CHARACTERIZATION OF BESSEL SEQUENCES

M. LAURA ARIAS, GUSTAVO CORACH, AND MIRIAM PACHECO

Abstract. Let $\mathcal{H}$ be a separable Hilbert space, $L(\mathcal{H})$ be the algebra of all bounded linear operators of $\mathcal{H}$ and $Bess(\mathcal{H})$ be the set of all Bessel sequences of $\mathcal{H}$. Fixed an orthonormal basis $E = \{e_k\}_{k\in\mathbb{N}}$ of $\mathcal{H}$, a bijection α_E : $Bess(\mathcal{H}) \longrightarrow L(\mathcal{H})$ can be defined. The aim of this paper is to characterize $\alpha_E^{-1}(\mathcal{A})$ for different classes of operators $\mathcal{A} \subseteq L(\mathcal{H})$. In particular, we characterize the Bessel sequences associated to injective operators, compact operators and Schatten p-classes.

1. Introduction

Let $\mathcal{H}$ be a separable Hilbert space, and let $L(\mathcal{H})$ be the algebra of all bounded linear operators of $\mathcal{H}$. A sequence $\{f_k\}_{k\in\mathbb{N}}$ in $\mathcal{H}$ is called a Bessel sequence if there exists a positive constant B for which

$$\sum_{k=1}^{\infty} |\langle x, f_k\rangle|^2 \leq B\|x\|^2$$

for all $x \in \mathcal{H}$. The bound of a Bessel sequence is the smallest B that satisfies the corresponding inequality. The set of all Bessel sequences of $\mathcal{H}$ will be denoted by $Bess(\mathcal{H})$. It is easy to check that $Bess(\mathcal{H})$ is a vector space. Moreover, $\|\{f_k\}_{k\in\mathbb{N}}\| = \sqrt{B}$ is a norm and $(Bess(\mathcal{H}), \| \,.\, \|)$ is a Banach space. Fixed an orthonormal basis $E = \{e_k\}_{k\in\mathbb{N}}$ of $\mathcal{H}$, consider the mapping:

$$\alpha_E : Bess(\mathcal{H}) \longrightarrow L(\mathcal{H})$$

$$F = \{f_k\}_{k\in\mathbb{N}} \longrightarrow T$$

where T is defined by $T(\sum_{k=1}^{\infty} \alpha_k e_k) = \sum_{k=1}^{\infty} \alpha_k f_k$. In section 2 we will show that α_E is well defined and that it is an invertible isometric bounded linear transformation. Observe that when E is the canonical orthonormal basis of l^2, $\alpha_E(F)$ is known as the analysis operator of F and its adjoint, $\alpha_E(F)^*$, is the synthesis operator of F. The aim of this note is to characterize several classes of bounded linear operators on $\mathcal{H}$ in terms of their corresponding Bessel sequences. More precisely, we characterize the subsets of $Bess(\mathcal{H})$ which correspond to injective operators, Fredholm operators, compact operators, and Schatten p-classes. This work is a kind of continuation of [5], where there is a geometric study of frames and epimorphisms. The paper [2] by P. Balasz contains several results which are in the same spirit of this one.

Key words and phrases. Bessel sequences, frames, bounded linear operators.
UBACYT 1030, ANPCYT PICT 03-09521, CONICET PIP 5272/05.

2. Preliminaries

Let us prove that the mapping $\alpha_E(F) = T$ is well defined, i.e., that $\sum_{k=1}^{\infty} \alpha_k f_k$ is convergent in $\mathcal{H}$ and if $T(\sum_{k=1}^{\infty} \alpha_k e_k) = \sum_{k=1}^{\infty} \alpha_k f_k$ then $T \in L(\mathcal{H})$.
If $n > m$, then

$$\begin{aligned}
\|\sum_{k=1}^{n} \alpha_k f_k - \sum_{k=1}^{m} \alpha_k f_k\| &= \|\sum_{k=m+1}^{n} \alpha_k f_k\| = \sup_{\|g\|=1} \left| \left\langle \sum_{k=m+1}^{n} \alpha_k f_k, g \right\rangle \right| \\
&\leq \sup_{\|g\|=1} \sum_{k=m+1}^{n} |\alpha_k \langle f_k, g\rangle| \\
&\leq \left(\sum_{k=m+1}^{n} |\alpha_k|^2 \right)^{1/2} \sup_{\|g\|=1} \left(\sum_{k=m+1}^{n} |\langle f_k, g\rangle|^2 \right)^{1/2} \\
&\leq \sqrt{B} \left(\sum_{k=m+1}^{n} |\alpha_k|^2 \right)^{1/2} .
\end{aligned}$$

