

Frames of subspaces and operators

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Abstract

We study the relationship between operators, orthonormal basis of subspaces and frames of subspaces (also called fusion frames) for a separable Hilbert space \mathcal{H} . We get sufficient conditions on an orthonormal basis of subspaces $\mathcal{E} = \{E_i\}_{i \in I}$ of a Hilbert space \mathcal{K} and a surjective $T \in L(\mathcal{K}, \mathcal{H})$ in order that $\{T(E_i)\}_{i \in I}$ is a frame of subspaces with respect to a computable sequence of weights. We also obtain generalizations of results in [J. A. Antezana, G. Corach, M. Ruiz and D. Stojanoff, Oblique projections and frames. Proc. Amer. Math. Soc. 134 (2006), 1031-1037], which related frames of subspaces (including the computation of their weights) and oblique projections. The notion of refinement of a fusion frame is defined and used to obtain results about the excess of such frames. We study the set of admissible weights for a generating sequence of subspaces. Several examples are given.

Keywords: frames, frames of subspaces, fusion frames, Hilbert space operators, oblique projections.

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1 Introduction

Let \mathcal{H} be a (separable) Hilbert space. A *frame* for \mathcal{H} is a sequence of vectors $\mathcal{F} = \{f_i\}_{i \in I}$ for which there exist numbers $A, B > 0$ such that

$$A\|f\|^2 \leq \sum_{i \in I} |\langle f, f_i \rangle|^2 \leq B\|f\|^2, \quad \text{for every } f \in \mathcal{H}.$$

This definition has been generalized to the notion of frames of subspaces by Casazza and Kutyniok [5] (see also [12] and [13]) in the following way: Let $\mathcal{W} = \{W_i\}_{i \in I}$ be a sequence of closed subspaces, and let $w = \{w_i\}_{i \in I} \in \ell_+^\infty(I)$ (i.e. $w_i > 0$ for every $i \in I$). We say that

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$\mathcal{W}_w = (w_i, W_i)_{i \in I}$ is a frame of subspaces (shortly: FS) for \mathcal{H} if there exist $A_{\mathcal{W}_w}, B_{\mathcal{W}_w} > 0$ such that

$$A_{\mathcal{W}_w} \|f\|^2 \leq \sum_{i \in I} w_i^2 \|P_{W_i} f\|^2 \leq B_{\mathcal{W}_w} \|f\|^2 \quad \text{for every } f \in \mathcal{H},$$

where each P_{W_i} denotes the orthogonal projection onto W_i . The relevance of this notion, as remarked in [5], is that it gives criteria for constructing a frame for \mathcal{H} , by joining sequences of frames for subspaces of \mathcal{H} (see Theorem 3.4 for details). In other words, to give conditions which assure that a sequence of “local” frames, can be pieced together to obtain a frame for the complete space.

Recently, the frames of subspaces have been renamed as *fusion frames*. This notion is intensely studied during the last years, and several new applications have been discovered. The reader is referred to Casazza, Kutyniok, Li [7], Casazza and Kutyniok [6], Gavrutu [14] and the references therein.

Given sequences $\mathcal{W}_w = (w_i, W_i)_{i \in I}$, consider, for each $i \in I$, an orthonormal basis $\{e_{ik}\}_{k \in K_i}$ of W_i . It was proved in [5] that \mathcal{W}_w is a FS for \mathcal{H} if and only if $\mathcal{E} = \{w_i e_{ik}\}_{i \in I, k \in K_i}$ is a frame for \mathcal{H} . Therefore, a FS can be thought as a frame (of vectors) such that some subsequences are required to be orthogonal and to have the same norm. Therefore, many objects associated to vector frames have a generalization for frames of subspaces (see [5] and [3]), for example, synthesis, analysis and frame operators. Also, some useful results concerning frames still hold in the FS setting. For instance, as it is shown in [3], a Parseval FS is an orthogonal projection of a orthonormal basis of subspaces of a larger Hilbert space containing \mathcal{H} , generalizing the well known result of D. Han and D. Larson.

As we mention before, if $\mathcal{W}_w = (w_i, W_i)_{i \in I}$ is a FS, the synthesis, analysis and frame operator can be defined, and the properties of \mathcal{W}_w can be study using these operators, as well as for frames of vectors ([5], [3]). In [5], the domain of the synthesis operator is defined as $\mathcal{K}_{\mathcal{W}} = \bigoplus_{i \in I} W_i$. So the subspaces $\{W_i\}_{i \in \mathbb{N}}$ are embedded in $\mathcal{K}_{\mathcal{W}}$ as an orthonormal basis of subspaces (see also [3] where other type of domain is used). Therefore, the frame of subspaces is the image of the orthonormal basis under the synthesis operator (which is a bounded surjective operator).

However, if $\mathcal{W}_w = (w_i, W_i)_{i \in I}$ is a FS for \mathcal{H} , its synthesis operator $T_{\mathcal{W}_w}$ satisfies that $T_{\mathcal{W}_w} g = w_i g$ for every g in the copy of each W_i into $\mathcal{K}_{\mathcal{W}}$ (see [5] or Definition 3.2 below). Hence, unlike the vector case, if one fix an orthonormal basis of subspaces $\mathcal{E} = \{E_i\}_{i \in I}$ of a Hilbert space \mathcal{K} , not every surjective operator $T \in L(\mathcal{K}, \mathcal{H})$ is the synthesis operator of a FS. Even worse, there exist surjective operators $T \in L(\mathcal{K}, \mathcal{H})$ such that $T(E_i)$ is closed for every $i \in I$, but the sequence $(w_i, T(E_i))_{i \in I}$ fails to be a FS for every $w \in \ell_+^\infty(I)$ (see Example 7.1).

The purpose of this work is to study the relationship between operators and frames of subspaces. Our aim is to get more flexibility in the use of operator theory techniques, with respect to the rigid definition of the synthesis operator. In this direction we get (sufficient) conditions on an orthonormal basis of subspaces $\mathcal{E} = \{E_i\}_{i \in I}$ of a Hilbert space \mathcal{K} and a surjective $T \in L(\mathcal{K}, \mathcal{H})$ in order to assure that they produce a frame of subspaces with respect to a computable sequence of weights (Theorem 3.6). We use then this result for describing

properties of equivalent frames of subspaces, and for studying the *excess* of such frames. We obtain generalizations of two results of [2], which relate FS (including the computation of their weights) and oblique projections (see also [3] and [7]). We also define the notion of refinement of sequences of subspaces and frames of subspaces. This allows us to describe the excess of frames of subspaces, obtaining results which are very similar to the known results in classical frame theory.

It is remarkable that several known results of frame theory are not valid in the FS setting. For example, we exhibit a frame of subspaces $\mathcal{W}_w = (w_i, W_i)_{i \in I}$ of \mathcal{H} such that, for every $G \in Gl(\mathcal{H})$, the sequence $(v_i, GW_i)_{i \in I}$ fails to be a Parseval FS for every $v \in \ell_+^\infty(I)$, including the case $G = S_{\mathcal{W}_w}^{-1/2}$, where $S_{\mathcal{W}_w}$ is the frame operator of \mathcal{W}_w (see Examples 7.5 and 7.6). Several of this facts are exposed in a section of (counter)examples.

Finally we begin with the study of that is, in our opinion, the key problem of the theory of frames of subspaces: given a generating sequence $\mathcal{W} = \{W_i\}_{i \in I}$ of closed subspaces of \mathcal{H} , to obtain a characterization of the set of its admissible weights,

$$\mathcal{P}(\mathcal{W}) = \{ w \in \ell_+^\infty(I) : \mathcal{W}_w = (w_i, W_i)_{i \in I} \text{ is a FS for } \mathcal{H} \} .$$

Particularly, we search for conditions which assure that a sequence \mathcal{W} satisfy that $\mathcal{P}(\mathcal{W}) \neq \emptyset$. We obtain some partial results about these problems, and we study an equivalent relation between weights, compatible with their admissibility with respect to a generating sequence. We give also several examples which illustrate the complexity of the problem.

The paper is organized as follows: Section 2 contains preliminary results about angles between closed subspaces, the reduced minimum modulus of operators, and frames of vectors. In section 3 we introduce the frames of subspaces and we state the first results relating these frames and Hilbert space operators. In Section 4 the set of admissible weights of a FS is studied. Section 5 contains the results which relate oblique projections and frames of subspaces. Section 6 is devoted to refinement of sequences of subspaces and it contains several results about the excess of a FS. In section 7 we present a large collection of examples.

Note: after completing this paper, the authors were pointed out of the existence of recent works on fusion frames [7], [6] and [14] . Thus, Corollary 3.9 appears in [7] and [14]. Also, Theorem 5.4 is related with Theorem 3.1. in [7]. Nevertheless, the proofs in general are quite different.

2 Preliminaries and Notations.

Let \mathcal{H} and \mathcal{K} be separable Hilbert spaces and $L(\mathcal{H}, \mathcal{K})$ the space of bounded linear operators $A : \mathcal{H} \rightarrow \mathcal{K}$ (if $\mathcal{K} = \mathcal{H}$ we write $L(\mathcal{H})$). The symbol $Gl(\mathcal{H})$ denotes the group of invertible operators in $L(\mathcal{H})$, and $Gl(\mathcal{H})^+$ the set of positive definite invertible operators on \mathcal{H} . For an operator $A \in L(\mathcal{H}, \mathcal{K})$, $R(A)$ denotes the range of A , $N(A)$ the nullspace of A , $A^* \in L(\mathcal{K}, \mathcal{H})$ the adjoint of A , and $\|A\|$ the operator norm of A .

We write $\mathcal{M} \subseteq \mathcal{H}$ to denote that \mathcal{M} is a **closed subspace** of \mathcal{H} . Given $\mathcal{M} \subseteq \mathcal{H}$, $P_{\mathcal{M}}$ is the orthogonal (i.e., selfadjoint) projection onto \mathcal{M} . If also $\mathcal{N} \subseteq \mathcal{H}$, we write $\mathcal{M} \ominus \mathcal{N} := \mathcal{M} \cap (\mathcal{M} \cap \mathcal{N})^\perp$.

Let I be a denumerable set. We denote by $\ell_+^\infty(I)$ the space of bounded sequences of positive numbers. We consider on $\ell_+^\infty(I)$ the usual product of $\ell^\infty(I)$ (i.e. coordinatewise product). With this product $\ell^\infty(I)$ is a von Neumann algebra. We denote by

$$\ell_+^\infty(I)^* = \{ \{w_i\}_{i \in I} \in \ell_+^\infty(I) : \inf_{i \in I} w_i > 0 \} = \ell_+^\infty(I) \cap Gl(\ell^\infty(I)) . \quad (1)$$

We shall recall the definition and basic properties of angles between closed subspaces of \mathcal{H} . We refer the reader to [1] for details and proofs. See also the survey by Deutsch [11] or the book by Kato [17].

Angle between subspaces and reduced minimum modulus.

We shall recall the definition of angle between closed subspaces of \mathcal{H} . We refer the reader to [1] (where the same notations are used) for details and proofs. See also the survey by Deutsch [11] or the book by Kato [17].

