mathbf{x}_0} = P_0$ and $\langle \mathbf{x}_0, \mathbf{x}_0\rangle = p_0$. Since $\mathbf{X}_{\mathcal{P}}$ is a homogeneous reductive space of $\mathcal{U}_{\mathcal{B}_{\mathcal{A}}(\mathbf{X})} \times \mathcal{U}_{\mathcal{A}}$, a tangent vector $\mathbf{v} \in T(\mathbf{X}_{\mathcal{P}})_{\mathbf{x}_0}$ is of the form $\mathbf{v} = Z(\mathbf{x}_0) - \mathbf{x}_0 z$, for a pair $(Z, z) \in \mathbb{H}_{\mathbf{x}_0}$. Then

$$\|\mathbf{v}\|^2 = \|\langle Z(\mathbf{x}_0) - \mathbf{x}_0 z, Z(\mathbf{x}_0) - \mathbf{x}_0 z\rangle\| = \|\langle Z(\mathbf{x}_0), Z(\mathbf{x}_0)\rangle + \langle(\mathbf{x}_0 z, \mathbf{x}_0 z\rangle\|,$$

because $\langle Z(\mathbf{x}_0), \mathbf{x}_0 z\rangle = \langle \mathbf{x}_0 z, Z(\mathbf{x}_0)\rangle = 0$. Indeed, since Z is P_0 co-diagonal, it maps $Z([\mathbf{x}_0]) \subset [\mathbf{x}_0]^{\perp}$. Then, since $\langle(\mathbf{x}_0 z, \mathbf{x}_0 z\rangle = z^* p_0 z \geq 0$,

$$\|\mathbf{v}\| \geq \|\langle Z(\mathbf{x}_0), Z(\mathbf{x}_0)\rangle\|^{1/2} = \|Z(\mathbf{x}_0)\|.$$

Note also that, in matrix form in terms of P_0, $Z = \begin{pmatrix} 0 & Y \\ -Y^* & 0 \end{pmatrix}$, where $Y : [\mathbf{x}_0]^{\perp} \to [\mathbf{x}_0]$ is of the form $Y = \theta_{\mathbf{x}_0,\mathbf{y}}$ for some $\mathbf{y} \in [\mathbf{x}_0]^{\perp}$. Moreover,

$$\theta_{\mathbf{x}_0,\mathbf{y}}(\mathbf{x})\mathbf{x}_0\langle \mathbf{y}, \mathbf{x}\rangle = \mathbf{x}_0 p_0\langle \mathbf{y}, \mathbf{x}\rangle = \mathbf{x}_0\langle \mathbf{y}p_0, \mathbf{x}\rangle,$$

i.e., we may choose $\mathbf{y} = \mathbf{y}p_0$. Thus,

$$\|Z(\mathbf{x}_0)\| = \|Y^*(\mathbf{x}_0)\| = \|\theta_{\mathbf{x}_0,\mathbf{y}}(\mathbf{x}_0)\| = \|\mathbf{y}p_0\| = \|\mathbf{y}\|.$$

On the other hand, the element $\mathbf{v} = Z(\mathbf{x}_0) - \mathbf{x}_0 z \in (T\mathbf{x}_{\mathcal{P}})_{\mathbf{x}_0}$ is clearly the velocity vector at $t = 0$ of the curve (in fact, geodesic) $\delta(t) = e^{t(Z,z)} \bullet \mathbf{x}_0 = e^{tZ}(\mathbf{x}_0 e^{-tz})$. Then

$$d\mathfrak{h}_{\mathbf{x}_0}(\mathbf{v}) = \frac{d}{dt}\mathfrak{h}(\delta(t))|_{t=0} = \frac{d}{dt} e^{tZ} P_0 e^{-tZ}|_{t=0} = ZP_0 - P_0 Z.$$

Note that

$$ZP_0 - P_0 Z = \begin{pmatrix} 0 & Y \\ -Y^* & 0 \end{pmatrix} P_0 - P_0 \begin{pmatrix} 0 & Y \\ -Y^* & 0 \end{pmatrix} = Z,$$

because $P_0 = \begin{pmatrix} Id & 0 \\ 0 & 0 \end{pmatrix}$. Then

$$\|d\mathfrak{h}_{\mathbf{x}_0}(\mathbf{v})\| = \|Z\|.$$

Clearly,

$$\|Z\| = \|Z^*Z\|^{1/2} = \| \begin{pmatrix} YY^* & 0 \\ 0 & Y^*Y \end{pmatrix} \|^{1/2} = \max\{\|YY^*\|^{1/2}, \|Y^*Y\|^{1/2}\} = \|Y\|.$$

The trivial inequality $\|Y\| = \|\theta_{\mathbf{x}_0,\mathbf{y}}\| \leq \|\mathbf{x}_0\|\|\mathbf{y}\| = \|\mathbf{y}\|$ finishes the proof. □

Corollary 5.10. *Let γ be a smooth curve in $\mathbf{X}_{\mathcal{P}}$, then*

$$length(\mathfrak{h}(\gamma)) \leq length(\gamma).$$

Below d denotes the Finsler metric induced in the corresponding spaces by the ambient norm.

Corollary 5.11. *Let $\mathbf{x}_0, \mathbf{x}_1 \in \mathbf{X}_{\mathcal{P}}$ be in the same connected component of $\mathbf{X}_{\mathcal{P}}$, with $P_0 = \mathfrak{h}(\mathbf{x}_0)$ and $P_1 = \mathfrak{h}(\mathbf{x}_1)$. Then*

$$d(P_0, P_1) \leq d(\mathbf{x}_0, \mathbf{x}_1).$$

Remark 5.12. The above result can be rephrased as follows. Let $\gamma(t)$ be a smooth curve in $\mathbf{X}_{\mathcal{P}}$, joining $\mathbf{x}_0$ and $\mathbf{x}_1$, and let $P_0 = \mathfrak{h}(\mathbf{x}_0)$, $P_1 = \mathfrak{h}(\mathbf{x}_1)$. The (geodesic) Finsler distance in the Grassmann manifold of a C*-algebra has several characterizations. For instance, it is well known that [2]

$$d(P_0, P_1) = \arcsin(\|P_0 - P_1\|).$$

By the Krein-Krasnoselski-Milman formula [13], $\|P_0 - P_1\| = \max\{\|P_0P_1 - P_1\|, \|P_1P_0 - P_0\|\}$. In the special case where $P_i = \theta_{\mathbf{x}_i,\mathbf{x}_i}$,

$$\|P_0P_1 - P_1\| = \|\theta_{\mathbf{x}_0\langle\mathbf{x}_0,\mathbf{x}_1\rangle,\mathbf{x}_1} - \theta_{\mathbf{x}_1,\mathbf{x}_1}\| = \|\theta_{\mathbf{x}_0\langle\mathbf{x}_0,\mathbf{x}_1\rangle - \mathbf{x}_1,\mathbf{x}_1}\|.$$

Note that if $\mathbf{w} \in \mathbf{X}$ and $\mathbf{x} \in \mathbf{X}_{\mathcal{P}}$, then $\|\theta_{\mathbf{w},\mathbf{x}}\| \leq \|\mathbf{w}\|\|\mathbf{x}\| = \|\mathbf{w}\|$. Also $\|\theta_{\mathbf{w},\mathbf{x}}\| \geq \|\theta_{\mathbf{w},\mathbf{x}}(\mathbf{x})\| = \|\mathbf{w}\langle\mathbf{x},\mathbf{x}\rangle\|$. In our case, $\mathbf{w} = \mathbf{x}_0\langle\mathbf{x}_0,\mathbf{x}_1\rangle - \mathbf{x}_1$ and $\mathbf{x} = \mathbf{x}_1$, this equals

$$\|(\mathbf{x}_0\langle\mathbf{x}_0,\mathbf{x}_1\rangle - \mathbf{x}_1)p_1\| = \|\mathbf{x}_0\langle\mathbf{x}_0,\mathbf{x}_1\rangle - \mathbf{x}_1\|,$$

because $\mathbf{x}_1 p_1 = \mathbf{x}_1$ and $\mathbf{x}_0\langle\mathbf{x}_0,\mathbf{x}_1\rangle p_1 = \mathbf{x}_0\langle\mathbf{x}_0,\mathbf{x}_1 p_1\rangle = \mathbf{x}_0\langle\mathbf{x}_0,\mathbf{x}_1\rangle$. Thus, $\|\theta_{\mathbf{x}_0\langle\mathbf{x}_0,\mathbf{x}_1\rangle-\mathbf{x}_1,\mathbf{x}_1}\| = \|\mathbf{x}_0\langle\mathbf{x}_0,\mathbf{x}_1\rangle - \mathbf{x}_1\|$. There is an analogous computation for $\|P_1P_0 - P_0\|$. Summarizing,

$$\|P_0 - P_1\| = \max\{\|\mathbf{x}_0\langle\mathbf{x}_0,\mathbf{x}_1\rangle - \mathbf{x}_1\|, \|\mathbf{x}_1\langle\mathbf{x}_1,\mathbf{x}_0\rangle - \mathbf{x}_0\|\}.$$

