Refinements of spectral resolutions and modelling of operators in II_1 factors

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Abstract

We study refinements between spectral resolutions in an arbitrary II_1 factor $\mathcal{M}$ and obtain diffuse (maximal) refinements of spectral resolutions. We construct models of operators with respect to diffuse spectral resolutions. As an application we obtain new characterizations of submajorization and spectral preorder between positive operators in $\mathcal{M}$ and new versions of some known inequalities involving these preorders.

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1 Introduction

The study of the norm closure of unitary orbits of self-adjoint operators in von Neumann algebras is a well established area of research. Some of the early results on this subject go back to the work of Weyl and von Neumann in the type I factor case. Kamei, in his development of majorization between operators in II_1 factors, obtained an interesting characterization of the norm closure of the unitary orbit of a positive operator in terms of its singular values. Recently, Arveson and Kadison have described these sets for self-adjoint operators in terms of spectral distributions [4] in the II_1 factor and Sherman [17] has obtained interesting descriptions of several closures of unitary orbits in von Neumann algebras under weak restrictions (see the introduction of [17] for a detailed account on the history of these problems and recent references). It turns out that even in the general setting of [17], the spectral data of operators play a fundamental role in these investigations.

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There are other notions closely related to unitary orbits, that are defined in terms of spectral data, such as majorization, sub-majorization and spectral dominance; the study of these notions has been considered in several research works like the papers of Kamei [16] and Hiai [9, 10], Hiai and Nakamura [11, 12] and the more recent papers of Kadison [13, 14, 15] and of Arveson and Kadison [4]. In this context one usually tries to describe operators in some set associated with (the norm closure of)

$$\mathcal{U}_{\mathcal{M}}(b) := \{u^*bu : \ u \in \mathcal{M} \text{ is a unitary operator}\}$$

where $\mathcal{M}$ is a semifinite von Neumann algebra with faithful semifinite trace τ and $b \in \mathcal{M}$ is a self-adjoint operator. For example, it is well known [16] that if $\mathcal{M}$ is a II_1 factor then $a \in \overline{\text{conv}(\mathcal{U}_{\mathcal{M}}(b))}$ if and only if a is majorized by b, which is a spectral relation. In this case the spectral data of a may be more complex (disordered) than that of b. This makes things difficult when trying to recover a as an element of $\overline{\text{conv}(\mathcal{U}_{\mathcal{M}}(b))}$ whenever we know that a is majorized by b. In order to overcome a similar difficulty, in [2] we considered an "diffuse" refinement of the (joint) spectral measure of an ordered n-tuple of mutually commuting self-adjoint elements of a II_1 factor $\mathcal{M}$.

In this work we consider a related construction to that obtained in [2] that, roughly speaking, allows us to represent every positive operator $a \in \mathcal{M}^+$ as Borel functional calculus (by an increasing left-continuous function) of a positive operator $a' \in \mathcal{M}$ with maximal disordered spectral resolution (with respect to a preorder called *refinement* that we shall introduce). Moreover, the operator $a' \in \mathcal{M}^+$ has the following property: *any* positive operator $b \in \mathcal{M}^+$ is, up to approximately unitary equivalence, Borel functional calculus of a' (by an increasing left-continuous function). These constructions are what we call *diffuse refinements of spectral resolutions* and *modelling of operators*. We also consider some relations between these constructions and maximal abelian subalgebras of $\mathcal{M}$. The idea of considering maximal (diffuse) refinements of spectral resolutions and of constructing some models of operators in finite factors has already been considered in [11, 12] although the notion of refinement introduced here has not. In this work we attempt a brief but systematic treatment of these concepts.

Our results are related to Kadison's study of Schur-type inequalities [15] and Arveson-Kadison's study of closed unitary orbits in II_1 factors [4]. Indeed our techniques provide alternative proofs to some of their results. Moreover, our refinements and modelling techniques are the basis for a version of the Schur-Horn type theorem in II_1 factors in [3].

As an application of these constructions we present characterizations of the sets

$$\{c \in \mathcal{M} : \ 0 \le c \le d \in \overline{\mathcal{U}_{\mathcal{M}}(a)}\}$$

and

$$\{c \in \mathcal{M} : \ 0 \le c \le d \in \overline{\text{conv}(\mathcal{U}_{\mathcal{M}}(a))}\}$$

in terms of spectral data. These characterizations are then applied to some recent spectral inequalities obtained in [1, 5, 7].

The paper is organized as follows. In section 2 we recall some definitions and facts regarding spectral relations (spectral preorder, majorization and submajorization). In section 3 we present our results on refinements of bounded right spectral resolutions in II_1 factors. In section 4 we consider the modelling of operators and use this construction to study spectral dominance and submajorization.

2 Preliminaries

Let $B(\mathcal{H})$ be the algebra of bounded operators on a Hilbert space $\mathcal{H}$. In what follows, the pair $(\mathcal{M}, \tau)$ shall denote a semifinite von Neumann algebra and a faithful normal semifinite (f.n.s.) trace on $\mathcal{M}$. In particular, if $\mathcal{M}$ is a finite factor then τ denotes the unique f.n.s. trace such that $\tau(1) = 1$. The real space of self-adjoint operators in $\mathcal{M}$ is denoted by $\mathcal{M}_{sa}$, the cone of positive operators by $\mathcal{M}^+$ and the unitary group by $\mathcal{U}_{\mathcal{M}}$. If $a \in \mathcal{M}_{sa}$ then $P^a(\Delta)$ denotes the spectral projection of a corresponding to the measurable set $\Delta \subseteq \mathbb{R}$. For simplicity of notation we shall write $P^a(\alpha, \beta)$ (instead of $P^a((\alpha, \beta))$) for a real interval $(\alpha, \beta) \subseteq \mathbb{R}$. $\mathcal{P}(\mathcal{M}) \subseteq \mathcal{M}_{sa}$ denotes the lattice of orthogonal projections in $\mathcal{M}$ endowed with the strong operator topology. For $a \in \mathcal{M}$, $R(a)$ denotes its range and $P_{\overline{R(a)}} \in \mathcal{P}(\mathcal{M})$ the orthogonal projection onto the closure of its range. By a decreasing function (resp. increasing) we mean a non-increasing function (resp. non-decreasing). If (X, ν) is a measure space then $L^\infty(\nu)^+$ denotes the cone of ν-essentially bounded nonnegative functions on X. The set of nonnegative numbers is denoted by $\mathbb{R}_0^+$.

2.1 Singular values, spectral preorder and (sub) majorization

The τ-*singular values* (or τ-singular numbers) [8] of $x \in \mathcal{M}$ are defined for each $t \in \mathbb{R}_0^+$ by

$$\mu_x(t) = \inf\{\|xe\| : \ e \in \mathcal{P}(\mathcal{M}),\ \tau(1-e) \le t\}. \tag{1}$$

The function $\mu_x : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is decreasing and right-continuous. If x, $y \in \mathcal{M}$ then

$$|\mu_x(t) - \mu_y(t)| \le \|x - y\| \tag{2}$$

which shows a continuous dependence of the singular values on the operator norm. If $a \in \mathcal{M}^+$, we have

$$\mu_a(t) = \min\{s \in \mathbb{R}_0^+ : \ \tau(P^a(s, \infty)) \le t\}.$$

This last characterization of the singular values of positive operators shows the following property: if a, $b \in \mathcal{M}^+$ are such that $\tau(P^a(s,\infty)) = \tau(P^b(s,\infty))$ for every $s \in \mathbb{R}_0^+$ then $\mu_a = \mu_b$. On the other hand, from (1) we see that $\mu_a = \mu_{uau^*}$ for every unitary operator $u \in \mathcal{U}_\mathcal{M}$. Moreover, from this last fact and the continuous dependence (2) we see that $\mu_a = \mu_b$, whenever $b \in \overline{\mathcal{U}_\mathcal{M}(a)}$, where $\overline{\mathcal{U}_\mathcal{M}(a)}$ denotes the norm closure of the unitary orbit

$$\mathcal{U}_\mathcal{M}(a) = \{uau^* : u \in \mathcal{U}_\mathcal{M}\}.$$

Kamei proved [16] a converse of this fact when $(\mathcal{M}, \tau)$ is a finite factor. We summarize these remarks in the following proposition.

