# $p$-Schatten commutators of projections

Esteban Andruchow and María Eugenia Di Iorio y Lucero

**Abstract.** Let $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ be a fixed orthogonal decomposition of the complex separable Hilbert space $\mathcal{H}$ in two infinite dimensional subspaces. We study the geometry of the set $\mathcal{P}^p$ of selfadjoint projections in the Banach algebra

$$\mathcal{A}^p = \{A \in \mathcal{B}(\mathcal{H}) : [A, E_+] \in \mathcal{B}_p(\mathcal{H})\},$$

where $E_+$ is the projection onto $\mathcal{H}_+$ and $\mathcal{B}_p(\mathcal{H})$ is the Schatten ideal of $p$-summable operators ($1 \leq p < \infty$). The norm in $\mathcal{A}^p$ is defined in terms of the norms of the matrix entries of the operators given by the above decomposition. The space $\mathcal{P}^p$ is shown to be a differentiable $C^\infty$ submanifold of $\mathcal{A}^p$, and a homogeneous space of the group of unitary operators in $\mathcal{A}^p$. The connected components of $\mathcal{P}^p$ are characterized, by means of a partition of $\mathcal{P}^p$ in nine classes, four discrete classes and five essential classes:

- the first two corresponding to finite rank or co-rank, with the connected components parametrized by theses ranks;
- the next two discrete classes carrying a Fredholm index, which parametrizes its components;
- the remaining essential classes, which are connected.

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## 1. Introduction

Let $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ be an orthogonal decomposition of the complex separable Hilbert space $\mathcal{H}$ in two infinite dimensional closed subspaces, with corresponding projections $E_+$ and $E_-$. Consider the algebra

$$\mathcal{A}^p := \{A \in \mathcal{B}(\mathcal{H}) : [A, E_+] = AE_+ - E_+A \in \mathcal{B}_p(\mathcal{H})\},$$

where $\mathcal{B}(\mathcal{H})$ denotes the algebra of bounded linear operators in $\mathcal{H}$, and $\mathcal{B}_p(\mathcal{H})$ is the Schatten ideal of $p$-summable operators ($1 \leq p < \infty$). $\mathcal{A}^p$ is a $*$-Banach algebra with a suitable norm ($*$ is the usual adjoint). The purpose of this paper is the study of the set $\mathcal{P}^p$ of selfadjoint projections in $\mathcal{A}^p$,

$$\mathcal{P}^p = \{P \in \mathcal{A}^p : P^2 = P^* = P\}.$$

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It is known [5, 12] that the set of idempotents ($Q$ such that $Q^2 = Q$) of a Banach algebra is a complemented submanifold of the algebra. Here we show that also the set of selfadjoint idempotents is a submanifold of the algebra, in the case of the algebra $\mathcal{A}^p$ (it can be proved to hold for arbitrary $*$-Banach algebras). The case $p = 2$ was extensively treated in [4].

We characterize the connected components of $\mathcal{P}^p$. First, we see that $\mathcal{P}^p$ is partitioned in nine classes, four discrete classes $\mathbb{D}_j$, $1 \le j \le 4$, and five essential classes $\mathbb{E}_j$, $1 \le j \le 5$. The first two discrete classes correspond to the projections of finite rank or finite co-rank, and its connected components are characterized by these numbers. The next two discrete classes are more interesting, and correspond to the so called *p-restricted Grassmannian*, associated to $E_+$, and to $E_-$, respectively. Projections in a restricted Grassmannian carry an integer Fredholm index, which in turn parametrizes the connected components of $\mathbb{D}_3$ and $\mathbb{D}_4$ (as with the former two, one passes from one class to the other with the symmetry $P \mapsto P^\perp = 1 - P$, and thus the geometric and topological of both pairs are similar). The remaining essential classes are shown to be connected.

Examples of discrete projections (in the $p = 1$ restricted Grassmannian) of the decompostition $L^2(\mathbb{T}) = H^2(\mathbb{D}) \oplus H^2_-(\mathbb{D})$, where $H^2(\mathbb{D})$ is the Hardy space of the disk, are the projections onto the the subspaces $\varphi H^2(\mathbb{D})$, for $\varphi$ a smooth function of modulus one. The index given by (minus) the winding number of $\varphi$.

Examples of essential projections, again for $p = 1$, are given for the decomposition $L^2(\mathbb{R}^n) = L^2(\Omega) \oplus L^2(\Omega^c)$, where $\Omega \subset \mathbb{R}^n$ is a measurable set with positive finite measure. In this setting, the projection $FE_+F^{-1}$ ($F$ = Fourier-Plancherel transform), onto the space of functions in $L^2(\mathbb{R}^n)$ with Fourier transform supported in $\Omega$, is an essential projection.

This study is a continuation of [2], where the ideal of compact operators was considered. Some of the techniques and results are similar in both contexts, the main difficulty in the case at hand ($p$-Schatten ideals) is that the structure algebra $\mathcal{A}^p$ is a Banach algebra, whereas in the compact case it is a C$^*$-algebra. For instance, we need to prove the smooth local structure of the group $\mathcal{U}_{\mathcal{A}^p}$ of unitary operators in $\mathcal{A}^p$, which acts in $\mathcal{P}^p$.

We do not know if the geodesics of the Grassmann manifold of $\mathcal{H}$, lying in $\mathcal{P}^p$ (that is, with initial velocity in $\mathcal{A}^p$), are short for the Finsler metric given by the norm of the Banach algebra $\mathcal{A}^p$. However, we show that for the case of the discrete classes $\mathbb{D}_3$ and $\mathbb{D}_4$ (corresponding to the $p$-restricted Grassmannian), the connected component containing a given projection $P$, is a submanifold of the affine space $P + \mathcal{B}_p(\mathcal{H})$, which carries naturally the Schatten $p$-norm. With the Finsler metric given by this norm, the geodesics of the full Grassmannian of $\mathcal{H}$, lying inside this component, are short.

## 2. Preliminaries

For $1 \le p < \infty$, we denote by $\mathcal{B}_p(\mathcal{H})$ the ideal of $p$-Schatten operators in $\mathcal{H}$, i.e., $\mathcal{B}_p(\mathcal{H}) = \{T \in \mathcal{B}(\mathcal{H}) : Tr(|T|^p) < \infty\}$, with its norm $\|T\|_p = Tr^{1/p}(|T|^p)$. We fix an orthogonal decomposition

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$$

with corresponding projections $E_+$ and $E_-$. We make the assumption that both $\mathcal{H}_+$ and $\mathcal{H}_-$ are infinite dimensional. Denote by $\mathcal{P}(\mathcal{H}) = \mathcal{P}$ the set of all orthogonal projections in $\mathcal{H}$. We are interested in the set

$$\mathcal{P}^p_{\mathcal{H}_+} = \mathcal{P}^p := \{P \in \mathcal{P} : [P, E_+] \in \mathcal{B}_p(\mathcal{H})\}.$$

Accordingly, we denote by

$$\mathcal{A}^p_{\mathcal{H}_+} = \mathcal{A}^p := \{A \in \mathcal{B}(\mathcal{H}) : [A, E_+] \in \mathcal{B}_p(\mathcal{H})\}.$$

With $\mathcal{A}^p_h$ and $\mathcal{A}^p_{ah}$ we denote, respectively, the sets of selfadjoint and anti-Hermitian elements of $\mathcal{A}^p$.

In terms of the decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, operators in $\mathcal{H}$ can be written as $2 \times 2$ matrices. It is clear that elements of $\mathcal{A}^p$ are characterized as those matrices

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

such that $A_{12} \in \mathcal{B}_p(\mathcal{H}_-, \mathcal{H}_+)$ and $A_{21} \in \mathcal{B}_p(\mathcal{H}_+, \mathcal{H}_-)$.