Therefore, for a smooth curve $\gamma \subset \mathbf{X}_{\mathcal{P}}$ joining $\mathbf{x}_0$ and $\mathbf{x}_1$, we have

$$length(\gamma) \geq \max\{\arcsin\|\mathbf{x}_0\langle\mathbf{x}_0,\mathbf{x}_1\rangle - \mathbf{x}_1\|, \arcsin\|\mathbf{x}_1\langle\mathbf{x}_1,\mathbf{x}_0\rangle - \mathbf{x}_0\|\}. \quad (1)$$

Another elementary computation shows that $\|\mathbf{x}_0\langle\mathbf{x}_0,\mathbf{x}_1\rangle - \mathbf{x}_1\| = \|p_1 - |\langle\mathbf{x}_0,\mathbf{x}_1\rangle|^2\|^{1/2}$, and $\|\mathbf{x}_1\langle\mathbf{x}_1,\mathbf{x}_0\rangle - \mathbf{x}_0\| = \|p_0 - |\langle\mathbf{x}_1,\mathbf{x}_0\rangle|^2\|^{1/2}$, which can be plugged in formula (1).

6 A minimality property in $\mathbf{X}_p$

In this section we prove a minimality result for curves in $\mathbf{X}_p$ satisfying prescribed initial conditions. Consider, similarly as in the previous section, the tangent spaces of $\mathbf{X}_p$ endowed with the usual norm of $\mathbf{X}$. Recall that $\mathbf{X}_p$ is a (complemented) submanifold of $\mathbf{X}$ (in fact, it is an open subset of $\mathbf{X}_{\mathcal{P}} \cap \mathbf{X} \cdot p$: clearly $\mathbf{X}_p = \{\mathbf{x} \in \mathbf{X}_{\mathcal{P}} \cap \mathbf{X} \cdot p : \langle \mathbf{x}, \mathbf{x} \rangle$ is invertible in $p\mathcal{A}p\}$).

Remark 6.1. Note the fact, brought up previously, that the structural results obtained for the unit sphere $\mathbf{X}_1$, are valid for $\mathbf{X}_p$, for any given projection in $\mathcal{A}$. Indeed, as we saw, $\mathbf{X}_p$ is a complemented submanifold of $\mathbf{X} \cdot p$, which in turn is a module over $p\mathcal{A}p$. In other words, $\mathbf{X}_p$ can be regarded as the unit sphere of the $p\mathcal{A}p$-module $\mathbf{X} \cdot p$. Therefore, for simplicity, we may suppose $p = 1$.

As we have seen above, $\mathbf{X}_1$ carries the left action of $\mathcal{U}_{\mathcal{B}_{\mathcal{A}}(\mathbf{X})}$: if $\mathbf{x} \in \mathbf{X}_1$, $U \cdot \mathbf{x} = U\mathbf{x} \in \mathbf{X}_1$. Fix $\mathbf{x}_0 \in \mathbf{X}_1$. This action induces the smooth submersion

$$\pi_{\mathbf{x}_0} : \mathcal{U}_{\mathcal{B}_{\mathcal{A}}(\mathbf{X})} \to \mathbf{X}_1, \ \pi_{\mathbf{x}_0}(U) = U\mathbf{x}_0,$$

whose tangent map at 1 is

$$\delta_{\mathbf{x}_0} = d(\pi_{\mathbf{x}_0})_1 : \mathcal{B}_{\mathcal{A}}(\mathbf{X})_{ah} \to (T\mathbf{X}_1)_{\mathbf{x}_0}, \ \delta_{\mathbf{x}_0}(X) = X\mathbf{x}_0.$$

As said above, in $(T\mathbf{X}_1)_{\mathbf{x}_0}$ we consider the usual norm of $\mathbf{X}$: if $v \in (T\mathbf{X}_1)_{\mathbf{x}_0}$, $\|v\| = \|\langle \mathbf{v}, \mathbf{v} \rangle\|^{1/2}$.

Given $\mathbf{x}_0 \in \mathbf{X}_1$ and $\mathbf{v} \in (T\mathbf{X}_1)_{\mathbf{x}_0}$, let us denote an element $X \in \mathcal{B}_{\mathcal{A}}(\mathbf{X})_{ah}$ a *lifting* of $\mathbf{v}$ if $\delta_{\mathbf{x}_0}(X) = X\mathbf{x}_0 = \mathbf{v}$. The fact that $\pi_{\mathbf{x}_0}$ is a submersion, implies that $\delta_{\mathbf{x}_0}$ is onto, and therefore there exist liftings of $\mathbf{v}$. Note that if X is a lifting of v, then

$$\|\mathbf{v}\| = \|X\mathbf{x}_0\| \leq \|X\|.$$

Since we are trying to minimize the length of curves, it is natural to consider *minimal liftings*:

Definition 6.2. *A lifting Z of $\mathbf{v} \in (T\mathbf{X}_1)_{\mathbf{x}_0}$ is called* minimal *if it satisfies*

$$\|Z\| = \min\{\|X\| : X \text{ is a lifting of } v\} = \min\{\|X\| : X \in \mathcal{B}_{\mathcal{A}}(\mathbf{X})_{ah} \text{ such that } X\mathbf{x}_0 = \mathbf{v}\}.$$

Lemma 6.3. *Let $\mathbf{x}_0 \in \mathbf{X}_1$ and $\mathbf{v} \in (T\mathbf{X}_1)_{\mathbf{x}_0}$. Then a lifing Z of $\mathbf{v}$ is minimal if and only if $\|Z\| = \|Z\theta_{\mathbf{x}_0,\mathbf{x}_0}\| = \|v\|$.*

Proof. Suppose first that $\|Z\| = \|Z\theta_{\mathbf{x}_0,\mathbf{x}_0}\|$. Then

$$\|Z\| = \|Z\theta_{\mathbf{x}_0,\mathbf{x}_0}\| = \|\theta_{Z\mathbf{x}_0,\mathbf{x}_0}\| \leq \|Z\mathbf{x}_0\|\|\mathbf{x}_0\| = \|\mathbf{v}\|.$$

Conversely, if $\|Z\| = \|\mathbf{v}\|$,

$$\|\mathbf{v}\| = \|Z\| \geq \|Z\theta_{\mathbf{x}_0,\mathbf{x}_0}\| = \|\theta_{Z\mathbf{x}_0,\mathbf{x}_0}\| = \|\theta_{\mathbf{v},\mathbf{x}_0}\| \geq \|\theta_{\mathbf{v},\mathbf{x}_0}(\mathbf{x}_0)\| = \|\mathbf{v}\|.$$

□

The elementary property above, allows us to reframe the problem of finding minimal liftings, as follows. Let X, Y be liftings of $\mathbf{v} \in (T\mathbf{X}_1)_{\mathbf{x}_0}$. Then $X\theta_{\mathbf{x}_0,\mathbf{x}_0} = \theta_{X\mathbf{x}_0,\mathbf{x}_0} = \theta_{Y\mathbf{x}_0,\mathbf{x}_0} = Y\theta_{\mathbf{x}_0,\mathbf{x}_0}$. Therefore, since X, Y are anti-Hermitian, $\theta_{\mathbf{x}_0,\mathbf{x}_0}X = \theta_{\mathbf{x}_0,\mathbf{x}_0}Y$. Therefore, if we write X and Y as 2×2 matrices in terms of $\theta_{\mathbf{x}_0,\mathbf{x}_0}$, we have that both X and Y are completions of the (incomplete) matrix

$$M_{\mathbf{v}} := \begin{pmatrix} a & b^* \\ -b & \dots \end{pmatrix},$$

with $a^* = -a$. The problem of completing $M_{\mathbf{v}}$ to an anti-Hermitian operator such that the norm of the completed matrix does not exceed the norm of the first row (or column) of $M_{\mathbf{v}}$ is classsically called the *extension problem*. It was posed and solved by M.G. Krein in [12] (see also [19]), for $\mathcal{B} = \mathcal{B}(\mathcal{H})$. In fact, the problem was considered for selfadjoint matrix operators, but it is clearly equivalent in this context. Afterwards several papers appeared, considering arbitrary operators, parametrizations of solutions (solutions are, in general, non unique), etc. See for instance [7]. For our result, we need existence of solutions of this problem. We shall prove this fact in the following Lemma. We follow step by step Krein's solution of the extension problem, as presented in the book [19] (chapter VIII, section 125, page 336), adapted to the context of Hilbert C*-modules. We state the result for selfadjoint operators.

