Geodesics of projections in von Neumann algebras

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Abstract

Let $\mathcal{A}$ be a von Neumann algebra and $\mathcal{P}_{\mathcal{A}}$ the manifold of projections in $\mathcal{A}$. There is a natural linear connection in $\mathcal{P}_{\mathcal{A}}$, which in the finite dimensional case coincides with the the Levi-Civita connection of the Grassmann manifold of $\mathbb{C}^n$. In this paper we show that two projections p, q can be joined by a geodesic, which has minimal length (with respect to the metric given by the usual norm of $\mathcal{A}$), if and only if

$$p \wedge q^{\perp} \sim p^{\perp} \wedge q,$$

where $\sim$ stands for the Murray-von Neumann equivalence of projections. It is shown that the minimal geodesic is unique if and only if $p \wedge q^{\perp} = p^{\perp} \wedge q = 0$. If $\mathcal{A}$ is a finite factor, any pair of projections in the same connected component of $\mathcal{P}_{\mathcal{A}}$ (i.e., with the same trace) can be joined by a minimal geodesic.

We explore certain relations with Jones' index theory for subfactors. For instance, it is shown that if $\mathcal{N} \subset \mathcal{M}$ are $\mathbf{II}_1$ factors with finite index $[\mathcal{M} : \mathcal{N}] = \mathbf{t}^{-1}$, then the geodesic distance $d(e_{\mathcal{N}}, e_{\mathcal{M}})$ between the induced projections $e_{\mathcal{N}}$ and $e_{\mathcal{M}}$ is $d(e_{\mathcal{N}}, e_{\mathcal{M}}) = \arccos(\mathbf{t}^{1/2})$.

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1 Introduction

If $\mathcal{A}$ is a C*-algebra, let $\mathcal{P}_{\mathcal{A}}$ denote the set of (selfadjoint) projections in $\mathcal{A}$. $\mathcal{P}_{\mathcal{A}}$ has a rich geometric structure, see for instante the papers [12] by H. Porta and L.Recht and [6] by G. Corach, H. Porta and L. Recht. In these works, it was shown that $\mathcal{P}_{\mathcal{A}}$ is a C$^\infty$ complemented submanifold of $\mathcal{A}_s$, the set of selfadjoint elements of $\mathcal{A}$, and has a natural linear connection, whose geodesics can be explicitly computed. A metric is introduced, called in this context a Finsler metric: since the tangent spaces of $\mathcal{P}_{\mathcal{A}}$ are closed and complemented linear subspaces of $\mathcal{A}_s$, they can be endowed with the norm metric. With this Finsler metric, Porta and Recht [12] showed that two projections $p, q \in \mathcal{P}_{\mathcal{A}}$ which satisfy that $\|p - q\| < 1$ can be joined by a unique geodesic, which is minimal for the metric (i.e., it is shorter than any other smooth curve in $\mathcal{P}_{\mathcal{A}}$ joining the same endpoints).

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In general, two projections p, q in $\mathcal{A}$ satisfy that $\|p-q\| \leq 1$, so that what remains to consider is what happens in the extremal case $\|p-q\| = 1$: under what conditions does there exist a geodesic, or a minimal geodesic, joining them.

In the case when $\mathcal{A} = \mathcal{B}(\mathcal{H})$ the algebra of all bounded linear operators in a Hilbert space $\mathcal{H}$, it is known (see for instance [4]) that there exists a geodesic joining p and q if and only if

$$\dim R(p) \cap N(q) = \dim N(p) \cap R(q).$$

The geodesic is unique if and only if these intersections are trivial.

The purpose of this note is to show that these facts remain valid if $\mathcal{A}$ is a von Neumann algebra, if we replace dim by the dimension relative to $\mathcal{A}$. Namely, it is shown that there exists a minimal geodesic joining p and q in $\mathcal{P}_\mathcal{A}$, if and only if

$$p \wedge q^\perp \sim p^\perp \wedge q.$$

Here $\wedge$ denotes the infimum of two projections, $p^\perp = 1 - p$, and $\sim$ is the Murray-von Neumann equivalence of projections. Also, it is shown that there exists a unique minimal geodesic if and only if

$$p \wedge q^\perp = 0 = p^\perp \wedge q.$$

We show that if $\mathcal{A}$ is a finite factor, any pair of projections in $\mathcal{A}$ in the same connected component of $\mathcal{P}_\mathcal{A}$ (i.e., with the same trace), can be joined by a minimal geodesic.

In the final section of this paper, we explore the relationship with the index theory of von Neumann factors, introduced by V. Jones in [10]. A pairing $\mathcal{N} \subset \mathcal{M}$ of factors of type $\mathbf{II}_1$, induces a sequence of projections, by means of the *basic construction.* We show that one recovers Jones index as a geodesic distance (minima of lengths of curves joining two given points): if e, f are two consecutive terms in the sequence of projections, then

$$d(e, f) = \arccos(\mathbf{t}^{1/2}),$$

where $\mathbf{t}^{-1} = [\mathcal{M} : \mathcal{N}]$. Also we show that if $\mathcal{N}_0, \mathcal{N}_1 \subset \mathcal{M}$ with Jones' projections e_0, e_1, satisfy that $\|e_0 - e_1\| < 1$, then the unique geodesic $\delta(t)$ induces a smooth path of conditional expectations between $\mathcal{M}$ and intermediate factors $\mathcal{N}_t$, and the parallel transport of this geodesic, induces a smooth path of normal $*$-isomorphimsms between $\mathcal{N}_0$ and $\mathcal{N}_t$.

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2 Preliminaries

The space $\mathcal{P}_\mathcal{A}$ is sometimes called the Grassmann manifold of $\mathcal{A}$. The reason for this name is that in the case when $\mathcal{A} = \mathcal{B}(\mathcal{H})$, $\mathcal{P}_{\mathcal{B}(\mathcal{H})}$ parametrizes the set of closed subspaces of $\mathcal{H}$: to each closed subspace $\mathcal{S} \subset \mathcal{H}$ corresponds the orthogonal projection $P_\mathcal{S}$ onto $\mathcal{S}$. Let us describe below the main features of the geometry of $\mathcal{P}_\mathcal{A}$ in the general case.

2.1 Homogeneous structure

Denote by $\mathcal{U}_\mathcal{A} = \{u \in \mathcal{A} : u^*u = uu^* = 1\}$ the unitary group of $\mathcal{A}$. It is a Banach-Lie group, whose Banach-Lie algebra is $\mathcal{A}_{as} = \{x \in \mathcal{A} : x^* = -x\}$. This group acts on $\mathcal{P}_\mathcal{A}$ by means of

$u \cdot p = upu^*,\ u \in \mathcal{U}_{\mathcal{A}},\ p \in \mathcal{P}_{\mathcal{A}}$. The action is smooth and locally transitive. It is known (see [12], [6]) that $\mathcal{P}_{\mathcal{A}}$ is what in differential geometry is called a homogeneous space of the group $\mathcal{U}_{\mathcal{A}}$. The local structure of $\mathcal{P}_{\mathcal{A}}$ is described using this action. For instance, the tangent space $(T\mathcal{P}_{\mathcal{A}})_p$ of $\mathcal{P}_{\mathcal{A}}$ at p is given by $(T\mathcal{P}_{\mathcal{A}})_p = \{x \in \mathcal{A}_s : x = px + xp\}$.