We endow $\mathcal{A}^p$ with the following norm

$$\|A\|_{\infty,p} := \|A_{11}\| + \|A_{22}\| + \|A_{12}\|_p + \|A_{21}\|_p. \tag{2.1}$$

It is easy to see that $\mathcal{A}_p$ is a Banach space with this norm. Also it is clear that it is an algebra. This can be seen by elementary matrix computations, or noting that if $A, B \in \mathcal{A}^p$, then

$$\begin{aligned}[AB, E_+] &= ABE_+ - AE_+B + AE_+B - E_+AB \\ &= A[B, E_+] - [A, E_+]B \ \in \ \mathcal{B}_p(\mathcal{H}).\end{aligned}$$

With the norm $\| \ \|_{\infty,p}$ just defined, it is elementary that

$$\|AB\|_{\infty,p} \leq \|A\|_{\infty,p}\|B\|_{\infty,p}$$

for $A, B \in \mathcal{A}^p$, i.e., $\mathcal{A}^p$ is a Banach algebra. Note also that if $A \in \mathcal{A}^p$ then $A^* \in \mathcal{A}^p$ and $\|A^*\|_{\infty,p} = \|A\|_{\infty,p}$. Is is also clear that $\| \ \|_{\infty,p}$ is not a C$^*$-norm. However, the inclusion

$$(\mathcal{A}^p, \| \ \|_{\infty,p}) \hookrightarrow (\mathcal{B}(\mathcal{H}), \| \ \|)$$

is continuous, so that $\mathcal{A}^p$ is a Banach subalgebra of $\mathcal{B}(\mathcal{H})$.

Let us denote by $\mathcal{G}_{\mathcal{A}^p}$ the group of invertible elements in $\mathcal{A}^p$. It is usually known in the literature [11] as one of the reduced groups (acting in the restricted Grassmannian, see for instance [4]). For instance, it is known that if $G \in \mathcal{G}_{\mathcal{A}^p}$ then its diagonal entries (in the $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ matrix) are $p$-Fredholm operators (i.e., operators which are invertible modulo the ideal $\mathcal{B}_p(\mathcal{H})$). As such, to $G \in \mathcal{G}_{\mathcal{A}^p}$ an index can be attached, namely, the Fredholm index of the $1,1$-entry. As a consequence $\mathcal{G}_{\mathcal{A}^p}$ is disconnected, and its connected components are parametrized by this index.

Let us denote by $\mathcal{U}_{\mathcal{A}^p} := \{U \in \mathcal{G}_{\mathcal{A}^p} : U \text{ is unitary in } \mathcal{H}\}$.

The following elementary property of $\mathcal{G}_{\mathcal{A}^p}$ shall be very useful, it states that the usual polar decomposition of (invertible) elements of $\mathcal{A}^p$ stays in $\mathcal{A}^p$.

**Proposition 2.1.** *If $G \in \mathcal{G}_{\mathcal{A}^p}$ and $G = U|G|$ is the polar decomposition, then $|G|, U \in \mathcal{G}_{\mathcal{A}^p}$ (i.e., $U \in \mathcal{U}_{\mathcal{A}^p}$). Moreover, $G$ and $U$ share the same index.*

*Proof.* It suffices to show that $|G| \in \mathcal{G}_{\mathcal{A}^p}$. Clearly $G^*G \in \mathcal{G}_{\mathcal{A}^p}$. Denote by $\sigma_{\mathcal{B}(\mathcal{H})}(G^*G)$ and $\sigma_{\mathcal{A}^p}(G^*G)$ the spectra of $G^*G$ in $\mathcal{B}(\mathcal{H})$ and $\mathcal{A}^p$, respectively. Since $\mathcal{A}^p \subset \mathcal{B}(\mathcal{H})$ is a Banach subalgebra, its follows that (see for instance [10])

$$\sigma_{\mathcal{B}(\mathcal{H})}(G^*G) \subset \sigma_{\mathcal{A}^p}(G^*G) \quad \text{and} \quad \partial\sigma_{\mathcal{A}^p}(G^*G) \subset \partial\sigma_{\mathcal{B}(\mathcal{H})}(G^*G).$$

Moreover, since $G^*G$ is positive and invertible, $\sigma_{\mathcal{B}(\mathcal{H})}(G^*G) \subset (0, +\infty)$. Then, $\sigma_{\mathcal{B}(\mathcal{H})}(G^*G) = \partial\sigma_{\mathcal{B}(\mathcal{H})}(G^*G)$. Thus

$$\sigma_{\mathcal{B}(\mathcal{H})}(G^*G) = \sigma_{\mathcal{A}^p}(G^*G) = \sigma \subset [\delta, +\infty)$$

for some $\delta > 0$. Denote by $\log(z)$ the usual complex log function (discontinuous in the negative real axis). Note that $\log(z)$ is analytic on an open neighbourhood of $\sigma$, and thus $\log(G^*G)$ is defined in $\mathcal{A}^p$ by means of the usual holomorphic functional calculus in Banach algebras. Let $C = \exp(\frac{1}{2}\log(G^*G)) \in \mathcal{A}^p$. Note that if one regards $G^*G$ as an element in $\mathcal{B}(\mathcal{H})$, $C \in \mathcal{B}(\mathcal{H})$ is the usual (positive) square root of $G^*G$, i.e., $C = (G^*G)^{1/2} = |G|$.

The set of positive elements in $\mathcal{G}_{\mathcal{A}^p}$ is convex, and therefore connected. Thus $G$ and $U$ lie in the same connected component of $\mathcal{G}_{\mathcal{A}^p}$. □

## 3. Regular structure of $\mathcal{P}^p$

In this section we show that the set $\mathcal{P}^p$ of orthogonal projections in $\mathcal{A}^p$ is a complemented $C^\infty$ submanifold of $\mathcal{A}^p_h$. It is known that the set of idempotents of a Banach algebra is a complemented submanifold of the algebra (see [5] or [12]). Here we are dealing with selfadjoint idempotents.

First, note that the group $\mathcal{U}_{\mathcal{A}^p}$ is a Banach-Lie group, whose Banach-Lie algebra is $\mathcal{A}^p_{ah}$:

**Theorem 3.1.** *The group $\mathcal{U}_{\mathcal{A}^p}$ is a Banach-Lie group and a complemented submanifold of $\mathcal{A}^p$. Its Banach-Lie algebra is $\mathcal{A}^p_{ah}$.*

*Proof.* The exponential map $\exp : \mathcal{A}^p \to \mathcal{G}_{\mathcal{A}^p}$, $exp(X) = e^X$, is a local diffeomorphism, there exists a radius $0 < r < 1$ such and an open subset $0 \in \mathcal{W} \subset \mathcal{A}^p$ such that

$$exp : \mathcal{W} \to \{G \in \mathcal{A}^p : \|G - 1\|_{\infty,p} < r\}$$

is a diffeomorphism. Its inverse is the usual log series. When restricted to $\mathcal{A}^p_{ah}$, it takes values in $\mathcal{U}_{\mathcal{A}^p}$, which is a complemented (real) subspace of $\mathcal{A}^p$. Thus, in order to obtain a local chart for $\mathcal{U}_{\mathcal{A}^p}$ around 1 it suffices to show that elements in $\mathcal{U}_{\mathcal{A}^p}$ close enough to 1, are of the form $e^X$ for some $X \in \mathcal{A}^p_{ah}$ close to 0. In fact, if $U \in \mathcal{U}_{\mathcal{A}^p}$ satisfies $\|U - 1\|_{\infty,p} < r(< 2)$, since

$$\|U - 1\| \leq \|U - 1\|_{\infty,p} < 2,$$

the spectrum $\sigma_{\mathcal{B}(\mathcal{H})}(U)$ is contained in an arc $\{e^{i\theta} : |\theta| \leq \theta_0 < \pi\}$. Thus, by a similar argument as in Proposition 2.1,

$$\sigma_{\mathcal{A}^p}(U) = \sigma_{\mathcal{B}(\mathcal{H})}(U).$$

It follows that $\log(U) \in \mathcal{A}^p_{ah} \cap \mathcal{W}$. One obtains local charts around other elements of $\mathcal{U}_{\mathcal{A}^p}$ translating this chart around 1, by means of the left action of this group on itself.