Lemma 6.4. (Krein's method)

Let $\mathbf{Y} \subset \mathbf{X}$ be a closed orthocomplemented Hilbert C-submodule, and $Z \in \mathcal{B}_{\mathcal{A}}(\mathbf{X})$, $Z^* = Z$. Then there exists $Z_0 \in \mathcal{B}_{\mathcal{A}}(\mathbf{X})$, $Z_0^* = Z_0$, such that $Z_0|_{\mathbf{Y}} = Z|_{\mathbf{Y}}$ with minimal norm: $\|Z_0\| = \|Z|_{\mathbf{Y}}\|$.*

Proof. Without loss of generality, we suppose that $\|Z|_{\mathbf{Y}}\| = 1$. In the orthogonal (module) decomposition $\mathbf{X} = \mathbf{Y} \oplus \mathbf{Y}^{\perp}$, the operator Z has matrix

$$Z = \begin{pmatrix} a & b^* \\ b & c \end{pmatrix}.$$

Since we wish to extend the first column $Z|_{\mathbf{Y}} = \begin{pmatrix} a \\ b \end{pmatrix}$, our goal is to replace the entry c in order to minimize the norm of the completed matrix. Since the first column of Z is a contraction, it follows that the first row $Z_1 : \mathbf{X} \to \mathbf{Y}$ is also a contraction (in other words, as in Krein's original method, we have obtained a contractive extension of $a : \mathbf{Y} \to \mathbf{Y}$ to the whole module $\mathbf{X}$). We take now the 2,1-entry $b : \mathbf{Y} \to \mathbf{Y}^{\perp}$, which is clearly also a contraction, and extend it to a contraction $Z_2 : \mathbf{X} \to \mathbf{Y}^{\perp}$.

Consider in $\mathbf{X}$ the inner product

$$\prec \mathbf{x}_1, \mathbf{x}_2 \succ := \langle \mathbf{x}_1, \mathbf{x}_2 \rangle - \langle Z_1\mathbf{x}_1, Z_1\mathbf{x}_2 \rangle = \langle (1 - Z_1^*Z_1)\mathbf{x}_1, \mathbf{x}_2 \rangle.$$

Clearly it is $\mathcal{A}$-valued and positive semi-definite, and induces a seminorm in $\mathbf{X}$, namely

$$[[\mathbf{x}]] = \| \prec \mathbf{x}, \mathbf{x} \succ \|^{1/2} = \|\langle (1 - Z_1^*Z_1)\mathbf{x}, \mathbf{x} \rangle\|^{1/2}.$$

Let $\mathbf{Z} = \{\mathbf{z} \in \mathbf{X} : [[\mathbf{z}]] = 0\}$. Therefore $[[\ \ ]]$ induces a (quotient) Hilbert-C*-module norm in $\mathbf{X}/\mathbf{Z}$ (see for instance [14], chapter 1). Note that $\mathbf{Z} = N((1 - Z_1^*Z_1)^{1/2}) = N(1 - Z_1^*Z_1)$. The fact that Z_1 is contractive means that $Z_1Z_1^* : \mathbf{Y} \to \mathbf{Y}$ is contractive, i.e.,

$$1 \geq Z_1Z_1^* = a^2 + b^*b.$$

On the other hand, $\mathbf{y} \in \mathbf{Y}$ belongs to $N(1 - Z_1^* Z_1)$ if and only if (in matrix form)

$$\begin{pmatrix} \mathbf{y} \\ 0 \end{pmatrix} = Z_1^* Z_1 \begin{pmatrix} \mathbf{y} \\ 0 \end{pmatrix} = \begin{pmatrix} a^2 & ab^* \\ ba & bb^* \end{pmatrix} \begin{pmatrix} \mathbf{y} \\ 0 \end{pmatrix},$$

i.e., $a^2\mathbf{y} = \mathbf{y}$ and $ba\mathbf{y} = 0$. Thus, if $\mathbf{y} \in N(1 - Z_1^* Z_1) \cap \mathbf{Y}$, the inequality $a^2 + b^* b \leq 1$ implies that

$$\langle \mathbf{y}, \mathbf{y} \rangle \geq \langle a^2\mathbf{y}, \mathbf{y} \rangle + \langle \mathbf{y}, \mathbf{y} \rangle + \langle b^* b\mathbf{y}, \mathbf{y} \rangle, \ i.e., \ \langle b^* b\mathbf{y}, \mathbf{y} \rangle = 0,$$

thus, $b\mathbf{y} = 0$. In other words, b induces a well defined operator in $\mathbf{Y}/(\mathbf{Z} \cap \mathbf{Y})$, which is contractive (with values in $\mathbf{Y}^\perp$). Let us denote it by β. Let $\mathbf{P} : \mathbf{X}/\mathbf{Z} \to \mathbf{Y}/(\mathbf{Z} \cap \mathbf{Y})$ the quotient map induced by the orthogonal projection of $\mathbf{X}$ onto $\mathbf{Y}$. Put $Z_2 = \beta \mathbf{P} \Pi_{\mathbf{Z}}$, where $\Pi_{\mathbf{Z}} : \mathbf{X} \to \mathbf{X}/\mathbf{Z}$ is the quotient map. The operator Z_2 is contractive, because it is a composition of contractive ($\mathcal{A}$-)linear (adjointable) operators, $Z_2 : \mathbf{X} \to \mathbf{Y}^\perp$. Also note that on $\mathbf{Y}$, Z_2 coincides with b.

Put $Z_3 := Z_1 + Z_2$. The first row of Z_3 is Z_1 by construction. The entry $2, 1$ is the restriction $Z_2|_{\mathbf{Y}}$, i.e., b. Therefore, in matrix form we have

$$Z_3 = \begin{pmatrix} a & b^* \\ b & d \end{pmatrix}.$$

Clearly $Z_3 : \mathbf{X} \to \mathbf{X}$ is contractive: Z_1 and Z_2 are contractions which take values in orthogonal submodules ($\mathbf{Y}$ and $\mathbf{Y}^\perp$). The operator Z_3 need not be self-adjoint (d may not be selfadjoint). However

$$Z_3^* = \begin{pmatrix} a & b^* \\ b & d^* \end{pmatrix}, \quad \text{so that } Z_0 := Re(Z_3) = \begin{pmatrix} a & b^* \\ b & \frac{1}{2}(d + d^*) \end{pmatrix}$$

is a selfadjoint contraction, and therefore solves our problem. □

Remark 6.5. As in the scalar case $\mathcal{A} = \mathbb{C}$, minimal liftings need not be unique (see [12], [7]).

In our context, where $\mathbf{Y} = [\mathbf{x}_0]$ for some $\mathbf{x}_0 \in \mathbf{X}_p$, Lemma 6.4 implies that

Corollary 6.6. *Let $\mathbf{x}_0 \in \mathbf{X}_p$ and $\mathbf{v} \in (T\mathbf{X}_p)_{\mathbf{x}_0}$. Then there exists a minimal lifting Z_0 of v: $Z_0^* = -Z_0$, $Z_0(\mathbf{x}_0) = \mathbf{v}$ and $\|Z_0\| = \|Z_0 \theta_{\mathbf{x}_0,\mathbf{x}_0}\| = \|\mathbf{v}\|$.*

Theorem 6.7. *Let $\mathbf{x}_0 \in \mathbf{X}_p$ and $\mathbf{v} \in (T\mathbf{X}_p)_{\mathbf{x}_0}$. Let $Z \in \mathcal{B}_{\mathcal{A}}(\mathbf{X}))_{ah}$ be a minimal lifting of $\mathbf{v}$. Then the curve*

$$\delta(t) = e^{tZ}\mathbf{x}_0$$

has minimal length for $|t|\|\mathbf{v}\| \leq \pi$.

Proof. Suppose, without loss of generality (in view of the commentary at the beginning of this section) that $p = 1$. Also we can suppose that $\|\mathbf{v}\| = \pi$. Denote $P_0 = \theta_{\mathbf{x}_0,\mathbf{x}_0}$. Let γ be a smooth curve in $\mathbf{X}_1$ joining $\gamma(0) = \mathbf{x}_0$ and $\gamma(1) = e^Z(\mathbf{x}_0) = \delta(1)$. By elementary C*-algebra theory, there exists a representation $\rho : \mathcal{B}_{\mathcal{A}}(\mathbf{X}) \to \mathcal{B}(\mathcal{H})$ ($\mathcal{H}$ with inner product $< \ , \ >$) and a unit vector $\xi \in \mathcal{H}$ which satisfy $\rho(P_0 Z^2 P_0)\xi = -\|P_0 Z^2 P_0\|\xi$. Indeed, note that $-P_0 Z^2 P_0$ is a positive (non nil) element of $\mathcal{B}_{\mathcal{A}}(\mathbf{X})$. Note that since Z is a minimal lifting of $\mathbf{v}$ (at $\mathbf{x}_0$), $\|Z^2\| = \|P_0 Z^2 P_0\| = \|Z P_0\|^2 = \|Z\mathbf{x}_0\|^2$.