Proposition 2.1. *Let $(\mathcal{M}, \tau)$ be a semifinite von Neumann algebra and let a, $b \in \mathcal{M}^+$.*

1. *If $b \in \overline{\mathcal{U}_\mathcal{M}(a)}$, then $\mu_a = \mu_b$.*
2. *(Kamei [16]) Assume further that $(\mathcal{M}, \tau)$ is a finite factor and $\mu_a = \mu_b$. Then $b \in \overline{\mathcal{U}_\mathcal{M}(a)}$.*

Next we recall the definitions of three different preorders that we shall consider in the sequel. If a, $b \in \mathcal{M}^+$ we say that b *spectrally dominates* a, and write $a \precsim b$, if any of the following (equivalent) statements holds:

a) $\mu_a(t) \leq \mu_b(t)$, for all $t \geq 0$.

b) $\tau(P^a(t,\infty)) \leq \tau(P^b(t,\infty))$, for all $t \geq 0$.

If in addition $(\mathcal{M}, \tau)$ is a semifinite factor

c) $P^a(t,\infty) \precsim P^b(t,\infty)$ in the Murray-von Neumann's sense.

We say that a is *sub-majorized* by b, and write $a \prec_w b$, if

$$\int_0^s \mu_a(t)\ dt \leq \int_0^s \mu_b(t)\ dt, \quad \text{for every } s \geq 0.$$

If in addition $\tau(a) = \tau(b)$ then we say that a is *majorized* by b and write $a \prec b$. It is well known that $a \leq b \Rightarrow a \precsim b \Rightarrow a \prec_w b$.

We shall need the following result due to Hiai and Nakamura [12], concerning functions in a finite measure space (X, ν). In this case, a function $g \in L^\infty(\nu)$ is considered as an operator in the finite von Neumann algebra $(L^\infty(\nu), \varphi)$ and singular values are defined with respect to the normal faithful finite trace φ induced by ν, i.e.

$$\varphi(g) := \int_X g\ d\nu, \quad g \in L^\infty(\nu). \tag{3}$$

Proposition 2.2. *Let (X, ν) be a probability space and let $f, g \in L^\infty(\nu)^+$. Then $f \prec_w g$ if and only if there exists $h \in L^\infty(\nu)^+$ such that $f \leq h \prec g$.*

Remark 2.3. If $(\mathcal{M}, \tau)$ is a finite factor and $a \in \mathcal{M}^+$, then let ν be the regular Borel probability measure given by $\nu(\Delta) = \tau(P^a(\Delta))$. For every $g \in L^\infty(\nu)^+$ let

$$g(a) = \int_{\sigma(a)} g \, dP^a \in \mathcal{M}^+$$

and note that $\mu_{g(a)} = \mu_g$. As a consequence we get that $\tau(g(a)) = \varphi(g)$, where φ is given by 3. Thus, if $h, g \in L^\infty(\nu)^+$, then $h(a) \precsim g(a)$ (resp. $h(a) \prec g(a)$, $h(a) \prec_w g(a)$) in $\mathcal{M}$ if and only if $h \precsim g$ (resp. $h \prec g$, $h \prec_w g$) in $L^\infty(\nu)$. $\square$

3 Refinements of spectral resolutions

Let $I = [\alpha, \beta] \subseteq \mathbb{R}$ be a closed interval, and recall that $\mathcal{P}(\mathcal{M})$ denotes the lattice of orthogonal projections in $\mathcal{M}$ endowed with the strong operator topology. If $p \in \mathcal{P}(\mathcal{M})$, we say that a map $E : I \to \mathcal{P}(\mathcal{M})$ is a *bounded right spectral resolution of* p (abbreviated "brsr of p") if E is decreasing and right-continuous, $E(\beta) = 0$ and $E(\alpha) = p$. If $p = 1$ then this notion agrees with the usual definition of brsr in $\mathcal{M}$. For example, any $a \in \mathcal{M}^+$ induces a brsr of $p = P_{\overline{R(a)}}$, by

$$E(\lambda) = P^a(\lambda, \infty), \quad \lambda \in [0, \|a\|]. \tag{4}$$

Given $E : I \to \mathcal{P}(\mathcal{M})$ a brsr (of $E(\alpha)$) then, we identify E with the family $\{E_\lambda\}_{\lambda \in I}$, where $E_\lambda = E(\lambda)$ for every $\lambda \in I$. If the set I is clear from the context, we simply write $\{E_\lambda\}$.

If $E : [\alpha, \beta] \to \mathcal{P}(\mathcal{M})$ is a brsr, we say that $\lambda_0 \in (\alpha, \beta]$ is an atom for $\{E_\lambda\}$, if the resolution is not continuous at λ_0; if $p \neq 1$ then α is considered as an atom. The set of atoms of $\{E_\lambda\}$ is denoted by $\mathrm{At}(\{E_\lambda\})$. We say that $\{E_\lambda\}$ is a *diffuse* brsr if the set $\mathrm{At}(\{E_\lambda\})$ is empty. It is clear that $\{E_\lambda\}$ is diffuse if and only if $E(\alpha) = 1$ and E is a continuous function (recall that $\mathcal{P}(\mathcal{M})$ is endowed with the SOT). We say that a positive operator $a \in \mathcal{M}^+$ has *continuous distribution* if the resolution induced by a (see (4)) is diffuse. Therefore, $a \in \mathcal{M}^+$ has continuous distribution if and only if $P_{\overline{R(a)}} = 1$ and $P^a(\{x\}) = 0$ for every $x \in \mathbb{R}$.

It is well known that given a brsr $\{E_\lambda\}_{\lambda \in I}$ in $\mathcal{M}$ then there exists a unique spectral measure F on I with values in $\mathcal{P}(\mathcal{M})$ such that $E_\lambda = F((\lambda, \infty))$ for every $\lambda \in I$. If $h : I \to \mathbb{C}$ is a uniformly bounded measurable function then we use the following notation

$$\int_I h(\lambda) \, dE_\lambda := \int_I h \, dF. \tag{5}$$

Definition 3.1. Let $\{E_\lambda\}_{\lambda\in I}$ and $\{E'_\lambda\}_{\lambda\in I'}$ be brsr's, where $I = [\alpha,\beta]$ and $I' = [\alpha',\beta']$. We say that $\{E'_\lambda\}$ *refines* $\{E_\lambda\}$ if there exists $f : I \to I'$ such that

(a) f is increasing, right-continuous and $f(\beta) = \beta'$;

(b) $E_\lambda = E'_{f(\lambda)}$ for every $\lambda \in I$.

We say that $\{E'_\lambda\}$ *strongly refines* $\{E_\lambda\}$ if f also satisfies

(c) $f(\lambda) \geq \lambda$ for every $\lambda \in I$, and

(d) $f(\lambda) - f(\mu) \geq \lambda - \mu$, for every $\lambda > \mu \in I$.