Consequently, $\{\sum_{k=1}^{n} \alpha_k f_k\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{H}$, and therefore it is convergent. Thus, T is well defined and clearly it is linear.
A similar calculation shows that T is bounded. It is straightforward, that $\|\alpha_E(F)\|^2$ is the optimal bound of the Bessel sequence F.
The mapping

$$L(\mathcal{H}) \longrightarrow Bess(\mathcal{H})$$

$$T \longrightarrow \{Te_k\}_{k\in\mathbb{N}}$$

is also well defined, since for every $x \in \mathcal{H}$

$$\sum_{k=1}^{\infty} |\langle x, Te_k\rangle|^2 = \|T^* x\| \leq \|T^*\|^2 \|x\|^2,$$

i.e, $\{Te_k\}_{k\in\mathbb{N}} \in Bess(\mathcal{H})$.
Moreover, this transformation is bounded and it obviously is the inverse of α_E.

Remark 2.1. The notion of Bessel sequence provides necessary and sufficient conditions for an infinite matrix to be the representation (induced by a fixed basis) of a bounded linear operator on $\mathcal{H}$. In general, it is hard to determine whether an infinite matrix arises from a bounded linear operator on $\mathcal{H}$ (see [9] p.23). However, according to what we have just proved, an infinite matrix corresponds to a bounded linear operator if and only if the sequence formed by its columns is a Bessel sequence in l^2.

We highlight some of the terminology and notation we need in these notes. Denote by $GL(\mathcal{H})$ the group of invertible operators and by $\mathcal{U}(\mathcal{H})$ the group of unitary operators. Given an operator $T \in L(\mathcal{H})$, $R(T)$ denotes the range of T, $N(T)$ the nullspace of T and T^* the adjoint of T.

Definition 2.2. A sequence of vectors $\{x_k\}_{k\in\mathbb{N}}$ of $\mathcal{H}$ is a *Schauder basis* for $\mathcal{H}$ if, for each $x \in \mathcal{H}$, there exist unique scalar coefficients $\{\alpha_k\}_{k\in\mathbb{N}}$ such that $x = \sum_{k=1}^{\infty} \alpha_k x_k$.

If this property holds only for each $x \in \overline{span}\{x_k\}_{k\in\mathbb{N}}$ then the sequence $\{x_k\}_{k\in\mathbb{N}}$ is called a *Schauder sequence*.

Definition 2.3. A sequence of vectors $\{x_k\}_{k\in\mathbb{N}}$ belonging to $\mathcal{H}$ is a *Riesz sequence* if there exist constants $0 < c < C$ such that for every scalar sequence $\{a_n\}_{n\in\mathbb{N}} \in l^2$ one has

$$c\left(\sum_{n=1}^{\infty}|a_n|^2\right) \le \left\|\sum_{n=1}^{\infty}a_n x_n\right\|^2 \le C\left(\sum_{n=1}^{\infty}|a_n|^2\right).$$

A Riesz sequence $\{x_k\}_{k\in\mathbb{N}}$ is called a *Riesz basis* for $\mathcal{H}$ if $\overline{span}\left\{\{x_k\}_{k\in\mathbb{N}}\right\} = \mathcal{H}$. It can be observed that $\{x_k\}_{k\in\mathbb{N}}$ is a Riesz sequence of $\mathcal{H}$ if and only if $\{x_k\}_{k\in\mathbb{N}}$ is a Riesz basis for $\overline{span}\left\{\{x_k\}_{k\in\mathbb{N}}\right\}$.

Definition 2.4. A Bessel sequence $\{f_k\}_{k\in\mathbb{N}}$ is called a *frame* if there exists a constant $A > 0$ such that

$$A\|x\|^2 \le \sum_{k=1}^{\infty}|\langle x, f_k\rangle|^2$$

for every $x \in \mathcal{H}$. If this relation holds for every $x \in \overline{span}\{f_k\}_{k\in\mathbb{N}}$ then $\{f_k\}_{k\in\mathbb{N}}$ is called a *frame sequence*. If the bounds A and B coincide the frame is called *tight*. Tight frames with bound equal to 1 are called *Parserval frames*.

For a general background on bases and frames the reader is referred to the paper by Duffin and Schaeffer [6], and the books by R. C. Young [12] and O. Christensen [3].