Consider the curve $\rho(\theta_{\gamma(t),\mathbf{x}_0})\xi$. Clearly it is a smooth $\mathcal{H}$-valued curve. Note that $\theta_{\gamma(t),\mathbf{x}_0}$ is a partial isometry with initial space $(\theta_{\gamma(t),\mathbf{x}_0})^* \theta_{\gamma(t),\mathbf{x}_0} = \theta_{\mathbf{x}_0,\gamma(t)} \theta_{\gamma(t),\mathbf{x}_0} = \theta_{\mathbf{x}_0,\mathbf{x}_0} = P_0$. Thus

$\rho(\theta_{\gamma(t),\mathbf{x}_0})$ is a curve of partial isometries with initial space $\rho(P_0)$. Clearly $\xi \in R(\rho(P_0))$, therefore $\rho(\theta_{\gamma(t),\mathbf{x}_0})\xi$ consists of unit vectors, i.e., it is a smooth curve in the unit sphere $\mathbb{S}_{\mathcal{H}}$ of $\mathcal{H}$. Its length is

$$length(\rho(\theta_{\gamma,\mathbf{x}_0})\xi) = \int_0^1 \|\frac{d}{dt}\rho(\theta_{\gamma(t),\mathbf{x}_0})\xi\|dt = \int_0^1 \|\rho(\theta_{\dot{\gamma}(t),\mathbf{x}_0})\xi\|dt \leq \int_0^1 \|\dot{\gamma}(t)\|dt \leq length(\gamma).$$

For the curve $\delta(t) = e^{tZ}\mathbf{x}_0$ we have

$$length(\rho(\theta_{\delta,\mathbf{x}_0})\xi) = \int_0^1 \|\rho(\theta_{e^{tZ}Z(\mathbf{x}_0),\mathbf{x}_0})\xi\|dt = \int_0^1 \|\rho(e^{tZ})\rho(\theta_{Z(\mathbf{x}_0),\mathbf{x}_0})\xi\|dt = \int_0^1 \|\rho(\theta_{Z(\mathbf{x}_0),\mathbf{x}_0})\xi\|dt.$$

Note that

$$\|\rho(\theta_{Z(\mathbf{x}_0),\mathbf{x}_0})\xi\|^2 =< \rho(\theta_{Z(\mathbf{x}_0),\mathbf{x}_0})\xi, \rho(\theta_{Z(\mathbf{x}_0),\mathbf{x}_0})\xi >=< \rho(\theta_{\mathbf{x}_0,Z(\mathbf{x}_0)}\theta_{Z(\mathbf{x}_0),\mathbf{x}_0})\xi, \xi >$$

$$=< \rho(P_0Z^*ZP_0)\xi, \xi >= - < \rho(P_0Z^2P_0)\xi, \xi >= \|Z\|^2.$$

Since $\|\dot{\delta}(t)\| = \|e^{tZ}Z(\mathbf{x}_0)\| = \|Z(\mathbf{x}_0)\| = \|Z\|$, we have that

$$length(\rho(\theta_{\delta,\mathbf{x}_0})\xi) = length(\delta).$$

We claim that $\rho(\theta_{\delta,\mathbf{x}_0})\xi$ is a minimal geodesic of $\mathbb{S}_{\mathcal{H}}$. First note that ξ is also a norming eigenvector for $\rho(Z^2)$: since $\xi \in R(\rho(P_0))$,

$$\rho(Z^2)\xi = \rho(Z^2P_0)\xi = \rho(P_0Z^2P_0)\xi + \rho(P_0^\perp Z^2P_0)\xi = \xi_1 = \|Z\|^2\xi.$$

Since

$$\|\rho(Z^2)\|^2 \geq \|\rho(Z^2)\xi\|^2 = \|\rho(P_0^\perp Z^2\xi)\|^2 + \|\rho(P_0Z^2)\xi\|^2 = \|\xi_1\|^2 + \|Z\|^4 \geq \|\rho(Z^2)\|^2,$$

it follows that $\xi_1 = 0$, and $\rho(Z^2)\xi = -\|Z\|^2\xi$. Then

$$\rho(\theta_\delta\ddot{(t)}, \mathbf{x}_0)\xi = \rho(\theta_{\ddot{\delta}(t),\mathbf{x}_0})\xi = \rho(\theta_{e^{tZ}Z^2(\mathbf{x}_0),\mathbf{x}_0})\xi = \rho(e^{tZ})\rho(Z^2P_0)\xi = -\|Z\|^2\rho(e^{tZ}P_0)\xi$$

$$= -\|Z\|^2\rho(\theta_{\delta(t),\mathbf{x}_0})\xi.$$

That is, $\rho(\theta_{\delta(t),\mathbf{x}_0})\xi = \Delta(t)$ satisfies the differential equation

$$\ddot{\Delta} = -\pi^2\Delta,$$

because $\|Z\| = \|v\| = \pi$. In other words, $\rho(\theta_{\delta(t),\mathbf{x}_0})\xi$ is a geodesic of $\mathbb{S}_{\mathcal{H}}$, which is minimal for $|t| \leq 1$, as claimed. Therefore

$$length(\delta) = length(\rho(\theta_{\delta,\mathbf{x}_0})\xi) \leq length(\rho(\theta_{\gamma,\mathbf{x}_0})\xi) \leq length(\gamma),$$

which finishes the proof. □

7 Examples

Let us consider the following examples:

7.1 $\mathbf{X} = \mathcal{A}^n$

First consider $n = 1$, $\mathbf{X} = \mathcal{A}$, ($\mathcal{A}$ unital) with the inner product $\langle a, b\rangle = a^*b$. The module norm is then the usual C*-norm of $\mathcal{A}$, and $\mathcal{B}_{\mathcal{A}}(\mathcal{A}) = \mathcal{A}$. The set $\mathbf{X}_{\mathcal{P}} = \{v \in \mathcal{A} : v^*v \text{ is a projection}\}$ is the set of partial isometries of $\mathcal{A}$, and, for p_0 a projection in $\mathcal{A}$, $\mathbf{X}_{p_0} = \{v \in \mathcal{A} : v^*v = p_0\}$ is the subset of partial isometries with initial space p_0. Note that in this case $\theta_{a,b}(x) = ab^*x$ is left multiplication by ab^*, i.e., $\theta_{a,b} \simeq ab^*$. The product map $\rho : \mathbf{X}_{\mathcal{P}} \to \mathcal{P}_{\mathcal{A}}$ is $\rho(v) = v^*v$, the initial projection of v, whereas the Hopf map $\mathfrak{h} : \mathbf{X}_{\mathcal{P}} \to \mathcal{P}_{\mathcal{B}_{\mathcal{A}}(\mathcal{A})} \simeq \mathcal{P}_{\mathcal{A}}$ is

$$\mathfrak{h}(v) = \theta_{v,v} \simeq vv^*$$

the final projection of v. Thus, we have proved that the set of partial isometries in $\mathcal{A}$ is a smooth complemented submanifold of $\mathcal{A}$, and that the initial and final projection maps are smooth locally trivial fiber bundles.