The isotropy subgroup of the action at p, i.e., the elements of $\mathcal{U}_{\mathcal{A}}$ which fix a given p, is $\mathcal{I}_p = \{v \in \mathcal{U}_{\mathcal{A}} : vp = pv\}$. The isotropy algebra $\mathfrak{I}_p$ at p is its Banach-Lie algebra $\mathfrak{I}_p = \{y \in \mathcal{A}_{as} : yp = py\}$.

It is useful, in order to describe and understand the geometry of $\mathcal{P}_{\mathcal{A}}$, to consider the *diagonal / co-diagonal* decomposition of $\mathcal{A}$ in terms of a fixed projection $p_0 \in \mathcal{P}_{\mathcal{A}}$. Elements $x \in \mathcal{A}$ which commute with p_0, or equivalently, commute with the symmetry $2p_0 - 1$, when written as 2×2 in terms of p_0, have diagonal matrices. Co-diagonal matrices correspond with elements in $\mathcal{A}$ which anti-commute with $2p_0 - 1$.

Then, the isotropy subgroup and the isotropy algebra $\mathcal{I}_{p_0}, \mathfrak{I}_{p_0}$ at p_0, are respectively the sets of diagonal unitaries and diagonal anti-Hermitian elements of $\mathcal{A}$. On the other side, the tangent space $(T\mathcal{P}_{\mathcal{A}})_{p_0}$ is the set of co-diagonal selfadjoint elements of $\mathcal{A}$.

2.2 Reductive structure

Given a homogeneous space, a *reductive* structure is a smooth distribution $p \mapsto \mathbf{H}_p \subset \mathcal{A}_{as}$, $p \in \mathcal{P}_{\mathcal{A}}$, of supplements of $\mathfrak{I}_p$ in $\mathcal{A}_{as}$, which is invariant under the action of $\mathcal{I}_p$. That is, a distribution $\mathbf{H}_p$ of closed linear subspaces of $\mathcal{A}_{as}$ verifying that $\mathbf{H}_p \oplus \mathfrak{I}_p = \mathcal{A}_{as}$; $v\mathbf{H}_p v^* = \mathbf{H}_p$ for all $v \in \mathcal{I}_p$; and the map $p \mapsto \mathbf{H}_p$ is smooth.

In the case of $\mathcal{P}_{\mathcal{A}}$, the choice of the (so called) *horizontal* subspaces $\mathbf{H}_p$ is natural. The horizontal $\mathbf{H}_p$ defined in [6] is $\mathbf{H}_p = \{\begin{pmatrix} 0 & z \\ -z^* & 0 \end{pmatrix} : z \in p\mathcal{A}p^{\perp}\}$, i.e., the set of co-diagonal anti-Hermitian elements of $\mathcal{A}$

As in classical differential geometry, a reductive structure on a homogeneous space defines a linear connection: if $X(t)$ is a smooth curve of vectors tangent to a smooth curve $p(t)$ in $\mathcal{P}_{\mathcal{A}}$, i.e., a smooth curve of selfadjoint elements of $\mathcal{A}$, which are pointwise co-diagonal with respect to $p(t)$, then the covariant derivative of the linear connection is given by

$$\frac{D}{dt}X(t) := \text{diagonal part w.r.t. } p(t) \text{ of } \dot{X}(t) = p(t)\dot{X}(t)p(t) + p^{\perp}(t)\dot{X}(t)p^{\perp}(t).$$

It is not difficult to deduce then that a *geodesic* starting at $p_0 \in \mathcal{P}_{\mathcal{A}}$ is given by the action of a one parameter group with horizontal (anti-Hermitian co-diagonal) velocity on p_0 . Namely, given the base point $p_0 \in \mathcal{P}_{\mathcal{A}}$, and a tangent vector $\mathbf{x} = \begin{pmatrix} 0 & x \\ x^* & 0 \end{pmatrix} \in (T\mathcal{P}_{\mathcal{A}})_{p_0}$, the unique geodesic δ of $\mathcal{P}_{\mathcal{A}}$ with $\delta(0) = p_0$ and $\dot{\delta}(0) = \mathbf{x}$ is given by

$$\delta(t) = e^{tz_{\mathbf{x}}} p_0 e^{-tz_{\mathbf{x}}},$$

where $z_{\mathbf{x}} := \begin{pmatrix} 0 & -x \\ x^* & 0 \end{pmatrix}$. The horizontal element $z_{\mathbf{x}}$ is characterized as the unique horizontal element at p_0 such that $[z_{\mathbf{x}}, p_0] = \mathbf{x}$.

2.3 Finsler metric

As we mentioned above, one endows each tangent space $(T\mathcal{P}_{\mathcal{A}})_p$ with the usual norm of $\mathcal{A}$. We emphasize that this (constant) distribution of norms is not a Riemannian metric (the C*-norm is not given by an inner product), neither is it a Finsler metric in the classical sense (the map $a \mapsto \|a\|$ is non differentiable). Therefore the minimality result which we describe below does not follow from general considerations. It was proved in [12] using ad-hoc techniques.

1. Given $p \in \mathcal{P}_{\mathcal{A}}$ and $\mathbf{x} \in (T\mathcal{P}_{\mathcal{A}})_p$, normalized so that $\|\mathbf{x}\| \leq \pi/2$, then the geodesic δ remains minimal for all t such that $|t| \leq 1$.
2. Given $p, q \in \mathcal{P}_{\mathcal{A}}$ such that $\|p - q\| < 1$, there exists a unique minimal geodesic δ such that $\delta(0) = p$ and $\delta(1) = q$.

We shall call these geodesics (with initial speed $\|\mathbf{x}\| \leq \pi/2$) *normalized* geodesics.

3 Von Neumann algebras

In this paper we consider the case when $\mathcal{A}$ is a von Neumann algebra. We shall suppose $\mathcal{A}$ acting in a Hilbert space $\mathcal{H}$ (i.e., $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$). As we shall see, this representation is auxiliary, and the results on the geometry of $\mathcal{P}_{\mathcal{A}}$ do not depend on the representation. The main assertion of this section is that the conditions of existence and uniqueness of minimal geodesics joining given projections $p, q \in \mathcal{P}_{\mathcal{A}}$ are the a natural generalization of the conditions valid in the case of $\mathcal{B}(\mathcal{H})$.

If $p, q \in \mathcal{P}_{\mathcal{A}}$, we denote by $p^{\perp} = 1 - p$, and by $p \wedge q$ the projection onto $R(p) \cap R(q)$ (which belongs to $\mathcal{P}_{\mathcal{A}}$); p and q are said to be *Murray - von Neumann equivalent*, in symbols $p \sim q$, if there exists $v \in \mathcal{A}$ (a partial isometry) such that $v^*v = p$ and $vv^* = q$. Our main result follows:

Theorem 3.1. *Let* $p, q \in \mathcal{P}_{\mathcal{A}}$.