Definition 2.1. Let $\mathcal{M}, \mathcal{N} \subseteq \mathcal{H}$. The **angle** between \mathcal{M} and \mathcal{N} is the angle in $[0, \pi/2]$ whose cosine is

$$c[\mathcal{M}, \mathcal{N}] = \sup \{ |\langle x, y \rangle| : x \in \mathcal{M} \ominus \mathcal{N}, y \in \mathcal{N} \ominus \mathcal{M} \text{ and } \|x\| = \|y\| = 1 \} .$$

If $\mathcal{M} \subseteq \mathcal{N}$ or $\mathcal{N} \subseteq \mathcal{M}$, we define $c[\mathcal{M}, \mathcal{N}] = 0$, as if they were orthogonal. The *sine* of this angle is denoted by $s[\mathcal{M}, \mathcal{N}] = (1 - c[\mathcal{M}, \mathcal{N}]^2)^{1/2}$. \blacktriangle

Now, we state some known results concerning angles (see [1] or [11]).

Proposition 2.2. Let $\mathcal{M}, \mathcal{N} \subseteq \mathcal{H}$. Then

1. $c[\mathcal{M}, \mathcal{N}] = c[\mathcal{N}, \mathcal{M}] = c[\mathcal{M} \ominus \mathcal{N}, \mathcal{N}] = c[\mathcal{M}, \mathcal{N} \ominus \mathcal{M}]$.
2. If $\dim \mathcal{M} < \infty$, then $c[\mathcal{M}, \mathcal{N}] < 1$.
3. $c[\mathcal{M}, \mathcal{N}] < 1$ if and only if $\mathcal{M} + \mathcal{N}$ is closed.
4. $c[\mathcal{M}, \mathcal{N}] = c[\mathcal{M}^\perp, \mathcal{N}^\perp]$
5. $c[\mathcal{M}, \mathcal{N}] = \|P_{\mathcal{M}}P_{\mathcal{N} \ominus \mathcal{M}}\| = \|P_{\mathcal{M} \ominus \mathcal{N}}P_{\mathcal{N}}\| = \|P_{\mathcal{M}}P_{\mathcal{N}} - P_{\mathcal{M} \cap \mathcal{N}}\|$.
6. $s[\mathcal{M}, \mathcal{N}] = \text{dist}(B_1(\mathcal{M} \ominus \mathcal{N}), \mathcal{N})$, where $B_1(\mathcal{M} \ominus \mathcal{N})$ is the unit ball of $\mathcal{M} \ominus \mathcal{N}$. \blacksquare

Definition 2.3. The *reduced minimum modulus* $\gamma(T)$ of $T \in L(\mathcal{H}, \mathcal{K})$ is defined by

$$\gamma(T) = \inf \{ \|Tx\| : \|x\| = 1, x \in N(T)^\perp \} \quad (2)$$

Remark 2.4. The following properties are well known (see [1]). Let $T \in L(\mathcal{H}, \mathcal{K})$.

1. $\gamma(T) = \gamma(T^*) = \gamma(T^*T)^{1/2}$.

2. $R(T) \subseteq \mathcal{K}$ if and only if $\gamma(T) > 0$.

3. If T is invertible, then $\gamma(T) = \|T^{-1}\|^{-1}$.

4. If $B \in L(\mathcal{K})$, then

$$\|B^{-1}\|^{-1}\gamma(T) \leq \gamma(BT) \leq \|B\|\gamma(T). \quad (3)$$

5. Suppose that $R(T) \subseteq \mathcal{K}$ and take $\mathcal{M} \subseteq \mathcal{H}$. Then

$$\gamma(T) s[N(T), \mathcal{M}] \leq \gamma(TP_{\mathcal{M}}) \leq \|T\| s[N(T), \mathcal{M}]. \quad (4)$$

In particular, $T(\mathcal{M}) \subseteq \mathcal{K}$ if and only if $c[N(T), \mathcal{M}] < 1$. ▲

Preliminaries on frames.

We introduce some basic facts about frames in Hilbert spaces. For a complete description of frame theory and its applications, the reader is referred to Daubechies, Grossmann and Meyer [10], the review by Heil and Walnut [15] or the books by Young [18] and Christensen [8].

Definition 2.5. Let $\mathcal{F} = \{f_n\}_{n \in \mathbb{N}}$ a sequence in a Hilbert space \mathcal{H} . \mathcal{F} is called a *frame* if there exist numbers $A, B > 0$ such that

$$A\|f\|^2 \leq \sum_{n \in \mathbb{N}} |\langle f, f_n \rangle|^2 \leq B\|f\|^2, \quad \text{for every } f \in \mathcal{H}. \quad (5)$$

The optimal constants $A_{\mathcal{F}}, B_{\mathcal{F}}$ for Eq. (5) are called the *frame bounds* for \mathcal{F} . The frame \mathcal{F} is called *tight* if $A_{\mathcal{F}} = B_{\mathcal{F}}$, and *Parseval* if $A_{\mathcal{F}} = B_{\mathcal{F}} = 1$. ▲

Definition 2.6. Let $\mathcal{F} = \{f_n\}_{n \in \mathbb{N}}$ be a frame in \mathcal{H} and let \mathcal{K} be a separable Hilbert space. Fix $\mathcal{B} = \{\varphi_n\}_{n \in \mathbb{N}}$ an orthonormal basis of \mathcal{K} . From Eq. (5), one can deduce that there exists a unique $T_{\mathcal{F}, \mathcal{B}} \in L(\mathcal{K}, \mathcal{H})$ such that $T_{\mathcal{F}, \mathcal{B}}(\varphi_n) = f_n$ for every $n \in \mathbb{N}$. We shall say that $T_{\mathcal{F}, \mathcal{B}}$ is a *preframe operator* for \mathcal{F} . Another consequence of Eq. (5) is that $T_{\mathcal{F}, \mathcal{B}}$ is surjective. If one takes the canonical basis \mathcal{E} of $\ell^2(\mathbb{N})$, then $T_{\mathcal{F}} = T_{\mathcal{F}, \mathcal{E}}$ is called the *synthesis operator* for \mathcal{F} . ▲

Remark 2.7. Let $\mathcal{F} = \{f_n\}_{n \in \mathbb{N}}$ be a frame in \mathcal{H} and $T_{\mathcal{F}, \mathcal{B}} \in L(\mathcal{K}, \mathcal{H})$ a preframe operator for \mathcal{F} , with $\mathcal{B} = \{\varphi_n\}_{n \in \mathbb{N}}$. Then $T_{\mathcal{F}, \mathcal{B}}^* \in L(\mathcal{H}, \mathcal{K})$ is given by $T_{\mathcal{F}, \mathcal{B}}^*(x) = \sum_{n \in \mathbb{N}} \langle x, f_n \rangle \varphi_n$, for $x \in \mathcal{H}$. It is an *analysis operator* for \mathcal{F} . The operator $S_{\mathcal{F}} = T_{\mathcal{F}, \mathcal{B}} T_{\mathcal{F}, \mathcal{B}}^* \in L(\mathcal{H})^+$, called the *frame operator* of \mathcal{F} , satisfies $S_{\mathcal{F}} f = \sum_{n \in \mathbb{N}} \langle f, f_n \rangle f_n$, for $f \in \mathcal{H}$. It follows from (5) that $A_{\mathcal{F}} I \leq S_{\mathcal{F}} \leq B_{\mathcal{F}} I$. So that $S_{\mathcal{F}} \in Gl(\mathcal{H})^+$. Note that the frame operator $S_{\mathcal{F}}$ does not depend on the preframe operator chosen. ▲

Proposition 2.8. Let $\mathcal{F} = \{f_j\}_{j \in J}$ be a frame sequence in \mathcal{H} . Then the optimal frame constants for \mathcal{F} are $A_{\mathcal{F}} = \gamma(T_{\mathcal{F}})^2$ and $B_{\mathcal{F}} = \|T_{\mathcal{F}}\|^2$. ■

Definition 2.9. Let $\mathcal{F} = \{f_n\}_{n \in \mathbb{N}}$ be a frame in \mathcal{H} . The cardinal number $E(\mathcal{F}) = \dim \ker T_{\mathcal{F}}$ is called the excess of the frame. Holub [16] and Balan, Casazza, Heil and Landau [4] proved that

$$E(\mathcal{F}) = \sup \left\{ |I| : I \subseteq \mathbb{N} \text{ and } \{f_n\}_{n \notin I} \text{ is still a frame for } \mathcal{H} \right\}. \quad (6)$$

This characterization justifies the name “excess of \mathcal{F} ”. For every preframe operator $T_{\mathcal{F}, \mathcal{B}} \in L(\mathcal{K}, \mathcal{H})$ of \mathcal{F} , it holds that $E(\mathcal{F}) = \dim \ker T_{\mathcal{F}, \mathcal{B}}$. The frame \mathcal{F} is called a *Riesz basis* (or exact) if $E(\mathcal{F}) = 0$, i.e., if the preframe operators of \mathcal{F} are invertible. \blacktriangle

3 Frames of subspaces, or fusion frames

Throughout this section, \mathcal{H} shall be a fixed separable Hilbert space, and $I \subseteq \mathbb{N}$ a fixed index set ($I = \mathbb{N}$ or $I = \mathbb{I}_n := \{1, \dots, n\}$ for $n \in \mathbb{N}$). Recall that $\ell_+^\infty(I)$ denotes the space of bounded sequences of (strictly) positive numbers, which will be considered as weights in the sequel. The element $e \in \ell_+^\infty(I)$ is the sequence with all its entries equal to 1.

Preliminaries

Following Casazza and Kutyniok [5], we define:

Definition 3.1. Let $\mathcal{W} = \{W_i\}_{i \in I}$ be a sequence of closed subspaces of \mathcal{H} , and let $w = \{w_i\}_{i \in I} \in \ell_+^\infty(I)$.

1. We say that $\mathcal{W}_w = (w_i, W_i)_{i \in I}$ is a *Bessel sequence* of subspaces (BSS) if there exists $B > 0$ such that

$$\sum_{i \in I} w_i^2 \|P_{W_i} f\|^2 \leq B \|f\|^2 \quad \text{for every } f \in \mathcal{H}. \quad (7)$$

where each $P_{W_i} \in L(\mathcal{H})$ is the orthogonal projection onto W_i .

2. We say that \mathcal{W}_w is a *frame of subspaces* (or a *fusion frame*) for \mathcal{H} , and write that \mathcal{W}_w is a FS (resp. FS for $\mathcal{S} \subseteq \mathcal{H}$) if there exist $A, B > 0$ such that

$$A \|f\|^2 \leq \sum_{i \in I} w_i^2 \|P_{W_i} f\|^2 \leq B \|f\|^2 \quad \text{for every } f \in \mathcal{H} \text{ (resp. } f \in \mathcal{S}), \quad (8)$$

The sharp constants for (8) are denoted by $A_{\mathcal{W}_w}$ and $B_{\mathcal{W}_w}$.

3. \mathcal{W} is a *minimal* sequence if

$$W_i \cap \overline{\text{span}} \{W_j : j \neq i\} = \{0\} \quad \text{for every } i \in I. \quad (9)$$

Suppose that \mathcal{W}_w is a fusion frame for \mathcal{H} . Then

4. \mathcal{W}_w is a *tight* frame if $A_{\mathcal{W}_w} = B_{\mathcal{W}_w}$, and *Parseval* frame if $A_{\mathcal{W}_w} = B_{\mathcal{W}_w} = 1$.