Remark 7.1. In this special case, the involution $* : \mathcal{A} \to \mathcal{A}$, which is skew-linear and isometric, defines an isometry in $\mathbf{X}_{\mathcal{P}}$, which intertwines the initial projection map ρ and the final projection map $\mathfrak{h}$. It follows that the metric properties of $\mathfrak{h}$ observed in the last part of Section 5, hold also for ρ. Thus we have:

Corollary 7.2. *Let v_0, v_1 be partial isometries in $\mathcal{A}$, in the same connected component of $\mathbf{X}_{\mathcal{P}}$, and let $p_i = v_i^*v_i$, $q_i = v_iv_i^*$, $i = 1, 2$. If $\nu(t)$ is a smooth curve of partial isometries in $\mathcal{A}$, with $\nu(0) = v_0$ and $\nu(1) = v_1$ then*

$$\int_0^1 \|\dot{\nu}(t)\|dt \geq \max\{\arcsin \|p_0 - p_1\|, \arcsin \|q_0 - q_1\|\}.$$

The result of the existence of minimal liftings and minimal curves with prescribed initial conditions (Corollary 6.6 and Theorem 6.7), in this context, gives us the following result:

Corollary 7.3. *Let v_0 be a partial isometry in $\mathcal{A}$, with inititial projection p_0 (i.e., $v_0 \in \mathbf{X}_{p_0}$), and $\mathbf{v} \in (T\mathbf{X}_{p_0})_{\mathbf{x}_0}$ (i.e., $\mathbf{v} \in \mathcal{A}$ satisfies $\mathbf{v}p_0 = \mathbf{v}$ and $\mathbf{v}^*v_0 + v_0^*\mathbf{v} = 0$). Then*

1. *There exists $z \in \mathcal{A}_{ah}$ such that $zv_0 = \mathbf{v}$ and $\|z\| = \|\mathbf{v}\|$.*
2. *For this element z, the curve $\delta(t) = e^{tz}v_0$, which verifies $\delta(0) = v_0$ and $\dot{\delta}(0) = \mathbf{v}$, has minimal length among smooth curves in $\mathbf{X}_{p_0}$ up to time t such that $|t|\|\mathbf{v}\| \leq \pi/2$.*

In the above result, the length of curves is measured using the usual norm of $\mathcal{A}$.

Let $\mathbf{X} = \mathcal{A}^n$, with the inner product

$$\langle \mathbf{x}, \mathbf{y}\rangle = \sum_{j=1}^n x_j^*y_j \ , \ \mathbf{x} = (x_1, \dots, x_n), \ \mathbf{y} = (y_1, \dots, y_n).$$

It will be useful to regard $\mathbf{x}$ and $\mathbf{y}$ as columns, and write

$$\langle \mathbf{x}, \mathbf{y}\rangle = \mathbf{x}^*\mathbf{y}.$$

Let us consider the particular case when $\mathcal{A} = \mathcal{B}(\mathcal{H})$ and $\mathbf{X} = \mathcal{B}(\mathcal{H})^n$.

Lemma 7.4. *Let $\mathcal{A} = \mathcal{B}(\mathcal{H})$, for an infinite dimensional Hilbert space $\mathcal{H}$, $\mathbf{X} = \mathcal{B}(\mathcal{H})^n$, $n \geq 2$, and p a projection in $\mathcal{B}(\mathcal{H})$ with range $p(\mathcal{H}) := \mathcal{H}_0$. Then each $\mathbf{x} \in \mathbf{X}_p$ defines an isometry $\mathbf{x} : \mathcal{H}_0 \to \mathcal{H}^n$ such that $R(\mathbf{x})^\perp$ is infinite dimensional.*

Proof. If $\xi \in \mathcal{H}_0$,

$$\|\mathbf{x}\xi\|^2 = \|x_1\xi\|^2 + \dots \|x_n\xi\|^2 = \langle x_1^* x_1\xi, \xi\rangle + \dots \langle x_n^* x_n \xi, \xi\rangle = \langle p\xi, \xi\rangle = \|\xi\|^2.$$

Let us check now that $R(\mathbf{x})^\perp$ is infinite dimensional. The claim is evident if $R(p) = \mathcal{H}_0$ is finite dimensional. Suppose that it is infinite dimensional.

Let $Q = \mathbf{x}\mathbf{x}^* \in \mathcal{B}(\mathcal{H}_0)^n$ be the projection on $R(\mathbf{x})$. The $n \times n$ matrix of Q with entries in $\mathcal{B}(\mathcal{H}_0)$ is

$$Q = \begin{pmatrix} x_1x_1^* & x_1x_2^* & \dots & x_1x_n^* \\ x_2x_1^* & x_2x_2^* & \dots & x_2x_n^* \\ \vdots & \vdots & & \vdots \\ x_nx_1^* & x_nx_2^* & \dots & x_nx_n^* \end{pmatrix}, \text{ so that } Q^\perp = \begin{pmatrix} p - x_1x_1^* & -x_1x_2^* & \dots & -x_1x_n^* \\ -x_2x_1^* & p - x_2x_2^* & \dots & -x_2x_n^* \\ \vdots & \vdots & & \vdots \\ -x_nx_1^* & -x_nx_2^* & \dots & p - x_nx_n^* \end{pmatrix}.$$

Therefore,

$$Tr(Q^\perp) = n\ Tr(p) - \sum_{i=i}^{n} Tr(x_i x_i^*) = n\ Tr(p) - \sum_{i=i}^{n} Tr(x_i^* x_i) = (n-1)Tr(p) = +\infty.$$

□

Let us prove that the space $\mathbf{X}_p$ is connected in this case.

Corollary 7.5. *Let $\mathcal{A} = \mathcal{B}(\mathcal{H})$, with $\dim \mathcal{H} = +\infty$, $p \in \mathcal{B}(\mathcal{H})$ and $\mathbf{X} = \mathcal{B}(\mathcal{H})^n$, for $n \geq 2$. Then the manifold $\mathbf{X}_p$ is connected.*

Proof. As above $\mathcal{H}_0 = R(p)$. Let $\mathbf{x}, \mathbf{y} \in \mathbf{X}_p$ be regarded, again, as isometries from $\mathcal{H}_0$ to $\mathcal{H}^n$. The connected components of the set of isometries between two Hilbert spaces (namely $\mathcal{H}_0$ and $\mathcal{H}^n$) are parametrized by the co-rank: two isometries belong to the same component if and only if they have the same co-rank. Indeed, If $\mathbf{x} : \mathcal{H}_0 \to R(\mathbf{x}) \subset \mathcal{H}^n$ and $\mathbf{y} : \mathcal{H}_0 \to R(\mathbf{y}) \subset \mathcal{H}^n$ have the same co-rank, one easily constructs a unitary operator in $\mathcal{H}^n$ such that $U\mathbf{x} = \mathbf{y}$. Take $U = U_1 \oplus U_2$, with $U_1 : R(\mathbf{y}) \to R(\mathbf{x})$ equal to $\mathbf{y}\mathbf{x}^*$, and U_2 any unitary operator from $R(\mathbf{x})^\perp$ to $R(\mathbf{y})^\perp$. It follows that there exists a selfadjoint operator $Z \in \mathcal{B}(\mathcal{H}^n)$ such that

$$e^{iZ}\mathbf{x} = \mathbf{y}.$$

Then $\mathbf{x}(t) = e^{itZ}\mathbf{x}$ is a continuous curve of isometries from $\mathcal{H}_0$ to $\mathcal{H}^n$ (i.e., a continuous curve in $\mathbf{X}_p$) such that $\mathbf{x}(0) = \mathbf{x}$ and $\mathbf{x}(1) = \mathbf{y}$. □

Remark 7.6.

1. For $n = 1$, this result does not hold in general: the same fact remarked above states that for the case $p = 1$, when $\mathbf{X}_1$ is the set of isometries of $\mathcal{H}$, the connected components of $\mathbf{X}_1$ are parametrized by the co-rank of the isometries.

2. Note that if $\dim R(p) = 1$, $p =< \ ,\xi_0 > \xi_0$ for some $\xi_0 \in \mathcal{H}$ with $\|\xi_0\| = 1$. If $\mathbf{x} \in \mathcal{B}(\mathcal{H})^n$ satisfies $\langle \mathbf{x}, \mathbf{x} \rangle = p$, then $x_i p = x_i$, for $1 \leq i \leq n$ and therefore $x_i =< \ ,\xi_0 > \eta_i$, for some $\eta_i \in \mathcal{H}$. Thus $\mathbf{x} \in \mathcal{B}(\mathcal{H})^n_p$ means that

$$p =< \ ,\xi_0 > \xi_0 = \sum_{i=1}^n x_i^* x_i = \left(\sum_{i=1}^n < \eta_i, \eta_i > \right) < \ ,\xi_0 > \xi_0,$$

i.e, $\eta = (\eta_1, \ldots, \eta_n)$ belongs to the (usual) unit sphere $\mathbb{S}(\mathcal{H}^n)$ of the Hilbert space $\mathcal{H}^n$. Clearly the map $\mathcal{B}(\mathcal{H})^n_p \ni \mathbf{x} \overset{\simeq}{\longmapsto} (\eta_1, \ldots, \eta_n) \in \mathbb{S}(\mathcal{H}^n)$ is a homeomorphism. Thus $\mathcal{B}(\mathcal{H})^n_p$ is contractible if $\dim \mathcal{H} = \infty$.