3. Relation between Bessel sequences and different classes of linear bounded operators

In this section we will characterize Bessel sequences related, through the map α_E, to different classes of bounded linear operators. The map α_E depends on the previously fixed orthonormal basis E, but it would be desirable that the characterization of the Bessel sequences be independent of E. The next proposition shows that this independence only holds for subsets $\mathcal{A}$ of $L(\mathcal{H})$ which are invariant by unitary right multiplication.

Proposition 3.1. *Let $\mathcal{A} \subset L(\mathcal{H})$. Then $\alpha_E^{-1}(\mathcal{A}) = \alpha_{\tilde{E}}^{-1}(\mathcal{A})$ for every pair of orthonormal bases of $\mathcal{H}$, $E = \{e_k\}_{k\in\mathbb{N}}$ and $\tilde{E} = \{\tilde{e}_k\}_{k\in\mathbb{N}}$, if and only if $\mathcal{A} = \mathcal{A}\mathcal{U}(\mathcal{H})$.*

Proof. Let $T \in \mathcal{A}$, $U \in \mathcal{U}(\mathcal{H})$ and $E = \{e_k\}_{k\in\mathbb{N}}$ be an orthonormal basis of $\mathcal{H}$. Then $\tilde{E} = U^*E = \{\tilde{e}_k\}_{k\in\mathbb{N}}$ is also an orthonormal basis of $\mathcal{H}$. Now, by hypotheses, there exists $\tilde{T} \in \mathcal{A}$ such that $Te_k = \tilde{T}\tilde{e}_k$, i.e., $T = \tilde{T}\,U^*$. Therefore, $TU = \tilde{T} \in \mathcal{A}$.
Conversely, let $\{f_k\}_{k\in N}$ be a Bessel sequence in $\alpha_E^{-1}(\mathcal{A})$. Then there exist $T \in \mathcal{A}$ such that $Te_k = f_k$. Let $\tilde{E} = \{\tilde{e}_k\}_{k\in\mathbb{N}}$ be an orthonormal basis of $\mathcal{H}$, and let $U \in \mathcal{U}(\mathcal{H})$, such that $U\tilde{e}_k = e_k$. Then, $f_k = Te_k = TU\tilde{e}_k$, and as $TU \in \mathcal{A}$, we obtain that $f_k \in \alpha_{\tilde{E}}(\mathcal{A})$. □

Even though the condition given in Proposition 3.1 is very restrictive, there exist many classes of operators that verify it. For example, invertible operators, injective operators and compact operators, as well as surjective operators, closed range operators, partial isometries, contractions, and so on.

The next proposition summarizes some well known characterizations. We include the proof of some items for the reader's convenience.

Proposition 3.2.

(1) *If* $\mathcal{A}=\mathcal{U}(\mathcal{H})$ *then* $\alpha_E^{-1}(\mathcal{A})=\{F=\{f_k\}_{k\in\mathbb{N}}\in Bess(\mathcal{H})$*: F is an orthonormal basis of* $\mathcal{H}\}$.

(2) *If* $\mathcal{A}=GL(\mathcal{H})$ *then* $\alpha_E^{-1}(\mathcal{A})=\{F=\{f_k\}_{k\in\mathbb{N}}\in Bess(\mathcal{H})$*: F is a Riesz basis of* $\mathcal{H}\}$.

(3) *If* $\mathcal{A}$ *is the set of all epimorphisms in* $L(\mathcal{H})$ *then* $\alpha_E^{-1}(\mathcal{A})=\{F=\{f_k\}_{k\in\mathbb{N}}\in Bess(\mathcal{H})$*: F is a frame of* $\mathcal{H}\}$.

(4) *If* $\mathcal{A}$ *is the set of all closed range operators then* $\alpha_E^{-1}(\mathcal{A})=\{F=\{f_k\}_{k\in\mathbb{N}}\in Bess(\mathcal{H})$*: F is a frame sequence of* $\mathcal{H}\}$.

(5) *If* $\mathcal{A}$ *is the set of all the partial isometries of* $\mathcal{H}$ *then* $\alpha_E^{-1}(\mathcal{A})=\{F=\{f_k\}_{k\in\mathbb{N}}\in Bess(\mathcal{H})$*: F is a Parserval frame sequence of* $\mathcal{H}\}$.

(6) *If* $\mathcal{A}$ *is the set of all the co-isometries of* $\mathcal{H}$ *then* $\alpha_E^{-1}(\mathcal{A})=\{F=\{f_k\}_{k\in\mathbb{N}}\in Bess(\mathcal{H})$*: F is a Parserval frame of* $\mathcal{H}\}$.