5. \mathcal{W}_w is an *orthonormal basis of subspaces* (shortly OBS) if $w = e$ and $W_i \perp W_j$ for $i \neq j$.
6. \mathcal{W}_w is *Riesz basis of subspaces* (shortly RBS) if \mathcal{W} is a minimal sequence. ▲

The notions of synthesis, analysis and frame operators can be defined for BSS. But with a different structure of the Hilbert space of frame sequences, which now relies strongly in the sequence of subspaces $\mathcal{W} = \{W_i\}_{i \in I}$.

Definition 3.2. Let $\mathcal{W}_w = (w_i, W_i)_{i \in I}$ be a BSS for \mathcal{H} . Define the Hilbert space

$$\mathcal{K}_{\mathcal{W}} = \bigoplus_{i \in I} W_i \quad \text{with the } \ell^2 \text{ norm} \quad \|g\|^2 = \sum_{i \in I} \|g_i\|^2, \quad \text{for } g = (g_i)_{i \in I} \in \mathcal{K}_{\mathcal{W}}.$$

The **Synthesis operator**: $T_{\mathcal{W}_w} \in L(\mathcal{K}_{\mathcal{W}}, \mathcal{H})$ is defined by

$$T_{\mathcal{W}_w}(g) = \sum_{i \in I} w_i g_i, \quad \text{for } g = (g_i)_{i \in I} \in \mathcal{K}_{\mathcal{W}}.$$

Its adjoint $T_{\mathcal{W}_w}^* \in L(\mathcal{H}, \mathcal{K}_{\mathcal{W}})$ is called the **Analysis operator** of \mathcal{W}_w . It is easy to see that $T_{\mathcal{W}_w}^*(f) = \{w_i P_{W_i} f\}_{i \in I}$, for $f \in \mathcal{H}$. The **Frame operator**: $S_{\mathcal{W}_w} = T_{\mathcal{W}_w} T_{\mathcal{W}_w}^* \in L(\mathcal{H})^+$ satisfies the formula $S_{\mathcal{W}_w} f = \sum_{i \in I} w_i^2 P_{W_i} f$, for $f \in \mathcal{H}$. ▲

Remark 3.3. Let $\mathcal{W} = \{W_i\}_{i \in I}$ be a sequence of closed subspaces of \mathcal{H} , and let $w \in \ell_+^\infty(I)$. In [5] the following results were proved:

1. $\mathcal{W}_w = (w_i, W_i)_{i \in I}$ is a BSS if and only if the synthesis operator $T_{\mathcal{W}_w}$ is well defined and bounded. In this case,

$$\mathcal{W}_w \text{ is a FS for } \mathcal{H} \text{ (resp. for } \mathcal{S} \subseteq \mathcal{H}) \iff T_{\mathcal{W}_w} \text{ is onto (resp } R(T_{\mathcal{W}_w}) = \mathcal{S}) .$$

This is also equivalent to the fact that $T_{\mathcal{W}_w}^*$ is bounded from below.

If \mathcal{W}_w is a FS for \mathcal{H} , then

2. $A_{\mathcal{W}_w} = \gamma(T_{\mathcal{W}_w})^2$ and $B_{\mathcal{W}_w} = \|T_{\mathcal{W}_w}\|^2$. So that $A_{\mathcal{W}_w} \cdot I \leq S_{\mathcal{W}_w} \leq B_{\mathcal{W}_w} \cdot I$.
3. \mathcal{W}_w is a RBS if and only if $T_{\mathcal{W}_w}$ is invertible (i.e. injective) and \mathcal{W}_w is an OBS if and only if $w = e$ and $T_{\mathcal{W}_w}^* T_{\mathcal{W}_w} = I_{\mathcal{K}_{\mathcal{W}}}$.
4. \mathcal{W}_w is tight if and only if $T_{\mathcal{W}_w} T_{\mathcal{W}_w}^* = A_{\mathcal{W}_w} \cdot I_{\mathcal{H}}$, and \mathcal{W}_w is Parseval if and only if $T_{\mathcal{W}_w}$ is an coisometry (i.e. $T_{\mathcal{W}_w} T_{\mathcal{W}_w}^* = I_{\mathcal{H}}$). ▲

We state another useful result proved in [5], which determines a relationship between frames of subspaces and frames of vectors.

Theorem 3.4. Let $\mathcal{W} = \{W_i\}_{i \in I}$ be a sequence of closed subspaces of \mathcal{H} and let $w \in \ell_+^\infty(I)$. For each $i \in I$, let $\mathcal{G}_i = \{f_{ij}\}_{j \in J_i}$ be a frame for W_i . Suppose that

$$0 < A = \inf_{i \in I} A_{\mathcal{G}_i} \quad \text{and} \quad B = \sup_{i \in I} B_{\mathcal{G}_i} < \infty .$$

Let $\mathcal{E}_i = \{e_{ik}\}_{k \in K_i}$ be an orthonormal basis for each W_i . Then the following conditions are equivalent.

1. $\mathcal{F} = \{w_i f_{ij}\}_{i \in I, j \in J_i} = \{w_i \mathcal{G}_i\}_{i \in I}$ is a frame for \mathcal{H} .
2. $\mathcal{E} = \{w_i e_{ik}\}_{i \in I, k \in K_i} = \{w_i \mathcal{E}_i\}_{i \in I}$ is a frame for \mathcal{H} .
3. $\mathcal{W}_w = (w_i, W_i)_{i \in I}$ is a frame of subspaces for \mathcal{H} .

In this case, the bounds of \mathcal{W}_w satisfy the inequalities

$$\frac{A_{\mathcal{F}}}{B} \leq A_{\mathcal{W}_w} = A_{\mathcal{E}} \quad \text{and} \quad B_{\mathcal{E}} = B_{\mathcal{W}_w} \leq \frac{B_{\mathcal{F}}}{A} . \quad (10)$$

Also $T_{\mathcal{E}} = T_{\mathcal{W}_w}$, using the orthonormal basis $\mathcal{B} = \{e_{ik}\}_{i \in I, k \in K_i}$ of $\mathcal{K}_{\mathcal{W}} = \bigoplus_{i \in I} W_i$. ■

Operators and frames

Our next purpose is to characterize frames of subspaces as images of OBS under an epimorphism with certain properties.

Definition 3.5. Let $\mathcal{W}_w = (w_i, W_i)_{i \in I}$ be a BSS for \mathcal{H} , with synthesis operator $T_{\mathcal{W}_w}$. The excess of \mathcal{W}_w is defined as: $E(\mathcal{W}_w) = \dim N(T_{\mathcal{W}_w})$.

Theorem 3.6. Let $\{E_i\}_{i \in I}$ be an OBS of \mathcal{K} and let $T \in L(\mathcal{K}, \mathcal{H})$ be surjective. Suppose that $0 < \inf_{i \in I} \frac{\gamma(TP_{E_i})}{\|TP_{E_i}\|}$. Let $0 < A, B < \infty$ be such that,

$$\frac{A}{B} \leq \frac{\gamma(TP_{E_i})^2}{\|TP_{E_i}\|^2} \quad \text{i.e. ,} \quad \frac{\|TP_{E_i}\|^2}{B} \leq \frac{\gamma(TP_{E_i})^2}{A} , \quad \forall i \in I . \quad (11)$$

Denote $W_i = T(E_i) \subseteq \mathcal{H}$, for $i \in I$. Let $w = \{w_i\}_{i \in I} \in \ell_+^\infty(I)$ such that

$$\frac{\|TP_{E_i}\|^2}{B} \leq w_i^2 \leq \frac{\gamma(TP_{E_i})^2}{A} \quad \text{for each } i \in I . \quad (12)$$

Then the following statements hold:

1. The sequence $\mathcal{W}_w = (w_i, W_i)_{i \in I}$ is a FS for \mathcal{H} .
2. Moreover, \mathcal{W}_w has bounds

$$\frac{\gamma(T)^2}{B} \leq A_{\mathcal{W}_w} \quad \text{and} \quad B_{\mathcal{W}_w} \leq \frac{\|T\|^2}{A} . \quad (13)$$

3. If $\ker T \cap E_i = \{0\}$ for every $i \in I$, then $E(\mathcal{W}_w) = \dim \ker T$.

Proof. Suppose that (11) and (12) hold for every $i \in I$.

1. Since $\gamma(TP_{E_i}) > 0$, then $W_i = TE_i$ is closed for every $i \in I$. Let $\{b_{ij}\}_{j \in J_i}$ be an orthonormal basis for each E_i . By Proposition 2.8, Eq. (11) and Eq. (12), every sequence $\mathcal{G}_i = \{w_i^{-1} T b_{ij}\}_{j \in J_i}$ is a frame for W_i with

$$A_{\mathcal{G}_i} = w_i^{-2} \gamma(TE_i)^2 \geq A \quad \text{and} \quad B_{\mathcal{G}_i} = w_i^{-2} \|TE_i\|^2 \leq B.$$

On the other hand, since $\{b_{ij}\}_{i \in I, j \in J_i}$ is a orthonormal basis for \mathcal{K} , and T an epimorphism, the sequence $\mathcal{F} = \{Tb_{ij}\}_{i \in I, j \in J_i}$ is a frame for \mathcal{H} . Finally, since $\mathcal{F} = \{w_i(w_i^{-1} Tb_{ij})\}_{i \in I, j \in J_i} = \{w_i \mathcal{G}_i\}_{i \in I}$, Theorem 3.4 implies that \mathcal{W}_w is a FS for \mathcal{H} .

2. Eq. (13) follows from Eq. (10) and the fact that $A_{\mathcal{F}} = \gamma(T)^2$ and $B_{\mathcal{F}} = \|T\|^2$.
3. Suppose that $\ker T \cap E_i = \{0\}$ for every $i \in I$. Then $\ker TP_{E_i} = E_i^\perp$ and $\gamma(TP_{E_i}) \|z\| \leq \|TP_{E_i} z\|$ for every $z \in E_i$. By Eq. (12), for every $x \in \mathcal{K}$ and $i \in I$,

$$A^{1/2} w_i \|P_{E_i} x\| \leq \gamma(TP_{E_i}) \|P_{E_i} x\| \leq \|TP_{E_i} x\| \leq B^{1/2} w_i \|P_{E_i} x\|,$$

and $\|x\|^2 = \sum_{i \in I} \|P_{E_i} x\|^2$. Let $\mathcal{K}_{\mathcal{W}} = \bigoplus_{i \in I} W_i$ (the domain of $T_{\mathcal{W}_w}$). Observe that $T(E_i) = W_i$ for every $i \in I$. Therefore the map

$$V : \mathcal{K} \rightarrow \mathcal{K}_{\mathcal{W}} \quad \text{given by} \quad Vx = (w_i^{-1} T(P_{E_i} x))_{i \in I}, \quad \text{for } x \in \mathcal{K},$$

is well defined, bounded and invertible. By the definition of the synthesis operator $T_{\mathcal{W}_w}$, and the fact that $x = \sum_{i \in I} P_{E_i} x$, for every $x \in \mathcal{K}$, we can deduce that $T_{\mathcal{W}_w} \circ V = T$. Therefore $\dim \ker T = \dim V^{-1}(\ker T_{\mathcal{W}_w}) = \dim \ker T_{\mathcal{W}_w} = E(\mathcal{W}_w)$. \blacksquare

Example 7.1 shows a surjective operator T and an OBS $\mathcal{E} = \{E_i\}_{i \in I}$ such that $\gamma(TP_{E_i}) > 0$ for every $i \in I$, but the sequence $\mathcal{W}_w = (w, \mathcal{W})$ fails to be a FS for every $w \in \ell_+^\infty(I)$. Hence T and \mathcal{E} do not satisfy Eq. (11).

