Automorphisms of toroidal Lie superalgebras

Dimitar Grantcharov and Arturo Pianzola*

Abstract

We give a description of the algebraic group $\mathbf{Aut}(\mathfrak{g})$ of automorphisms of a simple finite dimensional Lie superalgebra $\mathfrak{g}$ over an algebraically closed field k of characteristic 0, which is obtained by viewing $\mathfrak{g}$ as a module over a Levi subalgebra of its even part. As an application, we give a detailed description of the group of automorphism of the k-Lie superalgebra $\mathfrak{g} \otimes_k R$ for a large class of commutative rings R.

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1 Introduction

The group $\mathbf{Aut}(\mathfrak{g})$ of automorphisms of a finite dimensional simple Lie superalgebra over an algebraically closed field k of characteristic 0 has been described in [S] and [GP]. The automorphisms of the finite dimensional contragredient Lie superalgebras have been classified in [FSS1] and [vdL]. The results herein somehow refine and complement those of [S] and [GP], and provide a framework whereby the explicit nature of the abstract group of R-points of $\mathbf{Aut}(\mathfrak{g})$ can be determined for a large class of interesting rings R. In the Lie algebra case, the group $\mathbf{Aut}(\mathfrak{g})$ is a split extension of a finite constant group (the symmetries of the Dynkin diagram) by a simple group (the adjoint group, which is also the connected component of the identity of $\mathbf{Aut}(\mathfrak{g})$). By contrast, in the super case the analogous extension is not split, and the connected component of the identity of $\mathbf{Aut}(\mathfrak{g})$ need not even be reductive (let alone simple).

Our approach is to view $\mathfrak{g}$ as a module over a Levi subalgebra $\mathfrak{g}_0^{ss}$ of the even part of $\mathfrak{g}$. We will introduce three subgroups of $\mathbf{Aut}(\mathfrak{g})$; denoted by $\mathbf{Aut}(\mathfrak{g}; \mathfrak{g}_0^{ss})$, $\mathbf{Aut}(\mathfrak{g}, \Pi_0)$ and $\mathbf{H}$, which help clarify the nature of $\mathbf{Aut}(\mathfrak{g})$ and its outer part. These subgroups are interesting on their own right, and should prove useful in any future classification of multiloop algebras of $\mathfrak{g}$ via Galois cohomology (See [P4] and [P5] for

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the case of affine Lie algebras, and [GP] for the case of affine Lie superalgebras. See also [GiPi1] and [GiPi2] for toroidal Lie algebras).

As another application, we give an explicit description of the group of automorphisms of the (in general infinite dimensional) k-Lie superalgebra $\mathfrak{g} \otimes_k R$ for a large class of commutative ring extensions R/k. This result generalizes the simple Lie algebra situation studied in [P1] (which is considerably easier by comparison). Of particular interest is the "toroidal case", namely when $R = k[t_1^{\pm 1}, ..., t_n^{\pm 1}]$.

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2 Notation and conventions

Throughout k will be an algebraically closed field of characteristic zero. The category of associative commutative unital k-algebras will be denoted by *k-alg*. If V is a vector space over k, and R an object of *k-alg*, we set $V_R = V(R) := V \otimes_k R$. For a nilpotent Lie algebra $\mathfrak{a}$, a finite dimensional $\mathfrak{a}$-module M, and $\lambda \in \mathfrak{a}^*$, we denote by M^λ the subspace of M on which $a - \lambda(a)$ acts nilpotently for every $a \in \mathfrak{a}$. We have then $M = \bigoplus_{\lambda \in \mathfrak{a}^*} M^\lambda$.

In what follows, $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ will denote a simple finite dimensional Lie superalgebra over k (see [K] and [Sch] for details). A *Cartan subsuperalgebra* $\mathfrak{h} = \mathfrak{h}_{\bar{0}} \oplus \mathfrak{h}_{\bar{1}}$ of $\mathfrak{g}$, is by definition a selfnormalizing nilpotent subsuperalgebra. Then $\mathfrak{h}_{\bar{0}}$ is a Cartan (in particular nilpotent) subalgebra of $\mathfrak{g}_{\bar{0}}$, and $\mathfrak{h}_{\bar{1}}$ is the maximal subspace of $\mathfrak{g}_{\bar{1}}$ on which $\mathfrak{h}_{\bar{0}}$ acts nilpotently (see Proposition 1 in [PS] for the proof). We denote by $\Delta = \Delta_{(\mathfrak{g},\mathfrak{h})}$ the *roots of* $\mathfrak{g}$ *with respect to* $\mathfrak{h}$. Thus $\Delta = \{\alpha \in \mathfrak{h}_{\bar{0}}^*, \ \alpha \neq 0 \mid \mathfrak{g}^\alpha \neq 0\}$. For $\bar{\imath} \in \mathbb{Z}/2\mathbb{Z}$ we set $\Delta_{\bar{\imath}} = \{\alpha \in \mathfrak{h}_{\bar{0}} \mid \mathfrak{g}_{\bar{\imath}}^\alpha \neq 0\}$. Then $\Delta = \Delta_{\bar{0}} \cup \Delta_{\bar{1}}$. The *root lattice* $\mathbb{Z}\Delta$ of $(\mathfrak{g}, \mathfrak{h})$ will be denoted by $Q_{(\mathfrak{g},\mathfrak{h})}$.

A linear algebraic group $\mathbf{G}$ over k (in the sense of [B]) can be thought as a smooth affine algebraic group (in the sense of [DG]) via its functor of points $\mathrm{Hom}_k(k[\mathbf{G}], -)$. We will find both of these points of view useful, and will henceforth refer to them simply as "Algebraic Groups" (and trust that the reader will be able at all times to understand which of these two viewpoints is being taken).

Let $\mathrm{Aut}_k(\mathfrak{g})$ be the (abstract) group of automorphisms of $\mathfrak{g}$. We point out that by definition, all automorphisms of a Lie superalgebra preserve the given $\mathbb{Z}/2\mathbb{Z}$-grading. It is clear that $\mathrm{Aut}_k(\mathfrak{g})$ gives rise to a linear algebraic group over k, which we denote by $\mathbf{Aut}(\mathfrak{g})$, whose functor of points is given by $\mathbf{Aut}(\mathfrak{g})(R) = \mathrm{Aut}_R(\mathfrak{g}_R)$; the automorphisms of the R-Lie superalgebra $\mathfrak{g}_R = \mathfrak{g} \otimes_k R$.

We will make repeated use of the following affine k-groups: $\mathbf{Hom}(U,V) : R \to \mathrm{Hom}_{R\text{-}mod}(U \otimes R, V \otimes R)$ where U and V are finite dimensional k-spaces, $\mathbf{G}_a : R \to (R,+)$, and $\mathbf{G}_m : R \to R^\times$ (the units of R). In addition we will also use many of the classical groups $\mathbf{GL}$, $\mathbf{SL}$, etc; as well as the groups $\boldsymbol{\mu}_n$ defined by $\boldsymbol{\mu}_n(R) := \{r \in R \,|\, r^n = 1\}$.

Recall that there are three types of simple finite dimensional Lie superalgebras.[1] We use the notation of [Pen].

Type I: $\mathfrak{sl}(m|n)$, $\mathfrak{psl}(r|r)$, $\mathfrak{osp}(2|2n)$, and $\mathfrak{sp}(l)$ ($m > n$, $r \geq 2$, $l \geq 3$). Every Lie superalgebra $\mathfrak{g}$ of type I comes equipped with a $\mathbb{Z}$-grading $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with $\mathfrak{g}_{\bar{0}} = \mathfrak{g}_0$ and $\mathfrak{g}_{\bar{1}} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_1$. In addition, $\mathfrak{g}_{\bar{0}}$ is reductive and $\mathfrak{g}_{\pm 1}$ are irreducible $\mathfrak{g}_{\bar{0}}$-modules.

Type II: $\mathfrak{osp}(m|2n)$, $\mathfrak{psq}(l)$, $F(4)$, $G(3)$, and $D(\alpha)$ ($m \neq 2$, $l \geq 3$). For these Lie superalgebras $\mathfrak{g}_{\bar{0}}$ is reductive and $\mathfrak{g}_{\bar{1}}$ is an irreducible $\mathfrak{g}_{\bar{0}}$-module. We set $\mathfrak{g}_i := \mathfrak{g}_{\bar{i}}$, $i = 0, 1$.

Cartan type: $W(n)$, $S(m)$, $S'(2l)$, $H(r)$ ($n \geq 2$, $m \geq 3, l \geq 2, r \geq 5$). Every Lie superalgebra $\mathfrak{g}$ of Cartan type comes equipped with a choice of subspaces $\mathfrak{g}_i$ for each $i \in \mathbb{Z}$ (see the Appendix for details). The Lie algebra $\mathfrak{g}_{\bar{0}}$ is not reductive but admits a $\mathbb{Z}$-grading $\mathfrak{g}_{\bar{0}} = \mathfrak{g}_0 \oplus \mathfrak{g}_2 \oplus ... \oplus \mathfrak{g}_{2r}$ for which $\mathfrak{g}_0$ is reductive. The Lie superalgebra $\mathfrak{g}$ itself, for all $\mathfrak{g}$ except $\mathfrak{g} = S'(2l)$, admits a $\mathbb{Z}$-grading $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus ... \oplus \mathfrak{g}_s$ where $\mathfrak{g}_{-1}$ and $\mathfrak{g}_s$ are irreducible $\mathfrak{g}_0$-modules. Note that the notation $\mathfrak{g}_n$ for even n is not ambiguous (i.e., the same space appears as the degree n component of the $\mathbb{Z}$-gradings of $\mathfrak{g}_{\bar{0}}$ and of $\mathfrak{g}$ mentioned above).

We thus have a $\mathbb{Z}$-grading $\mathfrak{g}_0 \oplus \mathfrak{g}_2 \oplus ... \oplus \mathfrak{g}_{2r}$ of $\mathfrak{g}_{\bar{0}}$ in all cases ($r = 0$ if $\mathfrak{g}$ is of type I or II). Furthermore, $\mathfrak{g}$ admits a $\mathbb{Z}$-grading $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus ... \oplus \mathfrak{g}_s$ for $\mathfrak{g}$ of type I or Cartan type except for $\mathfrak{g} = S'(2l)$. For convenience, all of the above will be referred to as *standard $\mathbb{Z}$-gradings*.

3 Structure of $\mathfrak{g}$ with respect to $\mathfrak{g}_0^{ss}$

We set $\mathfrak{g}_0^{ss} := [\mathfrak{g}_0, \mathfrak{g}_0]$. This is the semisimple part of the reductive Lie algebra $\mathfrak{g}_0$, and a Levi subalgebra of $\mathfrak{g}_{\bar{0}}$. Henceforth we fix a Cartan subsuperalgebra $\mathfrak{h}$ of $\mathfrak{g}$ for which $\mathfrak{h}_{\mathfrak{g}_0^{ss}} := \mathfrak{h}_{\bar{0}} \cap \mathfrak{g}_0^{ss}$ is a Cartan subalgebra of $\mathfrak{g}_0^{ss}$. If $\mathfrak{g}_{\bar{0}}$ is reductive, then every Cartan subsuperlagebra of $\mathfrak{g}$ has this property. For the remaining cases, namely when $\mathfrak{g}$ is of Cartan type, the choice of $\mathfrak{h}$ is specified in the Appendix. The root system of

[1] These types are not mutually exclusive, and some overlap is indeed present in small rank.

$(\mathfrak{g}_0^{ss}, \mathfrak{h}_{\mathfrak{g}_0^{ss}})$ will be denoted by $\Delta_{\mathfrak{g}_0^{ss}}$ and the corresponding root lattice by $Q_{\mathfrak{g}_0^{ss}}$. We fix once and for all a base Π_0 of $\Delta_{\mathfrak{g}_0^{ss}}$. Let $p : \mathfrak{h}_{\bar{0}}^* \to \mathfrak{h}_{\mathfrak{g}_0^{ss}}^*$ be the canonical map (namely the transpose of the inclusion $\mathfrak{h}_{\mathfrak{g}_0^{ss}} \subset \mathfrak{h}_{\bar{0}}$).

Remark 3.1 Assume $V \subseteq \mathfrak{g}$ is an $\mathfrak{h}_{\bar{0}}$-module (under the adjoint action). Then V can be viewed as an $\mathfrak{h}_{\mathfrak{g}_0^{ss}}$-module as well. The generalized weight spaces V^λ are thus defined for both actions. Note that in the last case, i.e., for $\lambda \in \mathfrak{h}_{\mathfrak{g}_0^{ss}}$, we have $V^\lambda = \{x \in V \mid [h, x] = \lambda(x)v,\ \text{all } h \in \mathfrak{h}_{\mathfrak{g}_0^{ss}}\}$. Indeed, since $\mathfrak{h}_{\mathfrak{g}_0^{ss}}$ is a Cartan subalgebra of the semisimple Lie algebra $\mathfrak{g}_0^{ss}$ it acts semisimply on the $\mathfrak{g}_0^{ss}$-module $\mathfrak{g}$, hence also on V.

Lemma 3.2 *(i) The set of weights of the $(\mathfrak{g}_0^{ss}, \mathfrak{h}_{\mathfrak{g}_0^{ss}})$-module $\mathfrak{g}$ coincides with $p(\Delta)$. Moreover, $\mathfrak{g}^{p(\alpha)} = \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x,\ \text{for all } h \in \mathfrak{h}_{\mathfrak{g}_0^{ss}}\}$ whenever $\alpha \in \Delta$.*

(ii) The root lattice $Q_{\mathfrak{g}_0^{ss}}$ is a sublattice of $p(Q_\mathfrak{g})$.

Proof. (i) Follows from the previous remark.