It is clear that the group operations (multiplication and taking adjoint) are smooth: these operations are smooth in the whole Banach algebra $\mathcal{A}^p$. □

Note that if $P \in \mathcal{P}^p$, then $P^\perp := 1 - P$ also belongs to $\mathcal{P}^p$. Let $P \in \mathcal{P}^p$ and $A \in \mathcal{A}^p$, denote by

$$\mathbf{S}_{P,A} = AP + (1 - A)P^\perp \in \mathcal{A}^p.$$

**Lemma 3.2.** *There exists an open neighbourhood*

$$\mathcal{W}_P = \{A \in \mathcal{A}_h^p : \|A - P\|_{\infty,p} < r_P\}$$

*(for a given $r_P > 0$) of $P$ in $\mathcal{A}^p$, such that if $A \in \mathcal{W}_P$ then $\mathbf{S}_{P,A} \in \mathcal{G}_{\mathcal{A}^p}$.*

*Proof.* If $A = P$, then $\mathbf{S}_{P,P} = 1$, so for $A \in \mathcal{A}_h^p$ close enough to $P$, $\mathbf{S}_{P,A}$ remains in $\mathcal{G}_{\mathcal{A}^p}$, which is open in $\mathcal{A}^p$. □

If $A = Q \in \mathcal{P}^p$, then $\mathbf{S}_{P,Q}$ is a standard element used to intertwine $P$ and $Q$: clearly

$$\mathbf{S}_{P,Q}P = PQ = Q\mathbf{S}_{P,Q}.$$

If, additionally, $Q$ belongs to $\mathcal{W}_P$, then $\mathbf{S}_{P,Q}$ is invertible and

$$Q = \mathbf{S}_{P,Q}P\mathbf{S}_{P,Q}^{-1}.$$

It is also a standard procedure to use $\mathbf{U}_{P,Q}$, the unitary part in the polar decomposition $\mathbf{S}_{P,Q} = \mathbf{U}_{P,Q}|\mathbf{S}_{P,Q}|$, to obtain a unitary that intertwines

$$Q = \mathbf{U}_{P,Q}P\mathbf{U}_{P,Q}^{*}.$$

Thus, a continuous map is defined.

$$\mu_P : \mathcal{P}^p \cap \mathcal{W}_P \to \mathcal{U}_{\mathcal{A}^p} \ , \quad \mu(Q) = \mathbf{U}_{P,Q}. \tag{3.1}$$

*Remark* 3.3. Let us denote by

$$\pi_P : \mathcal{U}_{\mathcal{A}^p} \to \mathcal{P}^p, \quad \pi_P(U) = UPU^*.$$

Clearly $\pi_P$ is a continuous map, whose image is the unitary orbit

$$\mathcal{O}_P = \{UPU^* : U \in \mathcal{U}_{\mathcal{A}^p}\}$$

of $P$. The map $\mu_P$ of ((3.1)) is a continuous local cross section for $\pi_P$:

$$\pi_P(\mu_P(Q)) = Q, \text{ for } Q \in \mathcal{P}^p \cap \mathcal{W}_P.$$

By translating this section using the left action of $\mathcal{U}_{\mathcal{A}^p}$ on itself, one obtains local cross sections on neighbourhoods of any point $P'$ in $\mathcal{P}^p$.

By general topological considerations, this fact implies that the orbit $\mathcal{O}_P$ is open and closed in $\mathcal{P}^p$, and therefore a union of connected components. This union is necessarily discrete, because as we have seen, close projection belong to the same orbit.

Let us show that $\mathcal{P}^p$ is a complemented submanifold of $\mathcal{A}_h^p$ (and since $\mathcal{A}^p$ is a (real) complemented subspace of $\mathcal{A}^p$, $\mathcal{P}^p$ is also a complemented submanifold of $\mathcal{A}^p$). Also, the same argument shows that the map $\pi_P : \mathcal{U}_{\mathcal{A}^p} \to \mathcal{O}_P$ is a submersion. To prove this fact we shall use the next general result, which can be found in [9], and is a consequence of the implicit function theorem in Banach spaces.

**Lemma 3.4.** *Let $G$ be a Banach-Lie group acting smoothly on a Banach space $X$. For a fixed $x_0 \in X$, denote by $\pi_{x_0} : G \to X$ the smooth map $\pi_{x_0}(g) = g \cdot x_0$. Suppose that*

1. *$\pi_{x_0}$ is an open mapping, regarded as a map from $G$ onto the orbit*

$$\mathcal{O}_{x_0} := \{g \cdot x_0 : g \in G\}$$

*of $x_0$ (with the relative topology of $X$).*

2. *The differential* $d(\pi_{x_0})_1 : (TG)_1 \to X$ *splits: its null space and range are closed complemented subspaces.*

*Then the orbit* $\mathcal{O}_{x_0}$ *is a smooth submanifold of* $X$*, and the map*

$$\pi_{x_0} : G \to \mathcal{O}_{x_0}$$

*is a smooth submersion.*

Here smooth means $C^\infty$.

**Theorem 3.5.** $\mathcal{P}^p$ *is a* $C^\infty$ *complemented submanifold of* $\mathcal{A}^p_h$*, and for any* $P \in \mathcal{P}^p$ *the map*

$$\pi_P : \mathcal{U}_{\mathcal{A}^p} \to \mathcal{O}_P$$

*is a* $C^\infty$ *submersion.*

*Proof.* Let us use Lemma 3.4 in our context, namely: $X = \mathcal{A}^p_h$, $G = \mathcal{U}_{\mathcal{A}^p}$ and $x_0 = P$. Clearly, $\pi_P$ is an open mapping, because it has local continuous cross sections. Let us denote by $\Pi = (d\pi_P)_1 : \mathcal{A}^p_{ah} \to \mathcal{A}^p_h$ and prove that it splits. The cross section $\mu_P$ can be extended to a map defined in $\mathcal{W}_P$ (which is open in $\mathcal{A}^p_h$): let

$$\tilde{\mu}_P : \mathcal{W}_P \to \mathcal{U}_{\mathcal{A}^p}, \tilde{\mu}_p(A) = \mathbf{S}_{P,A}|\mathbf{S}_{P,A}|^{-1},$$

i.e. $\tilde{\mu}_P(A)$ is the unitary part in the polar decomposition of the invertible element $\mathbf{S}_{P,A}$. Clearly $\tilde{\mu}_P$ is a $C^\infty$ extension of $\mu_P$, defined on an open set in $\mathcal{A}^p_h$. Let us denote by $\Sigma = (d\tilde{\mu}_P)_P$. Clearly $\Sigma : \mathcal{A}^p_H \to \mathcal{A}^p_{ah}$. Note that, since $\pi_P$ takes values in $\mathcal{P}^p$, on a neighbourhood of $1 \in \mathcal{U}_{\mathcal{A}^p}$ one has

$$\pi_P \; \tilde{\mu}_P \; \pi_P = \pi_P \; \mu_P \; \pi_p = \pi_P.$$

Differentiating this identity at 1 one gets

$$\Pi \; \Sigma \; \Pi = \Pi \quad \text{in } \mathcal{A}^p_{ah}.$$

This implies that both $\Pi\Sigma$ and $\Sigma\Pi$ are idempotent operators, acting $\mathcal{A}^p_h$ and $\mathcal{A}^p_{ah}$, respectively. Thus $R(\Pi\Sigma) \subset \mathcal{A}^p_h$ is complemented, and note that

$$R(\Pi\Sigma) \subset R(\Pi) = R(\Pi\Sigma\Pi) \subset R(\Pi\Sigma),$$

i.e. $R(\Pi\Sigma) = R(\Pi)$. Similarly, $N(\Sigma\Pi) = N(\Pi)$ is complemented in $\mathcal{A}^p_{ah}$, i.e. $\Pi = (d\pi_P)_1$ splits. □

*Remark* 3.6. In particular, the tangent space $(T\mathcal{P}^p)_P$ at $P \in p^p$ is given by

$$(T\mathcal{P}^p)_P = \{[X, P] : X \in \mathcal{A}^p\}.$$

## 4. Spectral picture of projections in $\mathcal{P}^p$

If $P \in \mathcal{P}^p$ has matrix (in terms of $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$)

$$P = \begin{pmatrix} x & a \\ a^* & y \end{pmatrix},$$

then the fact that $P^2 = P \geq 0$ implies that $x, y \geq 0$, $x - x^2 = aa^*$, $y - y^2 = a^*a$, and $xa + ay = a$. Moreover, $a \in \mathcal{B}_p(\mathcal{H}_-, \mathcal{H}_+)$. Let us state the following elementary consequences of these relations:

**Lemma 4.1.** *With the above notations, one has that $\|a\| \le 1/2$, and the eigenvalues of $x$ and $y$ are of the form*

$$t^+ = \frac{1}{2} + \sqrt{\frac{1}{4} - s^2} \quad or \quad t^- = \frac{1}{2} - \sqrt{\frac{1}{4} - s^2}$$

*where $s \le \frac{1}{2}$ is a a singular value of $a$. One or both $t_+, t_-$ may occur.*

*Proof.* Clearly $\|x\| \le 1$ and $\|y\| \le 1$. Then

$$\|a\|^2 = \|aa^*\| = \|x - x^*\| = \sup\{t - t^2 : t \in \sigma(x)\} \le \sup\{t - t^2 : t \in [0,1]\} = \frac{1}{4}.$$

Again, $x - x^2 = aa^*$, and the fact that $aa^*$ is compact, imply that the elements $t$ in the spectrum of $x$ which are neither 0 nor 1 (which correspond to the spectral value 0 for $aa^*$) are (finite or countable many) eigenvalues. Moreover, if $t \ne 0, 1$ is an eigenvalue of $x$, then

$$t - t^2 = s^2,$$

for $s$ a singular value of $a$. Then either $t = t^+ = \frac{1}{2} + \sqrt{\frac{1}{4} - s^2}$ or $t = t^- = \frac{1}{2} - \sqrt{\frac{1}{4} - s^2}$. The same facts hold for $y$. □

Note that the biggest singular value $s = \frac{1}{2}$ of $a$ corresponds to $t^+ = t^- = \frac{1}{2}$.

The next result, which was proven for compact commutators in [2], holds also in this context, and clarifies the relation between (the multiplicities of) the eigenvalues of $x$ and $y$. We include the proof because it is elementary and straightforward.

**Lemma 4.2.** *If $\lambda \ne 0, 1$ is an eigenvalue of $y$, then $1 - \lambda$ is an eigenvalue of $x$, and the operator $a|_{N(y - \lambda 1_{\mathcal{H}_-})}$ maps $N(y - \lambda 1_{\mathcal{H}_-})$ isomorphically onto $N(x - (1-\lambda)1_{\mathcal{H}_+})$. Thus in particular, these eigenvalues have the same multiplicity. Moreover,*

$$aP_{N(y-\lambda 1_{\mathcal{H}_-})} = P_{N(x-(1-\lambda)1_{\mathcal{H}_-})}a.$$

*Proof.* Let $\xi \in \mathcal{H}$, $\xi \ne 0$, such that $y\xi = \lambda\xi$ (with $\lambda \ne 0, 1$). Then, using the relation $a = xa + ay$, one has

$$a\xi = xa\xi + ay\xi = xa\xi + \lambda a\xi, \quad \text{i.e. } xa\xi = (1-\lambda)a\xi.$$

Also note that

$$N(a) = N(a^*a) = N(y - y^2) = N(y) \oplus N(y - 1_{\mathcal{H}_-}),$$

and thus $a\xi \ne 0$ is an eigenvector for $x$, with eigenvalue $1 - \lambda$, and the map $a|_{N(y-\lambda 1_{\mathcal{H}_-})}$ is injective from $N(y - \lambda 1_{\mathcal{H}_-})$ to $N(x - (1-\lambda)1_{\mathcal{H}_+})$. Therefore

$$\dim(N(y - \lambda 1_{\mathcal{H}_-})) \le \dim(N(x - (1-\lambda)1_{\mathcal{H}_+}).$$

By a symmetric argument, using $a^*$ (and the relation $ya^* + a^*x = a^*$), one obtains equality.

Pick now an arbitrary $\xi \in \mathcal{H}_-$, $\xi = \xi_1 + \xi_2$, with $\xi_1 \in N(y - \lambda 1_{\mathcal{H}_-})$ and $\xi_2 \perp N(y - \lambda 1_{\mathcal{H}_-})$. Then

$$aP_{N(y-\lambda 1_{\mathcal{H}_-})}\xi = a\xi_1.$$

On the other hand

$$P_{N(x-(1-\lambda)1_{\mathcal{H}_+})}a\xi_1 = a\xi_1,$$

by the fact proven above. Let us see that $P_{N(x-(1-\lambda)1_{\mathcal{H}_+})}a\xi_2 = 0$, which would prove our claim. Since $\xi_2 \perp N(y - \lambda 1_{\mathcal{H}_-})$, $\xi_2 = \sum_{l\ge 2} \eta_l + \eta_0 + \eta_1$, where $\eta_l$, $l \ge 2$, are eigenvectors of $y$ corresponding to eigenvalues $\lambda_l$ different from 0, 1 and $\lambda$, $\eta_0 \in N(y)$, $\eta_1 \in N(y - 1_{\mathcal{H}_-})$ (where these two latter may be trivial). Note then that $\eta_0, \eta_1 \in N(a)$, and thus

$$a\xi_2 = \sum_{l\geq 2} a\eta_l,$$

where the (non nil) vectors $a\eta_l$ are eigenvectors of $x$ corresponding to eigenvalues $1 - \lambda_l$, different from $0, 1$ and $1 - \lambda$. Thus $P_{N(x-(1-\lambda)1_{\mathcal{H}_+})}a\xi_2 = 0$. □

*Remark* 4.3.
1. In the notation above, this Lemma says that to $t^+$ of $x$ corresponds $t^- = 1 - t^+$ of $y$, and vice versa, with the same multiplicity.
2. If $a$ has infinite rank, $s = s_n$ form a sequence in $\ell^p$. If there are infinitely many $t^-$, then they form a sequence in $\ell^{\frac{p}{2}}$. If there are infinitely many $t^+$, they form a sequence $t_n^+$ such that $1-t_n^+$ belongs to $\ell^{\frac{p}{2}}$. Indeed, note that near the origin, $f(s) = \frac{1}{2} - \sqrt{\frac{1}{4} - s^2} = s^2 + o(s^4)$

We will characterize the connected components of $\mathcal{P}^p$. To this effect, it shall be useful and clarifying to consider the $*$-homomorphism

$$\pi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$$

onto the Calkin algebra $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$. Note that $\pi(E_+), \pi(E_-)$ are non trivial projections with $\pi(E_+) + \pi(E_-) = 1$. Thus, elements in $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ can be written as $2 \times 2$ matrices in terms of this sum. Let us write

$$\pi(E_+) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \pi(E_-) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then, if $P \in \mathcal{P}^p$, it follows that

$$\pi(P) = \begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix},$$

where $e, f$ are projections in $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ with $e \leq \pi(E_+)$ and $f \leq \pi(E_-)$.

**Lemma 4.4.** *Let $P, Q \in \mathcal{P}^p$ in the same connected component, say*

$$\pi(P) = \begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix} \quad \textit{and} \quad \pi(Q) = \begin{pmatrix} e' & 0 \\ 0 & f' \end{pmatrix}.$$

*Then there exists a curve $\gamma(t) = \begin{pmatrix} e(t) & 0 \\ 0 & f(t) \end{pmatrix}$ of projections in $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ such that $\gamma(0) = \pi(P)$ and $\gamma(1) = \pi(Q)$.*

*Proof.* Let $P(t)$ be a continuous path in $\mathcal{P}^p$ with $P(0) = P$ and $P(1) = Q$. Note that in particular $P(t)$ is continuous in the norm topology of $\mathcal{B}(\mathcal{H})$, and clearly, $\pi(P(t))$ has diagonal matrix with respect to $\pi(E_+) + \pi(E_-) = 1$. □

Recall that there are three classes of projections in $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ modulo unitary equivalence: 0, 1 and $p \neq 0, 1$. Also, two projections are connected by a continuous path of projections if and only if they are unitarily equivalent.