Remark 7.7. Again in the case $\mathbf{X} = \mathcal{B}(\mathcal{H})^n$, an interesting case of elements in $\mathbf{X}_1$ is provided by n-tuples $(S_1, \ldots, S_n) \in \mathbf{X}$ satisfying the relations

$$S_i^* S_i = 1, \ 1 \leq i \leq n, \ \text{and} \ \sum_{i=1}^n S_i S_i^* = 1. \tag{2}$$

These relations were introduced by J. Cuntz [6], as generating a class of universal C*-algebras, usually denoted $\mathcal{O}_n$. Note that $\mathbf{x} = (S_1^*, \ldots, S_n^*) \in \mathbf{X}_1$. Easy computations show that $(S_1, \ldots, S_n)$ satisfy (2) if and only if $\mathbf{x} \in \mathbf{X}_1$ satisfies $\mathfrak{h}(\mathbf{x}) = \theta_{\mathbf{x},\mathbf{x}} = 1$. In particular, this implies that the set of generators of $\mathcal{O}_n$ (inside $\mathcal{B}(\mathcal{H})$), regarded as n-tuples, form a smooth submanifold of $\mathbf{X}$: recall that the C^∞-map $\mathfrak{h}$ is a submersion, and thus $\mathfrak{h}^{-1}(\{1\}) \subset \mathbf{X}_{\mathcal{P}} \subset \mathbf{X}$ is a chain of submanifolds.

The left and right actions of, respectively, $\mathcal{U}_{\mathcal{B}(\mathcal{H})}(\mathbf{X})$ and $\mathcal{U}_{\mathcal{B}(\mathcal{H})}$ restrict to this submanifold. If $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbf{X}_1$ satisfies that $\mathbf{x}_i^*$ are isometries, then

$$(\mathbf{x} \cdot u^*)_i^* = u x_i^* \text{ is also an isometry if } u \in \mathcal{U}_{\mathcal{B}(\mathcal{H})}$$

and for $U \in \mathcal{U}_{\mathcal{B}(\mathcal{H})}(\mathbf{X})$ (i.e., a unitary matrix in $M_n(\mathcal{B}(\mathcal{H}))$)

$$\mathfrak{h}(U\mathbf{x}) = \theta_{U\mathbf{x}, U\mathbf{x}} = U \theta_{\mathbf{x},\mathbf{x}} U^* = UU^* = 1.$$

Next, note that both actions are (separately) transitive and have global cross sections. More precisely, if $\mathbf{x}, \mathbf{y} \in \mathbf{X}_1$ satisfy that x_i^*, y_i^* are isometries ($1 \leq i \leq n$), then $\langle \mathbf{x}, \mathbf{y} \rangle \in \mathcal{U}_{\mathcal{B}(\mathcal{H})}$ and $\theta_{\mathbf{x},\mathbf{y}} \in \mathcal{U}_{\mathcal{B}(\mathcal{H})}(\mathbf{X})$:

$$\langle \mathbf{x}, \mathbf{y} \rangle \langle \mathbf{y}, \mathbf{x} \rangle = (\sum_{i=1}^n x_i^* y_i)(\sum_{j=1}^n y_j^* x_j) = \sum_{i,j} x_i^* y_i y_j^* x_j$$

which, since y_i^*, y_j^* are isometries onto orthogonal subspaces if $i \neq j$ (due to (2)), equals

$$\sum_{i=1}^n x_i^* y_i y_i^* x_i = \sum_{i=1}^n x_i^* x_i = 1.$$

Also, since $\theta_{\mathbf{x},\mathbf{y}}(\mathbf{y}) = \mathbf{x} \langle \mathbf{y}, \mathbf{y} \rangle = \mathbf{x}$,

$$\theta_{\mathbf{x},\mathbf{y}} \theta_{\mathbf{y},\mathbf{x}} = \theta_{\theta_{\mathbf{x},\mathbf{y}}(\mathbf{y}),\mathbf{x}} = \theta_{\mathbf{x},\mathbf{x}} = 1.$$

As seen in the line above, the unitary $\theta_{\mathbf{x},\mathbf{y}}$ carries $\mathbf{y}$ to $\mathbf{x}$. The unitary $\langle \mathbf{x}, \mathbf{y} \rangle$ also carries $\mathbf{y}$ to $\mathbf{x}$ with the right action:

$$(\mathbf{y} \cdot (\langle \mathbf{x}, \mathbf{y} \rangle)^*)_i = y_i \sum_{j=1}^n y_j^* x_j = y_i y_i^* x_i = x_i.$$

In other words, if we fix $\mathbf{x}_0 = (x_1, \ldots, x_n)$ such that $\mathfrak{h}(\mathbf{x}_0) = 1$, the map

$$\mathcal{U}_{\mathcal{B}(\mathcal{H})} \to \{\mathbf{x} \in \mathbf{X}_1 : \mathfrak{h}(\mathbf{x}) = 1\} \ , \quad u \mapsto \mathbf{x}_0 \cdot u = \mathbf{x}_0 u^*$$

is a diffeomorphism. In particular, this implies that the submanifold of n-tuples of isometries satisfying the relations (2), is contractible.

7.2 Conditional expectations

Let $\mathcal{A} \subset \mathcal{B}$, with the same unit, and let $E : \mathcal{B} \to \mathcal{A} \subset \mathcal{B}$ be a conditional expectation. Then $\mathcal{B}$ becomes an inner product $\mathcal{A}$-module by means of

$$\langle \mathbf{x}, \mathbf{y} \rangle_E = E(\mathbf{x}^* \mathbf{y}), \ \mathbf{x}, \mathbf{y} \in \mathcal{B}.$$

In general, $\mathcal{B}$ is not complete with the ensuing norm $\|\mathbf{x}\|_E = \|E(\mathbf{x}^*\mathbf{x})\|^{1/2}$. One may have to consider the completion $\mathbf{X} = \overline{\mathcal{B}}^E$. In [4] it was shown that the norm is complete if and only if the conditional expectation has finite Jones index [10]. This is the case, for instance, if $\mathcal{A} \subset \mathcal{B}$ is an inclusion of von Neumann factors, with finite index [10].

A particularly interesting module in this context is $\mathbf{X}_E = \{\mathbf{x} \in \mathcal{B} : E(\mathbf{x}) = 0\} = \ker E$. Clearly $\mathbf{X}_E$ is a closed and complemented $\mathcal{A}$-submodule of $\mathcal{B}$: if $\mathbf{x} \in \mathbf{X}_E$ and $a \in \mathcal{A}$, then $E(a\mathbf{x}) = aE(\mathbf{x}) = 0$; and for $b, c \in \mathcal{B}$,

$$\langle E(b), c \rangle = E(E(b)^* c) = E(b^*)E(c) = E(b^* E(c)) = \langle b, E(c) \rangle,$$

i.e., $E \in \mathcal{B}_{\mathcal{A}}(\mathcal{B})$ is the selfadoint projection onto the submodule $\mathcal{B} \subset \mathcal{A}$, thus $\mathbf{X}_E = N(E) = \mathcal{B}^{\perp}$.

For certain projections $p \in \mathcal{P}_{\mathcal{A}}$, $(\mathbf{X}_E)_p$ might be empty. Let $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ be an orthogonal decomposition of the Hilbert space in two infinite dimensional closed subspaces, and let $P_+ = P_{\mathcal{H}_+}$. Consider the C*-algebras

$$\mathcal{B} = \{T \in \mathcal{B}(\mathcal{H}) : [T, P_+] \text{ is compact}\}, \text{ and } \mathcal{A} = \{S \in \mathcal{B}(\mathcal{H}) : [S, P_+] = 0\}.$$

In matrix form, in terms of the given decomposition of $\mathcal{H}$, we have

$$\mathcal{B} = \{ \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} : T_{12}, T_{21} \text{ compact}\} \text{ and } \mathcal{A} = \{ \begin{pmatrix} S_{11} & 0 \\ 0 & S_{22} \end{pmatrix} \}.$$

Let

$$E : \mathcal{B} \to \mathcal{A}, \ E(\begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}) = \begin{pmatrix} T_{11} & 0 \\ 0 & T_{22} \end{pmatrix}.$$

Then $\mathbf{X}_E = \{ \begin{pmatrix} 0 & T_{12} \\ T_{21} & 0 \end{pmatrix} : T_{12}, T_{21} \text{ compact}\}$. Then $\langle \mathbf{x}, \mathbf{x} \rangle = T_{12}^* T_{12} + T_{21}^* T_{21}$ is compact. Thus, if $E \le P_+$ and $F \le P_+^{\perp}$ are projections, one of them of infinite rank, then $P = E + F \in \mathcal{P}_{\mathcal{A}}$ satisfies $(\mathbf{X}_E)_p = \emptyset$; whereas if both E and F have finite rank, $(\mathbf{X}_E)_P$ is non empty.