1. *There exists a geodesic* δ *of* $\mathcal{P}_{\mathcal{A}}$ *joining* p *and* q *if and only if*
$$p \wedge q^{\perp} \sim p^{\perp} \wedge q.$$
Moreover, the geodesic can be chosen minimal (i.e., normalized).
2. *There is a unique normalized geodesic if and only if* $p \wedge q^{\perp} = p^{\perp} \wedge q = 0$.

Proof. Existence: suppose first that $p \wedge q^{\perp} \sim p^{\perp} \wedge q$. Consider following projections which sum to 1 and commute both with p and q:

$$e_{11} = p \wedge q \ , \ e_{00} = p^{\perp} \wedge q^{\perp} \ , \ e_{10} = p \wedge q^{\perp} \ , \ e_{01} = p^{\perp} \wedge q \ , \ e_0 = 1 - \sum_{i,j=0,1} e_{i,j}.$$

It is straightforward to verify that e_{ij} commute with p and q, and thus e_0 also does. The decomposition of the Hilbert space induced by these projections is sometimes called the Halmos decomposition of the space, in the presence of two closed subspaces ($R(p)$ and $R(q)$); the last subspace $R(e_0)$, is called the generic part of p and q. We shall construct the exponent $\mathbf{x}$ of the geodesic joining p and q as a sum of anti-Hermitian elements in $\mathcal{A}$,

$$\mathbf{x} = \mathbf{x}' + \mathbf{x}'' + \mathbf{x}_0,$$

where $\mathbf{x}'$ acts in the range of $e_{11}+e_{00}$, $\mathbf{x}''$ acts in the range of $e_{10}+e_{01}$ and $\mathbf{x}_0$ acts in the range of e_0. Moreover, each of these elements is co-diagonal with respect to the corresponding reduction of p to these subspaces. First note that $pe_{ii}=qe_{ii}$ (on e_{00} they are both zero, on e_{11} they are both the identity). Thus the exponent $\mathbf{x}'$ can be chosen 0.

Let us consider next the part in e_0. Here we make use of the representation $\mathcal{A}\subset\mathcal{B}(\mathcal{H})$. Denote by $\mathcal{H}_0=R(e_0)$, and by $p_0=pe_0$, $q_0=qe_0$ the reductions of p,q to this subspace $\mathcal{H}_0$. Then, it is clear that p_0,q_0 lie in *generic position* ([9], [8]): their ranges and nullspaces intersect trivially. Thus, by a result by P. Halmos [9], there exist a Hilbert space $\mathcal{L}$, a positive operator $X\in\mathcal{B}(\mathcal{L})$ ($\|X\|\le\pi/2$) and a unitary isomorphism $\mathcal{H}_0\to\mathcal{L}\times\mathcal{L}$ which carries

$$p_0 \ \text{ to } \ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \ \text{ and } q_0 \ \text{ to } \ \begin{pmatrix} \cos^2(X) & \cos(X)\sin(X) \\ \cos(X)\sin(X) & \sin^2(X) \end{pmatrix}.$$

Between these operator matrices, one can find the (co-diagonal) exponent

$$Z=\begin{pmatrix} 0 & X \\ -X & 0 \end{pmatrix},$$

which satisfies

$$e^{-Z}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}e^{Z}=\begin{pmatrix} \cos^2(X) & \cos(X)\sin(X) \\ \cos(X)\sin(X) & \sin^2(X) \end{pmatrix}.$$

These are straightforwward verifications, and provide the exponent for a geodesic joining the two operator matrices. One loses track though of how elements of $\mathcal{A}$ are changed by the Halmos isomorphism. The key fact to relate these matrices to the former projections p_0,q_0 is the following elementary identity proved in [4]

$$e^{Z}=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}V, \tag{1}$$

where V is the unitary part in the polar decomposition of

$$B-1=\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}+\begin{pmatrix} \cos^2(X) & \cos(X)\sin(X) \\ \cos(X)\sin(X) & \sin^2(X) \end{pmatrix}-1,$$

i.e. $B-1=V|B-1|$. Again, this is an elementary matrix computation. Let $b_0=p_0+q_0$, and let v_0 be the isometric part in the polar decomposition (recall that e_0 is the unit in this part of the algebra)

$$b_0-e_0=v_0|b_0-e_0|.$$

Clearly $v_0\in\mathcal{A}$ and is carried by the Halmos isomorphism to V. Therefore, if one regards (1), it follows that the unitary element $v_0(2p_0-1)$ is carried by this ismorphism to e^Z, i.e. e^Z corresponds to an element of $\mathcal{A}$. Moreover,

$$\|Z\|=\|X\|\le\pi/2,$$

which implies that Z is the unique anti-Hermitian logarithm of e^Z with spectrum in $(-i\pi,i\pi)$. It follows that there exists a unique element $\mathbf{x}_0\in\mathcal{A}$ which corresponds to Z, and therefore satisfies

$$e^{-\mathbf{x}_0}p_0e^{\mathbf{x}_0}=q_0.$$

It remains to construct the exponent $\mathbf{x}''$ acting in $e_{10}+e_{01}$. Note that the reductions of p and q to this part are

$$p(e_{10}+e_{01}) = p(p\wedge q^{\perp} + p^{\perp}\wedge q) = p\wedge q^{\perp},$$

and similarly $q(e_{10}+e_{01}) = p^{\perp}\wedge q$. By hypothesis, there is a partial isometry $w\in\mathcal{A}$ such that

$$w^*w = p\wedge q^{\perp} \quad \text{and} \quad ww^* = p^{\perp}\wedge q.$$

Then $\mathbf{x}'' = i\frac{\pi}{2}(w+w^*)$ does the feat: since $p\wedge q^{\perp}\perp p^{\perp}\wedge q$, it follows that $\mathbf{x}''$ is co-diagonal with respect to the decomposition $p\wedge q^{\perp}+p^{\perp}\wedge q$. Clearly $\|\mathbf{x}''\| = \pi/2$. Note also that $w^2=0$, so that

$$e^{\mathbf{x}''} = i(w+w^*).$$

Finally,

$$e^{\mathbf{x}''}(p\wedge q^{\perp}) = i(w+w^*)(w^*w) = iww^*w = iww^*(w+w^*) = (p^{\perp}\wedge q)e^{\mathbf{x}''},$$

i.e., $e^{\mathbf{x}''}$ intertwines the reductions of p and q to this part.