*Remark* 4.5. The projections in $\mathcal{P}^p$ can be classified in the following nine types:

1. $P$ belongs to $\mathbb{D}_1$ if $\pi(P) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$;
2. $P$ belongs to $\mathbb{D}_2$ if $\pi(P) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$;

3. $P$ belongs to $\mathbb{D}_3$ if $\pi(P) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$;
4. $P$ belongs to $\mathbb{D}_4$ if $\pi(P) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$;
5. $P$ belongs to $\mathbb{E}_1$ if $\pi(P) = \begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix}$, where $e \neq 0, 1$;
6. $P$ belongs to $\mathbb{E}_2$ if $\pi(P) = \begin{pmatrix} 0 & 0 \\ 0 & f \end{pmatrix}$, where $f \neq 0, 1$;
7. $P$ belongs to $\mathbb{E}_3$ if $\pi(P) = \begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix}$, where $e \neq 0, 1$;
8. $P$ belongs to $\mathbb{E}_4$ if $\pi(P) = \begin{pmatrix} 1 & 0 \\ 0 & f \end{pmatrix}$, where $f \neq 0, 1$;
9. $P$ belongs to $\mathbb{E}_5$ if $\pi(P) = \begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix}$, where $e, f \neq 0, 1$.

We call the classes $\mathbb{D}_i$ *discrete* and $\mathbb{E}_j$ *essential*.

Summarizing, one has the following expression for an arbitrary projection in $\mathcal{P}^p$. We make here a slight change of notation. Without loss of generality, we assume that $\mathcal{H} = \mathcal{L} \times \mathcal{L}$, $\mathcal{H}_+ = \mathcal{L} \times 0$ and $\mathcal{H}_- = 0 \times \mathcal{L}$.

**Theorem 4.6.** *If* $P = \begin{pmatrix} x & a \\ a^* & y \end{pmatrix} \in \mathcal{P}^p \subset \mathcal{B}(\mathcal{L} \times \mathcal{L})$, *then*

$$P = \left( \begin{array}{c|c} \sum_n \alpha_n P_n + \sum_m \beta_m Q_m + E_1 & \sum_k \lambda_k \xi_k \otimes \xi'_k + \sum_l \mu_l \eta_l \otimes \eta'_l \\ \sum_k \lambda_k \xi'_k \otimes \xi_k + \sum_l \mu_l \eta'_l \otimes \eta_l & \sum_n (1-\alpha_n) P'_n + \sum_m (1-\beta_m) Q'_m + E'_1 \end{array} \right),$$

*where*

- *The spectrum of* $x$ *(in* $\mathcal{B}(\mathcal{L})$*) consists of two strictly monotone (eventually finite) sequences* $\alpha_n, \beta_m$*, such that* $\frac{1}{2} > \alpha_n \to 0$*,* $\frac{1}{2} \leq \beta_m \to 1$*, plus, eventually,* $0$ *and* $1$*, which may or may not be eigenvalues. The spectrum of* $y$ *consists of* $1-\alpha_n$*,* $1-\beta_m$ *and eventually* $0$ *and* $1$ *(with similar considerations).*
- $r(P_n) = r(P'_n)$*,* $r(Q_m) = r(Q'_m)$*. These multiplicities are finite.*
- $r(P_m)\alpha_m$ *and* $1 - r(Q_m)\beta_m$ *belong to* $\ell^{\frac{p}{2}}$*;* $\lambda_k = \sqrt{\alpha_k - \alpha_k^2}$ *and* $\mu_l = \sqrt{\beta_l - \beta_l^2}$*.*
- $E_1$ *and* $E'_1$ *denote the spectral projections of* $x$ *and* $y$*, respectively, corresponding to the spectral value* $1$*. They can be nil, finite or infinite, and are unrelated.*
- $\{\xi_k : k \geq 1\}, \{\xi'_k : k \geq 1\}, \{\eta_l : l \geq 1\}$*,* $\{\eta'_l : l \geq 1\}$ *are orthonormal systems which span, respectively*

$$\bigoplus_{n\geq 1} R(P_n), \quad \bigoplus_{n\geq 1} R(P'_n), \quad \bigoplus_{m\geq 1} R(Q_m) \quad \text{and} \quad \bigoplus_{m\geq 1} R(Q'_m),$$

*and consists of eigenvectors of* $x$ *and* $y$ *in the following manner:*

$$x\xi_k = \alpha_{n(k)}\xi_k \quad \text{and} \quad x\eta_l = \beta_{m(l)}\eta_l,$$

$$y\xi'_k = (1-\alpha_{n(k)})\xi'_k \quad \text{and} \quad y\eta'_l = (1-\beta_{m(l)})\eta'_l.$$

## 5. Halmos decomposition

Given two projections, in this case $P$ and $E_+$, the space $\mathcal{H}$ can be decomposed in 5 orthogonal subspaces which reduce $P$ and $E_+$, namely

$$\mathcal{H} = (R(P) \cap \mathcal{H}_+) \oplus (N(P) \cap \mathcal{H}_-) \oplus (R(P) \cap \mathcal{H}_-)\,(N(P) \cap \mathcal{H}_+) \oplus \mathcal{H}_0,$$

where $\mathcal{H}_0$, the orthogonal complement of the sum of the first 4, is usually called the *generic part* of $P$ and $E_+$. In [7], Halmos proved that there is a unitary isomorphism between $\mathcal{H}_0$ and a product space $\mathcal{L} \times \mathcal{L}$, and a positive operator $\Gamma$ with trivial null space and $\|\Gamma\| \leq \pi/2$, acting in $\mathcal{L}$, such that the reductions $E_+^0$ and $P_0$ of $E_+$ and $P$ to $\mathcal{H}_0$ are unitarily equivalent to (respectively)

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} C^2 & CS \\ CS & S^2 \end{pmatrix},$$

where $C = \cos(\Gamma)$ and $S = \sin(\Gamma)$. It can be shown that $P_0$ and $E_+^0$ are unitarily equivalent:

$$e^{iX} E_+^0 e^{-iX} = P_0,$$

where $X = X_{E_+^0, P_0} = \begin{pmatrix} 0 & -i\Gamma \\ i\Gamma & 0 \end{pmatrix}$.

In this decomposition of $\mathcal{H}$, the commutator has the form

$$[E_+, P] = 0 \oplus 0 \oplus 0 \oplus 0 \oplus \begin{pmatrix} 0 & CS \\ -CS & 0 \end{pmatrix}$$

Therefore:

**Proposition 5.1.** *$P \in \mathcal{P}^p$ if and only if $CS \in \mathcal{B}_p(\mathcal{L})$. Moreover, this means the spectrum of $\Gamma$ has the form $\{\gamma_n^+ : n \geq 1\} \cup \{\gamma_k^- : k \geq 1\}$, where $\pi/4 \leq \gamma_n^+ < \pi/2$ and $0 < \gamma_k^- < \pi/4$ are strictly monotone (eventually finite) sequences,*

$$\Gamma = \sum_{k \geq 1} \gamma_k^- G_k^- + \sum_{n \geq 1} \gamma_n^+ G_n^+,$$

*for $G_n^+, G_k^-$ mutually orthogonal projections in $\mathcal{L}$, of ranks $r(G_n^+) = r_n^+ < \infty$ and $r(G_k^-) = r_k^- < \infty$, and*

$$\{r_k^- \gamma_k^-\},\ \{\pi/2 - r_n^+ \gamma_k^+\} \in \ell^p.$$

*Proof.* The first assertion is clear. Note that $CS = \cos(\Gamma)\sin(\Gamma) = \frac{1}{2}\sin(2\Gamma)$. Thus, the facts that $\sin(2\Gamma) \in \mathcal{B}_p(\mathcal{L})$ and $0 \leq 2\Gamma \leq \pi$, means that the spectrum of $\Gamma$ consists of eigenvalues which accumulate only (eventually) at 0 and $\pi$. Since $C$ and $S$ have trivial null spaces, neither 0 nor $\pi/2$ are eigenvalues of $\Gamma$. □