(ii) Follows from (i), and the fact that $\mathfrak{g}_0^{ss}$ is a $\mathfrak{g}_0^{ss}$-submodule of $\mathfrak{g}$. □

Remark 3.3 We have three different lattices that define gradings of $\mathfrak{g}$:

- The root lattice $Q_\mathfrak{g}$ of $\mathfrak{g}$;
- The weight lattice $P_{(\mathfrak{g}_0^{ss}, \mathfrak{h}_{\mathfrak{g}_0^{ss}})}$ of $\mathfrak{g}_0^{ss}$;
- The sublattice $P_{(\mathfrak{g}_0^{ss}, \mathfrak{h}_{\mathfrak{g}_0^{ss}})}(\mathfrak{g}) := \mathbb{Z}p(\Delta)$ of $P_{(\mathfrak{g}_0^{ss}, \mathfrak{h}_{\mathfrak{g}_0^{ss}})}$ generated by the weights of the $(\mathfrak{g}_0^{ss}, \mathfrak{h}_{\mathfrak{g}_0^{ss}})$-module $\mathfrak{g}$.

We have $Q_{\mathfrak{g}_0^{ss}} \leq P_{(\mathfrak{g}_0^{ss}, \mathfrak{h}_{\mathfrak{g}_0^{ss}})}(\mathfrak{g}) \leq P_{(\mathfrak{g}_0^{ss}, \mathfrak{h}_{\mathfrak{g}_0^{ss}})}$. Note that $P_{(\mathfrak{g}_0^{ss}, \mathfrak{h}_{\mathfrak{g}_0^{ss}})}(\mathfrak{g})$ might be a proper sublattice of $P_{(\mathfrak{g}_0^{ss}, \mathfrak{h}_{\mathfrak{g}_0^{ss}})}$: for example, for $\mathfrak{g} = \mathfrak{psq}(n)$, $P_{(\mathfrak{g}_0^{ss}, \mathfrak{h}_{\mathfrak{g}_0^{ss}})}(\mathfrak{g}) = Q_{\mathfrak{g}_0^{ss}}$ and $\mathfrak{g}_0^{ss} \simeq \mathfrak{sl}_n$.

If V is a finite dimensional k-space, and $\sigma \in \mathrm{GL}_k(V)$, then $\sigma^* \in \mathrm{GL}_k(V^*)$ will denote the transpose inverse of σ. Thus, under the natural pairing $\langle - , - \rangle : V^* \times V \to k$ we have $\langle \sigma^*\alpha, x\rangle = \langle \alpha, \sigma^{-1}x\rangle$.

Lemma 3.4 *Assume $\sigma \in \mathrm{Aut}_k(\mathfrak{g}(R))$ stabilizes $\mathfrak{h}_{\mathfrak{g}_0^{ss}}$ and $\mathfrak{g}_0^{ss}(R)$. Then $(\sigma_{|\mathfrak{h}_{\mathfrak{g}_0^{ss}}})^*$ stabilizes $\Delta_{\mathfrak{g}_0^{ss}}$.*

Proof. A straightforward calculation shows that $\sigma((\mathfrak{g}_0^{ss})^\beta \otimes R) = (\mathfrak{g}_0^{ss})^{\sigma^*(\beta)} \otimes R$ for any root β in $\Delta_{\mathfrak{g}_0^{ss}}$. □

Lemma 3.5 *Let $i \in \mathbb{Z}$. Assume $\alpha_0 \in p(\Delta)$ is such that $\mathfrak{g}_i^{\alpha_0} \neq 0$. Then there exists a unique $\alpha \in \Delta$ with the property that $p(\alpha) = \alpha_0$ and $\mathfrak{g}^\alpha \cap \mathfrak{g}_i \neq 0$. Moreover, $\mathfrak{g}_i^{\alpha_0} = \mathfrak{g}^\alpha \cap \mathfrak{g}_i$.*

Proof. First observe that the spaces $\mathfrak{g}_i$ are naturally $\mathfrak{h}_{\mathfrak{g}_0^{ss}}$-modules, so the notation $\mathfrak{g}_i^\beta$ is meaningful for all $\beta \in \mathfrak{h}_{\mathfrak{g}_0^{ss}}$. The inclusion $\mathfrak{g}_i^{p(\alpha)} \supseteq \mathfrak{g}^\alpha \cap \mathfrak{g}_i$ is obvious for every $\alpha \in \Delta$ and $i \in \mathbb{Z}$. To establish the reverse inclusion, we consider first the case when $\mathfrak{h}_{\bar{0}} \subset \mathfrak{g}_0$. If $\mathfrak{h}_{\bar{0}} = \mathfrak{h}_{\mathfrak{g}_0^{ss}}$ (i.e. $\mathfrak{g}_0$ is semisimple), it is clear that $\mathfrak{g}_i^{p(\alpha)} = \mathfrak{g}^\alpha \cap \mathfrak{g}_i$ for every $\alpha \in \Delta$ and $i \in \mathbb{Z}$. Let us assume now that $\mathfrak{g}_0$ has a nontrivial center $\mathfrak{z}$. In this case we fix $z \in \mathfrak{z}$ such that $\mathfrak{z} = kz$ and $[z, y] = iy$ whenever $y \in \mathfrak{g}_i$ (see Proposition 1.2.12 in [K]). Considering $\mathfrak{g}_i$ as an $\mathfrak{h}_{\bar{0}}$-module we have that every nonzero x_0 in $\mathfrak{g}_i^{\alpha_0}$ decomposes as a sum $x_0 = \sum_{\alpha \in p^{-1}(\alpha_0) \cap \Delta} x_\alpha$ for some $x_\alpha \in \mathfrak{g}^\alpha \cap \mathfrak{g}_i$ (recall that the action of $\mathfrak{h}_{\mathfrak{g}_0^{ss}}$ on $\mathfrak{g}$ is semisimple). But since $[z, x_\alpha] = ix_\alpha$ we have $\alpha(z) = i$. Therefore $x_0 \in \mathfrak{g}^\alpha \cap \mathfrak{g}_i$, where $\alpha \in \Delta$ is determined uniquely by $\alpha_{|\mathfrak{h}_{\mathfrak{g}_0^{ss}}} = \alpha_0$ and $\alpha(z) = i$. In particular, $\mathfrak{g}_i^{\alpha_0} \subseteq \mathfrak{g}^\alpha \cap \mathfrak{g}_i$ and therefore $\mathfrak{g}_i^{\alpha_0} = \mathfrak{g}^\alpha \cap \mathfrak{g}_i$. If $\alpha' \in \Delta$ is another root with the properties described in the lemma, then for $0 \neq x' \in \mathfrak{g}^{\alpha'} \cap \mathfrak{g}_i$ we have $[z, x'] = ix'$ and thus $\alpha'(z) = i$. Hence $\alpha' = \alpha$.

It remains to consider the case when $\mathfrak{h}_{\bar{0}} \neq \mathfrak{h}_{\bar{0}} \cap \mathfrak{g}_0$, which is present for $\mathfrak{g} = H(n)$ only. Using the explicit description of $\mathfrak{h}_{\bar{0}}$ provided in the Appendix we see that if $h \in \mathfrak{h}_{\bar{0}}$, then $h = h_0 + h_n$ for some $h_0 \in \mathfrak{h}_{\mathfrak{g}_0^{ss}}$ and $h_n \in \mathfrak{g}^2 = \mathfrak{g}_2 \oplus ...\mathfrak{g}_{2r}$ (here $r = \left[\frac{n-3}{2}\right]$). Let x be a nonzero element in $\mathfrak{g}_i^{\alpha_0}$ and let α be any root in Δ such that $p(\alpha) = \alpha_0$. Then the identities $\alpha(h_n) = 0$ and $(\mathrm{ad}(h_0) - \alpha(h_0))(x) = 0$ imply that $(\mathrm{ad}(h) - \alpha(h))^N(x) = (\mathrm{ad}(h_n))^N(x)$ and, in particular, $(\mathrm{ad}(h) - \alpha(h))^N(x) \in \mathfrak{g}_{2N+i} \oplus \mathfrak{g}_{2N+i+2} \oplus ...$ from which it follows that $x \in \mathfrak{g}^\alpha \cap \mathfrak{g}_i$ and thus $\mathfrak{g}_i^{\alpha_0} = \mathfrak{g}^\alpha \cap \mathfrak{g}_i$. The uniqueness of α follows from the fact that if $x \in \mathfrak{g}_i^{\alpha_0}$ and $\alpha_0 = p(\alpha)$, then $x \in \mathfrak{g}^\alpha$.
□

If B is a basis of $Q_\mathfrak{g}$, then $p(B) \cap Q_{\mathfrak{g}_0^{ss}}$ is not necessarily a basis of $Q_{\mathfrak{g}_0^{ss}}$. However, this statement is true to some extent as the following result shows.

Lemma 3.6 *Suppose that either $\mathfrak{g} = \mathfrak{psl}(2|2)$, or $\mathfrak{g} \neq H(2k)$ with $\mathfrak{g}_0^{ss}$ simple.*

(i) There exists a basis B of $Q_\mathfrak{g}$ for which $p(B) \cap \Delta_{\mathfrak{g}_0^{ss}}$ is a base of $\Delta_{\mathfrak{g}_0^{ss}}$.

(ii) Every group homomorphism τ in $\mathrm{Hom}(Q_{\mathfrak{g}_0^{ss}}, R^\times)$ can be extended to a group homomorphism $\widehat{\tau}$ in $\mathrm{Hom}(Q_\mathfrak{g}, R^\times)$, i.e., $\widehat{\tau} = \tau \circ p$.

(iii) If $\alpha \in B$ is such that $p(\alpha) \in \Delta_{\mathfrak{g}_0^{ss}}$, then $\mathfrak{g}^\alpha \cap \mathfrak{g}_0 = \mathfrak{g}_0^{p(\alpha)} = (\mathfrak{g}_0^{ss})^{p(\alpha)}$.

Proof. (i) This statement follows by a case-by-case verification. We will use the notations and explicit description of the root systems provided in Appendix A of [Pen].

Case 1: $\mathfrak{g} = \mathfrak{psl}(2|2)$. We set $B = \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \delta_1, \delta_1 - \delta_2\}$.

Case 2: $\mathfrak{g} = \mathfrak{sl}(n|1)$. We set $B = \{\varepsilon_1 - \varepsilon_2, ..., \varepsilon_{n-1} - \varepsilon_n, \varepsilon_n - \delta\}$.

Case 3: $\mathfrak{g} = \mathfrak{osp}(m|2n)$, $m = 1, 2$. In the case $m = 1$ we have that $\Delta = \Delta_{\mathfrak{g}_0^{ss}}$. For $m = 2$ we set $B = \{\varepsilon_1 - \delta_1, \delta_1 - \delta_2, ..., \delta_{n-1} - \delta_n, 2\delta_n\}$.

Case 4: $\mathfrak{g} = \mathfrak{sp}(n)$. Set $B = \{-2\varepsilon_1, \varepsilon_1 - \varepsilon_2, ..., \varepsilon_{n-1} - \varepsilon_n\}$.

Case 5: $\mathfrak{g} = \mathfrak{psq}(n)$. In this case $\Delta = \Delta_{\mathfrak{g}_0^{ss}}$.

Case 6: $\mathfrak{g} = W(n), S(n)$*, or* $S'(n)$ *(*$n = 2l$ *in the last case).* Set $B = \{\varepsilon_1 - \varepsilon_2, ..., \varepsilon_{n-1} - \varepsilon_n, \varepsilon_n\}$.

Case 7: $\mathfrak{g} = H(2l + 1)$. We set $B = \{\varepsilon_1 - \varepsilon_2, ..., \varepsilon_{l-1} - \varepsilon_l, \varepsilon_l\}$.

The second assertion follows directly from (i). It remains to show (iii). This assertion follows from Lemma 3.5 by verifying case by case that $\mathfrak{g}^\alpha \cap \mathfrak{g}_0 \neq 0$ for $\alpha \in B$. This verification is trivial if $\mathfrak{h}_{\bar{0}} = \mathfrak{h}_{\mathfrak{g}_0^{ss}}$ (see the proof of Lemma 3.5). In the remaining three cases we proceed as follows.

If $\mathfrak{g} = \mathfrak{sl}(n|1)$ then $\mathfrak{g}^{\varepsilon_i - \varepsilon_j} \cap \mathfrak{g}_0 = \mathfrak{g}_0^{p(\varepsilon_i - \varepsilon_j)}$ is generated by the matrix $\begin{pmatrix} E_{ij} & 0 \\ 0 & 0 \end{pmatrix}$, where E_{ij} is the (i, j)-th elementary $n \times n$ matrix.

If $\mathfrak{g} = W(n)$ then $\mathfrak{g}^{\varepsilon_i - \varepsilon_j} \cap \mathfrak{g}_0 = \mathfrak{g}_0^{p(\varepsilon_i - \varepsilon_j)}$ is generated by the derivations $D_{\xi_i \xi_j}$.

If $\mathfrak{g} = H(2l + 1)$, then we use the description of the roots of $\mathfrak{g}$ provided in the Appendix. In particular, we verify that $\mathfrak{g}^{\varepsilon_i - \varepsilon_j} \cap \mathfrak{g}_0 = \mathfrak{g}_0^{p(\varepsilon_i - \varepsilon_j)}$ is generated by $D_{\eta_i \eta_j}$, and that $\mathfrak{g}^{\varepsilon_i} \cap \mathfrak{g}_0 = \mathfrak{g}_0^{p(\varepsilon_i)}$ is generated by $D_{\eta_i \eta_{2l+1}}$. □

Lemma 3.7 *Suppose that* $\mathfrak{g}_0^{ss}$ *is not simple and that* $\mathfrak{g} \neq \mathfrak{psl}(2|2)$*. Then* $\dim \mathfrak{g}^\alpha = \dim \mathfrak{g}^{-\alpha} = 1$ *for every* $\alpha \in \Delta$*. In addition, there exists a set* Π *of simple roots of* Δ *such that the root spaces* $\mathfrak{g}^{\pm\alpha}$*,* $\alpha \in \Pi$*, generate* $\mathfrak{g}$*.*

Proof. This follows from the classification of the contragredient finite-dimensional Lie superalgebras (see §2.5 and Theorem 3 in [K]). □

4 Some subgroups of $\mathbf{Aut}(\mathfrak{g})$

In this section we introduce and study an important list of subgroups of $\mathbf{Aut}(\mathfrak{g})$ for each simple Lie superalgebra $\mathfrak{g}$. These groups will be used in the next section for describing $\mathrm{Aut}_k(\mathfrak{g}(R))$.