If we put together $\mathbf{x} = \mathbf{x}'+\mathbf{x}''-\mathbf{x}_0$, which is an orthogonal sum, we have a p-co-diagonal anti-Hermitian element of $\mathcal{A}$, with $\|\mathbf{x}\|\le\pi/2$ (note that $\mathbf{x}''$ might be zero, if $p\wedge q^{\perp} = p^{\perp}\wedge q = 0$), which satisfies

$$e^{\mathbf{x}}pe^{-\mathbf{x}} = q.$$

Conversely, suppose that there exists a normalized geodesic wich joins p and q, i.e. there exists a p-co-diagonal anti-Hermitian element $\mathbf{x}\in\mathcal{A}$ with $\|\mathbf{x}\|\le\pi/2$ such that $e^{\mathbf{x}}pe^{-\mathbf{x}} = q$. We claim that $e^{\mathbf{x}}$ maps $R(p\wedge q^{\perp})$ onto $R(p^{\perp}\wedge q)$. Clearly $e^{\mathbf{x}}$ maps $R(p)$ onto $R(q)$. Pick $\xi\in R(p\wedge q^{\perp}) = R(p)\cap N(q)$. Then $e^{\mathbf{x}}\xi\in R(q)$. It was noted in [12], that the fact that $\mathbf{x}$ is p-co-diagonal means that $\mathbf{x}$ anti-commutes with $2p-1$. Thus,

$$(2p-1)e^{\mathbf{x}} = e^{-\mathbf{x}}(2p-1).$$

Then, since $(2p-1)\xi = \xi$ and $(2q-1)\xi = -\xi$,

$$(2p-1)e^{\mathbf{x}}\xi = e^{-\mathbf{x}}(2p-1)\xi = e^{-\mathbf{x}}\xi = -e^{-\mathbf{x}}(2q-1)\xi = -(2p-1)e^{-\mathbf{x}}\xi = -e^{\mathbf{x}}(2p-1)\xi = -e^{\mathbf{x}}\xi,$$

i.e., $e^{\mathbf{x}}\xi\in N(p)$, and thus $e^{\mathbf{x}}(R(p)\cap N(q))\subset R(q)\cap N(p)$ The other inclusion follows similarly (or by symmetry: in fact $\mathbf{x}$ is also q-co-diagonal, because $-\mathbf{x}$ is the initial velocity of the reversed geodesic which starts at q). It folllows that $w = e^{\mathbf{x}}(p\wedge q^{\perp})\in\mathcal{A}$ is a partial isometry with initial space $p\wedge q^{\perp}$ and final space $p^{\perp}\wedge q$.

Uniqueness: if $p\wedge q^{\perp} = p^{\perp}\wedge q = 0$, then $R(p)\cap N(q) = N(p)\cap R(q) = \{0\}$, and there exists a unique normalized geodesic in $\mathcal{P}_{\mathcal{B}(\mathcal{H})}$ joining p and q. By the first part of the proof, there is a normalized geodesic joining them in $\mathcal{P}_{\mathcal{A}}$. Thus, it is unique.

Conversely, suppose that there exists a unique geodesic joining p and q. Then necessarily $p\wedge q^{\perp}\sim p^{\perp}\wedge q$. Suppose that these projections are non zero. Then, there are infinitely many different partial isometries w such that $w^*w = p\wedge q^{\perp}$ and $ww^* = p^{\perp}\wedge q$. As in the first part of the proof, any such w give rise to different exponents $\mathbf{x}''$, and thus different $\mathbf{x}$, i.e. different geodesics joining p and q. $\square$

Remark 3.2. In the above result, it was shown in fact that the submanifold $\mathcal{P}_{\mathcal{A}}\subset\mathcal{P}_{\mathcal{B}(\mathcal{H})}$ is totally geodesic: the geodesics of $\mathcal{P}_{\mathcal{A}}$ are geodesics of the bigger manifold $\mathcal{P}_{\mathcal{B}(\mathcal{H})}$; if $p,q\in\mathcal{P}_{\mathcal{A}}$ are joined by a unique geodesic of $\mathcal{P}_{\mathcal{B}(\mathcal{H})}$, then this geodesic remains inside $\mathcal{P}_{\mathcal{A}}$.

Remark 3.3. Note that it follows from the computations in the above proof, that if $\|p-q\| < 1$ and z is the exponent of the unique geodesic joining p and q, then $\|z\| < \pi/2$, or equivalently, that $\|e^z - 1\| < \sqrt{2}$.

4 Hopf-Rinow theorem in finite factors

Two subspaces of dimension k in $\mathbb{C}^n$ can be joined by a minimal geodesic of the Levi-Civita connection in the Grassmann manifold. This fact can be proved using the projection formalism. That is, parametrizing subspaces with othogonal projections in $M_n(\mathbb{C})$, by means of

$$\mathbb{C}^n \supset \mathcal{S} \longleftrightarrow P_{\mathcal{S}} \in M_n(\mathbb{C}),$$

where $P_{\mathcal{S}}$ is the orthogonal projection onto $\mathcal{S}$. Two subspaces $\mathcal{S}, \mathcal{T} \subset \mathbb{C}^n$ have the same dimension if and only if the corresponding projections $P_{\mathcal{S}}, P_{\mathcal{T}}$ have the same rank, i.e.

$$Tr(P_{\mathcal{S}} - P_{\mathcal{T}}) = 0.$$

Let us see that in this case one has, automatically, that

$$\dim(\mathcal{S} \cap \mathcal{T}^{\perp}) = \dim(\mathcal{S}^{\perp} \cap \mathcal{T}).$$

This fact has an elementary proof. Let us prove it in a non totally elementary fashion, which will allow us to obtain a generalization. The operator $A = P_{\mathcal{S}} - P_{\mathcal{T}}$ is a selfadjoint contraction, and if $B = P_{\mathcal{S}} + P_{\mathcal{T}}$,

$$N(B-1) = \mathcal{S} \cap \mathcal{T}^{\perp} \ \oplus \ \mathcal{S}^{\perp} \cap \mathcal{T} = N(A-1) \oplus N(A+1).$$

On the subspace $N(B-1)^{\perp}$, $B-1$ is an invertible matrix, and the symmetry V in its polar decomposition $B - 1 = V|B-1|$ satisfies that $VP_{\mathcal{S}}V = P_{\mathcal{T}}$ in $N(B-1)^{\perp}$. Then A is reduced by $N(B-1)$, and

$$V(A\Big|_{N(B-1)^{\perp}})V = -A\Big|_{N(B-1)^{\perp}}.$$

This implies that the spectrum of $A\Big|_{N(B-1)^{\perp}}$ is symmetric with respect to the origin: if λ is an eigenvalue of A with $|\lambda| < 1$, then $-\lambda$ is also an eignevalue of A, and they have the same multiplicity: $\dim(N(A-\lambda)) = \dim(N(A+\lambda))$. Then

$$A = -P_{N(A+1)} + P_{N(A-1)} + A\Big|_{N(B-1)^{\perp}} = -P_{N(A+1)} + P_{N(A-1)} + \sum_{0<\lambda<1} \lambda(P_{N(A-\lambda)} - P_{N(A+\lambda)}).$$

Thus, the fact that $Tr(A) = 0$, means that $Tr(P_{N(A-1)}) = Tr(P_{N(A+1)})$. Thus $P_{\mathcal{S}}$ and $P_{\mathcal{T}}$ can be joined by a normalized geodesic. Remarkably, this geodesic is minimal for the Levi-Civita connection of the Grassmann manifold, but also, using the projection formalism, for the operator norm of $M_n(\mathbb{C})$, the p-Schatten norms ([3]), or more generally, for unitary invariant norms (see [5]).