However, Eq. (11) is not a necessary condition in order to assure that $\mathcal{P}(\mathcal{W}) \neq \emptyset$ (see Definition 4.1), if $\mathcal{W} = T\mathcal{E}$. In Example 7.2 we show a FS which is the image of an OBS under an epimorphism which doesn't satisfy Eq. (11).

Remark 3.7. If $\mathcal{W}_w = (w_i, W_i)_{i \in I}$ is a FS for \mathcal{H} , then its synthesis operator $T_{\mathcal{W}_w}$, defined as in Definition 3.2 clearly satisfies Eq. (11). Moreover, it holds that

$$T_{\mathcal{W}_w} g = w_i g \quad \text{for every } g \in E_i, \text{ the copy of } W_i \text{ in } \mathcal{K}_{\mathcal{W}}.$$

Hence $\gamma(T_{\mathcal{W}_w} P_{E_i}) = \|T_{\mathcal{W}_w} P_{E_i}\| = w_i$ for every $i \in I$. \blacktriangle

Remark 3.8. Let $\mathcal{W}_w = (w_i, W_i)_{i \in I}$ be a FS for \mathcal{H} , and let $G \in Gl(\mathcal{H})$. In [5], [7, Thm 2.11] and [14, Thm 2.4] it is proved that $G\mathcal{W}_w = (w_i, GW_i)_{i \in I}$ must be also a FS for \mathcal{H} . We give a short proof of this fact, including extra information about the bounds and the excess of $G\mathcal{W}_w$, in order to illustrate the techniques given by Theorem 3.6. \blacktriangle

Corollary 3.9. *Let $\mathcal{W}_w = (w_i, W_i)_{i \in I}$ be a FS for \mathcal{H} , and let $G \in L(\mathcal{H}, \mathcal{H}_1)$ be invertible. Then $G\mathcal{W}_w = (w_i, GW_i)_{i \in I}$ is a FS for \mathcal{H}_1 , which satisfies that $E(\mathcal{W}_w) = E(G\mathcal{W}_w)$,*

$$(\|G\| \|G^{-1}\|)^{-2} A_{\mathcal{W}_w} \leq A_{G\mathcal{W}_w} \quad \text{and} \quad B_{G\mathcal{W}_w} \leq (\|G\| \|G^{-1}\|)^2 B_{\mathcal{W}_w}.$$

Proof. Denote by E_i the copy of each W_i in $\mathcal{K}_{\mathcal{W}} = \bigoplus_{i \in I} W_i$. Define $T = GT_{\mathcal{W}_w} \in L(\mathcal{K}_{\mathcal{W}}, \mathcal{H}_1)$, which is clearly surjective (since $T_{\mathcal{W}_w}$ is). By Eq. (3) and Remark 3.7,

$$\gamma(TP_{E_i}) \geq \gamma(G) \cdot \gamma(T_{\mathcal{W}_w} P_{E_i}) = \gamma(G) w_i \quad \text{and} \quad \|TP_{E_i}\| \leq \|G\| \|T_{\mathcal{W}_w} P_{E_i}\| = \|G\| w_i,$$

for every $i \in I$. In particular, $T(E_i) \subseteq \mathcal{H}_1$. Then, we can apply Theorem 3.6 for T with constants $A = \gamma(G)^2$ and $B = \|G\|^2$. Indeed, for every $i \in I$, we have seen that

$$\frac{\gamma(G)^2}{\|G\|^2} \leq \frac{\gamma(TP_{E_i})^2}{\|TP_{E_i}\|^2} \quad \text{and} \quad \frac{\|TP_{E_i}\|^2}{\|G\|^2} \leq w_i^2 \leq \frac{\gamma(TP_{E_i})^2}{\gamma(G)^2}.$$

Therefore, $G\mathcal{W}_w = (w_i, GW_i)_{i \in I}$ is a FS for \mathcal{H}_1 by Theorem 3.6. In order to prove the bound inequalities, by Eq. (3) and item 2 of Remark 3.3 we have that

$$\gamma(GT_{\mathcal{W}_w}) \geq \gamma(G) \gamma(T_{\mathcal{W}_w}) = \|G^{-1}\|^{-1} A_{\mathcal{W}_w}^{1/2} \quad \text{and} \quad \|GT_{\mathcal{W}_w}\| \leq \|G\| \|T_{\mathcal{W}_w}\| = \|G\| B_{\mathcal{W}_w}^{1/2}.$$

Now apply Eq. (13) of Theorem 3.6 with our constants $A = \|G^{-1}\|^{-2}$ and $B = \|G\|^2$. It is easy to see that $\ker T = \ker T_{\mathcal{W}_w}$. Then $\ker T \cap E_i = \{0\}$ ($i \in I$). By Theorem 3.6, we deduce that $E(\mathcal{W}_w) = \dim \ker T_{\mathcal{W}_w} = \dim \ker T = E(G\mathcal{W}_w)$. \blacksquare

4 Admissible weights

Definition 4.1. We say that $\mathcal{W} = \{W_i\}_{i \in I}$ is a *generating* sequence of \mathcal{H} , if $W_i \subseteq \mathcal{H}$ for every $i \in I$, and $\overline{\text{span}} \{W_i : i \in I\} = \mathcal{H}$. In this case, we define

$$\mathcal{P}(\mathcal{W}) = \{w \in \ell_+^\infty(I) : \mathcal{W}_w = (w_i, W_i)_{i \in I} \text{ is a FS for } \mathcal{H}\} \subseteq \ell_+^\infty(I),$$

the set of *admissible* sequences of weights for \mathcal{W} . \blacktriangle

It is apparent that, if $\mathcal{W}_w = (w_i, W_i)_{i \in I}$ is a FS for \mathcal{H} , then $\mathcal{W} = \{W_i\}_{i \in I}$ is a generating sequence. Nevertheless, in Examples 7.1 and 7.3 we shall see that there exist generating sequences $\mathcal{W} = \{W_i\}_{i \in I}$ for \mathcal{H} such that $\mathcal{P}(\mathcal{W}) = \emptyset$. Recall that we denote by

$$\ell_+^\infty(I)^* = \{\{w_i\}_{i \in I} \in \ell_+^\infty(I) : \inf_{i \in I} w_i > 0\} = \ell_+^\infty(I) \cap Gl(\ell_+^\infty(I)). \quad (14)$$

Proposition 4.2. *Let $\mathcal{W} = \{W_i\}_{i \in I}$ be a generating sequence of \mathcal{H} .*

1. *If $w \in \mathcal{P}(\mathcal{W})$, then $aw \in \mathcal{P}(\mathcal{W})$ and $E(\mathcal{W}_w) = E(\mathcal{W}_{aw})$, for every $a \in \ell_+^\infty(I)^*$.*
2. *If $\mathcal{W}_w = (w, \mathcal{W})$ is a RBS, for some $w \in \ell_+^\infty(I)$, then $\mathcal{P}(\mathcal{W}) = \ell_+^\infty(I)^*$, and (a, \mathcal{W}) is still a RBS for every $a \in \ell_+^\infty(I)^*$. In particular, (e, \mathcal{W}) is a RBS.*

3. Let $G \in Gl(\mathcal{H})$. Then $\mathcal{P}(\mathcal{W}) = \mathcal{P}(\{GW_i\}_{i \in I})$. In other words, a sequence $w \in \ell_+^\infty(I)$ is admissible for \mathcal{W} if and only if it is admissible for $G\mathcal{W}$.

Proof. Let $\mathcal{K}_{\mathcal{W}} = \bigoplus_{i \in I} W_i$, and denote by $E_i \subseteq \mathcal{K}_{\mathcal{W}}$ the copy of each W_i in \mathcal{K} .

1. For every $a \in \ell_+^\infty(I)^*$, consider the SOT limit $D_a = \sum_{i \in I} a_i P_{E_i}$. Then $D_a \in Gl(\mathcal{K}_{\mathcal{W}})^+$.

Therefore, if $T_{\mathcal{W}_w} \in L(\mathcal{K}_{\mathcal{W}}, \mathcal{H})$ is the synthesis operator of \mathcal{W}_w , then $T_{\mathcal{W}_w} \circ D_a$ is, by definition, the synthesis operator of (aw, \mathcal{W}) . Since $T_{\mathcal{W}_w} D_a$ is bounded and surjective, then (aw, \mathcal{W}) is also a FS. Note that $N(T_{\mathcal{W}_{aw}}) = N(T_{\mathcal{W}_w} D_a) = D_a^{-1}(N(T_{\mathcal{W}_w}))$.

2. If \mathcal{W}_w is a RBS for \mathcal{H} , then $T_{\mathcal{W}_w}$ is invertible. Since $T_{\mathcal{W}_w} x = w_i x$ for $x \in E_i$, then $w_i \geq \gamma(T_{\mathcal{W}_w}) = A_{\mathcal{W}_w}^{1/2}$ for every $i \in I$. This implies that $w \in \ell_+^\infty(I)^*$. Observe that $w \ell_+^\infty(I)^* = \ell_+^\infty(I)^*$ (because $w^{-1} \in \ell_+^\infty(I)^*$). Then $\ell_+^\infty(I)^* \subseteq \mathcal{P}(\mathcal{W})$ by item (1). But, for every $a \in \mathcal{P}(\mathcal{W})$, we have that \mathcal{W}_a is a RBS, because \mathcal{W} is still minimal. Then $s \in \ell_+^\infty(I)^*$.

3. Apply Corollary 3.9 for G and G^{-1} . ■

Definition 4.3. Let $\mathcal{W} = \{W_i\}_{i \in I}$ be a generating sequence of \mathcal{H} . Given $v, w \in \mathcal{P}(\mathcal{W})$, we say that v and w are *equivalent* if there exists $a \in \ell_+^\infty(I)^*$ such that $v = a \cdot w$. ▲

Remarks 4.4. Let $\mathcal{W} = \{W_i\}_{i \in I}$ be a generating sequence of \mathcal{H} .