If $\{E'_\lambda\}$ (strongly) refines $\{E_\lambda\}$ we also say that $(\{E'_\lambda\}, f)$ is a (strong) refinement of $\{E_\lambda\}$, where f is as in Definition 3.1. It is easy to see that refinement is a preorder relation.

The following, which is the main result of this section, is related with the refinement of spectral measures of separable abelian C^*-subalgebras in a II_1 factor developed in [2].

Theorem 3.2. *Let $(\mathcal{M},\tau)$ be a II_1 factor and let $a \in \mathcal{M}^+$. Then there exists $a' \in \mathcal{M}^+$ with continuous distribution and such that the brsr induced by a' strongly refines the brsr induced by a. Further, if $a \in \mathcal{A}^+$, where $\mathcal{A}$ is a masa in $\mathcal{M}$, then a' can be selected from $\mathcal{A}$.*

In what follows we state some lemmas and use them to prove Theorem 3.2 at the end of this section. In the rest of the paper, the pair $(\mathcal{M},\tau)$ will always denote a II_1 factor. Let $I = [\alpha,\beta]$ and let $\{E_\lambda\}_{\lambda\in I}$ be a brsr of a projection $p \in \mathcal{P}(\mathcal{M})$. If $\lambda_0 \in (\alpha,\beta]$ is an atom for $\{E_\lambda\}$, then

$$\lim_{\lambda\to\lambda_0^-} E_\lambda = E_{\lambda_0} + p(\lambda_0), \quad p(\lambda_0) \neq 0. \tag{6}$$

In this case $p(\lambda_0) \in \mathcal{P}(\mathcal{M})$ is the *jump projection* of $\{E_\lambda\}$ at λ_0. If $p \neq 1$ then $\alpha \in \mathrm{At}(\{E_\lambda\})$ and the jump projection at α is by definition $p(\alpha) = 1 - p$. Note that the set of atoms $\mathrm{At}(\{E_\lambda\})$ is countable. Indeed, if λ_0, $\lambda_1 \in \mathrm{At}(\{E_\lambda\})$ and $\lambda_0 \neq \lambda_1$, then it is easy to see that $p(\lambda_0)\, p(\lambda_1) = 0$, i.e. $p(\lambda_0)$ and $p(\lambda_1)$ are orthogonal projections. Therefore

$$\mathcal{J}(\{E_\lambda\}) := \sum_{\lambda\in\mathrm{At}(\{\mathrm{E}_\lambda\})} \tau(p(\lambda)) = \tau\left(\sum_{\lambda\in\mathrm{At}(\{\mathrm{E}_\lambda\})} p(\lambda)\right) \leq 1 \tag{7}$$

and this implies that $\mathrm{At}(\{E_\lambda\})$ is countable. The real number $\mathcal{J}(\{E_\lambda\})$ is called the *total jump* of the resolution.

Lemma 3.3. *Let $\{E_\lambda\}_{\lambda\in I}$, $\{E'_\lambda\}_{\lambda\in I'}$ be brsr's in $\mathcal{M}$. If $\{E'_\lambda\}$ refines $\{E_\lambda\}$ then $\mathcal{J}(\{E_\lambda\}) \geq \mathcal{J}(\{E'_\lambda\})$.*

Proof. Let $\lambda_0 \in \mathrm{At}(\{E'_\lambda\})$ and consider $\mu_0 = \min\{\mu \in I : \ f(\mu) \geq \lambda_0\}$ which is well defined by (a) in Definition 3.1. Then by definition of μ_0, $f(\mu_0) \geq \lambda_0$ and $f(\mu) < \lambda_0$ if $\mu < \mu_0$. So

$$\lim_{\mu\to\mu_0^-} E_\mu - E_{\mu_0} = \lim_{\mu\to\mu_0^-} E'_{f(\mu)} - E'_{f(\mu_0)} \geq \lim_{\lambda\to\lambda_0^-} E'_\lambda - E'_{\lambda_0} \neq 0,$$

since λ_0 is an atom of $\{E'_\lambda\}$. Therefore $\mu_0 \in I$ is an atom of the resolution $\{E_\lambda\}$ and we have

$$\lim_{\mu\to\mu_0^-} \tau(E_\mu) = \lim_{\mu\to\mu_0^-} \tau(E'_{f(\mu)}) > \tau(E'_{\lambda_0}) \geq \tau(E'_{f(\mu_0)}) = \tau(E_{\mu_0}), \tag{8}$$

since $f(\mu) \to \lambda_1^- \leq \lambda_0$ when $\mu \to \mu_0^-$ and $\lambda_0 \in \mathrm{At}(\{E'_\lambda\})$. We consider the following relation in $\mathrm{At}(\{E'_\lambda\})$: if $\lambda_1, \lambda_2 \in \mathrm{At}(\{E'_\lambda\})$ then $\lambda_1 \approx \lambda_2$ if and only if there exists $\mu_0 \in \mathrm{At}(\{E_\lambda\})$ such that

$$\tau(E'_{\lambda_1}),\ \tau(E'_{\lambda_2}) \in \left[\tau(E_{\mu_0}),\ \lim_{\mu\to\mu_0^-} \tau(E_\mu)\right). \tag{9}$$

The inequality (8) shows that this relation is reflexive. On the other hand it is clearly symmetric. Note that if $\mu_1 < \mu_2$ then $\lim_{\mu\to\mu_2^-} \tau(E_\mu) \leq \tau(E_{\mu_1})$ and

$$[\tau(E_{\mu_2}),\ \lim_{\mu\to\mu_2^-} \tau(E_\mu)) \cap [\tau(E_{\mu_1}),\ \lim_{\mu\to\mu_1^-} \tau(E_\mu)) = \emptyset.$$

So, if $\lambda_1 \approx \lambda_2$ then there exists a unique $\mu_0 \in \mathrm{At}(\{E_\lambda\})$ such that (9) holds, so in particular $\approx$ is an equivalence relation. Therefore, for any equivalence class $Q \in \Pi = \mathrm{At}(\{E'_\lambda\})/\approx$, there exists a unique atom $\mu_Q \in \mathrm{At}(\{E_\lambda\})$ such that

$$\tau(E_\lambda) \in [\tau(E_{\mu_Q}),\ \lim_{\mu\to\mu_Q^-} \tau(E_\mu)) \quad \text{for all } \lambda \in Q.$$

Let $\lambda_1, \ldots, \lambda_n \in Q$ with $\lambda_1 < \ldots < \lambda_n$. Then, if $p'(\lambda_i)$ is the jump projection of the resolution $\{E'_\lambda\}$ at λ_i and $p(\mu_Q)$ is the jump projection of the resolution $\{E_\lambda\}$ at μ_Q, we have

$$\begin{aligned}
\sum_{i=1}^{n} \tau(p'(\lambda_i)) &= \sum_{i=1}^{n} (\lim_{\lambda\to\lambda_i^-} \tau(E'_\lambda) - \tau(E'_{\lambda_i})) \leq \lim_{\lambda\to\lambda_1^-} \tau(E'_\lambda) - \tau(E'_{\lambda_n}) \\
&\leq \lim_{\mu\to\mu_Q^-} \tau(E'_{f(\mu)}) - \tau(E'_{f(\mu_Q)}) = \tau(p(\mu_Q))
\end{aligned}$$

Taking limit over n if necessary, we get $\sum_{\lambda\in Q} \tau(p'(\lambda)) \leq \tau(p(\mu_Q))$. Therefore

$$\mathcal{J}(\{E'_\lambda\}) = \sum_{Q\in\Pi} \sum_{\lambda\in Q} \tau(p'(\lambda)) \leq \sum_{Q\in\Pi} \tau(p(\mu_Q)) \leq \mathcal{J}(\{E_\lambda\})$$

where the rearrangement is valid since we are considering series of positive terms. □

We introduce the following notation in order to state Lemma 3.5.