Let us focus now on the case of a finite index inclusion $\mathcal{A} \subset \mathcal{B}$ of $\mathbf{II}_1$-factors, arguably the most relevant case in this context. We recall in the following remark the so called *basic construction* (see [10], [18]):

Remark 7.8. Let $\mathcal{A} \subset \mathcal{B}$ be a finite index inclusion of $\mathbf{II}_1$-factors, denote by τ the unique trace of $\mathcal{B}$ (which restricts to the unique trace of $\mathcal{A}$). The unique trace invariant conditional expectation $E : \mathcal{B} \to \mathcal{A}$ extends to the orthogonal projection e in $L^2(\mathcal{B}, \tau)$, with range $L^2(\mathcal{A}, \tau)$. Let $\mathcal{B}_1 = \{\mathcal{B}, e\}'' \subset \mathcal{B}(L^2(\mathcal{B}, \tau))$ be the von Neumann algebra generated by $\mathcal{B}$ and e ($\mathcal{B} \subset \mathcal{B}(L^2(\mathcal{B}, \tau))$ as left multiplication operators). The following known facts hold:

1. $\mathcal{B}_1$ is also a $\mathbf{II}_1$-factor, $\mathcal{B} \subset \mathcal{B}_1$ has the same (finite) index as $\mathcal{A} \subset \mathcal{B}$.

2. For $b \in \mathcal{B}$, $ebe = E(b)e = eE(b)$.

3. Elements of the form $b_0 + \sum_{i=1}^n b_i e b_1'$, with $b_0, b_i, b_i' \in \mathcal{B}$ form a dense $*$-subalgebra of $\mathcal{B}_1$.

4. The index $\mathbf{t} = [\mathcal{B} : \mathcal{A}]$ can be recovered as the optimal lower bound
$$\|E(b^*b)\| \geq \mathbf{t}^{-1} \|b^*b\|, b \in \mathcal{B}.$$
In particular, since E is contractive, this means that the usual C*-norm and the modular norm are equivalent in $\mathcal{B}$.

5. The map $x \mapsto xe$ is a $*$-(multiplicative) isomorphism between $\mathcal{B}$ and $e\mathcal{B}_1 e$.

6. The same map $b \mapsto be$, but between $\mathcal{B}$ and $\mathcal{B}e$, is a linear isomorphism. Moreover, it is isometric if $\mathcal{B}$ is endowed with the C*-module norm:
$$\|be\| = \|eb^*be\|^{1/2} = \|E(b^*b)e\|^{1/2} = \|E(b^*b)\|^{1/2}.$$
where the last equality holds because of the previous item. In particular, $\mathcal{B}e$ is norm closed in $\mathcal{B}_1$.

Proposition 7.9. *The unit sphere $\mathbf{X}_1 = \{b \in \mathcal{B} : E(b^*b) = 1\}$ is diffeomorphic to the space $\{x \in \mathcal{B}_1 : x^*x = e\}$ of partial isometries in $\mathcal{B}_1$ with initial space e. The diffeomorphism is given by $x \mapsto xe$ (or equivalently, the inclusion $\mathcal{B} \hookrightarrow \mathcal{B}_1$). Moreover, $\{x \in \mathcal{B}_1 : x^*x = e\} = \mathcal{U}_{\mathcal{B}_1} \cdot e$.*

Proof. If $E(b^*b) = 1$, then $eb^*be = E(b^*b)e = e$, and thus be is a partial isometry in $\mathcal{B}_1$, with initial space b. Conversely, let $x \in \mathcal{B}_1$ such that $x^*x = e$. Then $x = xe$. We claim that there exists $b \in \mathcal{B}$ such that $be = x$. Indeed, elements of the form $b_0 + \sum_{i=1}^n b_i e b_i'$, with $b_0, b_i, b_i' \in \mathcal{B}$ are dense in $\mathcal{B}_1$, thus there exists a sequence x_n of these sums such that $x_n \to x$ (in norm). Then $x_n e \to xe = x$. Note that
$$(b_0 + \sum_{i=1}^n b_i e b_1')e = b_0 e + \sum_{i=1}^n b_i e b_i' e = b_0 e b_0 + \sum_{i=1}^n b_i E(b_i')e \in \mathcal{B} \cdot e,$$
i.e. $x_n e \in \mathcal{B}e$, which is closed in $\mathcal{B}_1$, and thus $x \in \mathcal{B}e$. Then $eb^*be = x^*x = e$, i.e., $E(b^*b)e = e$. Since the map $a \mapsto ae$ is an isomorphism between $\mathcal{A}$ and $e\mathcal{B}_1 e$, it follows that $E(b^*b) = 1$.

It remains to prove that $\{x \in \mathcal{B}_1 : x^*x = e\} = \mathcal{U}_{\mathcal{B}_1} \cdot e$. The inclusion $\mathcal{U}_{\mathcal{B}_1} \cdot e \subset \{x \in \mathcal{B}_1 : x^*x = e\}$ is clear. Let $x \in b_1$ such that $x^*x = e$. Then $xx^* = xex^*$, the final projection of x, is Murray-von Neumann equivalent to e. Since $\mathcal{B}_1$ is a $\mathbf{II}_1$ factor, there exists a unitary element $w \in \mathcal{U}_{\mathcal{B}_1}$ such that $wew^* = xx^*$. Then w^*x is a partial isometry with initial and final space e, and $w^*x + e^\perp \in \mathcal{U}_{\mathcal{B}_1}$. Thus $u = w(w^*x + e^\perp) = x + we^\perp \in \mathcal{U}_{\mathcal{B}_1}$ satisfies $ue = xe = x$. □

Corollary 7.10. *If $\mathcal{A} \subset \mathcal{B}$ is a finite index inclusion of $\mathbf{II}_1$ factors, $E : \mathcal{B} \to \mathcal{A}$ the unique trace invariant conditional expectation and $\mathbf{X} = \mathcal{B}$ as a right $\mathcal{A}$-module with the inner product given by E, then the unit sphere*

$$\mathbf{X}_1 = \{b \in \mathcal{B} : E(b^*b) = 1\}$$

is connected.

Proof. $\mathcal{U}_{\mathcal{B}_1}$ is connected. □

Remark 7.11. The diffeomorphism $\mathbf{X}_1 \to \{x \in \mathcal{B}_1 : x^*x = e\}$, $b \mapsto be$, is isometric. Indeed, since it is the restriction of a global lineal map, it coincides (formally) with its tangent map at every point. The tangent spaces of $\mathbf{X}_1$ are endowed with with the C*-module norm, the tangent spaces of the space $\{x \in \mathcal{B}_1 : x^*x = e\}$ of partial isometries in $\mathcal{B}_1$ are endowed with the usual norm of $\mathcal{B}_1$. Therefore, this assertion (that the map is isometric) is essentially item 6. of Remark 7.8.

7.3 The standard module $\mathbf{X} = H(\mathcal{A})$

Let $\mathbf{X} = H(\mathcal{A})$, the *standard* module, given by

$$H(\mathcal{A}) = \{\mathbf{x} = (x_j)_{j\geq 1} : \sum_{j=1}^{\infty} x_j^* x_j \text{ converges in } \mathcal{A}\},$$

with inner product

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{j=1}^{\infty} x_j^* y_j.$$

Denote by $H^0(\mathcal{A}) = \{\mathbf{x} = (x_j)_{j\geq 1} : x_j \neq 0 \text{ for finitely many } j\}$, the set of elements of $H(\mathcal{A})$ of *finite support*, and by $H^0(\mathcal{A})_{\mathcal{P}} = H^0(\mathcal{A}) \cap H(\mathcal{A})_{\mathcal{P}}$.