1. By Proposition 4.2, if $w \in \mathcal{P}(\mathcal{W})$, then its whole equivalence class $w \cdot \ell_+^\infty(I)^* \subseteq \mathcal{P}(\mathcal{W})$.
2. On the other hand, in Example 7.5 below we shall see that there exist generating sequences \mathcal{W} of \mathcal{H} with infinite not equivalent sequences $w \in \mathcal{P}(\mathcal{W})$.
3. If \mathcal{W}_w is a RBS for \mathcal{H} , then by Proposition 4.2 all the admissible sequences for \mathcal{W} are equivalent to w , since $\mathcal{P}(\mathcal{W}) = \ell_+^\infty(I)^*$. Since $\mathcal{W}_v = (v, \mathcal{W})$ is a RBS for \mathcal{H} for every $v \in \ell_+^\infty(I)^*$, from now on we will not mention the weights. We just say that the sequence of subspaces \mathcal{W} is a Riesz basis of subspaces.
4. By definition, if \mathcal{W} is a RBS, then it is a minimal sequence. Nevertheless, in Example 7.3, we shall see that there exist minimal sequences which are generating for \mathcal{H} , but with $\mathcal{P}(\mathcal{W}) = \emptyset$. ▲

Proposition 4.5. Let $\mathcal{E} = \{E_i\}_{i \in I}$ be a OBS for \mathcal{H} . Let $G \in L(\mathcal{H}, \mathcal{H}_1)$ be an invertible operator. Then the sequence $\mathcal{W} = (GE_i)_{i \in I}$ is a RBS for \mathcal{H}_1 .

Proof. It is a consequence of Corollary 3.9. ■

Remark 4.6. It is well known (and easy to verify) that for a frame $\mathcal{F} = \{f_i\}_{i \in I}$ in \mathcal{H} , the sequence $\{S_{\mathcal{F}}^{-1/2} f_i\}_{i \in I}$ is a Parseval frame. Nevertheless, if $\mathcal{W}_w = (w_i, W_i)_{i \in I}$ is a FS, then $S_{\mathcal{W}_w}^{-1/2} \mathcal{W}_w$ may be not a Parseval FS (see Example 7.5 below), neither allowing to change the sequence of weights. Even worse, there exist frames of subspaces $\mathcal{W}_w = (w, \mathcal{W})$ for \mathcal{H} such that the sequence $(v, G\mathcal{W})$ fails to be a Parseval FS for \mathcal{H} for every $G \in Gl(\mathcal{H})$ and

$v \in \ell_+^\infty(I)$ (see Example 7.6). In the next Proposition we shall see that the situation is different for a RBS of \mathcal{H} : ▲

Proposition 4.7. *Let $\mathcal{W} = \{W_i\}_{i \in I}$ be a RBS for \mathcal{H} . Then, for every $w \in \ell_+^\infty(I)^*$, the sequence $\{S_{\mathcal{W}_w}^{-1/2}W_i\}_{i \in I}$ is an orthonormal basis of subspaces.*

Proof. Let $\{e_{ik}\}_{k \in K_i}$ be an orthonormal basis of each W_i . According Theorem 3.4, the sequence $\mathcal{E} = \{w_i e_{ik}\}_{i \in I, k \in K_i}$ is a Riesz basis of \mathcal{H} and $T_{\mathcal{E}} = T_{\mathcal{W}_w}$. Hence the sequence $\{w_i S_{\mathcal{E}}^{-1/2} e_{ik}\}_{i \in I, k \in K_i}$ is an orthonormal basis for \mathcal{H} . Since $S_{\mathcal{W}_w} = S_{\mathcal{E}}$ and $\{w_i S_{\mathcal{W}_w}^{-1/2} e_{ik}\}_{k \in K_i}$ is a orthonormal basis of each subspace $S_{\mathcal{W}_w}^{-1/2}W_i$, then $\{S_{\mathcal{W}_w}^{-1/2}W_i\}_{i \in I}$ is an OBS for \mathcal{H} . ■

5 Projections and frames

In this section we obtain a generalization of two results of [2], which relates FS (including the computation of their weights) and oblique projections (see also [3]). Unlike for vector frames, all the results are in “one direction”. The converses fail in general (see Example 7.4 and Remarks 5.3 and 5.5).

Theorem 5.1. *Let $\mathcal{W}_w = (w_i, W_i)_{i \in I}$ be a FS for \mathcal{H} . Then there exists a Hilbert space $\mathcal{V} \supseteq \mathcal{H}$ and a Riesz basis of subspaces $\{B_i\}_{i \in I}$ for \mathcal{V} such that*

$$P_{\mathcal{H}}(B_i) = W_i \quad \text{and} \quad A_{\mathcal{W}_w}^{1/2} \|P_{\mathcal{H}}P_{B_i}\| \leq w_i \leq B_{\mathcal{W}_w}^{1/2} \|P_{\mathcal{H}}P_{B_i}\| \quad \text{for every } i \in I.$$

This means that the new sequence of weights $v_i = \|P_{\mathcal{H}}P_{B_i}\|$, $i \in I$, is equivalent to w . Also, we can compute $E(\mathcal{W}_w) = \dim \mathcal{V} \ominus \mathcal{H}$.

Proof. Denote by E_i the copy of each W_i in $\mathcal{K}_{\mathcal{W}} = \bigoplus_{i \in I} W_i$. Let $T_{\mathcal{W}_w} \in L(\mathcal{K}_{\mathcal{W}}, \mathcal{H})$ be the synthesis operator for \mathcal{W}_w . Denote by $\mathcal{N} = N(T_{\mathcal{W}_w})$ and $\mathcal{V} = \mathcal{H} \oplus \mathcal{N}$. We can identify \mathcal{H} with $\mathcal{H} \oplus \{0\} \subseteq \mathcal{V}$. Let —

$$U : \mathcal{K}_{\mathcal{W}} \rightarrow \mathcal{V} \quad \text{given by} \quad U(x) = T_{\mathcal{W}_w}x \oplus \gamma(T_{\mathcal{W}_w}) P_{\mathcal{N}} x, \quad x \in \mathcal{K}_{\mathcal{W}}. \quad (15)$$

Since $\mathcal{K}_{\mathcal{W}} = \mathcal{N}^\perp \oplus \mathcal{N}$ and $T_{\mathcal{W}_w}|_{\mathcal{N}^\perp} : \mathcal{N}^\perp \rightarrow \mathcal{H}$ is invertible, we can deduce that U is bounded and invertible. Moreover, it is easy to see that

$$\|U^{-1}\|^{-1} = \gamma(U) = \gamma(T_{\mathcal{W}_w}) = A_{\mathcal{W}_w}^{1/2} \quad \text{and} \quad \|U\| = \|T_{\mathcal{W}_w}\| = B_{\mathcal{W}_w}^{1/2}. \quad (16)$$

By Proposition 4.5, the sequence $\{B_i\}_{i \in I} = \{U(E_i)\}_{i \in I}$ is a RBS for \mathcal{V} . Observe that

$$P_{\mathcal{H}}(B_i) = P_{\mathcal{H}}U(E_i) = T_{\mathcal{W}_w}(E_i) \oplus \{0\} = W_i \oplus \{0\} \sim W_i, \quad \text{for every } i \in I.$$

Let y be an unit vector of $B_i = U(E_i)$. Then $y = Ux$ with $x \in E_i$. We have that

$$\gamma(U)\|x\| \leq \|Ux\| = \|y\| = 1 \leq \|U\| \|x\|.$$

Recall that E_i is the copy of W_i in \mathcal{K} . If $x \in E_i$, we denote by x_i its component in W_i (the others are zero). Using that $\|P_{\mathcal{H}} y\| = \|T_{\mathcal{W}_w} x\| = w_i \|x_i\| = w_i \|x\|$ and Eq. (16), we can conclude that for every such y (i.e. any unit vector of B_i),

$$A_{\mathcal{W}_w}^{1/2} \|P_{\mathcal{H}} y\| = \gamma(T_{\mathcal{W}_w}) \|P_{\mathcal{H}} y\| = w_i \gamma(U) \|x\| \leq w_i \implies A_{\mathcal{W}_w}^{1/2} \|P_{\mathcal{H}} P_{B_i}\| \leq w_i .$$

Similarly, $w_i \leq w_i \|U\| \|x\| = B_{\mathcal{W}_w}^{1/2} \|P_{\mathcal{H}} y\| \leq B_{\mathcal{W}_w}^{1/2} \|P_{\mathcal{H}} P_{B_i}\|$. ■

As a particular case of Theorem 5.1, we get a result proved by Asgari and Khosravi [3] (see also [7]), with some information extra:

Corollary 5.2. *Let $\mathcal{W}_w = (w_i, W_i)_{i \in I}$ be a Parseval FS for \mathcal{H} . Then there exists a Hilbert space $\mathcal{V} \supseteq \mathcal{H}$ and an **orthonormal basis** of subspaces $\{F_i\}_{i \in I}$ for \mathcal{V} such that*

$$P_{\mathcal{H}}(F_i) = W_i \quad \text{and} \quad w_i = c[\mathcal{H}, F_i] = \|P_{\mathcal{H}} P_{F_i}\| \quad \text{for every } i \in I .$$

Proof. We use the notations of the proof of Theorem 5.1. If \mathcal{W}_w is Parseval, then $A_{\mathcal{W}_w} = B_{\mathcal{W}_w} = 1$. By Eq. (16), this implies that the operator $U \in L(\mathcal{K}, \mathcal{V})$ defined in Eq. (15) becomes unitary (it is an invertible isometry). Hence, in this case, the sequence $\{F_i\}_{i \in I} = \{U(E_i)\}_{i \in I}$ is a orthonormal basis of subspaces for \mathcal{V} . Also, by Theorem 5.1, we have that $w_i = \|P_{\mathcal{H}} P_{F_i}\|$ for every $i \in I$. It is easy to see that $F_i \cap (\mathcal{H} \oplus \{0\}) \neq \{0\}$ implies that $w_i = 1$ and $F_i \subseteq (\mathcal{H} \oplus \{0\})$ (because U is unitary). Then, we can deduce that $\|P_{\mathcal{H}} P_{F_i}\| = c[\mathcal{H}, F_i]$ for every $i \in I$. ■

Remark 5.3. Although the converse of Corollary 5.2 fails in general, it holds with some special assumptions, based on Theorem 3.6: If $\mathcal{E} = \{E_i\}_{i \in I}$ is a OBS for $\mathcal{V} \supseteq \mathcal{H}$ such that $0 < \inf_{i \in I} \frac{\gamma(P_{\mathcal{H}} P_{E_i})}{\|P_{\mathcal{H}} P_{E_i}\|}$, then $\mathcal{P}(\mathcal{W}) \neq \emptyset$, where $W_i = P_{\mathcal{H}}(E_i)$, $i \in I$. Moreover, as in Theorem 3.6, it can be found a concrete $w \in \mathcal{P}(\mathcal{W})$. Nevertheless, we can not assure that \mathcal{W}_w is a Parseval FS. ▲

The following theorem is closely related with a result proved by Casazza, Kutyniok and Li in [7, Thm. 3.1].