Definition 3.4. *If $\{\alpha_k\}_{k\in\mathbb{N}} \in \ell^1(\mathbb{R}^+)$ we say that a sequence $(\{E^k_\lambda\}_{\lambda\in I_k})_{k\in\mathbb{N}}$ of brsr's in $\mathcal{M}$ is $\{\alpha_k\}_{k\in\mathbb{N}}$-*compatible *if the following conditions hold:*

1. *$\exists\alpha,\,\beta \in \mathbb{R}^+_0$ such that $I_k = [\alpha, \beta + \sum_{i=1}^k \alpha_i]$ for every $k \in \mathbb{N}$.*
2. *$(\{E^{k+1}_\lambda\}, f_k)$ is a strong refinement of $\{E^k_\lambda\}$ for every $k \in \mathbb{N}$.*
3. *$f_k(\lambda) - \lambda \le \alpha_k$, for every $\lambda \in I_k$ and for every $k \in \mathbb{N}$.*

Lemma 3.5. *Let $\{\alpha_k\}_{k\in\mathbb{N}} \in \ell^1(\mathbb{R}^+)$ and $(\{E^k_\lambda\}_{\lambda\in I_k})_{k\in\mathbb{N}}$ be $\{\alpha_k\}_{k\in\mathbb{N}}$-compatible. Then there exists a brsr $\{E_\lambda\}_{\lambda\in I}$ in $\mathcal{M}$ such that $\{E_\lambda\}$ strongly refines $\{E^k_\lambda\}$, for every $k \in \mathbb{N}$. Moreover, if $\mathcal{A} \subseteq \mathcal{M}$ is a masa and $\{E^k_\lambda\}$ is in $\mathcal{A}$ for each $k \in \mathbb{N}$, we can choose $\{E_\lambda\}$ also in $\mathcal{A}$.*

Proof. For simplicity, we shall assume that $\alpha = 0$. The general case follows from this by reparametrization. Let $I = [0, \beta + \sum_{i=1}^\infty \alpha_i]$ and for every $k \in \mathbb{N}$ let $f_k : I_k \to I_{k+1}$ be as in Definition 3.4. Note that, since $f_k(\lambda) \ge \lambda$ for $\lambda \in I_k$ (condition (c) in 3.1),

$$E^k_\lambda = E^{k+1}_{f_k(\lambda)} \le E^{k+1}_\lambda.$$

Therefore, for each $\lambda \in I$ the sequence $\{E^k_\lambda\}_{k\in\mathbb{N}}$ is increasing, where we set $E^k_\lambda = 0$ if $\lambda \notin I_k$. Let us define

$$E_\lambda = \bigvee_{k\in\mathbb{N}} E^k_\lambda = \lim_{k\to\infty} E^k_\lambda \in \mathcal{P}(\mathcal{M}), \quad \lambda \in I \tag{10}$$

where the limit is in the strong operator topology. Note that, if $\mathcal{A} \subseteq \mathcal{M}$ is a masa and $E^k_\lambda \in \mathcal{P}(\mathcal{A})$ for every $k \in \mathbb{N}$, then $E_\lambda \in \mathcal{A}$. To see that $\{E_\lambda\}_{\lambda\in I}$ is a brsr note first that $E_{\lambda_0} \ge E_\lambda$ if $\lambda_0 \le \lambda$. Thus $\exists \lim_{\lambda\to\lambda_0^+} E_\lambda \le E_{\lambda_0}$. If $\{\lambda_n\} \subseteq I$ is a decreasing sequence such that $\lim_{n\to\infty} \lambda_n = \lambda_0$ then

$$\begin{aligned} \tau(\lim_{n\to\infty} E_{\lambda_n}) &= \lim_{n\to\infty} \tau(E_{\lambda_n}) = \lim_{n\to\infty}\lim_{k\to\infty} \tau(E^k_{\lambda_n}) \\ &= \lim_{k\to\infty}\lim_{n\to\infty} \tau(E^k_{\lambda_n}) = \tau(\bigvee_{k\in\mathbb{N}} E^k_{\lambda_0}) = \tau(E_{\lambda_0}) \end{aligned}$$

where the change of order of the iterated limits is valid since the double sequence $\{\tau(E^k_{\lambda_n})\}_{n,k}$ is positive, bounded and increasing in each variable. Therefore $\lim_{\lambda\to\lambda_0^+} E_\lambda = E_{\lambda_0}$ and $\{E_\lambda\}_{\lambda\in I}$ is a brsr.

Fix $k \in \mathbb{N}$ and consider the sequence $\{u_n : I_k \to I_{k+n}\}_{n\in\mathbb{N}}$ of increasing right-continuous functions, given inductively by $u_1 = f_k$ and $u_n = f_{k+n-1} \circ u_{n-1}$ for $n \ge 2$. Then, it is easy to see that

1. $E^k_\lambda = E^{k+n}_{u_n(\lambda)}$,
2. $u_{n+1} \ge u_n$, $\|u_{n+1} - u_n\|_\infty \le \alpha_{n+k}$,

3. $u_n(\lambda) - u_n(\mu) \geq \lambda - \mu$ if λ, $\mu \in I_k$ and $\lambda \geq \mu$.

Let $h_k : I_k \to I$ be the uniform limit of the increasing sequence $\{u_n\}$. Then h_k is increasing right-continuous, $h_k(\lambda) \geq \lambda$ $(u_1 = f_k)$ and $h_k(\lambda) - h_k(\mu) \geq \lambda - \mu$ if $\lambda > \mu \in I_k$. Let $\lambda_0 \in [0, \beta + \sum_{i=1}^k \alpha_i)$ and note that $E^k_{\lambda_0} = E^{k+n}_{u_n(\lambda_0)} \geq E^{k+n}_{h_k(\lambda_0)}$, since $u_n(\lambda) \leq h_k(\lambda)$. Therefore

$$E^k_{\lambda_0} \geq \lim_{n\to\infty} E^{k+n}_{h_k(\lambda_0)} = E_{h_k(\lambda_0)}. \tag{11}$$

To see that equality holds in (11) we consider

$$\lambda_n := \min\{\lambda \in I_k : \ u_n(\lambda) \geq h_k(\lambda_0)\}.$$

By definition we have $u_n(\beta + \sum_{i=1}^k \alpha_i) = \beta + \sum_{i=1}^{k+n} \alpha_i$ so λ_n is well defined. Further, $\lambda_n \geq \lambda_{n+1} \geq \lambda_0$, since $\{u_n\}$ is an increasing sequence, and $\lambda_n \to \lambda_0^+$. Indeed, if $\lambda > \lambda_0$ and $\lambda - \lambda_0 = \epsilon$ then $h_k(\lambda) \geq h_k(\lambda_0) + \epsilon$ and there exists $n \in \mathbb{N}$ such that $u_n(\lambda) > h_k(\lambda_0)$, which implies that $\lambda_0 \leq \lambda_n \leq \lambda$. Finally, we have

$$E_{h_k(\lambda_0)} \geq E_{u_n(\lambda_n)} \geq E^{k+n}_{u_n(\lambda_n)} = E^k_{\lambda_n}, \ \forall n \in \mathbb{N}$$

which implies that $E_{h_k(\lambda_0)} \geq \lim_{n\to\infty} E^k_{\lambda_n} = E^k_{\lambda_0}$. □

Lemma 3.6. *Let $\{E_\lambda\}_{\lambda\in[\alpha,\beta]}$ be a brsr in $\mathcal{M}$. If $\lambda_0 \in At(\{E_\lambda\})$, then there exists a strong refinement $(\{E'\}_{\lambda\in I'}, f)$ of $\{E_\lambda\}$, where $I' = [\alpha, \beta + \tau(p(\lambda_0))]$ such that*

1. *$\mathcal{J}(\{E'_\lambda\}) = \mathcal{J}(\{E'_\lambda\}) - \tau(p(\lambda_0))$.*
2. *$f(\lambda) - \lambda \leq \tau(p(\lambda_0))$ for every $\lambda \in I$.*
3. *$At(\{E'_\lambda\}) = f(At(\{E_\lambda\} \setminus \lambda_0))$.*

Moreover, if $\mathcal{A} \subseteq \mathcal{M}$ is a masa and $\{E_\lambda\}$ is a brsr in $\mathcal{A}$ then we can choose $\{E'_\lambda\}$ also in $\mathcal{A}$.