4.1 The even superadjoint group $\mathbf{G}_{\bar{0}}^{\text{sad}}$

Lemma 4.1 *Let* $\mathbf{G}$ *be the Chevalley* k*-group of simply connected type corresponding to* $\mathfrak{g}_0^{ss}$*. The restriction* $\mathrm{ad}_{|\mathfrak{g}_0^{ss}} : \mathfrak{g}_0^{ss} \to \mathfrak{gl}(\mathfrak{g})$ *of the adjoint representation of* $\mathfrak{g}$*, lifts uniquely to a morphism* $\mathrm{Ad} : \mathbf{G} \to \mathbf{Aut}(\mathfrak{g})$ *of linear algebraic groups.*

Proof. Because $\mathbf{G}$ is simply connected, there exists a homomorphism $\mathrm{Ad} : \mathbf{G} \to \mathbf{GL}(\mathfrak{g})$ of linear algebraic k-groups whose differential is ad. The group $\mathbf{G}$ is generated by the root subgroups, namely by elements $\exp(z)$, where the elements z belong to the root spaces of $\mathfrak{g}_0^{ss}$ with respect to $\mathfrak{h}_{\mathfrak{g}_0^{ss}}$. We have $\mathrm{Ad}(\exp(z)) = \exp(\mathrm{ad}(z))$. Since the $\exp(\mathrm{ad}(z))$ are automorphisms of the Lie superalgebra $\mathfrak{g}$, the result follows. □

The image of Ad (in the schematic sense) is denoted by $\mathbf{G}_{\bar{0}}^{\text{sad}}$: For $R \in k$-alg and $\sigma \in \mathbf{Aut}(\mathfrak{g})(R)$ we have that $\sigma \in \mathbf{G}_{\bar{0}}^{\text{sad}}(R)$ if and only if there exists an *fppf* extension $\widetilde{R}/R$ and an element $x \in \mathbf{G}(\widetilde{R})$ such that $\mathrm{Ad}_{\widetilde{R}}(x) = \widetilde{\sigma}$, where $\widetilde{\sigma}$ is the image of σ under the map $\mathbf{Aut}(\mathfrak{g})(R) \to \mathbf{Aut}(\mathfrak{g})(\widetilde{R})$. The homomorphism Ad induces an isomorphism between the quotient group $\mathbf{G}/\ker(\mathrm{Ad})$ and $\mathbf{G}_{\bar{0}}^{\text{sad}}$. The structure of $\mathbf{G}_{\bar{0}}^{\text{sad}}$ is given in Table 1 at the end of the paper.

There is another morphism of adjoint type - the morphism $\mathrm{Ad}_{\mathfrak{g}}$ defined in [GP] which is used to describe the connected component of the identity $\mathbf{Aut}^0(\mathfrak{g})$ of $\mathbf{Aut}(\mathfrak{g})$. The morphism Ad defined in Lemma 4.1 can roughly be thought as the part of $\mathrm{Ad}_{\mathfrak{g}}$ that arises from the semisimple part of $\mathfrak{g}$. For a matter of completeness we recall the definitions of $\mathrm{Ad}_{\mathfrak{g}}$ for Lie subsuperalgebras $\mathfrak{g}$ of $\mathfrak{gl}(m|n)$ and $W(n)$. For $\mathfrak{g} \subseteq \mathfrak{gl}(m|n)$, $(X,Y) \in (\mathbf{GL}_m \times \mathbf{GL}_n)(k)$ and $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathfrak{g}$ we define

$$\mathrm{Ad}_{\mathfrak{g}}(X,Y) : \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \begin{pmatrix} XAX^{-1} & XBY^{-1} \\ YCX^{-1} & YDY^{-1} \end{pmatrix}.$$

If $\mathfrak{g} \subseteq W(n)$ and φ is an automorphism of the Grasmann (super)algebra $\Lambda(n) = \Lambda(\xi_1, ..., \xi_n)$ we set $\mathrm{Ad}_{\mathfrak{g}}(\varphi)(D) := \varphi D \varphi^{-1}$ for every $D \in \mathfrak{g}$.

4.2 The diagonal subgroup H

It is easier to describe $\mathbf{H}$ via its functor of points. We have $\mathbf{H}(R) := \mathrm{Hom}(Q_{\mathfrak{g}}, R^{\times})$. By definition, an element λ in $\mathbf{H}(R)$ fixes $\mathfrak{h} \otimes_k R$ pointwise, and acts on $\mathfrak{g}^{\alpha} \otimes R$ as a multiplication by $\lambda(\alpha)$. As an algebraic group, $\mathbf{H}$ is a split torus (hence connected). In the case of semisimple Lie algebra, $\mathbf{H}$ is a Cartan subgroup of adjoint type.

4.3 The unipotent group $\mathbf{N}$

The groups $\mathbf{N}$ appear only when $\mathfrak{g}$ is of Cartan type. If $\mathfrak{g}$ is not of Cartan type we set $\mathbf{N} = 1$. For the automorphisms of the Cartan type Lie superalgebras we refer the reader to [K]. Details can be found in Table 2 at the end of this paper and in §6.9 of [GP] as well. Every automorphism $\sigma \in \mathbf{Aut}(\mathfrak{g})$ of a Cartan type Lie superalgebra $\mathfrak{g} \subseteq W(n)$ is of the form $\mathrm{Ad}_{\mathfrak{g}}(\widehat{\sigma}) = \widehat{\sigma} D \widehat{\sigma}^{-1}$ for some automorphism $\widehat{\sigma}$ of $\Lambda(n)$. For any Cartan type Lie subsuperalgebra $\mathfrak{g}$ of $W(n)$ we set

$$\mathbf{N} := \{\mathrm{Ad}_{\mathfrak{g}}(\widehat{\varphi}) \in \mathbf{Aut}(\mathfrak{g}) \mid \widehat{\varphi}(x) - x \in \bigoplus_{j \geq 1} \Lambda(n)_{i+2j}, \text{ if } x \in \Lambda(n)_i, i \geq 0\}.$$

Alternatively, $\mathbf{N}$ is the unipotent group that corresponds to the nilpotent Lie algebra $\mathfrak{g}^2 := \mathfrak{g}_2 \oplus \ldots \oplus \mathfrak{g}_{2r}$, whenever $\mathfrak{g} \neq H(2l)$. If $\mathfrak{g} = H(2l)$, then $\mathbf{N}$ corresponds to $\widetilde{H}(2l)^2 := H(2l)^2 \oplus k D_{\xi_1 \ldots \xi_{2l}}$ where $D_f := \sum_{i=1}^{2l} \frac{\partial f}{\partial \xi_i} \frac{\partial}{\partial \xi_i}$ and $f \in \Lambda(2l)$. Note that $\mathfrak{g}^2$ is the radical of $\mathfrak{g}_{\bar{0}}$ if $\mathfrak{g} \neq W(n)$, and is the radical of $[\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{0}}]$ if $\mathfrak{g} = W(n)$.

Lemma 4.2 *Let $\mathfrak{g}$ be a Cartan type Lie superalgebra and let $\sigma \in \mathbf{Aut}(\mathfrak{g})(R)$. Then $\sigma = \sigma_0 \sigma_n$ where $\sigma_n \in \mathbf{N}(R)$ and σ_0 preserves the standard $\mathbb{Z}$-gradings of $\mathfrak{g}_{\bar{0}}(R)$ and $\mathfrak{g}(R)$ (the latter if $\mathfrak{g} \neq S'(2k)$). In particular, $\sigma_0(\mathfrak{g}_0(R)) = \mathfrak{g}_0(R)$.*

Proof. This follows from the explicit description of $\mathbf{Aut}(\mathfrak{g})$ given in §6.9 and Table 1 of [GP]. □

4.4 The group $\mathbf{SL}_2^{\mathrm{out}}$

Let $\mathfrak{g} = \mathfrak{psl}(2|2)$. Recall (see §6.4 in [GP] for details) that we have a closed embedding $\rho : \mathbf{SL}_2 \to \mathbf{Aut}(\mathfrak{g})$. The image of ρ will be denoted by $\mathbf{SL}_2^{\mathrm{out}}$. We recall for future use the explicit nature of ρ.

Let V_2 be the standard $\mathfrak{sl}_2$-module. We have that $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, where $\mathfrak{g}_0 = \mathfrak{g}_{\bar{0}} \simeq \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$, $\mathfrak{g}_{-1} \simeq V_2^* \otimes V_2$, and $\mathfrak{g}_1 \simeq V_2 \otimes V_2^*$. The $\mathfrak{g}_0$-modules $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$ are isomorphic and an explicit isomorphism $\phi : \mathfrak{g}_1 \to \mathfrak{g}_{-1}$ is determined by the linear transformation $\psi : \mathcal{M}_2 \to \mathcal{M}_2$, where $\psi(E) := -J E^t J^{-1}$ for $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ ($\mathcal{M}_2$ is the space of 2×2 matrices with entries in k). Then ρ is given by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \begin{pmatrix} A & aB + b\psi(C) \\ c\psi(B) + dC & D \end{pmatrix},$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(k)$.

4.5 The group $\mathbf{Aut}(\mathfrak{g};\mathfrak{g}_0^{ss})$

For a given Lie subalgebra $\mathfrak{s}$ of $\mathfrak{g}_0^{ss}$, we let $\mathbf{Aut}(\mathfrak{g};\mathfrak{s})$ be the subgroup of $\mathbf{Aut}(\mathfrak{g})$ consisting of those automorphisms that fix all elements of $\mathfrak{s}$. Clearly $\mathbf{Aut}(\mathfrak{g};\mathfrak{s})$ is a closed subgroup of $\mathbf{Aut}(\mathfrak{g})$, hence a linear algebraic group. Its functor of points is given by $\mathbf{Aut}(\mathfrak{g};\mathfrak{s})(R) = \{\varphi \in \mathrm{Aut}_R(\mathfrak{g}(R)) \mid \varphi_{|\mathfrak{s}(R)} = \mathrm{Id}\}$. Our main interest is the case when $\mathfrak{s} = \mathfrak{g}_0^{ss}$.

If $\mathfrak{g} = \bigoplus_{n\in\mathbb{Z}} \mathfrak{g}_n$ is $\mathbb{Z}$-graded, then each $c \in k^\times$ defines an automorphism $\delta_c \in \mathrm{Aut}_k(\mathfrak{g})$ via $\delta_{c|\mathfrak{g}_n} := c^n \,\mathrm{Id}$. This yields a closed embedding $\delta : \mathbf{G}_m \to \mathbf{Aut}(\mathfrak{g})$. Along similar lines, if $\mathfrak{g}$ is $\mathbb{Z}/m\mathbb{Z}$-graded, we have a closed embedding $\delta : \boldsymbol{\mu}_m \to \mathbf{Aut}(\mathfrak{g})$ (this presupposes a choice of primitive mth root of unity in k.) It is clear that for the standard $\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z}$ gradings of $\mathfrak{g}$ the resulting closed subgroups $\delta(\mathbf{G}_m)$ or $\delta(\boldsymbol{\mu}_2)$ of $\mathbf{Aut}(\mathfrak{g})$ lie inside $\mathbf{Aut}(\mathfrak{g};\mathfrak{g}_0^{ss})$.

If $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is of type I then the irreducible $\mathfrak{g}_0^{ss}$-modules $\mathfrak{g}_{-1}$ and $\mathfrak{g}_1$ are non-isomorphic for all $\mathfrak{g}$ except for $\mathfrak{g} = \mathfrak{psl}(2|2)$, $\mathfrak{g} = \mathfrak{sl}(2|1)$, and $\mathfrak{g} = \mathfrak{osp}(2|2n)$. In the case $\mathfrak{g} = \mathfrak{psl}(2|2)$ we defined an isomorphism $\phi : \mathfrak{g}_1 \to \mathfrak{g}_{-1}$ in §4.4. For $\mathfrak{g} = \mathfrak{sl}(2|1)$ and $\mathfrak{g} = \mathfrak{osp}(2|2n)$ there is an automorphism τ in $\mathbf{Aut}(\mathfrak{g};\mathfrak{g}_0^{ss})$ for which $\phi := \tau_{|\mathfrak{g}_1}$ is an isomorphism of $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$. We define τ as follows. For $\mathfrak{g} = \mathfrak{sl}(2|1)$, τ is the composition of $-\mathrm{Id}$, the supertransposition $X \mapsto X^{st}$, and the automorphism $\mathrm{Ad}_{\mathfrak{g}}(J, \mathrm{Id})$. In matrix terms we have

$$\tau : \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \begin{pmatrix} JA^tJ & JC^t \\ B^tJ & -D \end{pmatrix}, \mathrm{tr}(A) = D.$$

For $\mathfrak{g} = \mathfrak{osp}(2|2n)$ we set $\tau := \mathrm{Ad}_{\mathfrak{g}}\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, I\right)$. Here we use that

$$\mathfrak{osp}(m|2n) = \left\{\begin{pmatrix} A & B \\ J_nB^t & D \end{pmatrix} \mid A \in \mathfrak{so}_m, B \in \mathcal{M}_{m,2n}, D \in \mathfrak{sp}_{2n}\right\},\ J_n := \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix},$$

where $\mathcal{M}_{p,q}$ denote the space of $p \times q$ matrices with entries in k.