Proof.
(5) Let T be a partial isometry, and $y\in R(T)$. Hence, there exists $x\in N(T)^{\perp}$ such that $y=Tx$, and $T^*T=P_{N(T)^{\perp}}$. Then,
$\|y\|^2=\|\,Tx\|^2=\|x\|^2=\|\,T^*Tx\|^2=\sum_{k=1}^{\infty}|\langle\,T^*\,Tx,e_k\rangle|^2=\sum_{k=1}^{\infty}|\langle y,\;Te_k\rangle|^2$.
Therefore $f=\{Te_k\}_{k\in\mathbb{N}}$ is a normalized frame sequence.
Conversely, let $x\in\mathcal{H}$

$$\|Tx\|^2=\sum_{k=1}^{\infty}|\langle Tx,Te_k\rangle|^2=\|T^*Tx\|^2.$$

Then T^* is an isometry onto $R(T)=N(T^*)^{\perp}$, i.e, T^* is a partial isometry, and then T is a partial isometry.

(6) $T\in L(\mathcal{H})$ is a co-isometry if and only if $\|x\|^2=\|T^*x\|^2=\sum_{k=1}^{\infty}|\langle T^*x,e_k\rangle|^2=\sum_{k=1}^{\infty}|\langle x,Te_k\rangle|^2$ for every $x\in\mathcal{H}$, i.e., if and only if $\alpha_E^{-1}(f)$ is a normalized frame. □

The class of Fredholm operators satisfies the condition of Proposition 3.1. The following definition will be needed to characterize the Bessel sequences related to them.

Definition 3.3. Let $G=\{g_i\}_{i\in\mathbb{N}}$ be a sequence in $\mathcal{H}$.

(a) The deficit of G is

$$d\,(G)=inf\,\{|J|:J\subset\mathcal{H}\text{ and }\overline{span}\,(J\cup G)=\mathcal{H}\}\,.$$

(b) The excess of G is

$$e\,(G)=sup\,\{|J|:J\subset G\text{ and }\overline{span}\,(G-J)=\overline{gen}\,(G)\}\,.$$

See [10] for the relation of these concepts with Besselian frames and near-Riesz Bases.
The reader is referred to [1] for the proof of the next lemma.

Lemma 3.4. *Let* $F=\{f_k\}_{k\in\mathbb{N}}$ *be a Bessel sequence of* $\mathcal{H}$*. Then,*

(a) $d(F)=dim\,N(\alpha_E\,(F)^*)=dim\,R(\alpha_E\,(F))^{\perp}$,

(b) $e(F)\geq dim\,N(\alpha_E\,(F))$,

(c) If F is a frame then $e(F) = dim\, N(\alpha_E\,(F))$.

Therefore, applying Proposition 3.2 and Lemma 3.4, Bessel sequences related to Fredholm operators can be characterized as follows.

Proposition 3.5. *Let $F = \{f_k\}_{k\in\mathbb{N}}$ be a Bessel sequence of $\mathcal{H}$. $\alpha_E(F)$ is a Fredholm operator if and only if F is a frame sequence with finite excess and deficit.*

Our next goal is to characterize the Bessel sequences related to injective and injective and closed range operators.

Proposition 3.6. *Let $T \in L(\mathcal{H})$. T is an injective and closed range operator if and only if $\alpha_E^{-1}\,(T)$ is a Riesz sequence.*

Proof. Recall that $T \in L(\mathcal{H})$ is injective and has closed range if and only if there exists a constant $c > 0$ such that $c\,\|x\|^2 \le \|Tx\|^2$ for every $x \in \mathcal{H}$.
Then, consider $T \in L(\mathcal{H})$ an injective closed range operator and let c be a positive constant as above. Now, for every $\{a_n\}_{n\in\mathbb{N}} \in l^2$ let $x = \sum_{n=1}^{\infty} a_n e_n$. Then $c\sum_{n=1}^{\infty} |a_n|^2 \le \|\sum_{n=1}^{\infty} a_n T e_n\|^2$.
On the other hand, since $T \in L(\mathcal{H})$ then there exists C such that $\|Tx\|^2 \le C\,\|x\|^2$ for every $x \in \mathcal{H}$, or what is equivalent there exists $C > 0$ such that $\|\sum_{n=1}^{\infty} a_n T e_n\|^2 \le C\sum_{n=1}^{\infty} |a_n|^2$ for every $\{a_n\}_{n\in\mathbb{N}} \in l^2$.
Summarizing, there exist $c, C > 0$ such that

$$c\left(\sum_{n=1}^{\infty} |a_n|^2\right) \le \left\|\sum_{n=1}^{\infty} a_n T e_n\right\|^2 \le C\left(\sum_{n=1}^{\infty} |a_n|^2\right)$$