Proof. For simplicity, we assume that $I = [0, \beta]$ $(\alpha = 0)$. The general case follows from this by reparametrization. Let $\lambda_0 \in \mathrm{At}(\{E_\lambda\})$, $p_0 = p(\lambda_0)$ be the jump projection at λ_0 and $\alpha_0 = \tau(p_0)$.

It is well known [4, 15] that there exists $\{U_\lambda\}_{\lambda\in[0,\alpha_0]}$ a brsr of p_0 in $\mathcal{M}$ such that

$$\tau(U_\lambda) = \frac{\tau(p_0)(\alpha_0 - \lambda)}{\alpha_0}, \quad \lambda \in [0, \alpha_0]. \tag{12}$$

Moreover, if $\mathcal{A} \subseteq \mathcal{M}$ is a masa and $p_0 \in \mathcal{P}(\mathcal{A})$ then we can choose $\{U_\lambda\}$ to be in $\mathcal{A}$. Let

$$E'_\lambda = \begin{cases} E_\lambda & \text{if } \ 0 \leq \lambda < \lambda_0 \\ E_{\lambda_0} + U_{\lambda-\lambda_0} & \text{if } \ \lambda_0 \leq \lambda \leq \lambda_0 + \alpha_0 \\ E_{\lambda-\alpha_0} & \text{if } \ \lambda_0 + \alpha_0 < \lambda \leq \alpha_0 + \beta. \end{cases}$$

It is easy to see that $\{E'_\lambda\}_{\lambda\in I'}$, where $I' = [0, \beta + \alpha_0]$, is a brsr. Note that if $\{E_\lambda\}$ is in a masa $\mathcal{A} \subseteq \mathcal{M}$ then $p_0 \in \mathcal{A}$ and we can choose $\{U_\lambda\}$ in $\mathcal{A}$, so that $\{E'_\lambda\}$ is also in $\mathcal{A}$. The increasing, right-continuous function $f : I \to I'$ given by

$$f(\lambda) = \begin{cases} \lambda & \text{if } 0 \le \lambda < \lambda_0 \\ \lambda + \alpha_0 & \text{if } \lambda_0 \le \lambda \le \beta + \alpha_0 \end{cases} \tag{13}$$

satisfies $E_\lambda = E'_{f(\lambda)}$, $\lambda \in [0, \beta]$. Moreover $\mathrm{At}(\{E'_\lambda\}) = f(\mathrm{At}(\{E_\lambda\}) \setminus \{\lambda_0\})$ and $p(\lambda) = p'(f(\lambda))$ for every $\lambda \in \mathrm{At}(\{E_\lambda\}) \setminus \{\lambda_0\}$, where $p'(f(\lambda))$ is the jump projection of $\{E'_\lambda\}$ at $f(\lambda) \in \mathrm{At}(\{E'_\lambda\})$. Therefore

$$\mathcal{J}(\{E'_\lambda\}) = \sum_{\lambda\in \mathrm{At}(\{E_\lambda\})\setminus\{\lambda_0\}} \tau(p(f(\lambda))) = \mathcal{J}(\{E_\lambda\}) - \tau(p_0).$$

The rest of the properties of f follow directly from (13). □

Proof of Theorem 3.2. Let $a \in \mathcal{M}^+$ and consider the brsr induced by a (see (4)). Set $\beta = \|a\|$, let $I = [0, \beta]$ and let $\{\lambda_n\}_{n\in N}$ be an enumeration of the set $\mathrm{At}(\{E_\lambda\})$, where $N \subseteq \mathbb{N}$ is an initial segment, and let $\alpha_n = \tau(p(\lambda_n)) > 0$. By (7) we have $\sum_{n\in N} \alpha_n \le 1$. Let $I_1 := I$, $\{E^1_\lambda\} := \{E_\lambda\}$ and let $(\{E^2_\lambda\}_{\lambda\in I_2}, f_1)$ be the strong refinement obtained from $\{E^1_\lambda\}_{\lambda\in I_1}$ and the atom λ_1 as in Lemma 3.6. Recall that in this case $I_2 = [0, \beta + \tau(p_1)]$ and set $g_2 := f_1 : I_1 \to I_2$.

We proceed inductively: assume that for $1 \le t \le k-1$ we have brsr's $\{E^t_\lambda\}_{\lambda\in I_t}$, where $I_t = [0, \beta + \sum_{j=1}^{t-1} \alpha_j]$ and for $1 \le i \le k-2$ increasing right-continuous functions $f_i : I_i \to I_{i+1}$ such that $(\{E^{i+1}_\lambda\}, f_i)$ strongly refines $\{E^i_\lambda\}$ and such that $f_i(\lambda) - \lambda \le \alpha_i$ for $\lambda \in I_i$. Assume further that for $2 \le l \le k-1$ there exist injective functions $g_l : I \to I_l$ such that

$$\mathrm{At}(\{E^l_\lambda\}) = g_l(\mathrm{At}(\{E_\lambda\}) \setminus \{\lambda_1, \dots, \lambda_{l-1}\})$$

and

$$\mathcal{J}(\{E^l_\lambda\}) = \mathcal{J}(\{E_\lambda\}) - \sum_{j=1}^{l-1} \alpha_j.$$

Apply Lemma 3.6 to the brsr $\{E^{k-1}_\lambda\}_{\lambda\in I_{k-1}}$ and the atom $g_{k-1}(\lambda_{k-1})$. Then we obtain a brsr $\{E^k_\lambda\}_{\lambda\in I_k}$, $I_k = [0, \beta + \sum_{j=1}^{k-1} \alpha_j]$, and an increasing right-continuous function $f_{k-1} : I_{k-1} \to I_k$ such that $(\{E^k_\lambda\}, f_{k-1})$ is a strong refinement of $\{E^{k-1}_\lambda\}$; in this case we have $f_{k-1}(\lambda) - \lambda \le \alpha_{k-1}$. If we let $g_k = f_{k-1} \circ g_{k-1} : I \to I_k$ then g_k is injective and such that

$$\begin{aligned} \mathrm{At}(\{E^k_\lambda\}) &= f_{k-1}(\mathrm{At}(\{E^{k-1}_\lambda\}) \setminus \{g_{k-1}(\lambda_{k-1})\}) \\ &= g_k(\mathrm{At}(\{E_\lambda\}) \setminus \{\lambda_1, \dots, \lambda_{k-1}\}). \end{aligned}$$

Moreover, $\mathcal{J}(\{E^k_\lambda\}) = \mathcal{J}(\{E^{k-1}_\lambda\}) - \alpha_{k-1} = \mathcal{J}(\{E_\lambda\}) - \sum_{i=1}^{k-1} \alpha_i$.