If $\mathfrak{g} = H(2l)$, we have an extra "additive" group of automorphisms of $\mathfrak{g}$ which we now describe. For $a \in k$ we define an automorphism $\widehat{\beta}_a$ of $\Lambda(2l)$, by $\widehat{\beta}_a(\xi_i) := \xi_i + a\frac{\partial}{\partial\xi_i}(\xi_1...\xi_{2l})$. This automorphism lifts to an automorphism $\beta_a = \mathrm{Ad}_{\mathfrak{g}}(\widehat{\beta}_a)$ of $\mathfrak{g}$. Since $\widehat{\beta}_{a_1}\widehat{\beta}_{a_2} = \widehat{\beta}_{a_1+a_2}$, we have a closed embedding $\beta : \mathbf{G}_a \to \mathbf{Aut}(\mathfrak{g})$. Furthermore, if $\widehat{\delta}_c$ is the automorphism of $\Lambda(2l)$ corresponding to δ_c, then one easily checks that $\widehat{\delta}_c(\xi_i) = c\xi_i$, and therefore

$$\widehat{\beta}_a(\widehat{\delta}_c(\xi_i)) = \widehat{\delta}_c(\widehat{\beta}_a(\xi_i)) = c\xi_i + ac^{2l-1}\frac{\partial}{\partial\xi_i}(\xi_1...\xi_{2l}).$$

The latter identity easily implies that δ and β commute, and that the resulting homomorphism $\delta \times \beta : \mathbf{G}_m \times \mathbf{G}_a \to \mathbf{Aut}(\mathfrak{g};\mathfrak{g}_0^{ss})$ is injective.

Proposition 4.3 *Let $\mathfrak{g}$ be a finite dimensional simple Lie superalgebra over k.*

(i) Assume that $\mathfrak{g}$ is of type I and $\mathfrak{g} \neq \mathfrak{psl}(2|2), \mathfrak{sl}(2|1), \mathfrak{osp}(2|2n)$. The map $\delta : \mathbf{G}_m \to \mathbf{Aut}(\mathfrak{g}; \mathfrak{g}_0^{ss})$ resulting from the standard $\mathbb{Z}$-grading of $\mathfrak{g}$ is an isomorphism.

(ii) If $\mathfrak{g} = \mathfrak{sl}(2|1)$ or $\mathfrak{g} = \mathfrak{osp}(2|2n)$, then $\mathbf{Aut}(\mathfrak{g}; \mathfrak{g}_0^{ss}) = \delta(\mathbf{G}_m) \rtimes \langle\tau\rangle$ is isomorphic to $\mathbf{G}_m \rtimes \boldsymbol{\mu}_2$.

(iii) Assume that $\mathfrak{g}$ is of type II. The map $\delta : \boldsymbol{\mu}_2 \to \mathbf{Aut}(\mathfrak{g}; \mathfrak{g}_0^{ss})$ resulting from the standard $\mathbb{Z}/2\mathbb{Z}$-grading of $\mathfrak{g}$ is an isomorphism.

(iv) If $\mathfrak{g} = \mathfrak{psl}(2|2)$, then $\mathbf{Aut}(\mathfrak{g}; \mathfrak{g}_0^{ss}) = \mathbf{SL}_2^{\mathrm{out}}$.

(v) If $\mathfrak{g}$ is of Cartan type and $\mathfrak{g} \neq S'(2l)$, $\mathfrak{g} \neq H(2l)$, the map $\delta : \mathbf{G}_m \to \mathbf{Aut}(\mathfrak{g}; \mathfrak{g}_0^{ss})$ resulting from the standard $\mathbb{Z}$-grading of $\mathfrak{g}$ is an isomorphism.

(vi) If $\mathfrak{g} = H(2l)$, then $\delta \times \beta : \mathbf{G}_m \times \mathbf{G}_a \to \mathbf{Aut}(\mathfrak{g}; \mathfrak{g}_0^{ss})$ is an isomorphism.

(vii) Let $\mathfrak{g} = S'(2l)$. Then $\mathbf{Aut}(\mathfrak{g}; \mathfrak{g}_0^{ss}) \simeq \boldsymbol{\mu}_{2l}$, where $\boldsymbol{\mu}_{2l}$ corresponds to the group $\mathrm{Ad}_{\mathfrak{g}}(cI) \subset \mathrm{Aut}_k(\mathfrak{g})$, $c^{2l} = 1$.

Proof.[2]

We will make a repeated use of the explicit structure of $\mathfrak{g}$ as a $\mathfrak{g}_0^{ss}$-module via the adjoint action. For details we refer the reader to [K] and [FSS2].

(i) In this case $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$ are nonisomorphic irreducible $\mathfrak{g}_0^{ss}$-modules. We must show that the closed embedding $\delta : \mathbf{G}_m \to \mathbf{Aut}(\mathfrak{g}; \mathfrak{g}_0^{ss})$ is surjective. Let $\sigma \in \mathbf{Aut}(\mathfrak{g}; \mathfrak{g}_0^{ss})$. Then $\sigma(\mathfrak{g}_1)$ is an irreducible $\mathfrak{g}_0^{ss}$-submodule of $\mathfrak{g}_{\bar{1}}$. Therefore $\sigma(\mathfrak{g}_1) \simeq \mathfrak{g}_1$ or $\sigma(\mathfrak{g}_1) \simeq \mathfrak{g}_{-1}$. The latter case is impossible because $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$ are non-isomorphic $\mathfrak{g}_0^{ss}$-modules. Thus $\sigma(\mathfrak{g}_1) = \mathfrak{g}_1$. Similarly $\sigma(\mathfrak{g}_{-1}) = \mathfrak{g}_{-1}$.

By Schur's Lemma we conclude that $\sigma_{|\mathfrak{g}_{\pm 1}} = c_{\pm 1}\,\mathrm{Id}$ for some scalars $c_{\pm 1} \in k^\times$. But since $[\mathfrak{g}_1, \mathfrak{g}_{-1}] = \mathfrak{g}_0$, we obtain that $c_1 c_{-1} = 1$ (because σ fixes $\mathfrak{g}_0^{ss}$), hence σ fixes $\mathfrak{g}_0$. Thus $\sigma = \delta_{c_1}$ which completes the proof of (i).

(ii) As mentioned in the beginning of this subsection, for these Lie superalgebras, there is an isomorphism $\phi : \mathfrak{g}_1 \to \mathfrak{g}_{-1}$. Let $\sigma \in \mathbf{Aut}(\mathfrak{g}; \mathfrak{g}_0^{ss})$. We will show that $\sigma \in \delta(\mathbf{G}_m) \rtimes \langle\tau\rangle$. We will apply Schur's Lemma to $\pi_{-1}\sigma i_{-1}$, $\pi_1\sigma i_1$, $\phi^{-1}\pi_{-1}\sigma i_1$, and $\phi^{-1}\pi_1\sigma i_{-1}$, where $i_{\pm 1} : \mathfrak{g}_{\pm 1} \to \mathfrak{g}_{-1} \oplus \mathfrak{g}_1$ and $\pi_{\pm 1} : \mathfrak{g}_{-1} \oplus \mathfrak{g}_1 \to \mathfrak{g}_{\pm 1}$ are the natural inclusions and projections. Let us first consider the case $\mathfrak{g} = \mathfrak{sl}(2|1)$. We have $a, b, c, d \in k$ for which

$$\sigma \begin{pmatrix} A & B \\ C & 0 \end{pmatrix} = \begin{pmatrix} A & aB + bJC^t \\ cB^tJ + dC & 0 \end{pmatrix}, \mathrm{tr}(A) = 0.$$

Applying

$$\sigma \left[\begin{pmatrix} 0 & B_1 \\ C_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & B_2 \\ C_2 & 0 \end{pmatrix} \right] = \left[\sigma \begin{pmatrix} 0 & B_1 \\ C_1 & 0 \end{pmatrix}, \sigma \begin{pmatrix} 0 & B_2 \\ C_2 & 0 \end{pmatrix} \right]$$

[2]The crucial ideas within this proof are due to Serganova [S].

for any B_i, C_i, for which $C_1B_2 + C_2B_1 = 0$ we obtain $ad + bc = 1, ac = bd = 0$. Therefore we have either $a = d = 0, bc = 1$, or $b = c = 0, ad = 1$. These easily imply that either $\sigma = \delta_b \circ \tau$ or $\sigma = \delta_a$.

For $\mathfrak{g} = \mathfrak{osp}(2|2n)$ we reason in a similar way. Namely, using Schur's Lemma we find a matrix Y in $\mathcal{M}_{2,2}$ for which

$$\sigma \begin{pmatrix} 0 & B \\ J_nB^t & D \end{pmatrix} = \begin{pmatrix} 0 & YB \\ J_nB^tY^t & D \end{pmatrix}.$$

Now applying

$$\sigma \left[\begin{pmatrix} 0 & B \\ J_nB^t & 0 \end{pmatrix}, \begin{pmatrix} 0 & B \\ J_nB^t & 0 \end{pmatrix} \right] = \left[\sigma \begin{pmatrix} 0 & B \\ J_nB^t & 0 \end{pmatrix}, \sigma \begin{pmatrix} 0 & B \\ J_nB^t & 0 \end{pmatrix} \right]$$

for B such that $BJ_nB^t = 0$ we find that $Y^tY = I$. Thus $Y \in \mathbf{O}_2$. From this we easily conclude that $\sigma = \mathrm{Ad}_{\mathfrak{g}}(Y, I)$. Therefore the map $Y \mapsto \mathrm{Ad}_{\mathfrak{g}}(Y, I)$ defines an isomorphism $\mathbf{O}_2 \simeq \mathbf{Aut}(\mathfrak{g}; \mathfrak{g}_0^{ss})$. We complete the proof of (ii) by using that $\mathbf{O}_2 \simeq \mathbf{G}_m \rtimes \boldsymbol{\mu}_2$. More explicitly, the subgroup $\delta(\mathbf{G}_m)$ in $\mathbf{Aut}(\mathfrak{g}; \mathfrak{g}_0^{ss})$ consists of the elements $\delta_c = \mathrm{Ad}_{\mathfrak{g}} \left(\begin{pmatrix} \frac{c+c^{-1}}{2} & \frac{c-c^{-1}}{2\sqrt{-1}} \\ -\frac{c-c^{-1}}{2\sqrt{-1}} & \frac{c+c^{-1}}{2} \end{pmatrix}, I \right)$.

(iii) The reasoning is similar to that of (i) above.

(iv) It is clear from the definition that $\mathbf{SL}_2^{\mathrm{out}}$ is a subgroup of $\mathbf{Aut}(\mathfrak{g}; \mathfrak{g}_0^{ss})$, so (iv) comes down to showing that $\rho : \mathbf{SL}_2 \to \mathbf{Aut}(\mathfrak{g}; \mathfrak{g}_0^{ss})$ is surjective. Let $\sigma \in \mathbf{Aut}(\mathfrak{g}; \mathfrak{g}_0^{ss})$. As in (iii), we apply Schur's Lemma to $\pi_{-1}\sigma i_{-1}$, $\pi_1 \sigma i_1$, $\phi^{-1}\pi_{-1}\sigma i_1$, and $\phi^{-1}\pi_1\sigma i_{-1}$, where $i_{\pm 1} : \mathfrak{g}_{\pm 1} \to \mathfrak{g}_{-1} \oplus \mathfrak{g}_1$ and $\pi_{\pm 1} : \mathfrak{g}_{-1} \oplus \mathfrak{g}_1 \to \mathfrak{g}_{\pm 1}$. As a result we obtain that $\sigma \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & aB + b\psi(C) \\ c\psi(B) + dC & D \end{pmatrix}$, for some $a, b, c, d \in k$ (ψ is defined in §4.4). Now using that $\psi([B, C]) = [\psi(B), \psi(C)]$ and

$$\sigma \left(\left[\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix} \right] \right) = \left[\sigma \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}, \sigma \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix} \right]$$

we find $ad - bc = 1$. Therefore $\sigma \in \mathbf{SL}_2^{\mathrm{out}}$.

(v) In these cases we use the fact that the standard $\mathbb{Z}$-grading $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus ... \oplus \mathfrak{g}_s$ of $\mathfrak{g}$ is such that $\mathfrak{g}_i$ is an irreducible $\mathfrak{g}_0^{ss}$-module for $i = -1$ or s. Furthermore, for all such Lie superalgebras $\mathfrak{g}$, except $\mathfrak{g} = W(n)$, the $\mathfrak{g}_0^{ss}$-modules $\mathfrak{g}_{\bar{0}}$ and $\mathfrak{g}_{\bar{1}}$ (via the adjoint action) are multiplicity free. Let $\sigma \in \mathbf{Aut}(\mathfrak{g}; \mathfrak{g}_0^{ss})$. We have $\sigma(\mathfrak{g}_i) = \mathfrak{g}_i$ for $i = -1, ..., s$ if $\mathfrak{g} \neq W(n)$. If $\mathfrak{g} = W(n)$, then

$$W(n)^{\mathfrak{g}_0^{ss}} := \{D \in W(n) \mid [D, x] = 0 \text{ for all } x \in \mathfrak{g}_0^{ss}\} = kE,$$

where $E := \sum_{i=1}^n \xi_i \frac{\partial}{\partial \xi_i}$. Since $\sigma(W(n)^{\mathfrak{g}_0^{ss}}) = W(n)^{\mathfrak{g}_0^{ss}}$ we have $\sigma(E) = cE$ for some $c \in k^\times$. But since $\mathfrak{g}_{-1} \oplus ... \oplus \mathfrak{g}_{n-1}$ coincides with the E-eigenspace decomposition of

$\mathfrak{g}$, with $\mathfrak{g}_i$ corresponding to eigenvalue i, we have that $\sigma(x)$ is in the (ic)-eigenspace of E whenever $x \in \mathfrak{g}_i$. Therefore $c = 1$ and in particular $\sigma \in \mathbf{Aut}(\mathfrak{g}; \mathfrak{g}_0)$. Now observing that $\mathfrak{g} = W(n)$ is multiplicity free $\mathfrak{g}_0$-module we conclude again that $\sigma(\mathfrak{g}_i) = \mathfrak{g}_i$ for $i = -1, ..., n-1$.

By Schur's Lemma $\sigma_{|\mathfrak{g}_j} = c_j \operatorname{Id}$ for $j = -1, s$, and some $c_j \in k^\times$. Now using $[\mathfrak{g}_{i+1}, \mathfrak{g}_{-1}] = \mathfrak{g}_i$ for $i = s-1, s-2, ..., -1$ we find $\sigma_{|\mathfrak{g}_i} = c_i \operatorname{Id}$ and $c_i = c_s(c_{-1})^{s-i}$. Thus $c_i = c_1^i$ and $\sigma = \delta_{c_1}$. Hence δ is surjective.