Let us suppose now that $\mathcal{A}$ is a finite von Neumann factor, with trace τ. We shall see that the above argument holds (essentially unaltered):

Theorem 4.1. *Let $\mathcal{A}$ be a finite von Neumann factor with faithful normal trace τ. Two projections $p, q \in \mathcal{P}_{\mathcal{A}}$ with $p \sim q$ (i.e., unitarily equivalent, or equivalently, in the same connected component of $\mathcal{P}_{\mathcal{A}}$) can be joined by a normalized geodesic.*

Proof. Let $a = p - q$, and again note that $N(a-1) = R(p) \cap N(q)$. Following previous notations, $P_{N(a-1)} = p \wedge q^{\perp} = e_{10}$. Similarly, $P_{N(a+1)} = p^{\perp} \wedge q = e_{01}$. Therefore, $P_{N(b-1)} = e_{10} + e_{01} := e''$. Again, since e_{10} and e_{01} are eigenspaces of a, these projections reduce a, let a_0 be the reduction of a to $R(e'')^{\perp}$. Then, we have that

$$a = e_{10} - e_{01} + a_0.$$

The operator a_0 is a difference of projections: $a_0 = p_0 - q_0$, where p_0 and q_0 are the reductions of p and q to $R(e'')^{\perp}$, with $N(a_0 \pm e_0) = \{0\}$ (e_0 is the identity in $R(e'')^{\perp}$). It was shown by Chandler Davis [7] that there exists a symmetry v_0 ($v_0^* = v_0$, $v_0^2 = e_0$; namely, v_0 is the isometric part in the polar decomposition of $b_0 - e_0$) such that

$$v_0 a_0 v_0 = -a_0.$$

Let μ be the projection-valued spectral measure of a_0:

$$a_0 = \int_{-1}^{1} \lambda d\mu(\lambda).$$

As in the above argument in $M_n(\mathbb{C})$, the existence of the symmetry v_0 implies the symmetry of the spectral measure of a_0 with respect to the origin: if $\Lambda \subset [-1, 1]$ is a Borel subset, then

$$\mu(-\Lambda) = v_0 \mu(\Lambda) v_0.$$

Then

$$\tau(a_0) = \int_{-1}^{1} \lambda d\tau(\mu(\lambda)) = 0,$$

because the function $f(\lambda) = \lambda$ is odd and the measure $\tau\mu$ is symmetric with respect to the origin:

$$\tau(\mu(-\Lambda)) = \tau(v_0 \mu(\Lambda) v_0) = \tau(e_0 \mu(\Lambda)) = \tau(\mu(\Lambda)),$$

because $\mu \leq e_0$. Then, since $p \sim q$,

$$0 = \tau(p) - \tau(q) = \tau(a) = \tau(e_{10} - e_{01} + a_0) = \tau(e_{10}) - \tau(e_{01}),$$

so that $\tau(p \wedge q^{\perp}) = \tau(p^{\perp} \wedge q)$, i.e., $p \wedge q^{\perp} \sim p^{\perp} \wedge q$. Therefore, by Theorem 3.1, there exists a (minimal) normalized geodesic joining p and q in $\mathcal{P}_{\mathcal{A}}$. □

Remark 4.2. In [1], it was shown that in a finite algebra with faithful trace τ, the geodesics have minimal length also when measured with the ρ norms $\| \ \|_\rho$ of the trace, for $\rho \geq 2$ ($\|x\|_\rho = (\tau(x^*x)^{\rho/2})^{1/\rho}$). Namely, it was shown that if $\delta(t) = e^{t\mathbf{z}} p e^{-t\mathbf{z}}$ is a normalized geodesic ($\|\mathbf{z}\| \leq \pi/2$) with $\delta(1) = q$, and γ is any other smooth curve in $\mathcal{P}_{\mathcal{A}}$ with $\gamma(t_0) = p$ and $\gamma(t_1) = q$, then

$$\ell_\rho(\gamma) := \int_{t_0}^{t_1} \|\dot{\gamma}(t)\|_\rho dt \geq \ell_\rho(\delta) = \|\mathbf{z}\|_\rho.$$

As we have seen, on finite factors, a version of the the Hopf-Rinow is valid in $\mathcal{P}_\mathcal{A}$, and the geodesics are minimal for the usual norm of $\mathcal{A}$ at every tangent space. However, as a consequence of the fact in the above remark, we have that for the p-norms in the tangent space, including the pseudo-Riemannian case $p = 2$, there are no *normal* neighbourhoods if $\mathcal{A}$ is a type II_1 factor. Indeed, for $2 \le \rho < \infty$, denote by

$$d_\rho(p, q) = \inf\{\ell_\rho(\gamma) : \gamma \text{ is smooth and joins } p \text{ and } q \text{ in } \mathcal{P}_\mathcal{A}\}$$

the metric induced in $\mathcal{P}_\mathcal{A}$ by the ρ-norm.

Proposition 4.3. *Let $\mathcal{A}$ a type II_1 factor and $2 \le \rho < \infty$ Then there exist pairs of projections in $\mathcal{P}_\mathcal{A}$, which are arbitrarily close for the d_ρ metric, which can be joined by infinitely many geodesics.*

Proof. Given $0 < r \le \frac{1}{2}$, let $p \in \mathcal{P}_\mathcal{A}$ such that $\tau(p) = r$. Let $q \in \mathcal{P}_\mathcal{A}$ such that $q \le p^\perp$ and $\tau(q) = r$ (consider the reduced factor $p^\perp \mathcal{A} p^\perp$, which is also of type II_1, and pick there a projection q with (renormalized) trace $\frac{r}{1-r}$). Then, the Halmos decomposition given by p and q yields (following the notation of the preceding section)

$$e_{00} = 0,\ e_{11} = 0,\ e_{10} = p,\ e_{10} = p,\ e_{01} = q,\ e_0 = 0.$$

Since $p \sim q$, there exist infinitely many $v \in \mathcal{A}$ such that $v^*v = p$ and $vv^* = q$. Any of these v provides a geodesic joining p and q, given by (see the last part of the proof of Theorem 3.1) the exponent $\mathbf{x} = i\frac{\pi}{2}(v + v^*)$. The length of any of these geodesics is

$$\|\mathbf{x}\|_\rho = \tau((\mathbf{x}^*\mathbf{x})^{\rho/2})^{1/\rho} = \frac{\pi}{2}\tau((v^*v + vv^*)^{\rho/2})^{1/\rho} = \frac{\pi}{2}2^{1/\rho}\tau(p)^{1/\rho} = \pi 2^{1/\rho - 1} r^{1/\rho}.$$