We obtain in this way a sequence $\{E^k_\lambda\}_{\lambda\in I_k}$ of brsr's where $I_k = [0, \beta + \sum_{j=1}^{k-1} \alpha_j]$, and increasing right-continuous functions $\{f_k : I_k \to I_{k+1}\}$ for $k \in \mathbb{N}$ as in the hypothesis of Lemma 3.5. Thus, there exists a brsr $\{E'_\lambda\}_{\lambda\in I'}$ such that for every $k \in \mathbb{N}$ $\{E'_\lambda\}$ is a strong refinement of $\{E^k_\lambda\}$. In particular, $\{E'_\lambda\}$ is a strong refinement of $\{E_\lambda\} = \{E^1_\lambda\}$. By Lemma 3.3, $\mathcal{J}(\{E'_\lambda\}) \le \mathcal{J}(\{E^k_\lambda\})$ for every $k \in \mathbb{N}$ and therefore $\mathcal{J}(\{E'_\lambda\}) = 0$, i.e. $\{E'_\lambda\}$ is diffuse.

Note that if $a \in \mathcal{A}^+$ for some masa $\mathcal{A} \subseteq \mathcal{M}$ then $\{E_\lambda\}$ is a brsr in $\mathcal{A}$; by Lemma 3.6 we can construct each $\{E^k_\lambda\}$ also in $\mathcal{A}$ and so, by Lemma 3.5 then $\{E'_\lambda\}$ is in $\mathcal{A}$. Finally if we let $a' = \int_{I'} \lambda \, dE'_\lambda$ (see (5)) then $a' \in \mathcal{M}^+$ has the desired properties. □

4 Modelling of operators and applications

4.1 Modelling of operators

We begin with the following elementary lemmas about functions that we shall need in the sequel.

Lemma 4.1. *Let $I = [\alpha, \beta]$, $J = [\alpha', \beta'] \subseteq \mathbb{R}$ be closed intervals, $g : J \to [0, 1]$ a decreasing right-continuous function and let $h : I \to [0, 1]$ be a decreasing continuous function such that $h(\alpha) \ge g(\alpha')$ and $h(\beta) \le g(\beta')$. If we let $\tilde{g} : J \to I$ be given by*

$$\tilde{g}(x) = \max\{t \in I : \; g(x) = h(t)\}$$

then $\tilde{g}$ is an increasing right-continuous function and $g = h \circ \tilde{g}$.

Lemma 4.2. *Let $I = [\alpha, \beta]$, $J = [\alpha', \beta'] \subseteq \mathbb{R}$ and let $f : J \to I$ be an increasing right-continuous function such that $f(\beta') = \beta$. If $f^\dagger : I \to J$ is the function given by*

$$f^\dagger(\lambda) = \min\{t \in J : \; \lambda \le f(t)\}$$

then it is increasing, left-continuous and such that for every $t \in J$

$$\{\lambda \in I : \; \lambda > f(t)\} = \{\lambda \in I : \; f^\dagger(\lambda) > t\}. \tag{14}$$

If f is strictly increasing then $f^\dagger$ is continuous. Moreover, if $\tilde{J} := [\gamma, \delta] \subseteq J$ and $g : \tilde{J} \to I$ is increasing and right-continuous, $g(\delta) = \beta'$ and $f(t) \ge g(t)$ for every $t \in \tilde{J}$, then $g^\dagger \ge f^\dagger$.

Lemma 4.3. *Let $I = [\alpha, \beta]$, $J = [\alpha', \beta'] \subseteq \mathbb{R}$ and let $f : I \to J$ be an increasing left-continuous function such that $f(\alpha) = \alpha'$. If $f_\dagger : J \to I$ is the function given by*

$$f_\dagger(\lambda) = \max\{t \in I : \; \lambda \ge f(t)\}$$

then it is increasing, right-continuous and such that for every $t \in I$

$$\{\lambda \in J : \; \lambda < f(t)\} = \{\lambda \in J : \; f_\dagger(\lambda) < t\}. \tag{15}$$

The following theorem develops the modelling of positive operators and relates it with the refinement between the spectral resolutions induced by these operators.

Theorem 4.4. *Let $(\mathcal{M},\tau)$ be a II_1 factor, let $a\in\mathcal{M}^+$ with continuous distribution and let $I=[0,\|a\|]$. Then*

1. *If $b\in\mathcal{M}^+$, there exists a nonnegative increasing left-continuous function h_b on I such that if $\tilde b=h_b(a)$ then $\mu_b=\mu_{\tilde b}$.*
2. *The brsr induced by a refines the brsr induced by b if and only if $\tilde b=b$. Moreover, if the brsr induced by a strongly refines the brsr induced by b then h_b is continuous.*
3. *If $c^+\in\mathcal{M}$ then $c\precsim b$ (resp $c\prec_w b$, $c\prec b$) if and only if $h_c(a)\le h_b(a)$ (resp. $h_c(a)\prec_w h_b(a)$, $h_c(a)\prec h_b(a)$).*

Proof. Let $a\in\mathcal{M}^+$ with continuous distribution, let $I=[0,\|a\|]$ and let $h: I\to[0,1]$ be the decreasing continuous function defined by $h(t)=\tau(P^a(t,\infty))$. Note that $h(\|a\|)=0$ and, since a has continuous distribution, $h(0)=1$.

Let $b\in\mathcal{M}^+$, $J=[0,\|b\|]$ and let $g:J\to[0,1]$ be the decreasing right-continuous function defined by $g(s)=\tau(P^b(s,\infty))$. By Lemma 4.1, there exists an increasing right-continuous function $\tilde g: J\to I$, such that $g=h\circ\tilde g$, i.e.

$$\tau(P^b(s,\infty))=\tau(P^a(\tilde g(s),\infty)),\quad s\in J. \tag{16}$$

By Lemma 4.2 there exists an increasing (and therefore uniformly bounded measurable) left-continuous function $h_b:=\tilde g^\dagger: I\to J$ such that

$$\{\lambda\in I:\ h_b(\lambda)>s\}=\{\lambda\in I:\ \lambda>\tilde g(s)\},\quad s\in J. \tag{17}$$

Let $\tilde b=h_b(a)$ and note that $\tau(P^{\tilde b}(s,\infty))=\tau(P^b(s,\infty))$, which follows from (16) and (17). Therefore, b and $\tilde b$ have the same singular values.

To prove *2.* assume that the brsr induced by $b\in\mathcal{M}^+$ is refined by the brsr induced by a. Let $\tilde b=h_b(a)$ and note that $P^{\tilde b}(s,\infty)=P^a(\tilde g(s),\infty)$ and by hypothesis $P^b(s,\infty)=P^a(f(s),\infty)$ for some increasing right-continuous function $f: J\to I$. Then $P^{\tilde b}(s,\infty)\le P^b(s,\infty)$ or $P^b(s,\infty)\le P^{\tilde b}(s,\infty)$ and by (17) we have $\tau(P^b(s,\infty))=\tau(P^{\tilde b}(s,\infty))$ so $P^b(s,\infty)=P^{\tilde b}(s,\infty)$, $s\in J$. Therefore $b=\tilde b$. On the other hand, if $b=j(a)$ for any increasing left-continuous function $j: I\to J$, then by Lemma 4.3 there exists an increasing right-continuous function $f:=j_\dagger: J\to I$ such that

$$\begin{aligned} P^b(\lambda,\infty) &= P^a(\{t\in I:\ \lambda<j(t)\})\\ &= P^a(\{t\in I:\ f(\lambda)<t\})=P^a(f(\lambda),\infty), \end{aligned}$$

so the brsr induced by a refines the brsr induced by b. Finally assume that the brsr induced by a strongly refines the brsr induced by b. Then, by (d) in

Definition 3.1 f is strictly increasing and therefore, by Lemma 4.2 $h_b = f^\dagger$ is continuous.