Theorem 5.4. *Let $\mathcal{W}_w = (w_i, W_i)_{i \in I}$ be a FS for \mathcal{H} such that $1 \leq A_{\mathcal{W}_w}$. Denote by $\mathcal{V} = \mathcal{H} \oplus \mathcal{K}_{\mathcal{W}}$. Then there exist an **oblique** projection $Q \in L(\mathcal{V})$ with $R(Q) = \mathcal{H} \oplus \{0\}$ and an **orthonormal system** of subspaces $\{B_i\}_{i \in I}$ in \mathcal{V} , such that*

$$W_i \oplus 0 = Q(B_i) \quad \text{and} \quad w_i = \|Q P_{B_i}\| = \gamma(Q P_{B_i}) \quad \text{for every } i \in I .$$

*Moreover, if $E(\mathcal{W}_w) = \infty$, then the sequence $\{B_i\}_{i \in I}$ can be supposed to be an **orthonormal basis** of subspaces of \mathcal{V} .*

Proof. Write $T_{\mathcal{W}_w} = T$. By hypothesis, $TT^* = S_{\mathcal{W}_w} \geq A_{\mathcal{W}_w} I \geq I$. Denote by

$$X = (TT^* - I)^{1/2} \in L(\mathcal{H})^+ .$$

Consider the (right) polar decomposition $T = |T^*|V$, where $V \in L(\mathcal{K}_{\mathcal{W}}, \mathcal{H})$ is a partial isometry with initial space $N(T)^\perp$ and final space \mathcal{H} , so that $VV^* = I_{\mathcal{H}}$. Consider the “ampliation” $\tilde{T} \in L(\mathcal{K}_{\mathcal{W}}, \mathcal{V})$ given by $\tilde{T}x = Tx \oplus 0$. Then $\tilde{T}\tilde{T}^* = \begin{pmatrix} TT^* & 0 \\ 0 & 0 \end{pmatrix} \begin{smallmatrix} \mathcal{H} \\ \mathcal{K}_{\mathcal{W}} \end{smallmatrix} \in L(\mathcal{V})$.

Define

$$Q = \begin{pmatrix} I_{\mathcal{H}} & XV \\ 0 & 0 \end{pmatrix} \begin{smallmatrix} \mathcal{H} \\ \mathcal{K}_{\mathcal{W}} \end{smallmatrix} \in L(\mathcal{V}) .$$

Then it is clear that Q is an oblique projection with $R(Q) = \mathcal{H} \oplus 0$. Moreover,

$$QQ^* = \begin{pmatrix} I_{\mathcal{H}} + XX^* & 0 \\ 0 & 0 \end{pmatrix} = \tilde{T}\tilde{T}^* \implies |Q^*| = |\tilde{T}^*| .$$

Define $U \in L(\mathcal{K}_{\mathcal{W}}, \mathcal{V})$ by

$$Ux = VP_{N(T)^\perp}x \oplus P_{N(T)}x , \quad \text{for } x \in \mathcal{K}_{\mathcal{W}} . \quad (17)$$

Then U is an **isometry**, because the initial space of V is $N(T)^\perp$. Note that also $\tilde{T} = |\tilde{T}^*|U$. The partial isometry of the right polar decomposition of Q extends to an unitary operator W on \mathcal{V} , because $\dim N(Q) = \dim R(Q)^\perp$. Moreover, $Q = |Q^*|W$. Then

$$\tilde{T} = |\tilde{T}^*|U = |Q^*|U = QW^*U .$$

Therefore, if we consider the OBS $\{E_i\}_{i \in I}$ of $\mathcal{K}_{\mathcal{W}}$,

$$W_i = T(E_i) \sim T(E_i) \oplus 0 = \tilde{T}(E_i) = QW^*U(E_i) = Q(B_i) , \quad i \in I ,$$

where $\{B_i\}_{i \in I} = \{W^*UE_i\}_{i \in I}$, which is clearly an orthonormal system in \mathcal{V} . If $y \in B_i$ is an unit vector, then $y = W^*Ux$ for $x \in E_i$ with $\|x\| = 1$, and

$$w_i = \|Tx\| = \|QW^*Ux\| = \|Qy\| \implies w_i = \|Q P_{B_i}\| = \gamma(Q P_{B_i}) .$$

Suppose now that $\dim N(T) = \infty$. Then the isometry U defined in equation (17) can be changed to an unitary operator from $\mathcal{K}_{\mathcal{W}}$ onto \mathcal{V} , still satisfying that $\tilde{T} = |\tilde{T}^*|U$. Indeed, take

$$U'x = VP_{N(T)^\perp}x \oplus Y P_{N(T)}x , \quad \text{for } x \in \mathcal{K}_{\mathcal{W}} ,$$

where $Y \in L(\mathcal{K}_{\mathcal{W}})$ is a partial isometry with initial space $N(T)$ and final space $\mathcal{K}_{\mathcal{W}}$. It is easy to see that U' is unitary. Then the sequence $\{B'_i\}_{i \in I} = \{W^*U'E_i\}_{i \in I}$ turns to be an OBS for \mathcal{V} . \square

Remark 5.5. As in Remark 5.3, it holds a kind of converse for Theorem 5.4, i.e., if $\inf_{i \in I} \frac{\gamma(Q P_{B_i})}{\|Q P_{B_i}\|} > 0$, then $\mathcal{P}(\{Q(B_i)\}_{i \in I}) \neq \emptyset$. \blacktriangle

6 Refinements of frames of subspaces

In [5] it is shown by an example that a FS with $E(\mathcal{W}_w) > 0$ can be exact, i.e. $(w_i, W_i)_{i \in J}$ is not a FS, for every proper $J \subset I$. This situation is possible because the excess of the frame can be contained properly in some $W_i \in \mathcal{W}_w$, so if we “erase” any of the subspaces of \mathcal{W}_w , this new sequence is not generating anymore.

Then, the notion of “excess” is not the same as for vector frames, in the sense of Definition 2.9 and Eq. (6). In this section, we introduce the notion of refinements of subspace sequences, which shall work as the natural way to recover the connection between excess and erasures. The results of this section are closely related with those of [6, Section 4].

Definition 6.1. Let $\mathcal{W} = \{W_i\}_{i \in I}$ be a sequence of closed subspaces.

1. A *refinement* of \mathcal{W} is a sequence $\mathcal{V} = \{V_i\}_{i \in J}$ of closed subspaces such that

- (a) $J \subseteq I$.
- (b) $\{0\} \neq V_i \subseteq W_i$ for every $i \in J$.

In this case we use the following notations:

2. The excess of \mathcal{W} over \mathcal{V} is the cardinal number

$$E(\mathcal{W}, \mathcal{V}) = \sum_{i \in J} \dim(W_i \ominus V_i) + \sum_{i \notin J} \dim W_i .$$

3. If $w \in \mathcal{P}(\mathcal{W})$, we say that $\mathcal{V}_w = (w_i, V_i)_{i \in J}$ is a *FS refinement* (FSR) of \mathcal{W}_w if \mathcal{V}_w is a FS for \mathcal{H} . ▲

Remark 6.2. It is easy to see that, if \mathcal{V} is a refinement of \mathcal{W} and \mathcal{V}' is a refinement of \mathcal{V} , then \mathcal{V}' is a refinement of \mathcal{W} and $E(\mathcal{W}, \mathcal{V}') = E(\mathcal{W}, \mathcal{V}) + E(\mathcal{V}, \mathcal{V}')$. ▲

The next result uses basic Fredholm theory. We refer to J. B. Conway book [9, Ch. XI].

Lemma 6.3. Let $\mathcal{W}_w = (w_i, W_i)_{i \in I}$ be a FS for \mathcal{H} and let $\mathcal{V} = \{V_i\}_{i \in J}$ be a refinement of \mathcal{W} . We consider $\mathcal{K}_{\mathcal{V}} = \oplus_{i \in J} \mathcal{V}_i$ as a subspace of $\oplus_{i \in I} W_i = \mathcal{K}_{\mathcal{W}}$. Then

1. $E(\mathcal{W}, \mathcal{V}) = \dim \mathcal{K}_{\mathcal{V}}^\perp = \dim \ker P_{\mathcal{K}_{\mathcal{V}}}$.
2. $\mathcal{V}_w = (w_i, V_i)_{i \in J}$ a FS refinement of \mathcal{W}_w if and only if $T_{\mathcal{W}_w} P_{\mathcal{K}_{\mathcal{V}}}$ is surjective.

In this case, we have that

3. $E(\mathcal{W}, \mathcal{V}) \leq E(\mathcal{W}_w)$.
4. If $E(\mathcal{W}, \mathcal{V}) < \infty$, then $E(\mathcal{V}_w) = E(\mathcal{W}_w) - E(\mathcal{W}, \mathcal{V})$.

Proof. For each $i \in I$, denote by E_i (resp. F_i) the copy of \mathcal{W}_i (resp. \mathcal{V}_i , or $F_i = \{0\}$ if $i \notin J$) in $\mathcal{K}_{\mathcal{W}}$. Then $\mathcal{K}_{\mathcal{V}}^\perp = \oplus_{i \in I} E_i \ominus F_i$, showing (1). Denote by $P = P_{\mathcal{K}_{\mathcal{V}}}$. By construction, $T_{\mathcal{V}_w} = T_{\mathcal{W}_w}|_{\mathcal{K}_{\mathcal{V}}} = T_{\mathcal{W}_w}|_{R(P)} \in L(\mathcal{K}_{\mathcal{V}}, \mathcal{H})$. Then $R(T_{\mathcal{W}_w}P) = R(T_{\mathcal{V}_w}) = \mathcal{H}$ if and only if \mathcal{V}_w a FS refinement of \mathcal{W}_w . In this case, $\{0\} = \ker PT_{\mathcal{W}_w}^*$. Since $R(T_{\mathcal{W}_w}^*) = \ker T_{\mathcal{W}_w}^\perp$, then

$$\ker T_{\mathcal{W}_w}^\perp \cap \ker P = \{0\} \implies E(\mathcal{W}, \mathcal{V}) = \dim \ker P \leq \dim \ker T_{\mathcal{W}_w} = E(\mathcal{W}_w) .$$

Observe that $T_{\mathcal{W}_w}$ is a semi-Fredholm operator, with $\text{Ind}(T_{\mathcal{W}_w}) = \dim \ker T_{\mathcal{W}_w} - 0 = E(\mathcal{W}_w)$. If $E(\mathcal{W}, \mathcal{V}) < \infty$, then P is a Fredholm operator, with $\text{Ind}(P) = 0$. Hence, we have that $E(\mathcal{W}_w) = \text{Ind}(T_{\mathcal{W}_w}) + \text{Ind}(P) = \text{Ind}(T_{\mathcal{W}_w}P) = \dim \ker T_{\mathcal{W}_w}P$. Finally, since $T_{\mathcal{V}_w} = T_{\mathcal{W}_w}|_{\mathcal{K}_{\mathcal{V}}}$,

$$E(\mathcal{V}_w) = \dim \ker T_{\mathcal{V}_w} = \dim \ker T_{\mathcal{W}_w}P - \dim \ker P = E(\mathcal{W}_w) - E(\mathcal{W}, \mathcal{V}) ,$$

which completes the proof. ■

Lemma 6.4. *Let $\mathcal{W}_w = (w_i, W_i)_{i \in I}$ be a FS for \mathcal{H} with $E(\mathcal{W}_w) > 0$. Then there exists a FS refinement $\mathcal{V}_w = (w_i, V_i)_{i \in J}$ of \mathcal{W}_w with $E(\mathcal{W}, \mathcal{V}) = 1$.*