□

5 Applications to finite index subfactors

V.F.R. Jones introduced the theory of index for subfactors of a $\mathbf{II}_1$ factor in [10]. An inclusion $\mathcal{N} \subset \mathcal{M}$ of $\mathbf{II}_1$ factors is said to be of *finite index* if the relative dimension (or coupling constant)

$$[\mathcal{M} : \mathcal{N}] := \dim_\mathcal{N}(L^2(\mathcal{M}, \tau)) = \mathbf{t}^{-1}$$

is finite (see [11]). A sequence of projections arises in this circumstance, by means of Jones' *basic cosntruction.* Denote by τ the normalized trace of $\mathcal{M}$ (and of the subsequent finite extensions which will be considered). Let $e_\mathcal{N}$ be the orthogonal projection of $L^2(\mathcal{M}, \tau)$ onto $L^2(\mathcal{N}, \tau)$. This projection, restricted to $\mathcal{M} \subset L^2(\mathcal{M}, \tau)$, induces the unique trace invariant conditional expectation $E_\mathcal{N} : \mathcal{M} \to \mathcal{N}$. Jones proved that the von Neumann algebra $\mathcal{M}_1 =< \mathcal{M}, e_\mathcal{N} >$ generated in $\mathcal{B}(L^2(\mathcal{M}, \tau))$ by $\mathcal{M}$ and $e_\mathcal{N}$ is again a $\mathbf{II}_1$ factor, and that the inclusion $\mathcal{M} \subset \mathcal{M}_1$ has finite index, with $[\mathcal{M}_1 : \mathcal{M}] = [\mathcal{M} : \mathcal{N}]$. Thus, iterating the basic construction, a sequence of orthogonal projections arises: $e_1 = e_\mathcal{N}, e_2 = e_\mathcal{M}, \dots$. We shall be concerned only with the first two. These projections recover the index:

$$\tau(e_\mathcal{N}) = \tau(e_\mathcal{M}) = \mathbf{t} = [\mathcal{M} : \mathcal{N}]^{-1}.$$

In particular, they are unitarily equivalent in any factor of the tower of factors enabled by the basic construction, in which both lie. More precisely, Jones proved ([10], Proposition 3.4.1) that

$$e_{\mathcal{N}}e_{\mathcal{M}}e_{\mathcal{N}} = \mathbf{t}e_{\mathcal{N}} \ \text{ and } \ e_{\mathcal{N}}^{\perp} \wedge e_{\mathcal{M}} = \ e_{\mathcal{N}} \wedge e_{\mathcal{M}}^{\perp} = 0.$$

It follows that $e_{\mathcal{N}}$ and $e_{\mathcal{M}}$ can be joined by a unique geodesic which lies in $\mathcal{M}_2 =< \mathcal{M}, e_{\mathcal{N}}, e_{\mathcal{M}} >$. Thus, the finite index inclusion $\mathcal{N} \subset \mathcal{M}$ gives rise to a unique element $\mathbf{z}_{\mathcal{M},\mathcal{N}} \in \mathcal{M}_2$, the exponent of this geodesic:

$$\mathbf{z}_{\mathcal{M},\mathcal{N}}^* = -\mathbf{z}_{\mathcal{M},\mathcal{N}} \ , \ d(e_{\mathcal{N}}, e_{\mathcal{M}}) = \|\mathbf{z}_{\mathcal{M},\mathcal{N}}\| \leq \pi/2 \ , \mathbf{z}_{\mathcal{M},\mathcal{N}} \text{ is } e_{\mathcal{N}} \text{ and } e_{\mathcal{M}} \text{ co-diagonal,}$$

and

$$e^{\mathbf{z}_{\mathcal{M},\mathcal{N}}} e_{\mathcal{N}} e^{-\mathbf{z}_{\mathcal{M},\mathcal{N}}} = e_{\mathcal{M}}.$$

The index $[\mathcal{M} : \mathcal{N}] = \mathbf{t}^{-1}$ is related to the geodesic distance between $e_{\mathcal{N}}$ and $e_{\mathcal{M}}$, meaured with the usual norm of $\mathcal{M}_2$, or with the ρ-norms ($1 \leq \rho < \infty$):

Theorem 5.1. *With the above notations,*

$$d(e_{\mathcal{N}}, e_{\mathcal{M}}) = \|\mathbf{z}_{\mathcal{M},\mathcal{N}}\| = \arccos(\mathbf{t}^{1/2}) \quad \textit{and} \quad d_\rho(e_{\mathcal{N}}, e_{\mathcal{M}}) = \|\mathbf{z}_{\mathcal{M},\mathcal{N}}\|_\rho = \mathbf{t}^{1/\rho} \arccos(\mathbf{t}^{1/2}).$$

Proof. The projections $e_{\mathcal{N}}$ and $e_{\mathcal{M}}$ act in $L^2(\mathcal{M}_1, \tau)$. Denote by $e'_{\mathcal{N}}$ and $e'_{\mathcal{M}}$ the generic part of these projections, acting on the Hilbet space $\mathcal{H}' \subset L^2(\mathcal{M}_1, \tau)$. By Halmos' theorem, there exists an isometric isomorphism between $\mathcal{H}'$ and $\mathcal{L} \times \mathcal{L}$ such that $e'_{\mathcal{N}}$ and $e'_{\mathcal{M}}$ are carried, respectively, onto

$$P_{\mathcal{N}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \ \text{ and } \ P_{\mathcal{M}} = \begin{pmatrix} \cos^2(X) & \cos(X)\sin(X) \\ \cos(X)\sin(X) & \sin^2(X) \end{pmatrix},$$

where $0 \leq X \leq \pi/2$. Note that since the only (non trivial) non generic part of $e_{\mathcal{N}}$ and $e_{\mathcal{M}}$ is $e_{\mathcal{N}}^{\perp} \wedge e_{\mathcal{M}}^{\perp}$, on which both $e_{\mathcal{N}}$ and $e_{\mathcal{M}}$ act trivially, we have that $e'_{\mathcal{N}}e'_{\mathcal{M}}e'_{\mathcal{N}} = e_{\mathcal{N}}e_{\mathcal{M}}e_{\mathcal{N}} = \mathbf{t}e_{\mathcal{N}}$. Therefore,

$$\mathbf{t}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \mathbf{t}P_{\mathcal{N}} = P_{\mathcal{N}}P_{\mathcal{M}}P_{\mathcal{N}} = \begin{pmatrix} \cos^2(X) & \cos(X)\sin(X) \\ \cos(X)\sin(X) & \sin^2(X) \end{pmatrix},$$

i.e., $\cos(X) = \mathbf{t}^{1/2}1_{\mathcal{L}}$, and therefore X is a scalar multiple of the identity in $\mathcal{L}$: $X = \arccos(\mathbf{t}^{1/2})1_{\mathcal{L}}$. The unique exponent $\mathbf{z} = \mathbf{z}_{\mathcal{M},\mathcal{N}}$ of the geodesic joining $e_{\mathcal{N}}$ and $e_{\mathcal{M}}$, is zero on the non generic part $e_{\mathcal{N}}^{\perp} \wedge e_{\mathcal{M}}^{\perp}$, and in the generic part is related (via the Halmos' isomorphism) to the operator

$$Z = \begin{pmatrix} 0 & X \\ -X & 0 \end{pmatrix}.$$

Then $\mathbf{z}^*\mathbf{z}$ corresponds to

$$Z^* = \begin{pmatrix} X^2 & 0 \\ 0 & X^2 \end{pmatrix} = (\arccos(\mathbf{t}^{1/2}))^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Therefore $\mathbf{z}^*\mathbf{z} = (\arccos(\mathbf{t}^{1/2}))^2 e_{\mathcal{N}}$. The geodesic distance induced by the usual operator norm is

$$d(e_{\mathcal{N}}, e_{\mathcal{M}}) = \|\mathbf{z}\| = \|\mathbf{z}^*\mathbf{z}\|^{1/2} = \arccos(\mathbf{t}^{1/2}),$$

and the one induced by the ρ norm is

$$d_\rho(e_{\mathcal{N}}, e_{\mathcal{M}}) = \|\mathbf{z}\|_\rho = \arccos(\mathbf{t}^{1/2})(\tau(e_{\mathcal{N}}))^{1/\rho} = \mathbf{t}^{1/\rho} \arccos(\mathbf{t}^{1/2}).$$