Proposition 7.12. *Let $\mathbf{x} \in H(\mathcal{A})_{\mathcal{P}}$ with $\sum_{j=1}^{\infty} x_j^* x_j = p_0$. Then there exist $\mathbf{x}_N \in H^0(\mathcal{A})_{\mathcal{P}}$ such that $\mathbf{x}_N \to \mathbf{x}$*

Proof. Let $\mathbf{y}_N = (x_1, \ldots, x_N, 0, \ldots)$, the sequence $\mathbf{x}$ truncated at the N-th entry. Clearly $\mathbf{y}_N \to \mathbf{x}$. First note that there exists N_0, such that for $N \geq N_0$, $\langle \mathbf{y}_n, \mathbf{y}_n \rangle = \sum_{j=1}^{N} x_j^* x_j$ has closed range. Indeed, note that $1 - p_0 + \langle \mathbf{x}, \mathbf{x} \rangle = 1$ and that $1 - p_0 + \langle \mathbf{y}_N, \mathbf{y}_n \rangle \to 1$. Then there exists N_0 such that for $N \geq N_0$, $1 - p_0 + \langle \mathbf{y}_n, \mathbf{y}_n \rangle$ is invertible in $\mathcal{A}$. Note that

$$\langle \mathbf{y}_N, \mathbf{y}_n \rangle = \sum_{j=1}^{N} x_j^* x_j \leq \sum_{j=1}^{\infty} x_j^* x_j = \langle \mathbf{x}, \mathbf{x} \rangle = p_0.$$

It follows that (if we represent $\mathcal{A}$ faithfully in a Hilbert space) $\langle \mathbf{y}_N, \mathbf{y}_N \rangle$ acts trivially in $N(p_0)$. Since, for $N \geq N_0$, $1 - p_0 + \langle \mathbf{y}_N, \mathbf{y}_N \rangle$ is invertible, then $\langle \mathbf{y}_N, \mathbf{y}_N \rangle$ acts as an invertible operator in $R(p_0)$. Then, $\langle \mathbf{y}_N, \mathbf{y}_N \rangle$ has closed range for $N \geq N_0$. Let b_N be the Moore-Penrose pseudo-inverse of $\langle \mathbf{y}_N, \mathbf{y}_N \rangle$ ($N \geq N_0$). Note that $b_N \geq 0$ and $\mathbf{x}_N = \mathbf{y}_N b_n^{1/2}$ satisfies that

$$\langle \mathbf{x}_n, \mathbf{x}_n \rangle = b_N^{1/2} \langle \mathbf{y}_n, \mathbf{y}_n \rangle b_n^{1/2} = p_N := P_{R(\langle \mathbf{y}_N, \mathbf{y}_N \rangle)},$$

i.e., $\mathbf{x}_N \in \mathbb{H}^0(\mathcal{A})_{\mathcal{P}}$. We claim that $\mathbf{x}_N \to \mathbf{x}$. In order to prove this, it suffices to show that b_N converge to the pseudo-inverse of $\langle \mathbf{x}, \mathbf{x} \rangle = p_0$, which is again p_0. To this effect, we shall use a result by C. Apostol (Proposition 2.1 in [3]). We simplify the statement, to deal only with positive operators. Suppose that a is a positive operator, denote by γ_a the *reduced minimum modulus* of a:

$$\gamma_a = \inf\{t \in \sigma(a) : t \neq 0\}.$$

Note that if b is positive and invertible, then $\gamma_b = \|b^{-1}\|^{-1}$. Apostol defined the function $\gamma_a(\lambda) := \gamma_{a-\lambda 1}$. In Proposition 2.1 [3] he proved that if a has closed range (i.e., $\gamma_a > 0$), then the function $\gamma_a(\lambda)$ is continuous at $\lambda = 0$. This result has the following consequence (which is an elementary, and perhaps well known result):

Lemma 7.13. *Let $0 \leq b \leq a$ in $\mathcal{A}$, with both a, b of closed range. Then $\gamma_b \leq \gamma_a$.*

Proof. Let $r > 0$, then $a + r1$ is invertible, and thus $\gamma_a(-r) = \gamma_{a+r1} = \|(a + r1)^{-1}\|^{-1}$. Since a has closed range, by Apostol's result we have that

$$\|(a + r1)^{-1}\|^{-1} \to \gamma_a \text{ if } r \to 0.$$

The same happens for b. Next note that $b \leq a$ implies that $b + r1 \leq a + r1$, and thus, $(a + r1)^{-1} \leq (b + r1)^{-1}$. Therefore $\|(b + r1)^{-1}\|^{-1} \leq \|(a + r1)^{-1}\|^{-1}$. Taking limit $r \to 0$ completes the proof □

Returning to our previous claim, we have that $\langle \mathbf{y}_N, \mathbf{y}_N \rangle \leq p_0$, with $\langle \mathbf{y}_N, \mathbf{y}_N \rangle$ positive elements with closed range. Then

$$\gamma_{\langle \mathbf{y}_N, \mathbf{y}_N \rangle} \leq \gamma_{p_0} = 1.$$

It is known that the Moore-Penrose pseudo-inverse is continuous on sets of operators with bounded reduced minimum modulus (see for instance the paper by Harte and Mbekhta [9]). □

As a consequence, we have that

Corollary 7.14. *Let $\mathbf{x} \in H(\mathcal{A})_{\mathcal{P}}$. Then there exists a unitary operator U in $\mathcal{B}_{\mathcal{A}}(H(\mathcal{A}))$, and $\mathbf{x}_0 \in H^0(\mathcal{A})_{\mathcal{P}}$ (of finite support) such that $U\mathbf{x}_0 = \mathbf{x}$.*

Proof. By the above proposition, there exists $\mathbf{z} \in H^0(\mathcal{A})_{\mathcal{P}}$ such that $\|\mathbf{x} - \mathbf{z}\| < \frac{1}{2}$. Then there exists a unitary U in $\mathcal{B}_{\mathcal{A}}(H(\mathcal{A}))$ such that

$$\theta_{\mathbf{x},\mathbf{x}} = U\theta_{\mathbf{z},\mathbf{z}}U^* = \theta_{U\mathbf{z},U\mathbf{z}}.$$

As seen before, since both $\mathbf{x}$ and $U\mathbf{z}$ belong to $H(\mathcal{A})_p$, the element $v = \langle U\mathbf{z}, \mathbf{x} \rangle$ is a partial isometry in $\mathcal{A}$ such that

$$\mathbf{x} = (U\mathbf{z})v = U(\mathbf{z}v).$$

Moreover, v has initial space $\langle \mathbf{x}, \mathbf{x} \rangle$ and final space $\langle U\mathbf{z}, U\mathbf{z} \rangle = \langle \mathbf{z}, \mathbf{z} \rangle$. Note that $\mathbf{x}_0 = \mathbf{z}v$ has finite support, and satisfies that $\langle \mathbf{x}_0, \mathbf{x}_0 \rangle = v^*\langle \mathbf{z}, \mathbf{z} \rangle v = v^*v$ is a projection in $\mathcal{A}$. Thus $\mathbf{x}_0 \in H^0(\mathcal{A})_{\mathcal{P}}$ and $U\mathbf{x}_0 = \mathbf{x}$. □

Recall now a result by J. Mingo [16], which establishes that if $\mathcal{A}$ is a unital C*-algebra, then the unitary group of $\mathcal{B}_{\mathcal{A}}(H(\mathcal{A}))$ is contractible.

Corollary 7.15. *Let $\mathcal{A} = \mathcal{B}(\mathcal{H})$, and p a projection. Then $H(\mathcal{A})_p$ is connected.*

Proof. Let $\mathbf{x} \in H(\mathcal{A})_p \subset H(\mathcal{A})_\mathcal{P}$. Then there exists $\mathbf{x}_0$ in $H(\mathcal{A})_\mathcal{P}$ of finite support, and a unitary element U in $\mathcal{B}_\mathcal{A}(H(\mathcal{A}))$ such that $U\mathbf{x}_0 = \mathbf{x}$. Recall from the beginning of Section 3, the fact that the left action of the unitary group of $\mathcal{B}_\mathcal{A}(\mathbf{X})$ preserves the fibers of the product map ρ. Thus, $\mathbf{x}_0 \in H(\mathcal{A})_p$ as well. Moreover, by the result of Mingo [16], there exists a continuous path U_t of unitaries in $\mathcal{B}_\mathcal{A}(H(\mathcal{A}))$, $t \in [0,1]$, such that $U_0 = 1$ and $U_1 = U$. Since $\mathbf{x}_0$ has finite support, there exists $n < \infty$ such that $p = \langle \mathbf{x}_0, \mathbf{x}_0 \rangle = \sum_{i=1}^n x_i^* x_i$, i.e., if we denote by $\mathbf{x}'_0 = (x_1, \dots, x_n)$, $\mathbf{x}'_0 \in \mathcal{B}(\mathcal{H})^n_p$. Recall from Corollary 7.5 that $\mathcal{B}(\mathcal{H})^n_p$ is connected when $n \geq 2$. Thus, for instance, there is a continuous path $\mathbf{x}'(t)$ in $\mathcal{B}(\mathcal{H})^n_2$ with $\mathbf{x}'(0) = \mathbf{x}'_0$ and $\mathbf{x}'(1) = (p, 0, \dots, 0)$. Clearly $\mathbf{x}(t) = (\mathbf{x}'(t), \dots)$ is a continuous path in $H(\mathcal{A})_p$ joining $(p, 0, \dots)$ and $\mathbf{x}_0$. Thus, any element in $H(\mathcal{A})_p$ can be joined to $(p, 0, \dots)$ with a continuous path. □

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