### 5.1. Examples

1. Let $\mathcal{H} = L^2(\mathbb{T})$, $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ with normalized Lebesgue measure, and $\mathcal{H}_+ = H^2(\mathbb{D})$ the Hardy space of the disk. Let $\varphi : \mathbb{T} \to \mathbb{C}$ be non vanishing and $C^1$. Then the projection $P_{\varphi H^2(\mathbb{D})}$ onto $\varphi H^2(\mathbb{D})$ belongs to the restricted Grassmannian given by the subspace $\mathcal{H}^2(\mathbb{D})$ (see [11]), with $[P_+, P_{\varphi H^2(\mathbb{D})}] \in \mathcal{B}_1(L^2(\mathbb{T}))$. Thus $P_{\varphi H^2(\mathbb{D})} \in \mathbb{D}_3$.
2. Let $\mathcal{H} = L^2(\mathbb{R}^n)$ (with Lebesgue measure), $\Omega \subset \mathbb{R}^n$ a measurable set with $|\Omega| < \infty$ and $\mathcal{H}_+ = L^2(\Omega)$, regarded a the subspace of $L^2(\mathbb{R}^n)$ of classes of functions with essential support contained in $\Omega$. Denote by $F : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ the Fourier-Plancherel transform. Put $P = FE_+F^{-1}$, which projects

onto functions whose Fourier transform is supported in $\Omega$. It is known (see for instance [6]) that $E_+P \in \mathcal{B}_1(L^2(\mathbb{R}^n))$. Thus, in particular,

$$[E_+, P] = E_+P - PE_+ = E_+P - (E_+P)^* \in \mathcal{B}_1(L^2(\mathbb{R}^n)).$$

Moreover, Lenard proved in [8] that

$$R(E_+) \cap R(P) = R(E_+) \cap N(P) = N(E_+) \cap R(P) = \{0\},$$

and that $N(E_+) \cap N(P)$ is infinite dimensional. Therefore, since $E_+PE_+$, $E_+PE_-$ and $E_-PE_+$ are compact, it follows that $\pi(P)$ is of the form

$$\pi(P) = \begin{pmatrix} 0 & 0 \\ 0 & p \end{pmatrix},$$

with $p \neq 0, 1$, because $P$ is an infinite rank projection, with $N(P)$ infinite dimensional. That is, $P \in \mathbb{E}_2$ .

3. Let $B \in \mathcal{B}_p(\mathcal{L})$, and put $\mathcal{H} = \mathcal{L} \times \mathcal{L}$ and $\mathcal{H}_+ = \mathcal{L} \times 0$. Consider the idempotent (non orthogonal projection)

$$E_B = E = \begin{pmatrix} 1 & B \\ 0 & 0 \end{pmatrix}$$

with $R(E) = \mathcal{H}_+$. Consider $P_{R(E^*)}$ the orthogonal projection onto $R(E^*)$. It is known (see for instance [1]), that if $Q$ is an idempotent operator, then $P_{R(Q)} = Q(Q+Q^*-1)^{-1}$. In our case, note that $(E^*+E-1)^2 = \begin{pmatrix} 1+BB^* & 0 \\ 0 & 1+B^*B \end{pmatrix}$, so that

$$\begin{aligned} P = P_{R(E^*)} &= E^*(E^* + E - 1)^{-1} \\ &= E^*(E^* + E - 1)(E^* + E - 1)^{-2} \\ &= \begin{pmatrix} (1+BB^*)^{-1} & B(1+B^*B)^{-1} \\ B^*(1+BB^*)^{-1} & B^*B(1+B^*B)^{-1} \end{pmatrix}. \end{aligned}$$

Note that the $1,2$ entry $B(1+B^*B)^{-1}$ belongs to $\mathcal{B}_p(\mathcal{H})$, with singular values which have the same asymptotic behaviour as those of $B$. Clearly, $\pi(P)$ is of the form $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, i.e. $P \in \mathbb{D}_3$. The index (of the $1,1$ entry) of $P$ is $0$.

## 6. The classes $\mathbb{D}_3$ and $\mathbb{D}_4$

Let us show that $\mathbb{D}_3$ coincides with the $p$-restricted Grassmannian induced by $E_+$. Recall that (see for instance [11]) the *$p$-restricted Grassmannian* $Gr^p_{res}(E_+)$ relative to $E_+$, is the space projections $P$ in $\mathcal{H}$ such that

- $E_+|_{R(P)} : R(P) \to \mathcal{H} \in \mathcal{B}(R(P), \mathcal{H})$ is a $p$-Fredholm operator (i.e., there exist $S \in \mathcal{B}(\mathcal{H}, R(P))$ such that $SE_+|_{R(P)} = 1 + M$ and $E_+|_{R(P)}S = 1 + N$, for $M \in \mathcal{B}_p(R(P))$, $N \in \mathcal{B}_p(\mathcal{H})$), and
- $E_-|_{R(P)} : R(P) \to \mathcal{H} \in \mathcal{B}_p(R(P), \mathcal{H})$.

The components of the restricted Grassmannian are parametrized by $k \in \mathbb{Z}$, where $k$ is the index of the operator $E_+|_{R(P)} : R(P) \to \mathcal{H} \in \mathcal{B}(R(P), \mathcal{H})$,

$$Gr^p_{res,k}(E_+) = \{P \in Gr^p_{res} : \text{ind}(E_+|_{R(P)} : R(P) \to \mathcal{H}) = k\}.$$

In particular, note that $E_+ \in Gr^p_{res,0}(E_+)$.

The coincidence of the $p$-restricted Grassmannian of $\mathcal{H}_+$ and $\mathbb{D}_3$ follows from this result:

**Theorem 6.1.** *Denote by $\mathcal{O}(E_+)$ the unitary orbit of $E_+$ under the action of $\mathcal{U}_{\mathcal{A}^p}$, $\mathcal{O}(E_+) = \{UE_+U^* : U \in \mathcal{U}_{\mathcal{A}^p}\}$. Then*

$$\mathcal{O}(E_+) = \{P \in \mathcal{P}^p : P - E_+ \in \mathcal{B}_p(\mathcal{H})\} = \{P \in \mathcal{P} : P - E_+ \in \mathcal{B}_p(\mathcal{H})\}.$$

*Proof.* If $P = UE_+U^*$ for some $U \in \mathcal{U}_{\mathcal{A}^p}$, then

$$P - E_+ = UE_+U^* - E_+ = (UE_+ - E_+U)U^* = [U, E_+]U^* \in \mathcal{B}_p(\mathcal{H}).$$

Clearly $\{P \in \mathcal{P}^p : P - E_+ \in \mathcal{B}_p(\mathcal{H})\} \subset \{P \in \mathcal{P} : P - E_+ \in \mathcal{B}_p(\mathcal{H})\}$. Suppose that $P \in \mathcal{P}$ such that $P - E_+ \in \mathcal{B}_p(\mathcal{H})$. It is easy to see that

$$N(P - E_+) = R(P) \cap \mathcal{H}_+ \oplus N(P) \cap \mathcal{H}_-.$$

Also it is clear that both summands reduce $P$ and $E_+$. Then $\mathcal{H}' = N(P - E_+)^\perp$ reduces $P$ and $E_+$. Denote by $P'$ and $E'_+$ the reductions. It is straightforward that $[P', E'_+] \in \mathcal{B}_p(\mathcal{H}')$. Since $P' - E'_+$ is selfadjoint and has trivial null space, if one performs the polar decomposition

$$P' - E'_+ = V'|P' - E'_+|,$$

the isometric part $V'$ is a symmetry (a selfadjoint unitary) in $\mathcal{H}'$. Also, the fact that $S' = P' - E'_+$ satisfies $S'E'_+ = E'_+S'$ implies that $V'$ intertwines $E'_+$ and $P'$: $V'E'_+V' = P'$. Then, it also follows that $V' \in$

$$\{X' \in \mathcal{B}(\mathcal{H}') : [X', E'_+] \in \mathcal{B}_p(\mathcal{H}')\},$$

the algebra $\mathcal{A}^p$ in $\mathcal{H}'$ corresponding to the reduced projection $E'_+$. Indeed,

$$V'E'_+ - E'_+V' = (V'E'_+V' - E'_+)V' \in \mathcal{B}_p(\mathcal{H}').$$

Consider now the unitary operator (in fact symmetry) $V$ of $\mathcal{H}$, which is given in terms of the decomposition $\mathcal{H} = \mathcal{H}' \oplus (R(P) \cap \mathcal{H}_+) \oplus (N(P) \cap \mathcal{H}_-)$ is given by

$$V' \oplus 1 \oplus 1.$$

Note that in this same decomposition, $P$ and $E_+$ are given by

$$P = P' \oplus 1 \oplus 0 \text{ and } E_+ = E'_+ \oplus 1 \oplus 0.$$

Then

$$[V, E_+] = (V'E'_+ - E'_+V') \oplus 0 \oplus 0 \in \mathcal{B}_p(\mathcal{H}),$$