(vi) In this case $\mathfrak{g} = \mathfrak{g}_{-1} \oplus ... \oplus \mathfrak{g}_{2l-3}$ and $\mathfrak{g}_i$ are irreducible $\mathfrak{g}_0^{ss}$-modules with $\mathfrak{g}_i \simeq \mathfrak{g}_j$ iff $i + j = 2l - 4$. Let $\sigma \in \mathbf{Aut}(\mathfrak{g}; \mathfrak{g}_0^{ss})$. Set $\iota := (\beta_1 - \operatorname{Id})_{|\mathfrak{g}_{-1}}$. Then $\iota : \mathfrak{g}_{-1} \to \mathfrak{g}_{2l-3}$ is an isomorphism of $\mathfrak{g}_0^{ss}$-modules. Then by Schur's Lemma, the restriction of σ on $\mathfrak{g}_{2l-3}$ is of the form $a \operatorname{Id} + b\iota^{-1}$ for $a, b \in k$. Since $\sigma(\mathfrak{g}_{l-2}) = \mathfrak{g}_{l-2}$, $[\mathfrak{g}_{l-2}, \mathfrak{g}_{-1}] \neq 0$, and $[\mathfrak{g}_{l-2}, \mathfrak{g}_{2l-3}] = 0$ we conclude that $b = 0$ and thus $a \in k^\times$. Again by Schur's Lemma, the restriction of σ on $\mathfrak{g}_{-1}$ is of the form $c \operatorname{Id} + d\iota$ for some $c, d \in k$. Now using that $[\mathfrak{g}_i, \mathfrak{g}_{-1}] = \mathfrak{g}_{i-1}$ for $i = 2l-3, 2l-4, ..., 0$ we obtain $\sigma_{|\mathfrak{g}_i} = ac^{2l-i-3} \operatorname{Id}$. In particular, $ac^{2l-3} = 1$, because $\mathfrak{g}_0 = \mathfrak{g}_0^{ss}$. Hence $\sigma \circ \delta_c$ is identity on all $\mathfrak{g}_i$, $i \geq 0$ and $\sigma(\delta_c(x)) = x + c^{-1} d\iota(x)$ for $x \in \mathfrak{g}_{-1}$. Thus $\sigma = \beta_{c^{-1}d} \circ \delta_{c^{-1}}$ which implies that $\delta \times \beta$ is surjectiive.

(vii) Let $\sigma \in \mathbf{Aut}(\mathfrak{g}; \mathfrak{g}_0^{ss})$. Recall that $S'(2l) = (1-\theta)S(2l)$, where $\theta := \xi_1 ... \xi_{2l}$. We have $\mathfrak{g} = S'(2l) = \bigoplus_{i=-1}^{2l-2} \mathfrak{g}_i$, where $\mathfrak{g}_i = S(2l)_i = \mathfrak{g} \cap W(2l)_i$, $i \geq 0$, and $\mathfrak{g}_{-1} := (1-\theta)S(n)_{-1}$. Since $S'(2l)$ is multiplicity free $\mathfrak{g}_0^{ss}$-module and $\mathfrak{g}_i$ are irreducible we conclude that $\sigma(\mathfrak{g}_i) = \mathfrak{g}_i$ for all $i \geq -1$. By Schur's Lemma we find constants $c_i \in k$, $i = -1, ..., 2l-2$ for which $\sigma_{|\mathfrak{g}_i} = c_i \operatorname{Id}$ (thus $c_0 = 1$). Since $\mathfrak{g}_{-1} \subset W(2l)_{-1} + W(2l)_{2l-1}$, we have $[\mathfrak{g}_i, \mathfrak{g}_{-1}] \subset \mathfrak{g}_{i-1}$ for $i > 0$ and thus $c_i = c_1^i$. However $[\mathfrak{g}_{-1}, \mathfrak{g}_{-1}] \subset \mathfrak{g}_{n-2}$ and hence $c_1^{-2} = c_1^{2l-2}$. Therefore $c_1^{2l} = 1$ and $\sigma = \operatorname{Ad}_{\mathfrak{g}}(\widehat{\sigma})$, where $\widehat{\sigma}(\xi_i) = c_1 \xi_i$. This completes the proof. $\square$

4.6 The group $\mathbf{Aut}(\mathfrak{g}, \Pi_0)$

This group measures the automorphisms of $\mathfrak{g}$ that induce symmetries of the Dynkin diagram of $(\mathfrak{g}_0^{ss}, \mathfrak{h}_{\mathfrak{g}_0^{ss}})$ with respect to the chosen base Π_0 of $\Delta_{\mathfrak{g}_0^{ss}}$. By definition $\mathbf{Aut}(\mathfrak{g}, \Pi_0) := \{\varphi \in \mathbf{Aut}(\mathfrak{g}) \mid \varphi(\mathfrak{g}_0^{ss}) = \mathfrak{g}_0^{ss}, \varphi(\mathfrak{h}_{\mathfrak{g}_0^{ss}}) = \mathfrak{h}_{\mathfrak{g}_0^{ss}}, \text{ and } \varphi^*(\Pi_0) = \Pi_0\}$.

Proposition 4.4 *Let $\mathbf{Aut}^0(\mathfrak{g})$ be the connected component of the identity of $\mathbf{Aut}(\mathfrak{g})$, and let $\mathbf{F}_k = \mathbf{Aut}(\mathfrak{g})/\mathbf{Aut}^0(\mathfrak{g})$ be the corresponding (finite constant) group of connected components of $\mathbf{Aut}(\mathfrak{g})$ (see Theorem 4.1 in [GP]).*

(i) If $\sigma \in \mathbf{Aut}^0(\mathfrak{g})$ is such that $\sigma(\mathfrak{g}_0^{ss}) = \mathfrak{g}_0^{ss}$, then $\sigma_{|\mathfrak{g}_0^{ss}}$ is in the connected component of the identity of $\mathbf{Aut}(\mathfrak{g}_0^{ss})$ (namely the Chevalley group of adjoint type of $\mathfrak{g}_0^{ss}$).

(ii) $\mathbf{Aut}(\mathfrak{g}, \Pi_0) \cap \mathbf{Aut}^0(\mathfrak{g}) = \mathbf{H}\big(\mathbf{Aut}(\mathfrak{g}; \mathfrak{g}_0^{ss}) \cap \mathbf{Aut}^0(\mathfrak{g})\big)$.

(iii) The restriction of the canonical map $j : \mathbf{Aut}(\mathfrak{g}) \to \mathbf{F}_k$ to $\mathbf{Aut}(\mathfrak{g}, \Pi_0)$ is surjective.

Proof. (i) We follow the descriptions of $\mathbf{Aut}^0(\mathfrak{g})$ and $\mathbf{G}_{\bar{0}}^{sad}$ provided in Table 1 of [GP] and the first table at the end of this paper, respectively. Let $\sigma \in \mathbf{Aut}^0(\mathfrak{g})$ be such that $\sigma(\mathfrak{g}_{\bar{0}}^{ss}) = \mathfrak{g}_{\bar{0}}^{ss}$. We first consider the case when $\mathfrak{g}$ is of type I or II. Then σ is a product of three automorphisms: an element σ_0 of $\mathbf{G}_{\bar{0}}^{sad}$; an element $j_k(c)$ of $\mathbf{G}_m$, $c \in k^\times$ (in fact, $j_k(c) = \delta_c$); and (in the case $\mathfrak{g} = \mathfrak{psl}(2|2)$) an element θ of $\mathbf{SL}_2^{\mathrm{out}}$. But $j_k(c)_{|\mathfrak{g}_{\bar{0}}^{ss}} = \theta_{|\mathfrak{g}_{\bar{0}}^{ss}} = \mathrm{Id}$. On the other hand, the restriction of $\mathbf{G}_{\bar{0}}^{sad}$ to $\mathfrak{g}_{\bar{0}}^{ss}$ is precisely the connected component of the identity of $\mathbf{Aut}(\mathfrak{g}_{\bar{0}}^{ss})$, from which the assertion follows. Let now $\mathfrak{g}$ be of Cartan type. Then $\sigma = \sigma_c \sigma_0 \sigma_n$, where (for $\mathfrak{g} = H(2l)$) $\sigma_c \in \mathbf{G}_m$ corresponds to the multiplication by $c \in k^\times$ in $\Lambda(2l)$ (in fact, $\sigma_c = \delta_c$), $\sigma_0 \in \mathbf{G}_{\bar{0}}^{sad}$, and $\sigma_n \in \mathbf{N}$. Then since σ, σ_c, and σ_0 leave invariant $\mathfrak{g}_{\bar{0}}^{ss}$, so it does σ_n. Thus $\sigma_{n|\mathfrak{g}_{\bar{0}}^{ss}} = \mathrm{Id}$, and, as before, we conclude that $\sigma_{|\mathfrak{g}_{\bar{0}}^{ss}} = \sigma_{0|\mathfrak{g}_{\bar{0}}^{ss}}$ is in the connected component of the identity of $\mathbf{Aut}(\mathfrak{g}_{\bar{0}}^{ss})$.

(ii) We first check that $\mathbf{H}$ is a subgroup of $\mathbf{Aut}(\mathfrak{g}, \Pi_0)$. Let $\sigma \in \mathbf{H}$. Since $\mathfrak{h}_{\mathfrak{g}_{\bar{0}}^{ss}} \subset \mathfrak{g}^0$ and $\sigma_{|\mathfrak{g}^0} = \mathrm{Id}$ we have that $\sigma_{|\mathfrak{h}_{\mathfrak{g}_{\bar{0}}^{ss}}} = \mathrm{Id}$. In particular, $\sigma^*(\Pi_0) = \Pi_0$. Since $\mathfrak{g}_{\bar{0}}^{ss} = \mathfrak{h}_{\mathfrak{g}_{\bar{0}}^{ss}} \oplus \left(\bigoplus_{\alpha_0 \in \Delta_{\mathfrak{g}_{\bar{0}}^{ss}}} (\mathfrak{g}_{\bar{0}}^{ss})^{\alpha_0}\right)$ and $(\mathfrak{g}_{\bar{0}}^{ss})^{\alpha_0} = \mathfrak{g}^\alpha \cap \mathfrak{g}_{\bar{0}}$ (this last by Lemma 3.5) we see that σ acts as a multiplication by a constant $\lambda(\alpha_0) \in k^\times$ on each $(\mathfrak{g}_{\bar{0}}^{ss})^{\alpha_0}$. In particular, $\sigma(\mathfrak{g}_{\bar{0}}^{ss}) \subseteq \mathfrak{g}_{\bar{0}}^{ss}$. This shows that $\mathbf{H} \subset \mathbf{Aut}(\mathfrak{g}, \Pi_0)$.

Clearly $\mathbf{H}$ normalizes $\mathbf{Aut}(\mathfrak{g}; \mathfrak{g}_{\bar{0}}^{ss}) \cap \mathbf{Aut}^0(\mathfrak{g})$. Thus $\mathbf{H}(\mathbf{Aut}(\mathfrak{g}; \mathfrak{g}_{\bar{0}}^{ss}) \cap \mathbf{Aut}^0(\mathfrak{g}))$ is a closed subgroup of $\mathbf{Aut}(\mathfrak{g})$. The inclusion $\mathbf{H}(\mathbf{Aut}(\mathfrak{g}; \mathfrak{g}_{\bar{0}}^{ss}) \cap \mathbf{Aut}^0(\mathfrak{g})) \subset \mathbf{Aut}(\mathfrak{g}, \Pi_0) \cap \mathbf{Aut}^0(\mathfrak{g})$ is clear because $\mathbf{H}$ is connected. Let $\sigma \in \mathbf{Aut}(\mathfrak{g}, \Pi_0) \cap \mathbf{Aut}^0(\mathfrak{g})$. By (i) we see that $\sigma_{|\mathfrak{g}_{\bar{0}}^{ss}}$ is an inner automorphisms of $\mathfrak{g}_{\bar{0}}^{ss}$ that stabilizes $\mathfrak{h}_{\mathfrak{g}_{\bar{0}}^{ss}}$ and Π_0 (this last via the * action). Thus $\sigma_{|\mathfrak{g}_{\bar{0}}^{ss}} = \mathrm{Ad}(x)_{|\mathfrak{g}_{\bar{0}}^{ss}}$ for some $x \in \mathbf{T}$, where $\mathbf{T}$ is the maximal torus of $\mathbf{G}$ corresponding to $\mathfrak{h}_{\mathfrak{g}_{\bar{0}}^{ss}}$. Note that $\mathrm{Ad}(\mathbf{T}) \subset \mathbf{H}$. Because $\mathbf{H}$ is connected, we have $\mathbf{H} \subset \mathbf{Aut}^0(\mathfrak{g})$. Thus $\mathrm{Ad}(x)^{-1}\sigma \in \mathbf{Aut}(\mathfrak{g}; \mathfrak{g}_{\bar{0}}^{ss}) \cap \mathbf{Aut}^0(\mathfrak{g})$ as desired.

(iii) This statement follows by a case-by-case verification. For example, for $\mathfrak{g} = \mathfrak{sl}(m|n)$, we see that the negative supertransposition $S : \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \begin{pmatrix} -A^t & C^t \\ -B^t & -D^t \end{pmatrix}$ is in $\mathbf{Aut}(\mathfrak{g}, \Pi_0)$, and that $j(S)$ generates $\mathbf{F}_k$. □

Corollary 4.5 $\mathbf{Aut}(\mathfrak{g}, \Pi_0)/\mathbf{H}(\mathbf{Aut}(\mathfrak{g}; \mathfrak{g}_{\bar{0}}^{ss}) \cap \mathbf{Aut}^0(\mathfrak{g})) \simeq \mathbf{F}_k$.