□

Next, we consider the case of two projections arising from two subfactors $\mathcal{N}_0, \mathcal{N}_1 \subset \mathcal{M}$. These give rise to two orthogonal projections e_0, e_1 in $\mathcal{B}(L^2(\mathcal{M}, \tau))$. We make the assumption that both inclusions have finite index: $[\mathcal{M} : \mathcal{N}_0], [\mathcal{M} : \mathcal{N}_1] < \infty$. If both projections lie in the same $\mathbf{II}_1$ factor $\mathcal{M}_0 \supset \mathcal{M}$ (with trace τ extending the trace of $\mathcal{M}$), then a necessary and sufficient condition for the existence of a geodesic joining e_0 and e_1 is $\tau(e_0) = \tau(e_1)$.

Lemma 5.2. *Let $\mathcal{N}_0, \mathcal{N}_1 \subset \mathcal{M}$ be finite index subfactors. Then there exists a $\mathbf{II}_1$ factor $\mathcal{M}_0$ such that $\mathcal{M} \subset \mathcal{M}_0$ has finite index, and $e_0, e_1 \in \mathcal{M}_0$.*

Proof. Let $E_i : \mathcal{M} \to \mathcal{N}_i$, $i = 0, 1$, be the unique trace preserving conditional expectations, giving rise to the orthogonal projections e_0, e_1. Let $\mathcal{M}_1 =< \mathcal{M}, e_0 >$, and $F : \mathcal{M}_1 \to \mathcal{M}$ the corresponding expectation. Note that $F_1 = E_1 F : \mathcal{M}_1 \to \mathcal{N}_1$ is a conditional expectation, which is trace invariant (for the trace of $\mathcal{M}_1$), and which corresponds to the finite index inclusion $\mathcal{N}_1 \subset \mathcal{M}_1$: $[\mathcal{M}_1 : \mathcal{N}_1] = [\mathcal{M}_1 : \mathcal{M}][\mathcal{M} : \mathcal{N}_1]$. Let f_1 be the orthogonal projection in $\mathcal{B}(L^2(\mathcal{M}_1))$ induced by this inclusion, and $\mathcal{M}_0 =< \mathcal{M}_1, f_1 >$, which is a finite factor with

$$[\mathcal{M}_0 : \mathcal{M}_1] = [\mathcal{M}_1 : \mathcal{M}][\mathcal{M} : \mathcal{N}_1] < \infty.$$

We claim that $f_1 = e_1$. Denote by $[x]$ the element $x \in \mathcal{M}_1$ regarded as a vector in $L^2(\mathcal{M}_1)$. Then, if $x \in \mathcal{M}$,

$$f_1([x]) = [F_1(x)] = [E_1(F(x))] = [E_1(x)] = e_1([x]).$$

If $\xi \in \mathcal{M}^\perp$, then $f_1(\xi) = e_1(\xi) = 0$. □

Remark 5.3. Note that $\mathcal{M}_0 \subset \{\mathcal{M}, e_0, e_1\}'' \subset \mathcal{B}(L^2(\mathcal{M}_1))$. However, $\mathcal{M}$, e_0 and e_1 act also on $L^2(\mathcal{M})$. Thus, the algebra $\{\mathcal{M}, e_0, e_1\}'' \subset \mathcal{B}(L^2(\mathcal{M}_1))$ is $*$-isomomorphic (by means of a normal isomorphism, given by restriction to $L^2(\mathcal{M})$) to the von Neumann $\mathbf{II}_1$ factor $\mathcal{M}_{1,2} :=< \mathcal{M}, e_0, e_1 >\subset \mathcal{B}(L^2(\mathcal{M}))$.

Proposition 5.4. *Let $\mathcal{N}_0, \mathcal{N}_1 \subset \mathcal{M}$ be finite index subfactors. Then there exists a geodesic joining e_0 and e_1 if and only if $[\mathcal{M} : \mathcal{N}_0] = [\mathcal{M} : \mathcal{N}_1]$.*

Proof. $[\mathcal{M} : \mathcal{N}_0] = [\mathcal{M} : \mathcal{N}_1] = \mathbf{t}^{-1}$ if and only if $\tau_{\mathcal{M}_0}(e_0) = \tau_{\mathcal{M}_0}(e_1) = \mathbf{t}$. □

With the same notations as above, we have the following:

Lemma 5.5. *Suppose that $\|e_0 - e_1\| < 1$, and let $\delta(t) = e^{tz} e_0 e^{-tz}$, $t \in [0, 1]$, be the unique geodesic of $\mathcal{P}_{\mathcal{M}_0}$ joining $\delta(0) = e_0$ and $\delta(1) = e_1$. Then*

$$E_t = \delta_t|_{\mathcal{M}} : \mathcal{M} \to \mathcal{N}_t := e^{tz} \mathcal{N}_0 e^{-tz} \subset \mathcal{M}$$

is a pointwise smooth path of conditional expectations joining $E_{\mathcal{N}_0}$ and $E_{\mathcal{N}_1}$ (i.e., the map $[0, 1] \ni t \mapsto E_t(a) \in \mathcal{M}$ is C^1 for all $a \in \mathcal{M}$).

Proof. We shall use repeatedly the following argument. Suppose that $m \in \mathcal{M}_0$ is normal, and satisfies that $m[\mathcal{M}] \subset [\mathcal{M}]$ and $m^*[\mathcal{M}] \subset [\mathcal{M}]$, i.e., m and m^* as operators acting in $L^2(\mathcal{M})$, leave the dense linear manifold $[\mathcal{M}] = \{[x] : x \in \mathcal{M}\}$ invariant, and let f be a continuous function on the spectrum $\sigma(m)$ of m. Then $f(m)$ also leaves $[\mathcal{M}]$ invariant. Indeed, let $p_k(z, \bar{z})$ be polynomials in z and $\bar{z}$ which converge uniformly to $f(z)$ in $\sigma(m)$. Clearly $p_k(m, m^*)$ leave $[\mathcal{M}]$ invariant. Let $x \in \mathcal{M}$. Then

$$\|p_k(m, m^*)x - p_j(m, m^*)x\|_2 \le \|(p_k(m, m^*) - p_j(m, m^*))\|_{\mathcal{B}(L^2(\mathcal{M}))} \|x\|_2.$$

It follows that $p_k(m,m^*)x$ is a Cauchy sequence in $L^2(\mathcal{M})$, and therefore $f(m)x \in L^2(\mathcal{M})$. On the other hand, $f(m)x$ is bounded in the norm of $\mathcal{M}_0$. It follows that $f(m)x \in \mathcal{M}$.