To prove *3.* assume that $c \in \mathcal{M}^+$ is such that $\tau(P^c(s,\infty)) \le \tau(P^b(s,\infty))$ for all $s \ge 0$ and therefore $\|c\| \le \|b\|$. As before, let $k : [0, \|c\|] \to [0,1]$ be the function given by $k(s) = \tau(P^c(s,\infty))$, $\tilde{k}$ obtained from k as in Lemma 4.1, and $h_c = \tilde{k}^\dagger$ obtained from $\tilde{k}$ as in Lemma 4.2. Then, $\tilde{g}(t) \le \tilde{k}(t)$ for every $t \in [0, \|c\|]$ and, by Lemma 4.2, we conclude that $h_c = \tilde{k}^\dagger \le \tilde{g}^\dagger = h_b$. From this it follows that $\tilde{c} \le \tilde{b}$, where $\tilde{b} = h_b(a)$, $\tilde{c} = h_c(a)$. The rest of the statement is a consequence of the fact that $\mu_b = \mu_{\tilde{b}}$ and $\mu_c = \mu_{\tilde{c}}$. □

We say that $c \in \mathcal{M}^+$ is a *model* of $b \in \mathcal{M}^+$ with respect to $a \in \mathcal{M}^+$, if there exists a nonnegative, left-continuous and increasing function h such that $c = h(a)$ and $\mu_c = \mu_b$. Thus, with the notations of the proof of Theorem 4.4, we see that $\tilde{b} \in \mathcal{M}^+$ is a model of $b \in \mathcal{M}^+$ with respect to a. As an immediate consequence of *2.* in Proposition 2.1, we conclude that the model $\tilde{b}$ is approximately unitarily equivalent to b in $\mathcal{M}$.

Remark 4.5. In [15] Kadison solved the following problem in a II_1 factor $(\mathcal{M}, \tau)$: given a masa $\mathcal{A} \subseteq \mathcal{M}$, $a \in \mathcal{A}_{sa}$ and $t \in [0,1]$ find a projection $p \in \mathcal{A}$ and $\lambda \in \mathbb{R}$ such that $\tau(p) = t$, $ap \ge \lambda p$ and $a(I-p) \le \lambda(I-p)$. Note that Theorems 3.2 and 4.4 give an alternative proof of this statement in the case $a \in \mathcal{A}^+$. Indeed, let $a' \in \mathcal{A}^+$ be as in Theorem 3.2 and h_a be as in Theorem 4.4. Then, if we let $p = P^{a'}(\alpha,\infty)$ with $\tau(p) = t$ (such α always exists since a' has continuous distribution) and $\lambda = h_a(\alpha)$ then p and λ have the desired properties, since h_a is an increasing function. □

As a final comment let us note that a variation of the proof of Theorem 4.4 implies that if $a \in \mathcal{M}^+$ has continuous distribution and if ν is any regular Borel probability measure of compact support in the real line then, there exists $h : [0, \|a\|] \to \mathbb{R}$ such that $\nu(\Delta) = \tau(P^{h(a)}(\Delta))$. Indeed, we just have to replace the function $\tau(P^b(\lambda,\infty))$ by $\nu((\lambda,\infty))$ in the proof of *2.* In particular, if $\mathcal{A} \subseteq \mathcal{M}$ is a masa and we consider $a \in \mathcal{A}^+$ then this argument gives a different proof of Proposition 5.2 in [4].

4.2 Some applications of the modelling technique

The following application of Theorem 4.4 provides new characterizations of spectral preorder and sub-majorization between positive operators in II_1 factors. Note that these re-formulations have an inequality-type form.

Theorem 4.6. *Let $(\mathcal{M}, \tau)$ be a II_1 factor and let a, $b \in \mathcal{M}^+$. Then*

1. *b spectrally dominates a if and only if*

$$\text{there exists} \quad c \in \overline{\mathcal{U}_{\mathcal{M}}(b)} \quad \text{with} \quad a \le c \tag{18}$$

or, equivalently, if

$$\text{there exists} \quad d \in \overline{\mathcal{U}_{\mathcal{M}}(a)} \quad \text{with} \quad d \le b. \tag{19}$$

Moreover, we can assume that a and c commute and that b and d commute.

2. *b sub-majorizes a if and only if there exists $c \in \mathcal{M}^+$ such that*

$$a \le c \prec b. \tag{20}$$

Moreover, we can assume that a and c commute.

Proof. Recall that for positive operators a, $b \in \mathcal{M}^+$, $a \le b$ implies $a \precsim b$. Thus, the existence of a sequence of unitary operators satisfying (18) or (19) implies spectral domination. Analogously, the existence of an operator satisfying (20) implies sub-majorization. Next show that the reverse implications are also true.

To prove the first part of *1.* let a, $b \in \mathcal{M}^+$ such that $a \prec b$. By Theorem 3.2 there exists $a' \in \mathcal{M}^+$ with continuous distribution such that the brsr induced by a' (strongly) refines the brsr induced by a. By Theorem 4.4 there exists an increasing left-continuous function h_b such that, if $\tilde{b} = h_b(a')$, $\mu_b = \mu_{\tilde{b}}$. By *2.* in Proposition 2.1, this implies that $\tilde{b} \in \overline{\mathcal{U}_{\mathcal{M}}(b)}$. Since by hypothesis $\mu_a \le \mu_b$, by *2.* and *3.* in Theorem 4.4 we have $\tilde{b} = h_b(a') \ge h_a(a') = a$. Thus, we obtain (18) with $c = \tilde{b}$. The proof of the second part follows a similar path, considering the model of a with respect to a refinement of b.

To prove *2.*, let a and a' be as in the first part of the proof. Let $b \in \mathcal{M}^+$ be such that $a \prec_w b$ and let ν denote the regular Borel probability measure on $I' = [0, \|a'\|]$ given by $\nu(\Delta) = \tau(P^{a'}(\Delta))$. Then, if h_a, h_b are as in Theorem 4.4 we have (see Remark 2.3) that $h_a \prec_w h_b$ in $L^\infty(\nu)$. Therefore, by Proposition 2.2 there exists $h \in L^\infty(\nu)$ such that $h_a \le h \prec h_b$. If we let $c = h(a')$ then $a \le c \prec b$ by construction, since $a = h_a(a')$. □

The first part of *1.* in Theorem 4.6 gives a partial affirmative solution to the following problem posed in [6, 7]: given a $(\mathcal{M}, \tau)$ a II_1 factor and a, $b \in \mathcal{M}^+$ such that $a \precsim b$, is there any automorphism of $\mathcal{M}$, Θ, such that $\Theta(b) \ge a$? Our considerations above lead to a sequence of τ-preserving automorphisms $(\text{Ad}_{u_n})_{n \in \mathbb{N}}$, where $u_n \in \mathcal{U}_{\mathcal{M}}$, such that *in the limit* the above statement is true.