Remark 4.6 We have $\mathbf{Aut}(\mathfrak{g}; \mathfrak{g}_{\bar{0}}^{ss}) \subset \mathbf{Aut}^0(\mathfrak{g})$ for all $\mathfrak{g} \neq \mathfrak{sl}(2|1), \mathfrak{osp}(2|2n), \mathfrak{psq}(n)$. For $\mathfrak{g} = \mathfrak{sl}(2|1)$ and $\mathfrak{g} = \mathfrak{osp}(2|2n)$, $\tau \in \mathbf{Aut}(\mathfrak{g}; \mathfrak{g}_{\bar{0}}^{ss})$ and $\tau \notin \mathbf{Aut}^0(\mathfrak{g})$. If $\mathfrak{g} = \mathfrak{psq}(n)$, then $\delta_{-1} \in \mathbf{Aut}(\mathfrak{g}; \mathfrak{g}_{\bar{0}}^{ss})$ and $\delta_{-1} \notin \mathbf{Aut}^0(\mathfrak{g})$.

This follows from a case-by-case verification using the Table 1 at the end of the paper. If $\mathbf{Aut}(\mathfrak{g}; \mathfrak{g}_{\bar{0}}^{ss})$ is connected there is nothing to proof. Otherwise $\mathfrak{g} = \mathfrak{sl}(2|1)$, $\mathfrak{g} = \mathfrak{osp}(2|2n)$, $\mathfrak{g}$ is of type II, or $\mathfrak{g} = S'(2l)$. In the cases for which $\mathbf{Aut}(\mathfrak{g}; \mathfrak{g}_{\bar{0}}^{ss}) \simeq \boldsymbol{\mu}_2$ or

$\mathbf{Aut}(\mathfrak{g};\mathfrak{g}_0^{ss}) \simeq \boldsymbol{\mu}_{2l}$, $\mathbf{Aut}(\mathfrak{g};\mathfrak{g}_0^{ss})$ is generated by δ_{-1} or $\mathrm{Ad}_{\mathfrak{g}}(cI)$, $c^{2l}=1$. If $\mathfrak{g} \neq \mathfrak{psq}(n)$ then δ_{-1} is of the form $\mathrm{Ad}_{\mathfrak{g}}(x)$. For example, if $\mathfrak{g} = D(\alpha)$, then $x = (I, I, -I)$. Furthermore, it is easy to check that for $\mathfrak{g} = \mathfrak{psq}(n)$, $\delta_{-1} = \mathrm{Ad}_{\mathfrak{g}}(x)$ has no solutions for $x \in \mathbf{SL}_n$.

5 Abstract automorphisms of $\mathfrak{g}(R)$ and its universal central extension

Recall that the *supercentroid* $\mathrm{Ctd}(\mathfrak{g})$ of $\mathfrak{g}$ is defined by

$$\mathrm{Ctd}_k(\mathfrak{g}) := \{\chi \in \mathrm{End}_k(\mathfrak{g}) \mid \chi([x,y]) = [\chi(x), y] \text{ for all } x, y \in \mathfrak{g}\}.$$

We call $\mathfrak{g}$ *central* if $\mathrm{Ctd}_k(\mathfrak{g}) = \lambda_{\mathfrak{g}}(k)$, where $\lambda_{\mathfrak{g}} : k \to \mathrm{End}_k(\mathfrak{g})$ is defined by $\lambda_{\mathfrak{g}}(a)(x) := ax$ for $a \in k$ and $x \in \mathfrak{g}$.

Lemma 5.1 $\mathrm{Aut}_k(\mathfrak{g}(R)) = \mathrm{Aut}_R(\mathfrak{g}(R)) \rtimes \mathrm{Aut}_k(R)$.

Proof. Proposition 7.1 in [GP] implies that $\mathfrak{g}$ is central. Since $\mathfrak{g}$ is also perfect (i.e. $[\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$), the Lemma follows from a superversion of Lemma 4.4 in [ABP] (see also Corollary 2.28 in [BN]). More precisely, for a k-algebra automorphism $\varphi : \mathfrak{g}(R) \to \mathfrak{g}(R)$ there exists a unique $\widetilde{\varphi} \in \mathrm{Aut}_k(R)$ for which $\varphi(rx) = \widetilde{\varphi}(r)\varphi(x)$ for all $x \in \mathfrak{g}$ and $r \in R$. It is also evident that every element of $\mathrm{Aut}_k(R)$ lifts naturally to and element of $\mathrm{Aut}_k(\mathfrak{g}(R))$. This leads to the split exact sequence

$$1 \to \mathrm{Aut}_R(\mathfrak{g}(R)) \to \mathrm{Aut}_k(\mathfrak{g}(R)) \to \mathrm{Aut}_k(R) \to 1.$$

$\square$

Let $\mathbf{G}$ and Ad be as in Lemma 4.1. The (abstract) group $\mathrm{Ad}_R\ \mathbf{G}(R) \subset \mathbf{Aut}(\mathfrak{g})(R) = \mathrm{Aut}_R(\mathfrak{g}(R))$ is in general much smaller than the group of R-points of the the quotient group $\mathbf{G}/\ker(\mathrm{Ad}) \simeq \mathbf{G}_{\bar{0}}^{\mathrm{sad}}$. While the group $\mathrm{Ad}\ \mathbf{G}(R)$ is quite explicit, $\mathbf{G}_{\bar{0}}^{\mathrm{sad}}(R)$ is not. The following Theorem shows that, for a large class of objects in k-*alg*, an explicit and concrete description of $\mathrm{Aut}_k(\mathfrak{g}(R))$ can still be achieved.

Theorem 5.2 *Assume the object R in k-alg is such that Spec(R) is connected, admits a rational point, and has trivial Picard group (for example R factorial). Then* $\mathrm{Aut}_k(\mathfrak{g}(R))$ *is generated by the subgroups* $\mathrm{Aut}_k(R)$, $\mathrm{Ad}\,\mathbf{G}(R)$, $\mathbf{Aut}(\mathfrak{g},\Pi_0)(R)$, $\mathbf{N}(R)$, *and* $\mathbf{Aut}(\mathfrak{g};\mathfrak{g}_0^{ss})(R)$.

Proof. Let $\sigma \in \mathrm{Aut}_k(\mathfrak{g}(R))$. By Lemma 5.1, we may assume that $\sigma \in \mathrm{Aut}_R(\mathfrak{g}(R))$. Let $\mathfrak{g}_{\bar{0}} = \mathfrak{g}_0 \oplus \mathfrak{g}_2 \oplus ... \oplus \mathfrak{g}_{2r}$ be the fixed $\mathbb{Z}$-grading of $\mathfrak{g}_{\bar{0}}$. Using Lemma 4.2, we can write σ as a product $\sigma_0\sigma_n$, where $\sigma_0(\mathfrak{g}_0(R)) = \mathfrak{g}_0(R)$ and $\sigma_n \in \mathbf{N}(R)$. Replacing σ by $\sigma\sigma_n^{-1}$, we may then assume that $\sigma(\mathfrak{g}_0(R)) \subseteq \mathfrak{g}_0(R)$. Since $\mathfrak{g}_0^{ss} = [\mathfrak{g}_0, \mathfrak{g}_0]$, we conclude that $\sigma(\mathfrak{g}_0^{ss}(R)) \subseteq \mathfrak{g}_0^{ss}(R)$. Given our assumptions on R, and by taking Lemma 4.1 into consideration, we can appeal to the conjugacy theorem of regular maximal abelian k-diagonalizable subalgebras of $\mathfrak{g}_0^{ss}(R)$ ([P2] Theorem 1(ii)(a)) for the existence of an element of $\mathrm{Ad}\,\mathbf{G}(R)$ taking $\sigma(\mathfrak{h}_{\mathfrak{g}_0^{ss}})$ to $\mathfrak{h}_{\mathfrak{g}_0^{ss}}$. We may thus assume that σ stabilizes $\mathfrak{h}_{\mathfrak{g}_0^{ss}}$. Lemma 3.4 implies that the contragradient automorphism σ^* of $\mathfrak{h}^*_{\mathfrak{g}_0^{ss}}$ stabilizes $\Delta_{\mathfrak{g}_0^{ss}}$. By means of the Weyl group of $(\mathfrak{g}_0^{ss}, \mathfrak{h}_{\mathfrak{g}_0^{ss}})$, whose elements we can recreate as restrictions to $\mathfrak{h}_{\mathfrak{g}_0^{ss}}$ of elements of $\mathrm{Ad}\,\mathbf{G}(k)$, we may further assume that $\sigma^*(\Pi_0) = \Pi_0$. Let $\mathfrak{m}$ be a maximal ideal of R for which $R/\mathfrak{m} \simeq k$. Then $\sigma \otimes 1 \in \mathrm{Aut}_R((\mathfrak{g} \otimes R) \otimes_R R/\mathfrak{m}) \simeq \mathrm{Aut}_k(\mathfrak{g})$. Clearly $\sigma \otimes 1$, when viewed as an element of $\mathbf{Aut}(\mathfrak{g})$, is in fact an element of $\mathbf{Aut}(\mathfrak{g}, \Pi_0)$. Let $\widetilde{\sigma}$ denote the R-linear extension of this element to $\mathrm{Aut}_R(\mathfrak{g}(R))$. Then, after replacing σ by $\widetilde{\sigma}^{-1}\sigma$, we may assume that σ fixes $\mathfrak{h}_{\mathfrak{g}_0^{ss}}$ pointwise, hence that σ stabilizes $\mathfrak{g}^\alpha \otimes R$ for every α in $\Delta_{\mathfrak{g}_0^{ss}}$ (see Lemma 3.4).

We now proceed by a case-by-case reasoning using Lemmas 3.6 and 3.7.

Case 1: $\mathfrak{g} = \mathfrak{psl}(2|2)$*; or* $\mathfrak{g}_0^{ss}$ *is simple,* $\mathfrak{g} \neq H(2l)$. In this case we use Lemma 3.6 (i) and fix a basis B of $Q_\mathfrak{g}$ such that $p(B) \cap \Delta_{\mathfrak{g}_0^{ss}}$ is a base of $\Delta_{\mathfrak{g}_0^{ss}}$. By multiplying σ with an element of $\mathbf{H}(R) \subset \mathbf{Aut}(\mathfrak{g}, \Pi_0)(R)$ (see Lemma 4.4 (ii)) we may assume that σ fixes a set of generators e_α of $(\mathfrak{g}_0^{ss})^{p(\alpha)} = \mathfrak{g}_0 \cap \mathfrak{g}^\alpha$ (see Lemma 3.6 (iii)), for any $\alpha \in B$. Since σ is R-linear, it fixes $\mathfrak{g}_0^{ss}(R)$, and thus is in $\mathbf{Aut}(\mathfrak{g}; \mathfrak{g}_0^{ss})(R)$.

Case 2: $\mathfrak{g}_0^{ss}$ *is not simple and* $\mathfrak{g} \neq \mathfrak{psl}(2|2)$. Now we use Lemma 3.7. For our chosen base $\Pi = \{\alpha_1, \alpha_2, ..., \alpha_l\}$ of Δ, we fix $e_i \in \mathfrak{g}^{\alpha_i}$, $f_i \in \mathfrak{g}^{-\alpha_i}$, and $\alpha_i^\vee = [e_i, f_i]$. Since the spaces $\mathfrak{g}^{\alpha_i}$ are 1-dimensional, after multiplying by an element of $\mathbf{H}(R)$ we may assume that σ fixes e_i, f_i, and $\alpha_i^\vee$. Since these generate $\mathfrak{g}$ and σ is R linear, we have $\sigma = \mathrm{Id}$.

Case 3: $\mathfrak{g} = H(2l)$, $l \geq 3$. In this case using the description of Δ provided in the Appendix we see that $\mathfrak{g}_{-1} \cap \mathfrak{g}^\alpha \neq 0$ iff $\alpha = \pm\varepsilon_i$, $i = 1, ..., l$. Moreover, the spaces $\mathfrak{g}_{-1} \cap \mathfrak{g}^{\pm\varepsilon_i} = \mathfrak{g}_{-1}^{p(\pm\varepsilon_i)}$ are one-dimensional. Multiplying σ by an element of $\mathbf{H}(R)$ we may assume that $\sigma_{|\mathfrak{g}_{-1} \cap \mathfrak{g}^{\varepsilon_i}} = \mathrm{Id}$ for every $i = 1, ..., l$. Let $r \in R^\times$ be such that $\sigma_{|\mathfrak{g}_{-1} \cap \mathfrak{g}^{-\varepsilon_1}} = r\,\mathrm{Id}$ (strictly speaking $\sigma_{|\mathfrak{g}_{-1}(R) \cap \mathfrak{g}^{-\varepsilon_1}(R)} = r\,\mathrm{Id}$). From $[\mathfrak{g}_0, \mathfrak{g}_{-1}] = \mathfrak{g}_{-1}$ and the fact that the spaces $\mathfrak{g}_i \cap \mathfrak{g}^\alpha$, $i = -1, 0$, are at most one-dimensional, we easily conclude that $\sigma_{|\mathfrak{g}_{-1} \cap \mathfrak{g}^{-\varepsilon_i}} = \sigma_{|\mathfrak{g}_0 \cap \mathfrak{g}^{-\varepsilon_i - \varepsilon_j}} = r\,\mathrm{Id}$, $\sigma_{|\mathfrak{g}_0 \cap \mathfrak{g}^{\varepsilon_i - \varepsilon_j}} = \mathrm{Id}$, and $\sigma_{|\mathfrak{g}_0 \cap \mathfrak{g}^{\varepsilon_i + \varepsilon_j}} = r^{-1}\,\mathrm{Id}$ for $1 \leq i \neq j \leq l$. This completely determines $\sigma \in \mathrm{Aut}_R(\mathfrak{g}(R))$ since every automorphism of $\mathfrak{g}$ is uniquely determined by its restriction on $\mathfrak{g}_{-1}$ (see [S]). In order to explicitly express σ we apply the change of coordinates $\eta_i := \frac{1}{\sqrt{2}}(\xi_i + \sqrt{-1}\xi_{i+l})$; $\eta_{i+l} := \frac{1}{\sqrt{2}}(\xi_i - \sqrt{-1}\xi_{i+l})$. Then $D_{\eta_{i+l}} \in \mathfrak{g}_{-1} \cap \mathfrak{g}^{\varepsilon_i}$ and $D_{\eta_i} \in \mathfrak{g}_{-1} \cap \mathfrak{g}^{\varepsilon_{i+l}}$ as explained at the end of the Appendix.