Consider now the element $e_0 + e_1 - 1 \in \mathcal{M}_0$. Note that $\|e_0 - e_1\| < 1$ implies that $e_0 + e_1 - 1$ is invertible. Clearly, $e_0 + e_1 - 1$ (which is selfadjoint) leaves $[\mathcal{M}]$ invariant:

$$(e_0 + e_1 - 1)([x]) = [E_0(x) + E_1(x) - 1] \in [\mathcal{M}]$$

for all $x \in \mathcal{M}$. By the above argument, it follows that $|e_0 + e_1 - 1|^{-1}$ leaves $[\mathcal{M}]$ invariant. Thus

$$e^z = (2e_0 - 1)(e_0 + e_1 - 1)|e_0 + e_1 - 1|^{-1}$$

leaves $[\mathcal{M}]$ invariant. Note that the same happens for

$$(e^z)^* = e^{-z} = |e_0 + e_1 - 1|(e_0 + e_1 - 1)^{-1}(2e_0 - 1).$$

On the other hand, as pointed in Remark 3.3, the fact that $\|e_0 - e_1\| < 1$ also implies that $\|e^z - 1\| < \sqrt{2} < 2$ (or equivalently, that $\|z\| < \pi/2$). It follows that there is a continuous logarithm defined in the spectrum of e^z, $arg : \sigma(e^z) \to (-i\pi/2, i\pi/2)$. Therefore, again using the argument at the beginning of this proof (e^z and $(e^z)^*$ leave $[\mathcal{M}]$ invariant), it follows that z and $z^* = -z$ leave $[\mathcal{M}]$ invariant. Therefore, e^{tz} leave $[\mathcal{M}]$ invariant for $t \in [0,1]$. It follows that $\delta(t)$, restricted to $\mathcal{M}$, induce the linear mappings

$$\delta(t)|_{\mathcal{M}} = e^{tz}e_0e^{-tz}|_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}.$$

The range of $\delta(t)|_{\mathcal{M}}$ is $e^{tz}L^2(\mathcal{N}_0)e^{-tz} \cap \mathcal{M} = e^{tz}\mathcal{N}_0e^{-tz} = \mathcal{N}_t$. Clearly these maps are idempotents, $*$-preserving, normal, and contractive for the norm of $\mathcal{M}$. Thus, by the theorem of Tomiyama [13], they are normal conditional expectations, interpolating between E_0 and E_1. The fact that the path is strongly smooth is also clear. □

Remark 5.6. Let us recall Theorem 2.6 of [2]:

Let $\mathcal{A}$ be a unital C*-algebra and suppose that for $t \in [0,1]$ one has subalgebras $1 \in \mathcal{B}_t \subset \mathcal{A}$ and conditional expectations $E_t : \mathcal{A} \to \mathcal{B}_t$. Assume that for each $a \in \mathcal{A}$, the map $t \mapsto E_t(a) \in \mathcal{A}$ is continuously differentiable. Denote by $dE_t : \mathcal{A} \to \mathcal{A}$ the derivative of E_t: $dEt(a) = \frac{d}{dt}E_t(a)$. For each fixed t, the operator $dE_t : \mathcal{A} \to \mathcal{A}$ is bounded. Consider the differential equation, for $a \in \mathcal{A}$,

$$\begin{cases} \dot{\alpha}(t) = [dE_t, E_t](\alpha(t)) \\ \alpha(0) = a \end{cases} . \tag{2}$$

We call this equation the *parallel transport equation.* Denote by Γ_t the propagator of this equation, i.e., the map $\Gamma_t : \mathcal{A} \to \mathcal{A}$ given by the solutions: $\Gamma_t(a) = \alpha(t)$ with $\alpha(0) = a$. Then (Theorem 2.6 of [2])

- $\Gamma_t E_0 \Gamma_{-t} = E_t$, and
- $\Gamma_t|_{\mathcal{B}_0} : \mathcal{B}_0 \to \mathcal{B}_t$ is a C*-algebra isomorphism.

Corollary 5.7. *Let $\mathcal{N}_0, \mathcal{N}_1 \subset \mathcal{M}$ be subfactors with $\|e_0 - e_1\| < 1$. Then the exponentials $\Gamma_t = e^{tz}$ which induce the unique geodesic $\delta(t) = e^{tz}e_0e^{-tz}$ joining e_0 and e_1 in $\mathcal{P}_{\mathcal{M}_0}$ ($\mathcal{M}_0 =< \mathcal{M}, e_0, e_1 >''$), satisfy that*

$$\Gamma_t|_{\mathcal{N}_0} : \mathcal{N}_0 \to \mathcal{N}_t = e^{tz}\mathcal{N}_0e^{-tz}$$

are normal $*$-isomorphisms. In particular, $\mathcal{N}_0$ and $\mathcal{N}_1$ are isomorphic.

Proof. It is straightforward to verify that e^{tz} are the propagators Γ_t of equation (2) in this case: since $E_t = \delta(t)|_{\mathcal{M}} = e^{tz}E_0e^{-tz}|_{\mathcal{M}}$, we have that (the maps below are restricted to $\mathcal{M}$):

$$dE_t = ze^{tz}E_0e^{-tz} - e^{tz}E_0e^{-tz}z$$

and after straightforward computations

$$[dE_t, E_t] = ze^{tz}E_0e^{-tz} - 2e^{tz}E_0zE_0e^{-tz} + e^{tz}E_0e^{-tz}z.$$

Since z is e_0 co-diagonal, it maps $L^2(\mathcal{N}_0)$ into $L^2(\mathcal{N}_0)^\perp$, and therefore $E_0zE_0 = 0$. Thus,

$$[dE_t, E_t]e^{tz} = ze^{tz}E_0 + e^{tz}E_0z = e^{tz}(zE_0 + E_0z).$$

The fact that the element z is e_0 co-diagonal, also means that

$$z = e_0z(1 - e_0) + (1 - e_0)ze_0 = e_0z - 2e_0ze_0 + ze_0 = ze_0 + e_0z.$$

Restricted to $\mathcal{M}$ gives $zE_0 + E_0z = z$. Therefore, for $x \in \mathcal{M}$,

$$[dE_t, E_t]e^{tz}x = e^{tz}(zE_0 + E_0z)zx = e^{tz}zx = (e^{tz}x)^{\cdot}.$$

That is, $\alpha(t) = e^{tz}x$ is the solution of (2) with $\alpha(0) = x$, i.e. $\Gamma_t = e^{tz}|_{\mathcal{M}}$ is the propagator of this equation, and the proof follows using Theorem 2.6 of [2]. □

Remark 5.8. Suppose that e_0, e_1 as above satisfy the condition $e_0 \wedge e_1^\perp = 0 = e_0^\perp \wedge e_1$ (weaker that $\|e_0 - e_1\| < 1$), then there exists a unique geodesic $\delta(t) = e^{tz}e_0e^{-tz}$ joining e_0 and e_1. We would like to know if also in this case, the propagators Γ_t of the parallel transport equation induce as in the above case, a curve of automorphisms. Following the same argument as above, it amounts to knowing if the projections $\delta(t)$ induce conditional expectations onto the intermediate algebras $e^{tz}\mathcal{N}_0e^{-tz}$.

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