for every $\{a_n\}_{n\in\mathbb{N}} \in l^2$, i.e., $\{Te_k\}_{k\in\mathbb{N}} = \alpha_E^{-1}\,(T)$ is a Riesz sequence.
Conversely, let $F = \{f_k\}_{k\in\mathbb{N}}$ be a Riesz sequence in $\mathcal{H}$. By the upper bound condition, $T = \alpha_E\,(F)$ is well defined, i.e., $T \in L(\mathcal{H})$. By the lower bound condition there exists $c > 0$ such that $c\,\|x\|^2 \le \|Tx\|^2$ for every $x \in \mathcal{H}$, then T is an injective closed range operator. □

Corollary 3.7. *Let $T \in L(\mathcal{H})$ and $E = \{e_k\}_{k\in\mathbb{N}}$ be an orthonormal basis of $\mathcal{H}$. Then $\{Te_k\}_{k\in\mathbb{N}}$ is a frame in $\mathcal{H}$ if and only if $\{T^*e_k\}_{k\in\mathbb{N}}$ is a Riesz sequence.*

Proof. It follows easily from Propositions 3.2 and 3.6, and the fact that an operator is surjective if and only if its adjoint is injective and has closed range. □

Remark 3.8. Considering the matrix representation (induced by a fixed basis) of a bounded linear operator, the last corollary can be rephrased as follows:
The sequence of columns of a matrix forms a Riesz sequence if and only if the sequence of rows forms a frame.

Proposition 3.9. *Let $F = \{f_k\}_{k\in\mathbb{N}}$ be a Bessel sequence in $\mathcal{H}$. Then, $T = \alpha_E\,(F) \in L(\mathcal{H})$ is an injective operator if and only if $\{f_k\}_{k\in\mathbb{N}}$ is a Schauder sequence.*

Proof. It suffices to observe that $T \in L(\mathcal{H})$ is an injective operator if and only if there exists unique scalars α_k, $k \in \mathbb{N}$, such that $x = \sum_{k=1}^{\infty} \alpha_k T e_k$.
□

Remark 3.10. In the last proposition, the hypothesis that F is a Bessel sequence is necessary. In fact, there exist Schauder bases that are not Bessel sequences. For example, consider $x_k = \frac{1}{\sqrt{k}} \sum_{n=1}^{k} e_n$. It is easily seen that it is a Schauder basis. Let us prove that it is not a Bessel sequence. In fact, if $x = \sum_{n=1}^{\infty} n^{-1} e_n$ then $\langle x, x_k \rangle = \frac{1}{\sqrt{k}} \sum_{n=1}^{k} \frac{1}{n}$ and therefore $\sum_{k=1}^{\infty} |\langle x, x_k \rangle|^2 > \sum_{k=1}^{\infty} \frac{1}{k}$ which proves that $(x_k)_{k\in\mathbb{N}}$ is not a Bessel sequence.
The other implication is also false, i.e, there exist Bessel sequences that are not Schauder basis. For example, the sequence $\{e_1, e_1, e_2, e_2, ...\}$ is a Bessel sequence (moreover, it is a frame) and it is not a Schauder basis.

Another right unitary invariant subset of $L(\mathcal{H})$ is the set of compact operators. Different characterizations of compact operators allow the following results.

Proposition 3.11. *Let $T \in L(\mathcal{H})$. Then, T is compact if and only if $\| Tx_n \| \underset{n\to\infty}{\longrightarrow} 0$ for every orthonormal sequence $\{x_n\}_{n\in\mathbb{N}}$ in $\mathcal{H}$.*

Proof. See [7] p. 263. □

Corollary 3.12. *Let $F = \{f_k\}_{k\in\mathbb{N}}$ be a Bessel sequence of $\mathcal{H}$. If $\alpha_E(F)$ is a compact operator then $F \in c_0(\mathcal{H}) = \left\{ \{f_n\}_{n\in\mathbb{N}} : \|f_n\| \underset{n\to\infty}{\longrightarrow} 0 \right\}$.*

Remark 3.13. Observe that the converse of the last corollary is false in general. Fixed an orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$ the sequence $(f_n)_{n\in\mathbb{N}}$ defined by $e_1, \frac{e_2}{\sqrt{2}}, \frac{e_2}{\sqrt{2}}, \frac{e_3}{\sqrt{3}}, \frac{e_3}{\sqrt{3}}, \frac{e_3}{\sqrt{3}}, ...$ is a normalized tight frame. Therefore, $T = \alpha_E(f)$ is a co-isometry, i.e., $TT^* = id$. Hence, T is not a compact operator, however, $\| Te_n \| \underset{n\to\infty}{\longrightarrow} 0$.