Corollary 4.7. *Let a, $b \in \mathcal{M}^+$. Then the following statements are equivalent:*

1. *b spectrally dominates a.*

2. *There exists a brsr $\{E_\lambda\}_{\lambda \in I}$, where $I = [0, \|a\|]$ such that $\tau(E_\lambda) = \tau(P^a(\lambda, \infty))$ for every $\lambda \in I$ and*

$$\lambda E_\lambda \le E_\lambda \, b \, E_\lambda, \quad \forall \lambda \ge 0. \tag{21}$$

Proof. Assume *1.* and note that, by Theorem 4.6 there exists a sequence $(v_n)_n \subseteq \mathcal{U}_\mathcal{M}$ such that $\lim_{n\to\infty} \|d - v_n^* a v_n\| = 0$ and $d \le b$ for some $d \in \mathcal{M}^+$. Then $\tau(p(a)) = \tau(p(d))$ for every polynomial $p \in \mathbb{C}[x]$ and, using monotone convergence, we have $\tau(P^a(\lambda,\infty)) = \tau(P^d(\lambda,\infty))$, $\lambda \ge 0$. Moreover,

$$\lambda P^d(\lambda,\infty) \le P^d(\lambda,\infty)\, d \le P^d(\lambda,\infty)\, b\, P^d(\lambda,\infty).$$

Then, if we set $E_\lambda = P^d(\lambda,\infty)$, $\{E_\lambda\}_{\lambda\in[0,\,\|a\|]}$ is the desired brsr. Conversely, assume that there exists a brsr $\{E_\lambda\}_{\lambda\in[0,\|a\|]}$ as in item 2. Given $\epsilon > 0$, let $b_\epsilon = b + \epsilon I$ and note that $\lambda\, E_\lambda < E_\lambda b_\epsilon E_\lambda$, so $P^{E_\lambda\, b_\epsilon\, E_\lambda}(\lambda,\infty) = E_\lambda$. In [8] Fack proved the following interlacing-like inequality: for every orthogonal projection $p \in \mathcal{M}$, $p\, b\, p \precsim b$. Then we have

$$\tau(P^a(\lambda,\infty)) = \tau(E_\lambda) = \tau(P^{E_\lambda\, b_\epsilon\, E_\lambda}(\lambda,\infty)) \le \tau(P^{b_\epsilon}(\lambda,\infty)).$$

The inequality above shows that $\mu_a \le \mu_{b_\epsilon}$ for every $\epsilon > 0$. The corollary is now a consequence of the fact that $\lim_{\epsilon\to 0^+} \mu_{b_\epsilon}(t) = \mu_b(t)$ for every $t \ge 0$. □

We end with some applications of our previous results. These are mostly re-statements of some inequalities with respect to spectral preorder and submajorization obtained in [1, 4, 5, 7], using Theorem 4.6.

Corollary 4.8. *Let $(\mathcal{M},\tau)$ be a II_1 factor.*

1. *(Young-type inequalities) Let x, $y \in \mathcal{M}$ and let p, q be conjugated indices. Then there exist sequences $(u_n)_{n\in\mathbb{N}}, (v_n)_{n\in\mathbb{N}} \subseteq \mathcal{U}_\mathcal{M}$ such that*

$$|xy^*| \le \lim_{n\to\infty} u_n^*(p^{-1}|x|^p + q^{-1}|y|^q)u_n$$

and

$$\lim_{n\to\infty} v_n^*|xy^*|v_n \le p^{-1}|x|^p + q^{-1}|y|^q$$

2. *(Jensen-type inequalities) Let $\mathcal{A}$ be a unital C^*-algebra, $\Phi : \mathcal{A} \to \mathcal{M}$ be a positive unital map, $a \in \mathcal{A}^+$ and $f : \sigma(a) \to \mathbb{R}$ be a convex function.*

   (a) If f is increasing, there exist sequences $(u_n)_{n\in\mathbb{N}}$, $(v_n)_{n\in\mathbb{N}} \subseteq \mathcal{U}_\mathcal{M}$ with

$$f(\Phi(a)) \le \lim_{n\to\infty} u_n^*\Phi(f(a))u_n$$

   and

$$\lim_{n\to\infty} v_n^* f(\Phi(a))v_n \le \Phi(f(a)).$$

   (b) If f is an arbitrary convex function, there exists $c \in \mathcal{M}^+$ such that

$$f(\Phi(a)) \le c \prec \Phi(f(a)).$$

   Moreover, we can choose c so that it commutes with $f(\Phi(a))$.

Proof. In [7], Farenick and Manjegani proved that if p, q, x, y are as above, then $|xy^*| \precsim p^{-1}|x|^p + q^{-1}|y|^q$. On the other hand, in [1] it was shown that if Φ, f, a are as above then, $f(\Phi(a)) \precsim \phi(f(a))$ if f is increasing and in general, $f(\Phi(a)) \prec_w \Phi(f(a))$ for an arbitrary convex function f. The corollary follows from these facts and Theorem 4.6. □

The proofs of Theorem 4.6 and Corollary 4.7 show a possible interplay between Theorems 3.2 and 4.4 to get an interesting tool to deal with problems regarding spectral relations. As far as we know, the conclusions of Corollary 4.8 are not possible using the previous literature.

Some of our results extend to certain classes of (unbounded) measurable operators affiliated with $\mathcal{M}$. Also, note that there is still the problem of finding characterizations of spectral order and sub-majorization similar to those in Theorem 4.6, for general semifinite factors; these characterizations may depend on generalizations of both Theorems 3.2 and 4.4. We shall investigate these matters elsewhere.

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References

[1] J. Antezana, P. Massey, and D. Stojanoff, *Jensen's Inequality and Majorization*, preprint (http://xxx.lanl.gov/abs/math.FA/0411442).

[2] M. Argerami and P. Massey, *The local form of doubly stochastic maps and joint majorization in II_1 factors*, preprint.

[3] M. Argerami and P. Massey, *On a Schur-Horn type theorem in II_1 factors*, preprint (http://arxiv.org/abs/math.OA/0604120).

[4] W. Arveson and R. Kadison, *Diagonals of self-adjoint operators*, preprint (http://xxx.lanl.gov/abs/math.OA/0508482).

[5] J.S. Aujla and F.C. Silva, *Weak majorization inequalities and convex functions*, Linear Algebra Appl. 369 (2003), 217-233.

[6] D.R. Farenick, private communication.

[7] D.R. Farenick and S.M. Manjegani, *Young's Inequality in Operator Algebras*, to appear in J. of the Ramanujan Math. Soc. (http://xxx.lanl.gov/abs/math.OA/0303318).

[8] T. Fack, *Sur la notion de valeur caractéristique*, J. Operator Theory (1982), 307-333.

[9] F. Hiai, *Majorization and Stochastic maps in von Neumann algebras*, J. Math. Anal. Appl. **127** (1987), no. 1, 18–48.

[10] F. Hiai, *Spectral majorization between normal operators in von Neumann algebras*, Operator algebras and operator theory (Craiova, 1989), 78–115, Pitman Res. Notes Math. Ser., 271, Longman Sci. Tech., Harlow, 1992.

[11] F. Hiai, Y. Nakamura, *Distance between unitary orbits in von Neumann algebras*, Pacific J. of Math, vol. 138 (1989) 259-294.

[12] F. Hiai, Y. Nakamura, *Closed Convex Hulls of Unitary Orbits in von Neumann Algebras*, Trans. Amer. Math. Soc. **323** (1991), 1-38.

[13] R.V. Kadison, *The Pythagorean theorem I: the finite case*. Proc. N.A.S. (USA),99(7):4178-4184, 2002.

[14] R.V. Kadison, *The Pythagorean theorem II: the infinite case*. Proc. N.A.S. (USA),99(8):5217-5222, 2002.

[15] R. V. Kadison, *Non-commutative conditional expectations and their applicactions*, volume 365 of Contemporary Mathematics, pages 143-179. Amer. Math Soc. 2004. Operator algebras, quantization and non-commutative geometry.

[16] E. Kamei,*Majorization in finite factors*, Math. Japonica 28, No. 4 (1983), 495-499.

[17] D. Sherman, *Unitary orbits of normal operators in von Neumann algebras*, to appear in Crelle's Journal.

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