and

$$VE_+V = (V'E'_+V') \oplus 1 \oplus 0 = P. \qquad \square$$

**Corollary 6.2.** $\mathbb{D}_3 = \mathcal{O}(E_+)$

*Proof.* $P \in \mathcal{O}(E_+)$, if and only if $P - E_+ \in \mathcal{B}_p(\mathcal{H})$, and then $\pi(P - E_+) = 0$, i.e. $\pi(P) = \pi(E_+)$. Conversely, in $\pi(P) = \pi(E_+)$, then $P - E_+$ is compact. Using a suitable unitary isomorphism we may suppose (as in Theorem 4.6) $\mathcal{H} = \mathcal{L} \times \mathcal{L}$ and $\mathcal{H}_+ = \mathcal{L} \times 0$, and use the spectral picture of $P \in \mathcal{P}^p$. Note that the assumption that $P - E_+$ is compact implies that $x - 1$ and $y$ are compact. Thus, following the notation of Theorem 4.6, one has that there are finitely many $\alpha_n$ and that $1 - \beta_n$ is a sequence in $\ell^{\frac{p}{2}}$. It follows that $P - E_+ \in \mathcal{B}_p(\mathcal{H})$. $\square$

*Remark* 6.3. Note that in particular, this facts imply that $\mathbb{D}_3 \in E_+ + \mathcal{B}_p(\mathcal{H})$, i.e. $\mathbb{D}_3$ is contained in the affine space obtained as a translation of $\mathcal{B}_p(\mathcal{H})$. Thus, the tangent spaces belong naturally inside $\mathcal{B}_p(\mathcal{H})$. We shall profit from this condition, in order to endow the manifold $\mathbb{D}_3$ with the natural Finsler metric, which consists in considering the $p$-norm at every tangent space.

*Remark* 6.4. In a similar fashion (or using the symmetry $P \mapsto P^\perp$), one proves that

$$\mathbb{D}_4 = \mathcal{O}(E_-) = \{P \in \mathcal{P} : P - E_- \in \mathcal{B}_p(\mathcal{H})\},$$

which coincides with the $p$-restricted Grassmannian $G^p_{res}(E_-)$ induced by $E-$. Similarly, one can consider the Finsler $p$-norm structure in $\mathbb{D}_4$.

In general, if $P$ and $Q$ are projections, $\|P - Q\| \leq 1$.

**Proposition 6.5.** *If $P \in \mathcal{P}^p$ satisfies that $\|P - E_+\| < 1$, then $P \in \mathbb{D}_3$. Similarly, if $\|P - E_-\| < 1$, then $P \in \mathbb{D}_4$.*

*Proof.* Recall Halmos decomposition. Clearly, $\|P - E_+\| < 1$ implies that $R(P) \cap \mathcal{H}_- = N(P) \cap \mathcal{H}_+ = \{0\}$. Indeed, a unit vector $\xi \in R(P) \cap \mathcal{H}_-$ satisfies $\|(P - E_+)\xi\| = \|\xi\| = 1$, and thus $R(P) \cap \mathcal{H}_- = \{0\}$; and similarly for the other intersection. Note that

$$P - E_+ = 0 \oplus 0 \oplus 0 \oplus 0 \oplus \begin{pmatrix} -S^2 & CS \\ CS & S^2 \end{pmatrix}$$

and therefore $(P - E_+)^2 = \begin{pmatrix} S^2 & 0 \\ 0 & S^2 \end{pmatrix}$. Then $\|P - E_+\| < 1$ implies that the spectrum of $\Gamma$ cannot accumulate at $\pi/2$, and therefore only accumulates only at the origin. Thus, analysing the spectral picture of $P = \begin{pmatrix} C^2 & CS \\ CS & S^2 \end{pmatrix}$ according to Theorem 4.6, it is clear that the eigenvalues of $x = C^2$ accumulate only at 1 and the eigenvalues of $y = S^2$ accumulate only at the origin.

The proof for the case $\|P - E_-\| < 1$ is analogous. □

**Corollary 6.6.** *If $P \in \mathbb{E}_j$, $1 \leq j \leq 5$, then $\|P - E_+\| = \|P - E_-\| = 1$.*

### 6.1. Finsler metric in the discrete classes

In this subsection we examine the relationship between the $p$-Finsler metric of $\mathbb{D}_3$ and $\mathbb{D}_4$, i.e., the metric which arises when endowing each tangent space of these manifolds with the $p$-norm, with the ambient metric given by the norm $\| \ \|_{\infty,p}$ of $\mathcal{A}^p$.

Let us reason with $\mathbb{D}_3$, the same facts hold analogously for $\mathbb{D}_4$.

Let us first compare the Finsler metric with the metric induced by the $p$ norm of the affine space $E_+ + \mathcal{B}_p(\mathcal{H})$ (if $X = E_+ + A, Y = E_+ + B \in E_+ + \mathcal{B}_p(\mathcal{H})$, the $p$ distance $\|X - Y\|_p = \|A - B\|_p < \infty$ is defined).

In [3] it was proven that if $P, Q$ lie in the same component of the $p$-restricted Grassmannian $G^p_{res,k}(E_+) = \mathbb{D}_3$, then there exists a minimal geodesic of the form $\delta(t) = e^{itX} P e^{-itX}$, with $X^* = X \in \mathcal{B}_p(\mathcal{H})$ $P$-codiagonal and $\|X\| \leq \pi/2$, such that $\delta(1) = Q$, so that the geodesic distance is $d_p(P, Q) = \|X\|_p$. We recall that $d_p$ is formally defined as

$$d_p(P, Q) = \inf \left\{ \int_I \|\dot\gamma(t)\|_p \, dt : \gamma : I \to \mathcal{P}^p \text{ is smooth with endpoints } P, Q \right\}.$$

**Proposition 6.7.** *With the current notations, if $P, Q$ lie in the same component of $\mathbb{D}_3$ (resp. $\mathbb{D}_4$), then*

$$\frac{2}{\pi} d_q(P,Q) \leq \|P-Q\|_p \leq d_p(P,Q).$$

*Proof.* The inequality $\|P-Q\| \leq d_p(P,Q)$ is clear: if one takes the infimum among all smooth curves with values in $E_+ + \mathcal{B}_p(\mathcal{H})_h$, which is an affine space, one obtains the norm distance $\|P-Q\|_p$.

Let $X = X^* \in \mathcal{B}_p(\mathcal{H})$ be the exponent of the geodesic joining $P$ and $Q$: $\|X\| \leq \pi/2$, $X$ is $P$-codiagonal, and $Q = e^{iX} P e^{-iX}$. Note that

$$\|P-Q\|_p = \|P - e^{iX}Pe^{-iX}\|_p = \|(Pe^{iX} - e^{iX}P)e^{-iX}\|_p = \|[P, e^{iX}]\|_p.$$

Since $X$ is $P$-codiagonal, $P$ commutes the the even powers of $X$, and thus

$$[P, X^{2n+1}] = PX^{2n}X - X^{2n}XP = X^{2n}(PX - XP) = X^{2n}[P,X].$$

It follows that

$$\begin{aligned}
[P, e^{iX}] &= [P, 1 + iX - \frac{1}{2}X^2 - \frac{i}{3!}X^3 + \frac{1}{4!}X^4 + \ldots] \\
&= i\{[P,X] - \frac{1}{3!}[P,X^3] + \frac{1}{5!}[P,X^5] - \ldots\} \\
&= i\{1 - \frac{1}{3!}X^2 + \frac{1}{5!}X^4 - \ldots\}[P,X] \\
&= i \ \mathrm{sinc}(X)[P,X],
\end{aligned}$$

where sinc denotes the *cardinal sine* function, which is the entire function given by $\mathrm{sinc}(t) = \frac{\sin(t)}{t}$ ($\mathrm{sinc}(0) = 1$). It $|t| \leq \pi/2$, this function verifies that

$$\frac{2}{\pi} \leq \mathrm{sinc}(t) \leq 1.$$

In particular, since $X$ is selfadjoint with spectrum in $[-\pi/2, \pi/2]$, $S = \mathrm{sinc}(X)$ is an invertible operator, with $\|S^{-1}\| \leq \frac{\pi}{2}$. Therefore

$$\|X\|_p = \|S^{-1}SX\|_p \leq \|S^{-1}\|\|SX\|_p \leq \frac{\pi}{2}\|SX\|_p,$$

i.e.,

$$\frac{2}{\pi}\|X\|_p \leq \|SX\|_p = \|i \ \mathrm{sinc}(X)[P,X]\|_p = \|P-Q\|_p. \qquad \square$$