The next well known result will be needed in Proposition 3.15.

Proposition 3.14. *$T \in L(\mathcal{H})$ is a compact operator if and only if for every orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$ of $\mathcal{H}$ holds that $P_n T \underset{n\to\infty}{\longrightarrow} T$ where $P_n = P_{span\{e_1 \dots e_n\}}$.*

Proof. See [4] p. 43 □

Proposition 3.15. *Let $F = \{f_k\}_{k\in\mathbb{N}}$ be a Bessel sequence of $\mathcal{H}$ and $F^N = \{f_k\}_{k>N}$. Then, $\alpha_E(F)$ is a compact operator if and only if $\left\|\alpha_E(F^N)\right\| \underset{N\to\infty}{\longrightarrow} 0$.*

Proof. Let $T = \alpha_E(F)$ be a compact operator, then T^* is also a compact operator. Consider $P_N = P_{span\{e_1,\dots,e_N\}}$. Then $\|P_N T^* - T^*\| \underset{N\to\infty}{\longrightarrow} 0$.
Let $x \in \mathcal{H}$,

$$(P_N T^* - T^*)\, x = -\sum_{k=N+1}^{\infty} \langle T^* x, e_k \rangle\, e_k = -\sum_{k=N+1}^{\infty} \langle x, Te_k \rangle\, e_k$$

Then,

$$\sum_{k=N+1}^{\infty} |\langle x, Te_k \rangle|^2 = \|(P_N T^* - T^*)\, x\|^2 \leq \|P_N T^* - T^*\|^2 \|x\|^2$$

Thus, $F^N = \{Te_k\}_{k>N} \in Bess(\mathcal{H})$ and $\left\|\alpha_E(F^N)\right\|^2 = \|P_N T^* - T^*\|^2 \underset{N\to\infty}{\longrightarrow} 0$.
Conversely, let $F = \{f_k\}_{k\in\mathbb{N}}$ be a Bessel sequence such that $\left\|\alpha_E(F^N)\right\|^2 \underset{N\to\infty}{\longrightarrow} 0$. Let $T = \alpha_E(F)$, and let show that T^* is compact. Following the same idea as

before,

$$\|(P_N T^* - T^*)\, x\|^2 = \sum_{k=N+1}^{\infty} |\langle x, Te_k\rangle|^2 \leq \left\|\alpha_E(f^N)\right\|^2 \|x\|^2$$

Then $\|P_N T^* - T^*\|^2 \leq \left\|\alpha_E(F^N)\right\|^2 \underset{N\to\infty}{\longrightarrow} 0$. So, T^* is compact and therefore T is compact.

□

Finally, we want to study the Schatten p-classes of operators. First, we recall some properties. (For more details see [11])

Definition 3.16. A compact operator $T \in L(\mathcal{H}, \mathcal{K})$ is said to be in the Schatten p-class $\sum_p$ $(1 \leq p < \infty)$ if $(\lambda_n)_{n\in\mathbb{N}} \in l^p$ where $(\lambda_n)_{n\in\mathbb{N}}$ is the sequence of positive eigenvalues of $|T| = (T^*T)^{1/2}$ arranged in decreasing order and repeated according to multiplicity.
The $\sum_1$ and $\sum_2$ classes are usually called the trace class and the Hilbert-Schmidt class, respectively.

If $T \in \sum_p$ and $S \in L(\mathcal{H})$ then $ST \in \sum_p$ and $TS \in \sum_p$. Therefore, $\sum_p \mathcal{U} = \sum_p$.
The following proposition gives conditions on $\alpha_E(f) \in \sum_p$ depending on the value of p. The reader is referred to [8] p. 95 for the proof of the next result.

Proposition 3.17. *Let $E = \{e_k\}_{k\in\mathbb{N}}$ be an orthonormal basis of $\mathcal{H}$ and $F = \{f_n\}_{n\in\mathbb{N}} \in Bess(\mathcal{H})$. Then:*

1. *If $p \leq 2$ and $\{\|f_n\|\}_{n\in\mathbb{N}} \in l^p$ then $\alpha_E(F) \in \sum_p$.*
2. *If $2 \leq p$ and $\alpha_E(F) \in \sum_p$ then $\{\|f_n\|\}_{n\in\mathbb{N}} \in l^p$.*

As a consequence of the last proposition, given a Bessel sequences $F = \{f_n\}_{n\in\mathbb{N}}$, $\alpha_E(F) \in \sum_2$ if and only if $\{\|f_n\|\}_{n\in\mathbb{N}} \in l^2$.
The following result may be proved in much the same way as Proposition 3.15.