In terms of the new coordinates we have that $\sigma = \mathrm{Ad}_{\mathfrak{g}}(\widehat{\sigma})$, where $\widehat{\sigma}$ is the linear automorphism of $\Lambda(2l)(R)$ given by the matrix $A_\sigma = \begin{pmatrix} r^{-1}I & 0 \\ 0 & I \end{pmatrix}$, i.e. $\widehat{\sigma}(\eta) = A_\sigma \eta$. It then follows that $\sigma \in \mathbf{Aut}(\mathfrak{g}, \Pi_0)(R)$. □

Remark 5.3 The proof of Theorem 5.2 yields an analogous result in the case when $\mathfrak{g}$ is a finite dimensional simple Lie algebra: Under the assumptions of R therein, the group $\mathrm{Aut}_k(\mathfrak{g}(R))$ is generated by $\mathrm{Aut}_k(R)$, $\mathrm{Ad}\,\mathbf{G}(R)$, $\mathbf{H}(R)$, and $\mathbf{Aut}(\mathfrak{g}, \Pi_0)$. If, furthermore, the automorphism is R-linear, then the three last groups suffice.

In what follows we give an example of a Dedekind domain R over $\mathbb{C}$ for which these results fails.

Let $k = \mathbb{C}$. We first look at the case of Lie algebras. Let $\mathfrak{g} = \mathfrak{sl}_2$. Then $\mathbf{Aut}(\mathfrak{g}) = \mathbf{PGL}_2$ and $\mathbf{G} = \mathbf{SL}_2$. The subgroup $\mathbf{H}$ of $\mathbf{Aut}(\mathfrak{g})$ is the "standard" (split) torus of $\mathbf{PGL}_2$. Thus, for R in $\mathbb{C}$-*alg*, $\mathbf{H}(R) = R^\times$, and $r \in R^\times$ acts as an automorphism of $\mathfrak{sl}_2(R) = Rf \oplus Rh \oplus Re$ via $f \mapsto r^{-1}f$, $h \mapsto h$, and $e \mapsto re$. We will denote this automorphism by $\widehat{r}$.

Consider the adjoint representation $\mathrm{Ad} : \mathbf{GL}_2 \to \mathbf{Aut}(\mathfrak{g})$. Let $\mathbf{S}_R$ be the subgroup of $\mathbf{Aut}(\mathfrak{g})(R)$ generated by $\mathrm{Ad}\,\mathbf{SL}_2(R)$ and $\mathbf{H}(R)$. Theorem 5.2 asserts that $\mathbf{Aut}_R(\mathfrak{g}(R)) = \mathbf{S}_R$. We will see that this may fail if $\mathrm{Pic}(R)$ is not trivial.

For $r \in \mathbf{H}(R) \simeq R^\times$ we have that $\widehat{r} \in \mathrm{Ad}(\mathbf{SL}_2(R))$ if and only if $r \in (R^\times)^2$. All automorphisms $\widehat{r}$, however, do belong to $\mathrm{Ad}(\mathbf{GL}_2(R))$. Indeed, $\widehat{r}$ is conjugation by the matrix $\begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix}$. Thus, if Theorem 5.2 were to hold, then the natural map $\mathbf{GL}_2(R) \to \mathbf{PGL}_2(R)$ would be surjective. From a theorem of Rosenberg and Zelinsky one knows that for a Dedekind domain R we have

$$\mathbf{PGL}_2(R)/\,\mathrm{Ad}(\mathbf{GL}_2(R)) \simeq \ {}_2\mathrm{Pic}(R),$$

the 2-torsion of the Picard group of R ([RZ], see also [KO] and [I]).

Let X be an irreducible complete smooth curve over $\mathbb{C}$ of genus $g > 0$. If $\mathrm{Spec}(R)$ is a nonempty open affine subscheme of X then R is a Dedekind domain whose Picard group has n-torsion for all n ([Ha], pp. 429 and 427).

The above shows that for this type of Dedekind domains the Lie algebra version of Theorem 5.2 fails for $\mathfrak{g} = \mathfrak{sl}_2$. Similar considerations apply to $\mathfrak{sl}_n$ for all $n > 2$. The obstruction lies in the fact that the matrix algebra over R may have non-inner automorphisms. In fact by taking the genus of X arbitrary large, we can make the order of ${}_n\mathrm{Pic}(R)$, hence also the index of $\mathrm{Ad}(\mathbf{GL}_n(R))$ in $\mathbf{PGL}_n(R)$, arbitrary large.

These Dedekind domains also provide counterexamples to Theorem 5.2 for the Lie superalgebras of type $\mathfrak{psq}(n)$. This follows from the above reasoning when taking into consideration the explicit nature of the isomorphism

$$\mathbf{Aut}(\mathfrak{psq}(n)) \simeq \mathbf{PGL}_n \rtimes \mathbb{Z}/4\mathbb{Z}$$

described in §6.6 of [GP].

Remark 5.4 Let $r \in R^\times$, and consider the quadratic extension $\widetilde{R} = R[r^{\frac{1}{2}}]$ of R. Then $\widetilde{R}/R$ is finite étale (in fact Galois). Let $\mathfrak{g} = H(2l)$ and $\widetilde{\sigma} = \mathrm{Ad}_{\mathfrak{g}} \begin{pmatrix} r^{\frac{1}{2}} I & 0 \\ 0 & r^{-\frac{1}{2}} I \end{pmatrix} \delta_{r^{\frac{1}{2}}}$. Then $\widetilde{\sigma} \in \mathbf{Aut}(\mathfrak{g})(\widetilde{R})$ is such that $\widetilde{\sigma}$ stabilizes $\mathfrak{g}(R)$ and $\widetilde{\sigma}_{|\mathfrak{g}(R)} = \sigma$, where σ is the automorphism of $\mathfrak{g}(R)$ determined by the matrix A_σ in Case 3 of the proof of Theorem 5.2. According to Proposition 4.4 we should be able to write σ as an R-point of the product of the groups $\mathbf{H}$ and $\mathbf{Aut}(\mathfrak{g}; \mathfrak{g}_0^{ss})$. The R-points of the product group are in general a larger group than the naive product of the R-points of the respective groups. In fact, $\mathrm{Ad}_{\mathfrak{g}} \begin{pmatrix} r^{\frac{1}{2}} I & 0 \\ 0 & r^{-\frac{1}{2}} I \end{pmatrix} \in \mathbf{H}(\widetilde{R})$ and $\delta_{r^{\frac{1}{2}}} \in \mathbf{Aut}(\mathfrak{g}, \mathfrak{g}_0^{ss})(\widetilde{R})$, which shows that σ is an R-point of the product group.

Remark 5.5 Let $\widetilde{\mathfrak{g}(R)}$ be the universal central extension of the k-Lie superalgebra $\mathfrak{g}(R)$. A result of Neher (Corollary 2.8 in [Ne]) implies that $\mathrm{Aut}_k(\mathfrak{g}(R)) = \mathrm{Aut}_k(\widetilde{\mathfrak{g}(R)})$. Theorem 5.2 can also be applied to this last group.

The following corollary (of the proof) of Theorem 5.2, is useful for the representation theory of $\mathfrak{g}(R)$ (notably in the case of $R = k[t, t^{-1}]$, which corresponds to the untwisted affine Kac-Moody superalgebras). It provides a description of the subgroup of the k-automorphisms of $\mathfrak{g}(R)$ that leave invariant the category of weight $(\mathfrak{g}(R), \mathfrak{h}_{\mathfrak{g}_0^{ss}})$-modules, i.e., all $\mathfrak{g}(R)$-modules M for which which $M = \bigoplus_{\lambda \in \mathfrak{h}^*_{\mathfrak{g}_0^{ss}}} M^\lambda$.

Corollary 5.6 *Let R be as in Theorem 5.2. Let $\mathbf{T}$ be the maximal torus of $\mathbf{G}$ corresponding to $\mathfrak{h}_{\mathfrak{g}_0^{ss}}$. The subgroup of $\mathrm{Aut}_k(\mathfrak{g}(R))$ consisting of all automorphisms σ for which $\sigma(\mathfrak{h}_{\mathfrak{g}_0^{ss}}) = \mathfrak{h}_{\mathfrak{g}_0^{ss}}$, is generated by $\mathrm{Ad}(N_{\mathbf{G}}(\mathbf{T}))(k)$ together with $\mathrm{Aut}_k(R)$, $\mathbf{Aut}(\mathfrak{g}, \Pi_0)(R)$, and $\mathbf{Aut}(\mathfrak{g}; \mathfrak{g}_0^{ss})(R)$.*

Proof. We may assume that $\sigma \in \mathrm{Aut}_R(\mathfrak{g}(R))$. The group $\mathrm{Ad}(N_{\mathbf{G}}(\mathbf{T}))$ accounts for the action of the Weyl group used in the proof of Theorem 5.2. Now if σ stabilizes $\mathfrak{h}_{\mathfrak{g}_0^{ss}}$, we reason as in the proof of Theorem 5.2 to conclude that σ belongs to the subgroup of $\mathrm{Aut}_R\, \mathfrak{g}(R)$ which is generated by the subgroups prescribed by the Corollary. □

Tables

1. The groups $\mathbf{G}_{\bar{0}}^{sad}$ and $\mathbf{Aut}(\mathfrak{g}; \mathfrak{g}_{\bar{0}}^{ss})$.

$\mathfrak{g}$	$\mathbf{G}_{\bar{0}}^{sad}$	$\mathbf{Aut}(\mathfrak{g}; \mathfrak{g}_{\bar{0}}^{ss})$
$\mathfrak{sl}(m\|n), (m,n) \neq (2,1)$	$(\mathbf{SL}_m \times \mathbf{SL}_n)/(\boldsymbol{\mu}_m \times \boldsymbol{\mu}_n)$	$\mathbf{G}_m$
$\mathfrak{sl}(2\|1)$	$\mathbf{SL}_2/\boldsymbol{\mu}_2$	$\mathbf{G}_m \rtimes \boldsymbol{\mu}_2$
$\mathfrak{psl}(n\|n), n > 2$	$(\mathbf{SL}_n \times \mathbf{SL}_n)/(\boldsymbol{\mu}_n \times \boldsymbol{\mu}_n)$	$\mathbf{G}_m$
$\mathfrak{psl}(2\|2)$	$(\mathbf{SL}_2 \times \mathbf{SL}_2)/\boldsymbol{\mu}_2$	$\mathbf{SL}_2$
$\mathfrak{sp}(n)$	$\mathbf{SL}_n/\boldsymbol{\mu}_n$	$\mathbf{G}_m$
$\mathfrak{psq}(n)$	$\mathbf{SL}_n/\boldsymbol{\mu}_n$	$\boldsymbol{\mu}_2$
$\mathfrak{osp}(2l\|2n), l > 1$	$(\mathbf{SO}_{2l} \times \mathbf{Sp}_{2n})/\boldsymbol{\mu}_2$	$\boldsymbol{\mu}_2$
$\mathfrak{osp}(2\|2n)$	$(\mathbf{SO}_2 \times \mathbf{Sp}_{2n})/\boldsymbol{\mu}_2$	$\mathbf{G}_m \rtimes \boldsymbol{\mu}_2$
$\mathfrak{osp}(2l+1\|2n)$	$\mathbf{SO}_{2l+1} \times \mathbf{Sp}_{2n}$	$\boldsymbol{\mu}_2$
$F(4)$	$(\mathbf{Spin}_7 \times \mathbf{SL}_2)/\boldsymbol{\mu}_2$	$\boldsymbol{\mu}_2$
$G(3)$	$\mathbf{G}_2 \times \mathbf{SL}_2$	$\boldsymbol{\mu}_2$
$D(\alpha)$	$(\mathbf{SL}_2 \times \mathbf{SL}_2 \times \mathbf{SL}_2)/(\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2)$	$\boldsymbol{\mu}_2$
$W(n)$	$\mathbf{GL}_n$	$\mathbf{G}_m$
$S(n)$	$\mathbf{SL}_n$	$\mathbf{G}_m$
$S'(2l)$	$\mathbf{SL}_{2l}$	$\boldsymbol{\mu}_{2l}$
$H(2l)$	$\mathbf{SO}_{2l}/\boldsymbol{\mu}_2$	$\mathbf{G}_a \times \mathbf{G}_m$
$H(2l+1)$	$\mathbf{SO}_{2l+1}$	$\mathbf{G}_m$

2. The groups $\mathbf{N}$.

Two alternative definitions of the unipotent groups $\mathbf{N}$ have been provided in §4.3. The table below gives an explicit description of $\mathbf{N}$ in terms of the n-dimensional vector space $V := k\xi_1 \oplus ... \oplus k\xi_n$, where $\xi_1, ..., \xi_n$ are odd variables, i.e. $\xi_i^2 = 0, \xi_i\xi_j = -\xi_j\xi_i$ if $i \neq j$. Recall that the additive affine group of a finite dimensional k-space U is denoted by U_a, that is $U_a = \mathrm{Hom}_k(S(U^*), -)$. For the proofs we refer the reader to Lemmas 6.1 and 6.2 in [GP].

$\mathfrak{g}$	$\mathbf{N}$
$W(n)$	$\mathbf{Hom}(V, \bigoplus_{i\geq 1} \Lambda^{2i} V)$
$S(n)$	$(\bigoplus_{i\geq 1}(V^* \otimes \Lambda^{2i+1} V)/\Lambda^{2i} V)_a$
$S'(n), n = 2l$	$(\bigoplus_{i\geq 1}(V^* \otimes \Lambda^{2i+1} V)/\Lambda^{2i} V)_a$
$H(n)$	$(\bigoplus_{i\geq 2} \Lambda^{2i} V)_a$

Appendix: Cartan subsuperalgebras and root systems of the Cartan type Lie superalgebras

We first recall some generalities about the Cartan type Lie superalgebras. By $W(n)$ we denote the (super)derivations of the Grassmann algebra $\Lambda(n) := \Lambda(\xi_1, ..., \xi_n)$ over k. Every element D of $W(n)$ is of the form $D = \sum_{i=1}^n P_i(\xi_1, ..., \xi_n)\frac{\partial}{\partial \xi_i}$ where by definition $\frac{\partial}{\partial \xi_i}(\xi_j) = \delta_{ij}$. Both $\Lambda(n)$ and $W(n)$ have natural gradings $\Lambda(n) = \bigoplus_{i=0}^n \Lambda(n)_i$ and $W(n) = \bigoplus_{j=-1}^{n-1} W(n)_j$, where $\Lambda(n)_i := \{P(\xi_1, ..., \xi_n) \mid \deg P = i\}$ and $W(n)_j := \{\sum_{i=1}^n P_i \frac{\partial}{\partial \xi_i} \mid \deg P_i = j+1\}$.