Proof. For each $i \in I$, denote by E_i the copy of \mathcal{W}_i in $\mathcal{K}_{\mathcal{W}}$. Suppose that there is no FS refinement \mathcal{V}_w of \mathcal{W}_w with $E(\mathcal{W}, \mathcal{V}) = 1$. Then, by Lemma 6.3, for every $i \in I$ and every unit vector $e \in E_i$, it holds that $R(T_{\mathcal{W}_w}P_{\{e\}^\perp}) \neq \mathcal{H}$. By Proposition 2.2 and Eq. (3),

$$c[N(T_{\mathcal{W}_w}), \{e\}^\perp] = c[N(T_{\mathcal{W}_w})^\perp, \text{span}\{e\}] < 1 \implies R(T_{\mathcal{W}_w}P_{\{e\}^\perp}) \subseteq \mathcal{H} .$$

Take $x_e \in R(T_{\mathcal{W}_w}P_{\{e\}^\perp})^\perp = \ker P_{\{e\}^\perp}T_{\mathcal{W}_w}^*$ a unit vector. Then $0 \neq T_{\mathcal{W}_w}^*x_e \in \text{span}\{e\}$, i.e., $e \in R(T_{\mathcal{W}_w}^*)$. This implies that $\cup_{i \in I} E_i \subseteq R(T_{\mathcal{W}_w}^*)$ (which is closed), so that $T_{\mathcal{W}_w}^*$ is surjective and $E(\mathcal{W}_w) = 0$. ■

Theorem 6.5. *Let $\mathcal{W}_w = (w_i, W_i)_{i \in I}$ be a FS for \mathcal{H} . Then*

$$E(\mathcal{W}_w) = \sup \left\{ E(\mathcal{W}, \mathcal{V}) : \mathcal{V}_w = (w_i, V_i)_{i \in J} \text{ is a FS refinement of } \mathcal{W}_w \right\} . \quad (18)$$

In particular, if $E(\mathcal{W}_w) = \infty$, then, for every $n \in \mathbb{N}$, there exists a FS refinement $\mathcal{V}_w = (w_i, V_i)_{i \in J}$ of \mathcal{W}_w such that $E(\mathcal{W}, \mathcal{V}) = n$.

Proof. Denote by α the supremum of Eq. (18). Observe that item 3 of Lemma 6.3 says that $\alpha \leq E(\mathcal{W}_w)$. If $E(\mathcal{W}_w) < \infty$, combining Remark 6.2, Lemma 6.4 and item 4 of Lemma 6.3, one obtains an inductive argument which shows that $\alpha \geq E(\mathcal{W}_w)$. If $E(\mathcal{W}_w) = \infty$, a similar inductive argument shows that, for every $n \in \mathbb{N}$, there exists a FS refinement \mathcal{V}_w of \mathcal{W}_w such that $E(\mathcal{W}, \mathcal{V}) = n$. ■

Corollary 6.6. *Let $\mathcal{W}_w = (w_i, W_i)_{i \in I}$ be a FS of \mathcal{H} such that $E(\mathcal{W}_w) < \infty$. Then*

1. *The sequence $w \in \ell_+^\infty(I)^*$.*
2. *There exists a FS refinement $\mathcal{V}_w = (w_i, V_i)_{i \in J}$ of \mathcal{W}_w such that:*
 - (a) *\mathcal{V} is a RBS for \mathcal{H} .*

$$(b) \ E(\mathcal{W}, \mathcal{V}) = E(\mathcal{W}_w).$$

Proof. By Theorem 6.5, there exists a FS refinement $\mathcal{V}_w = (w_i, V_i)_{i \in J}$ of \mathcal{W}_w such that $E(\mathcal{W}, \mathcal{V}) = E(\mathcal{W}_w)$. By item 4 of Lemma 6.3, $E(\mathcal{V}_w) = 0$. This means that \mathcal{V}_w is a RBS for \mathcal{H} . Then, by Proposition 4.2, the sequence $\{w_i\}_{i \in J} \in \ell_+^\infty(J)^*$. Since $E(\mathcal{W}, \mathcal{V}) < \infty$, then $I \setminus J$ is finite, and we get that also $w \in \ell_+^\infty(I)^*$. ■

Corollary 6.7. *Let $\mathcal{W}_w = (w_i, W_i)_{i \in I}$ be a FS for \mathcal{H} such that $E(\mathcal{W}_w) < \infty$. Then*

$$\mathcal{P}(\mathcal{W}) = \ell_+^\infty(I)^* \quad \text{and} \quad E(\mathcal{W}_v) = E(\mathcal{W}_w) \quad \text{for every other } v \in \mathcal{P}(\mathcal{W}).$$

Proof. By Corollary 6.6, we know that $w \in \ell_+^\infty(I)^*$. By Proposition 4.2, we deduce that $\ell_+^\infty(I)^* \subseteq \mathcal{P}(\mathcal{W})$. Let $\mathcal{V}_w = (w_i, V_i)_{i \in J}$ be a FS refinement of \mathcal{W}_w which is a RBS for \mathcal{H} , provided by Corollary 6.6. Let $v \in \mathcal{P}(\mathcal{W})$. We claim that the sequence $\mathcal{V}_v = (v_i, V_i)_{i \in J}$ is a FS refinement of \mathcal{W}_v .

Indeed, consider $T_{\mathcal{V}_v} = T_{\mathcal{W}_v}|_{\mathcal{K}_{\mathcal{V}}} \in L(\mathcal{K}_{\mathcal{V}}, \mathcal{H})$. By Lemma 6.3, $\dim \mathcal{K}_{\mathcal{V}}^\perp = E(\mathcal{W}, \mathcal{V}) < \infty$. As in the proof of Lemma 6.4, this implies that $R(T_{\mathcal{V}_v}) = R(T_{\mathcal{W}_v} P_{\mathcal{K}_{\mathcal{V}}}) \sqsubseteq \mathcal{H}$. On the other hand, $\text{span}\{\cup_{i \in J} V_i\} \subseteq R(T_{\mathcal{V}_v})$. But $\text{span}\{\cup_{i \in J} V_i\}$ is dense in \mathcal{H} , because $T_{\mathcal{V}_w}$ is surjective (recall that \mathcal{V}_w is a FS). This shows that also $T_{\mathcal{V}_v}$ is surjective, i.e. \mathcal{V}_v is a FS as claimed. In other words, we have that \mathcal{V} is a RBS, and $v_J = \{v_i\}_{i \in J} \in \mathcal{P}(\mathcal{V})$. By Proposition 4.2, $v_J \in \ell_+^\infty(J)^*$. As before, this implies that $v \in \ell_+^\infty(I)^*$. Using Proposition 4.2 again, we conclude that $E(\mathcal{W}_v) = E(\mathcal{W}_w)$. ■

Theorem 6.8. *Let $\mathcal{W}_w = (w_i, W_i)_{i \in I}$ be a FS for \mathcal{H} . Then*

$$E(\mathcal{W}_v) = E(\mathcal{W}_w) \quad \text{for every other } v \in \mathcal{P}(\mathcal{W}).$$

Proof. If $E(\mathcal{W}_w) < \infty$, apply Corollary 6.7. If $E(\mathcal{W}_w) = \infty$ and $v \in \mathcal{P}(\mathcal{W})$, then also $E(\mathcal{W}_v) = \infty$, since otherwise we could apply Corollary 6.7 to \mathcal{W}_v . ■

7 Examples

Observe that, if $\{E_i\}_{i \in I}$ is an OBS of \mathcal{K} and $T \in L(\mathcal{K}, \mathcal{H})$ is a surjective operator such that $T(E_i) \sqsubseteq \mathcal{H}$ for every $i \in I$, then $\mathcal{W} = \{TE_i\}_{i \in I}$ is a generating sequence for \mathcal{H} . Nevertheless, our first example shows that, in general, such a sequence \mathcal{W} may have $\mathcal{P}(\mathcal{W}) = \emptyset$, i.e. \mathcal{W}_w fails to be a FS for \mathcal{H} , for any sequence $w \in \ell_+^\infty(I)$ of weights.

Example 7.1. Take $\mathcal{B} = \{e_n\}_{n \in \mathbb{N}}$ an orthonormal basis of \mathcal{H} . For every $k \in \mathbb{N}$, consider the space $E_k = \text{span}\{e_{2k-1}, e_{2k}\}$. Observe that E_k is an OBS for \mathcal{H} . Consider the (densely defined) operator $T : \mathcal{H} \rightarrow \mathcal{H}$ given by

$$Te_n = \begin{cases} 2^{-k} e_1 & \text{if } n = 2k - 1 \\ e_{k+1} & \text{if } n = 2k \end{cases}.$$

Then, T can be extended to a bounded surjective operator T , since the sequence $\{Te_k\}_{k \in \mathbb{N}}$ is easily seen to be a tight frame for \mathcal{H} . We shall see that the sequence of closed subspaces

$$\mathcal{W} = \{W_k\}_{k \in \mathbb{N}} \quad \text{given by} \quad W_k = T(E_k) = \text{span}\{e_1, e_{k+1}\} \quad , \quad k \in \mathbb{N}$$

satisfies that $\mathcal{P}(\mathcal{W}) = \emptyset$. Indeed, suppose that $w \in \mathcal{P}(\mathcal{W})$. Then by Eq. (8) applied to $f = e_1 \in \bigcap_{k \in \mathbb{N}} W_k$, we would have that $w \in \ell^2(\mathbb{N})$. But this contradicts the existence of a lower frame bound $A_{\mathcal{W}_w}$ for $\mathcal{W}_w = (w_k, W_k)_{k \in \mathbb{N}}$, because for every $k \in \mathbb{N}$,

$$A_{\mathcal{W}_w} = A_{\mathcal{W}_w} \|e_{k+1}\|^2 \leq \sum_{j \in \mathbb{N}} w_j^2 \|P_{W_j} e_{k+1}\|^2 = w_k^2 \xrightarrow{k \rightarrow \infty} 0 \quad .$$

Observe that, by definition, $\frac{\gamma(TP_{E_k})}{\|TP_{E_k}\|} = \frac{2^{-k}}{1} \xrightarrow{k \rightarrow \infty} 0$. ▲

The operator T and the OBS $\mathcal{E} = \{E_n\}_{n \in \mathbb{N}}$ of the last Example do not satisfy Eq. (11) in Theorem 3.6. Still, Eq. (11) is not a necessary condition in order to assure that $\mathcal{P}(\mathcal{W}) \neq \emptyset$, if $\mathcal{W} = T\mathcal{E}$. Next example shows a FS which is the image of an OBS under an epimorphism which does not satisfy Eq. (11).