Proposition 3.18. *If $F = \{f_k\}_{k\in\mathbb{N}}$ is a Bessel sequence and $\left\{\left\|\alpha_E(F^N)\right\|^2\right\}_{N\in\mathbb{N}} \in l^p$ where $F^N = \{f_k\}_{k>N}$ then $\alpha_E(F) \in \sum_p$.*

Proof. An easy computation shows that $\lambda_{n+1} = inf\{\|T - B\| : B \in L(\mathcal{H}, \mathcal{K})$ and $dim(R(B)) \leq n\}$. Then, following the same idea as in Proposition 3.15 the result is obtained. □

The converse of last proposition is not true. The next example illustrates it:

Example 3.19. Consider $\alpha \in \mathbb{R}$ and $e = \left\{\frac{\alpha}{n}\right\}_{n\in\mathbb{N}} \in l^2$ such that $\|e\| = 1$. Let $T \in L(l^2)$ be defined by $Tx = \langle x, e\rangle\, e$. Thus, T is the orthogonal projection onto $span\,\{e\}$, and so $T \in \sum_p$ for every $p > 0$. In particular, $T \in \sum_1$. Consider now $E = \{e_n\}_{n\in\mathbb{N}}$ the canonical orthonormal basis of l^2 and let P_N be the orthogonal projection onto $span\,\{e_1, ..., e_N\}^{\perp}$. Therefore, if $F = \{f_k\}_{k\in\mathbb{N}} = \alpha_E^{-1}(T)$ and $F^N = \{f_k\}_{k>N}$ then, $\left\|\alpha_E(F^N)\right\|^2 = \|TP_N\|^2 = \|P_N e\|^2 = \sum_{k=N}^{\infty} \left(\frac{1}{k}\right)^2 \approx \frac{1}{N}$ and so $\left\{\|\alpha_E(F^N)\|^2\right\}_{N\in\mathbb{N}} \notin l^1$.
This example is also useful to prove that the converse of Proposition 3.17.1 is false in general. In fact, $\{\|f_n\|\}_{n\in\mathbb{N}} = \{\|Te_n\|\}_{n\in\mathbb{N}} = \left\{\frac{\alpha}{n}\right\}_{n\in\mathbb{N}} \notin l^1$

Proposition 3.20. *Let $F = \{f_n\}_{n\in\mathbb{N}} \in Bess(\mathcal{H})$. $\alpha_E(F) \in \sum_p$ if and only if there exist an orthonormal basis $\{\beta_k\}_{k\in\mathbb{N}}$ of $\overline{span}\{f_n\}_{n\in\mathbb{N}}$, an orthonormal sequence $\{\psi_k\}_{k\in\mathbb{N}}$ of $\mathcal{H}$ and $\{\lambda_k\}_{k\in\mathbb{N}} \in l^p$ with $0 < \lambda_{k+1} \leq \lambda_k$ such that*

$$\sum_{n=1}^{\infty} \langle \psi_j, e_n\rangle \langle f_n, \beta_k\rangle = \lambda_k \delta_{j,k} \tag{3.1}$$

and

$$\sum_{n=1}^{\infty} \langle \psi, e_n\rangle \langle f_n, \beta_k\rangle = 0 \ if \ \psi \in span\{\psi_k\}_{k\in\mathbb{N}}^{\perp} \tag{3.2}$$

Proof. Recall that $\alpha_E(F)(x) = \sum_{n=1}^{\infty} \langle x, e_n\rangle f_n$.
Let $\alpha_E(F) \in \sum_p$. By the spectral theorem (see [4]), $\alpha_E(F)x = \sum_{n=1}^{\infty} \lambda_n \langle x, \psi_n\rangle \beta_n$, where $(\psi_k)_{k\in\mathbb{N}}$ is an orthonormal basis of $N(\alpha_E(F))^{\perp}$, $(\beta_k)_{k\in\mathbb{N}}$ is an orthonormal basis of $\overline{R(\alpha_E(F))} = \overline{span}\{f_n\}_{n\in\mathbb{N}}$ and $(\lambda_n)_{n\in\mathbb{N}} \in l^p$, $0 < \lambda_{k+1} \leq \lambda_k$.
Then,