In this appendix we will use the following explicit description of the Cartan type Lie subsuperalgebras $S(n)$, $S'(n)$, $H(n)$, and $\widetilde{H}(n)$, of $W(n)$:

$$\begin{aligned}
S(n) &= \mathrm{Span}_k \left\{ \frac{\partial f}{\partial \xi_i}\frac{\partial}{\partial \xi_j} + \frac{\partial f}{\partial \xi_j}\frac{\partial}{\partial \xi_j} \mid f \in \Lambda(n), i, j = 1, ..., n \right\}, \\
S'(n) &= \mathrm{Span}_k \left\{ (1 - \xi_1 ... \xi_n) \left(\frac{\partial f}{\partial \xi_i}\frac{\partial}{\partial \xi_j} + \frac{\partial f}{\partial \xi_j}\frac{\partial}{\partial \xi_j} \right) \mid f \in \Lambda(n), i, j = 1, ..., n \right\}, \\
\widetilde{H}(n) &= \mathrm{Span}_k \left\{ D_f := \sum_i \frac{\partial f}{\partial \xi_i}\frac{\partial}{\partial \xi_i} \mid f \in \Lambda(n), f(0) = 0, i, j = 1, ..., n \right\}, \\
\widetilde{H}(n) &= H(n) \oplus k D_{\xi_1 ... \xi_n}.
\end{aligned}$$

We have also that $[D_f, D_g] = D_{\{f,g\}}$ where $\{f, g\} := (-1)^{\deg f} \sum_{i=1}^n \frac{\partial f}{\partial \xi_i}\frac{\partial g}{\partial \xi_j}$. For any of the Lie subsuperalgebras $\mathfrak{s}$ of $W(n)$ listed above, except for $\mathfrak{s} = S'(n)$, we set $\mathfrak{s}_j := \mathfrak{s} \cap W(n)_j$.

In what follows we give an explicit description of specific Cartan subsuperalgebras of $W(n), S(n), S'(n)$, and $H(n)$. This description can serve as a complement to the root structures provided in Appendix A in [Pen].

Case 1: $\mathfrak{g} = W(n)$, $n \geq 2$. In this case the elements $h_i = \xi_i \frac{\partial}{\partial \xi_i}$ form a basis for a Cartan subsuperalgebra $\mathfrak{h}$ of $\mathfrak{g}$. In particular, $\mathfrak{h} = \mathfrak{h}_{\bar{0}}$ is a subalgebra of $\mathfrak{g}_0 \simeq \mathfrak{gl}_n$. The elements in $\mathfrak{h}_{\bar{0}}^*$ that form the dual basis to h_i will be denoted by ε_i. The root system of $\mathfrak{g}$ is

$$\Delta = \{\varepsilon_{i_1} + ... + \varepsilon_{i_k}, \varepsilon_{i_1} + ... + \varepsilon_{i_k} - \varepsilon_j \mid i_r \neq i_s, j \neq i_r, 0 \leq k \leq n-1, 1 \leq j \leq n\}$$

Case 2: $\mathfrak{g} = S(n), S'(n)$ *($n = 2l$ in the second case),* $n \geq 3$. Now we consider the Cartan subsuperalgebra $\mathfrak{h}$ spanned by $h_i - h_{i+1} = \xi_i \frac{\partial}{\partial \xi_i} - \xi_{i+1} \frac{\partial}{\partial \xi_{i+1}}$, $i = 1, ..., n-1$. We have again that $\mathfrak{h} = \mathfrak{h}_{\bar{0}}$ is a subalgebra of $\mathfrak{g}_0 \simeq \mathfrak{sl}_n$. Denote by ε_i the images in $\mathfrak{h}_{\bar{0}}^*$ of the basis dual to h_i, $\varepsilon_1 + ... + \varepsilon_n = 0$.

The root system of $\mathfrak{g}$ is

$$\Delta = \{\varepsilon_{i_1}+...+\varepsilon_{i_k}, \varepsilon_{i_1}+...+\varepsilon_{i_l}-\varepsilon_j \mid i_r \neq i_s, j \neq i_r, 1 \leq k \leq n-2, 0 \leq l \leq n-1, 1 \leq j \leq n\}$$

Case 3: $\mathfrak{g} = H(2l), l \geq 3$. We fix $\mathfrak{h}$ to be the Cartan subsuperalgebra of $\mathfrak{g}$ spanned by

$$\{D_{\xi_{i_1}...\xi_{i_r}\xi_{i_1+l}...\xi_{i_r+l}} \mid 1 \leq r \leq l-1, 1 \leq i_1 < ... < i_r \leq l\}.$$

In this case we have $\mathfrak{h} = \mathfrak{h}_{\bar{0}}$ and $\mathfrak{h}_{\mathfrak{g}_0^{ss}} := \mathfrak{h} \cap \mathfrak{g}_0 = \mathrm{Span}_k\{D_{\xi_i\xi_{i+l}} \mid 1 \leq i \leq l\}$ is a Cartan subalgebra of $\mathfrak{g}_0 \simeq \mathfrak{so}_{2l}$. Moreover, $\mathfrak{h}_{\bar{0}} = \mathfrak{h}_{\mathfrak{g}_0^{ss}} \oplus \mathfrak{h}^2$, where $\mathfrak{h}^2 \subset \mathfrak{g}^2 = \bigoplus_{i\geq 1}\mathfrak{g}_{2i}$. Let $h_i := \sqrt{-1}D_{\xi_i\xi_{i+l}}$ and define $\varepsilon_i \in \mathfrak{h}_{\bar{0}}^*$ by $\varepsilon_i(h_j) = \delta_{ij}$ and $\varepsilon_{i|\mathfrak{h}^2} = 0$. The root system of $\mathfrak{g}$ is

$$\Delta = \{\varepsilon_{i_1} + ... + \varepsilon_{i_t} - \varepsilon_{j_1} - ... - \varepsilon_{j_s} \mid i_r \neq i_p, j_r \neq j_p, i_r \neq j_p, 0 \leq t, s \leq l\}.$$

Case 4: $\mathfrak{g} = H(2l+1), l \geq 2$. We fix $\mathfrak{h}$ to be the Cartan subsuperalgebra of $\mathfrak{g}$ spanned by

$$\{D_{\xi_{i_1}...\xi_{i_r}\xi_{i_1+l}...\xi_{i_r+l}\xi_{2l+1}}, D_{\xi_{i_1}...\xi_{i_s}\xi_{i_1+l}...\xi_{i_s+l}} \mid 0 \leq r \leq l-1, 1 \leq s \leq l, 1 \leq i_1 < ... < i_r \leq l\}.$$

We have $\mathfrak{h} = \mathfrak{h}_{\bar{0}} \oplus \mathfrak{h}_{\bar{1}}$ where $\mathfrak{h}_{\bar{0}}$ is spanned by the elements $D_{\xi_{i_1}...\xi_{i_s}\xi_{i_1+l}...\xi_{i_s+l}}$, while $\mathfrak{h}_{\bar{1}}$ is spanned by the elements $D_{\xi_{i_1}...\xi_{i_s}\xi_{i_1+l}...\xi_{i_s+l}\xi_{2l+1}}$. In particular, $\mathfrak{h}_{\mathfrak{g}_0^{ss}} := \mathfrak{h} \cap \mathfrak{g}_0 = \mathrm{Span}_k\{D_{\xi_i\xi_{i+l}} \mid 1 \leq i \leq l\}$ is a Cartan subalgebra of $\mathfrak{g}_0 \simeq \mathfrak{so}_{2l+1}$. We have again that $\mathfrak{h}_{\bar{0}} = \mathfrak{h}_{\mathfrak{g}_0^{ss}} \oplus \mathfrak{h}^2$, where $\mathfrak{h}^2 \subset \mathfrak{g}^2 = \bigoplus_{i\geq 1}\mathfrak{g}_{2i}$. As in Case 3 we set $h_i := \sqrt{-1}D_{\xi_i\xi_{i+l}}$ and define $\varepsilon_i \in \mathfrak{h}_{\bar{0}}^*$ by $\varepsilon_i(h_j) = \delta_{ij}$ and $\varepsilon_{i|\mathfrak{h}^2} = 0$. Then the root system of $\mathfrak{g}$ is

$$\Delta = \{\varepsilon_{i_1} + ... + \varepsilon_{i_t} - \varepsilon_{j_1} - ... - \varepsilon_{j_s} \mid i_r \neq i_p, j_r \neq j_p, i_r \neq j_p, 0 \leq t, s \leq l\}.$$

In what follows we describe the graded root spaces $\mathfrak{g}^\alpha \cap \mathfrak{g}_i$ of $\mathfrak{g} = H(2l)$ and $\mathfrak{g} = H(2l+1)$. For this description it is convenient to use the following coordinate change: $\eta_i := \frac{1}{\sqrt{2}}(\xi_i + \sqrt{-1}\xi_{i+l})$, $\eta_{i+l} := \frac{1}{\sqrt{2}}(\xi_i - \sqrt{-1}\xi_{i+l})$, $\eta_{2l+1} := \xi_{2l+1}$ (the last in the case $\mathfrak{g} = H(2l+1)$). Using the new coordinates we have that if $\mathfrak{g} = H(2l+\epsilon)$, $\epsilon = 0$ or 1, then $D_f = \sum_{i=1}^l \frac{\partial f}{\partial\eta_{i+l}}\frac{\partial}{\partial\eta_i} + \sum_{i=1}^l \frac{\partial f}{\partial\eta_i}\frac{\partial}{\partial\eta_{i+l}} + \epsilon\frac{\partial f}{\partial\eta_{2l+1}}\frac{\partial}{\partial\eta_{2l+1}}$, and $\{f,g\} = (-1)^{\deg f}\left(\sum_{i=1}^l \frac{\partial f}{\partial\eta_{i+l}}\frac{\partial g}{\partial\eta_i} + \sum_{i=1}^l \frac{\partial f}{\partial\eta_i}\frac{\partial g}{\partial\eta_{i+l}} + \epsilon\frac{\partial f}{\partial\eta_{2l+1}}\frac{\partial g}{\partial\eta_{2l+1}}\right)$.

For $I = (i_1, ..., i_t)$, $1 \leq i_1 < ... < i_t \leq l$, we set $\widehat{I} := (i_1+l, ..., i_t+l)$, $|I| := i_1+...+i_t$, $\varepsilon_I := \varepsilon_{i_1}+...+\varepsilon_{i_t}$, and $\eta_I := \eta_{i_1}...\eta_{i_t}$. Fix now I and J such that $I \neq J$. For $\mathfrak{g} = H(2l)$ we have that $D_{\eta_I\eta_{\widehat{J}}} \in \mathfrak{g}^{\varepsilon_I-\varepsilon_J} \cap \mathfrak{g}_{|I|+|J|-|I\cap J|-1}$. If $I \cap J = \emptyset$, then the set

$$\mathcal{B}_{I,J} := \left\{D_{\eta_I\eta_K\eta_{\widehat{J}}\eta_{\widehat{K}}} \mid K \cap I = \emptyset, K \cap J = \emptyset\right\}$$

forms a basis of $\mathfrak{g}^{\varepsilon_I-\varepsilon_J}$ and the set

$$\mathcal{B}_{I,J,i} := \mathcal{B}_{I,J} \cap \mathfrak{g}^i = \left\{ D_{\eta_I \eta_K \eta_{\widehat{J}} \eta_{\widehat{K}}} \mid K \cap I = \emptyset, K \cap J = \emptyset, |K| = \frac{i-1-|I|-|J|}{2} \right\}$$

forms a basis of $\mathfrak{g}^{\varepsilon_I-\varepsilon_J} \cap \mathfrak{g}_i$ (we have $\mathfrak{g}^{\varepsilon_I-\varepsilon_J} \cap \mathfrak{g}_i \neq 0$ iff $i-1-|I|-|J|$ is even).

In the case of $\mathfrak{g} = H(2l+1)$ and $I \cap J = \emptyset$, the set $\mathcal{B}_{I,J} \cup \mathcal{B}'_{I,J}$ forms a basis of $\mathfrak{g}^{\varepsilon_I-\varepsilon_J}$, where

$$\mathcal{B}'_{I,J} := \left\{ D_{\eta_I \eta_K \eta_{\widehat{J}} \eta_{\widehat{K}} \eta_{2l+1}} \mid K \cap I = \emptyset, K \cap J = \emptyset \right\}.$$

Furthermore, $\mathfrak{g}^{\varepsilon_I-\varepsilon_J} \cap \mathfrak{g}_i$ has $\mathcal{B}_{I,J,i}$ as a basis if $i-1-|I|-|J|$ is even and

$$\mathcal{B}'_{I,J,i} := \mathcal{B}'_{I,J} \cap \mathfrak{g}_i = \left\{ D_{\eta_I \eta_K \eta_{\widehat{J}} \eta_{\widehat{K}} \eta_{2l+1}} \mid K \cap I = \emptyset, K \cap J = \emptyset, |K| = \frac{i-|I|-|J|}{2} \right\}$$

as a basis if $i-1-|I|-|J|$ is odd.

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Dimitar Grantcharov
Department of Mathematics
San Jose State University
San Jose, CA 95192-0103
USA
E-mail: grantcharov@math.sjsu.edu

Arturo Pianzola
Department of Mathematical and Statistical Sciences
University of Alberta
Edmonton, Alberta T6G 2G1
CANADA

and

Instituto Argentino de Matemática
Saavedra 15
(1083) Buenos Aires
Argentina
E-mail a.pianzola@math.ualberta.ca