*Remark* 6.8. By the above remarks, if $P, Q \in \mathbb{D}_3$, then $P - Q \in \mathcal{B}_p(\mathcal{H})_h$. Denote $P - Q = A = \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{pmatrix}$. Since $A_{ij} = E_{ij}A$ for appropriate elementary (partial isometric) operators, it is clear that $A_{ij} \in \mathcal{B}_p(\mathcal{H})$. Moreover $\|A_{ij}\|_p = \|E_{ij}A\|_p \leq \|E_{ij}\|\|A\|_p = \|A\|_p$. Then

$$\begin{aligned}
\|P-Q\|_{\infty,p} &= \|A_{11}\| + \|A_{22}\| + \|A_{12}\|_p + \|A_{12}^*\|_p \\
&\leq \|A_{11}\|_p + \|A_{22}\|_p + \|A_{12}\|_p + \|A_{12}^*\|_p \\
&\leq 4\|P-Q\|_p.
\end{aligned}$$

However, these two metrics are not equivalent in $\mathbb{D}_3$ Indeed, fix an orthonormal basis $\{f_n\}$ for $\mathcal{H}_-$, and consider $P = E_+ + D$ and $Q = E_+ + F$, where $D, F \leq E_-$, project onto mutually orthogonal subspaces generated by finite (disjoints) subsets of the basis $\{f_n\}$. Then $\|P-Q\|_{\infty,p} = \|D-F\| = 1$, whereas $\|P-Q\|_p = (\mathrm{rank}(E) + \mathrm{rank}(F))^{1/p}$. Since these ranks are arbitrary, the metrics are non equivalent.

## 7. Connectedness of the essential classes

In this section we prove our main result in the classes $\mathbb{E}_i, 1 \le i \le 5$, namely, that each of these spaces is connected. The proof of this result is similar to the proof of the analogous result in [2]. We shall sketch the argument, emphasizing only the necessary modifications.

Recall from Theorem 4.6, the form of a projection $P = \left(\begin{smallmatrix} x & a \\ a^* & y \end{smallmatrix}\right) \in \mathcal{P}^p$:

$$P = \left( \begin{array}{c|c} \sum_n \alpha_n P_n + \sum_m \beta_m Q_m + E_1 & \sum_k \lambda_k \xi_k \otimes \xi_k' + \sum_l \mu_l \eta_l \otimes \eta_l' \\ \sum_k \lambda_k \xi_k' \otimes \xi_k + \sum_l \mu_l \eta_l' \otimes \eta_l & \sum_n (1-\alpha_n) P_n' + \sum_m (1-\beta_m) Q_m' + E_1' \end{array} \right),$$

where the relevant facts we need now are that that $r(P_n) = r(P_n') < \infty$, $r(Q_m) = r(Q_m') < \infty$, and $\alpha_n$, $1-\beta_m$ belong to $\ell^{\frac{p}{2}}$. Consider the projection

$$P_0 = \left( \begin{array}{c|c} \sum_n P_n + \sum_m Q_m + E_1 + N & 0 \\ 0 & \sum_n P_n' + \sum_m Q_m' + E_1' + N' \end{array} \right),$$

where $N$ and $N'$ are the projections onto the nullspaces of $x$ and $y$, respectively.

The first step of the argument is the following.

**Lemma 7.1.** *The operator $B = P + P_0 - 1$ is invertible in $\mathcal{A}^p$, and belongs to the connected component of the identity.*

*Proof.* In [2], Lemma 5.1, it was shown that $B$ is invertible, that its commutator with $E_+$ is compact, and that its $1,1$ entry is invertible. Here, $[B, P_+]$ is the $P_+$-codiagonal matrix whose non nil entries are those of $P$, and therefore $B \in \mathcal{A}^p$. Clearly, it belongs to the component of zero index. □

The operator $B$ is selfadjoint, and satisfies $BP = P_0 B$, which is equivalent to $BPB^{-1} = P_0$. Then the unitary part $V$ in the polar decomposition $B = V|B|$, satisfies $VPV^* = P_0$. Clearly, $V$ also belongs to the connected component of the identity in $\mathcal{U}_{\mathcal{A}^p}$. It follows that $P$ and $P_0$ belong to the same connected component of $\mathcal{P}^p$.

The next step is to show that any pair of *diagonal* essential projections in the same class $\mathbb{E}_i$, are conjugate by an element in the connected component of the identity of $\mathcal{U}_{\mathcal{A}^p}$:

Let $F, G$ be two projections, which are diagonal with respect to $E_+$, both in the same essential class.

- If $F, G \in \mathbb{E}_1$, are of the form

$$F = \begin{pmatrix} P_+ & 0 \\ 0 & F_- \end{pmatrix}, \quad G = \begin{pmatrix} P_+' & 0 \\ 0 & G_- \end{pmatrix}$$

where $P_+, P_+'$ are projections of infinite rank and co-rank, and $F_-, G_-$ are of finite rank. One can show that $F$ and $G$ are unitarily equivalent to the projection

$$\begin{pmatrix} P_+ & 0 \\ 0 & 0 \end{pmatrix},$$

with a unitary in operator in $\mathcal{A}^p$, which belongs to the connected component of the identity. First note that with an unitary of the form $\begin{pmatrix} U_+ & 0 \\ 0 & 1 \end{pmatrix}$, one can connect $G$ with $\begin{pmatrix} P_+ & 0 \\ 0 & G_- \end{pmatrix}$. That is, we may suppose $P'_+ = P_+$. Next we construct the unitary operator $U$ given in the proof of Lemma 5.2 of [2], which is a finite rank perturbation of the identity, and therefore also in the connected component of the identity of the invertible group of $\mathcal{A}^p$. This unitary connects $F$ to $\begin{pmatrix} P_+ & 0 \\ 0 & 0 \end{pmatrix}$, and then $\mathbb{E}_1$ is connected.

- The connectedness of $\mathbb{E}_3$ can be obtained by noting that the map
$$P \mapsto P^\perp = 1 - P$$
transforms $\mathbb{E}_1$ into $\mathbb{E}_3$.
- The case of $\mathbb{E}_2$ is analogous to the case of $\mathbb{E}_1$, and therefore the case of $\mathbb{E}_4$ also follows.
- If $F, G \in \mathbb{E}_5$, they are of the form
$$F = \begin{pmatrix} F_+ & 0 \\ 0 & F_- \end{pmatrix}, \quad G = \begin{pmatrix} G_+ & 0 \\ 0 & G_- \end{pmatrix},$$
with $F_\pm, G_\pm$ of infinite rank and co-rank. Clearly these projections are unitarily equivalent with a diagonal unitary matrix, which therefore belong to the connected component of the identity in the invertible group of $\mathcal{A}^p$.

Thus, we have the following:

**Theorem 7.2.** *The classes $\mathbb{E}_i$, $1 \le i \le 5$ are connected. Each of these spaces is the orbit of a fixed (diagonal) projection in the corresponding class, under the action of the the unitary operators in the connected component of the identity in $\mathcal{A}^p$.*

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Esteban Andruchow
Instituto de Ciencias,
Universidad Nacional de Gral. Sarmiento,
J.M. Gutierrez 1150,
(1613) Los Polvorines,
Argentina
and
Instituto Argentino de Matemática, 'Alberto P. Calderón',
CONICET,
Saavedra 15, piso 3,
(1083) Buenos Aires,
Argentina.
e-mail: `eandruch@ungs.edu.ar`

María Eugenia Di Iorio y Lucero
Instituto Argentino de Matemática, 'Alberto P. Calderón',
CONICET,
Saavedra 15, piso 3,
(1083) Buenos Aires,
Argentina.
e-mail: `mdiiorio@fi.uba.ar`