$$\sum_{n=1}^{\infty} \langle \psi_j, e_n\rangle \langle f_n, \beta_k\rangle = \langle \alpha_E(F)(\psi_j), \beta_k\rangle = \langle \sum_{n=1}^{\infty} \lambda_n \langle \psi_j, \psi_n\rangle \beta_n, \beta_k\rangle = \lambda_k \delta_{j,k},$$

thus equation (1) holds.
On the other hand, if $\psi \in span\{\psi_k\}^{\perp} = N(\alpha_E(F))$ then

$$\sum_{n=1}^{\infty} \langle \psi, e_n\rangle \langle f_n, \beta_k\rangle = \langle \alpha_E(F)(\psi), \beta_k\rangle = 0,$$

i.e., equation (2) holds.
Conversely, let $F = \{f_n\}_{n\in\mathbb{N}} \in Bess(\mathcal{H})$ such that equation (1) and (2) are verified. Complete $\{\psi_k\}_{k\in\mathbb{N}}$ to an orthonormal basis of $\mathcal{H}$. Denote by $\left\{\tilde{\psi}_k\right\}_{k\in\mathbb{N}}$ this completation.
Observe that,

$$\alpha_E(F)\tilde{\psi}_j = \sum_{k=1}^{\infty} \langle \alpha_E(F)\tilde{\psi}_j, \beta_k\rangle \beta_k = \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} \langle \tilde{\psi}_j, e_n\rangle \langle f_n, \beta_k\rangle \beta_k.$$

Therefore, by equation (1) and (2), we get that if $\tilde{\psi}_j \in \{\psi_k\}_{k\in\mathbb{N}}$ then $\alpha_E(F)\tilde{\psi}_j = \lambda_j \beta_j$, and if $\tilde{\psi}_j \notin \{\psi_k\}_{k\in\mathbb{N}}$ then $\alpha_E(F)\tilde{\psi}_j = 0$.
Now, consider $x = \sum_{n=1}^{\infty} \langle x, \tilde{\psi}_n\rangle \tilde{\psi}_n \in \mathcal{H}$. Then, $\alpha_E(F)x = \sum_{n=1}^{\infty} \langle x, \tilde{\psi}_n\rangle \alpha_E(F)\tilde{\psi}_n = \sum_{n=1}^{\infty} \lambda_n \langle x, \psi_n\rangle \beta_n$ where $\{\lambda_k\}_{k\in\mathbb{N}} \in l^p$. Thus, $\alpha_E(F) \in \sum_p$.
□

References

1. R. Balan, P.G. Casazza, C. Heil and Z. Landau, Deficits and excesses of frames, Advances in Computational Mathematics 18 (2003), 93-116.
2. R. P. Balazs, Matrix representation of operators using frames, Sampling Theory in signal and image processing, to appear.
3. O. Christensen, An introduction to frames and Riesz bases, Birkhuser, Boston, 2003.
4. J.B. Conway, A course in functional analysis, Springer-Verlag New York, 1985.
5. G. Corach, M. Pacheco, D. Stojanoff, Geometry of epimorphisms and frames, Proceedings of the American Mathematical Society 132 (2004), 2039-2049.

6. R. J. Duffin and A. C. Schaeffer; A class of nonharmonic Fourier series. Trans. Amer. Math. Soc. 72, (1952). 341-366.
7. P. A. Fillmore and J. P. Williams, On operator ranges, Advances in Mathematics, 7(1971), 254-281, .
8. I. C. Gohberg and M. G. Krein, Introduction to the theory of linear non-self adjoint operators, AMS Translation 18, Rhode Island, 1969.
9. P. R. Halmos, A Hilbert space problem book, Second edition, Springer-Verlag, New York, 1982.
10. J.R.Holub, Pre-frame operators, Besselian frames, and near-Riesz bases in Hilbert Spaces, Proc. Amer. Math. Soc. 122, (1994), 779-785.
11. B. Simon, Trace ideals and their applications, Cambridge University Press, London, 1979.
12. R. Young, An introduction to nonharmonic Fourier series, Revised first edition, Academic Press, San Diego, 2001.

Instituto Argentino de Matematica (CONICET), Buenos Aires, Argentina
E-mail address: `ml_arias@uolsinectis.com.ar`

Instituto Argentino de Matematica (CONICET) and Facultad de Ingeniera, Universidad de Buenos Aires, Buenos Aires, Argentina
E-mail address: `gcorach@fi.uba.ar`

Facultad de Ingeniera, Universidad Nacional de la Patagonia San Juan Bosco, Comodoro Rivadavia, Argentina
E-mail address: `mpacheco@ing.unp.edu.ar`