Example 7.2. Let $\{e_k\}_{k \in \mathbb{N}}$ be an orthonormal basis for \mathcal{H} and consider the frame (of vectors)

$$\mathcal{F} = \{f_n\}_{n \in \mathbb{N}} \quad \text{given by} \quad f_n = \begin{cases} e_k & \text{if } n = 2k - 1 \\ \frac{e_{k+1}}{\sqrt{k+1}} & \text{if } n = 2k \end{cases} \quad .$$

Let $T = T_{\mathcal{F}} \in L(\ell^2(\mathbb{N}), \mathcal{H})$ be its synthesis operator (which is surjective). If $\{b_n\}_{n \in \mathbb{N}}$ is the canonical basis of $\ell^2(\mathbb{N})$, then $Tb_n = f_n$. For each $k \in \mathbb{N}$ we set $E_k = \text{span}\{b_{2k-1}, b_{2k}\}$. Then, by construction, $\{E_k\}_{k \in \mathbb{N}}$ is an OBS of $\ell^2(\mathbb{N})$. Take the sequences

$$w = e \in \ell_+^\infty(\mathbb{N}) \quad \text{and} \quad W_k = TE_k = \text{span}\{e_k, e_{k+1}\} \quad , \quad k \in \mathbb{N} \quad .$$

By Theorem 3.4, $\mathcal{W}_w = (w_k, W_k)_{k \in \mathbb{N}}$ is a FS for \mathcal{H} . Nevertheless, T does not satisfy Eq. (11), since $\gamma(TP_{E_k}) = \frac{1}{\sqrt{k+1}}$, while $\|TP_{E_k}\| = 1$, for every $k \in \mathbb{N}$. ▲

The key argument in Example 7.1 was that $\bigcap_{i \in I} W_i \neq \{0\}$. This fact is sufficient for the emptiness of $\mathcal{P}(\mathcal{W})$ if $\text{span}\{W_i : 1 \leq i \leq n\} \neq \mathcal{H}$ for every $n \in \mathbb{N}$. Nevertheless, next example shows a minimal and generating sequence \mathcal{W} of finite dimensional subspaces such that $\mathcal{P}(\mathcal{W}) = \emptyset$.

Example 7.3. Fix an orthonormal basis $\mathcal{B} = \{e_i\}_{i \in \mathbb{N}}$ for \mathcal{H} . Consider the unit vector $g = \sum_{k=1}^{\infty} \frac{e_{2k}}{2^{k/2}} \in \mathcal{H}$. For every $n \in \mathbb{N}$, denote by $P_n \in L(\mathcal{H})$ the orthogonal projection onto

$\mathcal{H}_n = \text{span}\{e_1, e_2, \dots, e_n\}$. Consider the generating sequence $\mathcal{W} = \{W_k\}_{k \in \mathbb{N}}$ given by

$$W_k = \text{span}\{P_{2k} g, e_{2k-1}\} = \text{span}\left\{\sum_{j=1}^k \frac{e_{2j}}{2^{j/2}}, e_{2k-1}\right\}, \quad k \in \mathbb{N}.$$

Straightforward computations show that \mathcal{W} is a minimal sequence. The problem is that $c[W_i, W_j] \xrightarrow{i, j \rightarrow \infty} 0$ exponentially, and for this reason $\mathcal{P}(\mathcal{W}) = \emptyset$. Indeed, suppose that $w \in \mathcal{P}(\mathcal{W})$, and that $\mathcal{W}_w = (w, \mathcal{W})$ is a FS. Then

$$B_{\mathcal{W}_w} = B_{\mathcal{W}_w} \|g\|^2 \geq \sum_{k \in \mathbb{N}} w_k^2 \|P_{W_k} g\|^2 = \sum_{k \in \mathbb{N}} w_k^2 \|P_{2k} g\|^2 = \sum_{k \in \mathbb{N}} w_k^2 (1 - 2^{-k}), \quad (19)$$

which implies that $w_k \xrightarrow{k \rightarrow \infty} 0$. On the other hand, for every $k \in \mathbb{N}$,

$$A_{\mathcal{W}_w} = A_{\mathcal{W}_w} \|e_{2k-1}\|^2 \leq \sum_{i \in \mathbb{N}} w_i^2 \|P_{W_i} e_{2k-1}\|^2 = w_k^2 \implies A_{\mathcal{W}_w} = 0, \quad (20)$$

a contradiction. So $\mathcal{P}(\mathcal{W}) = \emptyset$. ▲

It is well known that $\{f_j\}_{j \in \mathbb{N}}$ is a Parseval frame in \mathcal{H} if and only if there exists a Hilbert \mathcal{K} containing \mathcal{H} such that $f_j = P_{\mathcal{H}} b_j$ for every $j \in \mathbb{N}$, where $\{b_j\}_{j \in \mathbb{N}}$ is an orthonormal basis for \mathcal{K} . One may think that a similar result is true for tight frames of subspaces, where we replace orthonormal basis by OBS. In section 4 we proved one implication (a Parseval FS is an orthogonal projection of an OBS) but the converse it is not true:

Example 7.4. Let $\{e_k\}_{k \in \mathbb{N}}$ be an orthonormal basis for \mathcal{H} . Consider the unit vector

$$g = \sum_{k \in \mathbb{N}} \frac{e_{2k-1}}{2^{k/2}}, \quad \text{and take } \mathcal{M} = \overline{\text{span}}\{g\} \cup \{e_{2k} : k \in \mathbb{N}\}.$$

On the other hand, take the sequence $\mathcal{E} = \{E_k\}_{k \in \mathbb{N}}$ given by $E_k = \text{span}\{e_{2k-1}, e_{2k}\}$ ($k \in \mathbb{N}$). Then \mathcal{E} is an OBS for \mathcal{H} . Take the sequence

$$\mathcal{W} = \{W_k\}_{k \in \mathbb{N}} \quad \text{given by} \quad W_k = P_{\mathcal{M}} E_k = \text{span}\{g, e_{2k}\}, \quad \text{for every } k \in \mathbb{N}.$$

Then $\mathcal{P}(\mathcal{W}) = \emptyset$ by same reason as in Example 7.1, because $g \in \bigcap_{k \in \mathbb{N}} W_k \neq \{0\}$. ▲

Example 7.5. Let $\mathcal{E} = \{e_n\}_{n \in \mathbb{N}}$ be an orthonormal basis of \mathcal{H} . Consider the sequence $\mathcal{W} = \{W_k\}_{k \in \mathbb{N}}$ given by

$$W_1 = \overline{\text{span}}\{e_k : k \geq 2\} = \{e_1\}^\perp \quad \text{and} \quad W_k = \text{span}\{e_1, e_k\}, \quad \text{for } k \geq 2.$$

Observe that $\mathcal{P}(\mathcal{W}) = \ell_+^2(\mathbb{N})$. Indeed, one inclusion is clear, and

$$w \in \mathcal{P}(\mathcal{W}) \implies \sum_{k=2}^{\infty} w_k^2 = \sum_{k=2}^{\infty} w_k^2 \|P_{W_k} e_1\|^2 \leq B_{\mathcal{W}_w} \implies w \in \ell_+^2(\mathbb{N}).$$

Now we shall see that \mathcal{W}_w can not be a **tight** FS for any $w \in \mathcal{P}(\mathcal{W})$. Indeed, if \mathcal{W}_w were a A -tight frame, then for every $k \geq 2$,

$$A = A\|e_k\|^2 = \sum_{i \in \mathbb{N}} w_i^2 \|P_{W_i} e_k\|^2 = w_1^2 + w_k^2 \implies w_k^2 = A - w_1^2 ,$$

which contradicts the fact that $w \in \ell_+^2(\mathbb{N})$. Our next step is to show that the frame operator $S_{\mathcal{W}_w} \in L(\mathcal{H})$ is diagonal with respect to \mathcal{E} , for every $w \in \mathcal{P}(\mathcal{W})$. Indeed,

$$T_{\mathcal{W}_w}^* e_1 = \{w_k P_{W_k} e_1\}_{k \in \mathbb{N}} = 0 \oplus \{w_k e_1\}_{k \geq 2} \implies S_{\mathcal{W}_w} e_1 = T_{\mathcal{W}_w} T_{\mathcal{W}_w}^* e_1 = \left(\sum_{k=2}^{\infty} w_k^2 \right) e_1 .$$

On the other hand, if E_k is the copy of each W_k in $\mathcal{K}_{\mathcal{W}}$, then for every $k \in \mathbb{N}$ and $j \geq 2$,

$$P_{E_k} (T_{\mathcal{W}_w}^* e_j) = \begin{cases} w_1 e_j & \text{if } k = 1 \\ w_j e_j & \text{if } k = j \\ 0 & \text{if } k \neq 1, j \end{cases} \implies S_{\mathcal{W}_w} e_j = T_{\mathcal{W}_w} T_{\mathcal{W}_w}^* e_j = (w_1^2 + w_j^2) e_j .$$

In particular, $S_{\mathcal{W}_w}^{-1/2}$ is also diagonal. This implies that $S_{\mathcal{W}_w}^{-1/2} \mathcal{W} = \mathcal{W}$, which we have seen that can not be tight for any sequence of weights.

Another property of this example is the following: \mathcal{W}_w is a FS for \mathcal{H} , but the sequence $(w_k, W_k)_{k > 1}$ is not a frame sequence of subspaces (i.e. a FS for $\overline{\text{span}}\{W_k : k > 1\}$). This can be proved by the same argument as in Example 7.1, using that $\cap_{k > 1} W_k \neq \{0\}$. \blacktriangle

Example 7.6. Let $\mathcal{B}_4 = \{e_n\}_{n \leq 4}$ be an orthonormal basis of \mathbb{C}^4 . Consider the sequence

$$W_1 = \text{span}\{e_1, e_2\} , \quad W_2 = \text{span}\{e_1, e_3\} \quad \text{and} \quad W_3 = \text{span}\{e_4\} .$$

We shall see that, for every invertible $G \in \mathcal{M}_4(\mathbb{C})$, and every $w \in \mathbb{R}_+^3$, the sequence $G\mathcal{W}_w = (w_k, GW_k)_{k \in \mathbb{I}_3}$ fails to be a Parseval FS. Take orthonormal basis of each GW_i

$$GW_1 = \text{span}\{g_1, g_2\} , \quad GW_2 = \text{span}\{g_1, g_3\} \quad \text{and} \quad GW_3 = \text{span}\{g_4\} ,$$

where $g_1 = \frac{Ge_1}{\|Ge_1\|}$, and similarly for g_4 . If $G\mathcal{W}_w$ were a Parseval FS, then the frame

$$\mathcal{E} = \{T_{G\mathcal{W}_w} g_k\}_{k \in \mathbb{I}_5} = \{w_1 g_1, w_1 g_2, w_2 g_1, w_2 g_3, w_3 g_4\} ,$$

would be also Parseval. Consider the matrix $T \in \mathcal{M}_{4,5}(\mathbb{C})$ with the vectors of \mathcal{E} as columns. After a unitary change of coordinates, T has the form

$$T = \begin{pmatrix} w_1 & w_2 & \vec{v} \\ 0 & 0 & V \end{pmatrix} \begin{matrix} \mathbb{C} \\ \mathbb{C}^3 \end{matrix} \quad \text{with } \vec{v} = (0, 0, a) \in \mathbb{C}^3 \text{ and } V \in \mathcal{M}_3(\mathbb{C}) .$$

Since $TT^* = I_4$, it is easy to see that $V \in \mathcal{U}(3)$. But this is impossible because the first two columns of V have norms $\|w_1 g_2\| = w_1$ and $\|w_2 g_3\| = w_2$, while $1 = w_1^2 + w_2^2 + |a|^2$. \blacktriangle

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