A SCHUR-HORN THEOREM IN II_1 FACTORS

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ABSTRACT. Given a II_1 factor $\mathcal{M}$ and a diffuse abelian von Neumann subalgebra $\mathcal{A} \subset \mathcal{M}$, we prove a version of the Schur-Horn theorem, namely

$$\overline{E_{\mathcal{A}}(\mathcal{U}_{\mathcal{M}}(b))}^{\sigma\text{-sot}} = \{a \in \mathcal{A}^{sa} : \ a \prec b\}, \quad b \in \mathcal{M}^{sa},$$

where $\prec$ denotes spectral majorization, $E_{\mathcal{A}}$ the unique trace-preserving conditional expectation onto $\mathcal{A}$, and $\mathcal{U}_{\mathcal{M}}(b)$ the unitary orbit of b in $\mathcal{M}$. This result is inspired by a recent problem posed by Arveson and Kadison.

1. Introduction

In 1923, I. Schur [18] proved that if $A \in M_n(\mathbb{C})^{sa}$ (i.e., A is selfadjoint) then

$$\sum_{j=1}^{k} \alpha_j^{\downarrow} \le \sum_{j=1}^{k} \beta_j^{\downarrow}, \ k = 1, \dots, n,$$

with equality when $k = n$ (denoted $\alpha \prec \beta$), where $\alpha = \operatorname{diag}(A) \in \mathbb{R}^n$, $\beta = \lambda(A) \in \mathbb{R}^n$ the spectrum (counting multiplicity) of A, and $\alpha^{\downarrow}, \beta^{\downarrow} \in \mathbb{R}^n$ are obtained from α, β by reordering their entries in decreasing order.

In 1954, A. Horn [12] proved the converse: given $\alpha, \beta \in \mathbb{R}^n$ with $\alpha \prec \beta$, there exists a selfadjoint matrix $A \in M_n(\mathbb{C})$ such that $\operatorname{diag}(A) = \alpha$, $\lambda(A) = \beta$. Since every selfadjoint matrix is diagonalizable, the results of Schur and Horn can be combined in the following assertion: if $\mathcal{D}$ denotes the diagonal masa in $M_n(\mathbb{C})$ and $E_{\mathcal{D}}$ is the compression onto $\mathcal{D}$, then

$$E_{\mathcal{D}}(\{U\, M_\beta\, U^* : U \in M_n(\mathbb{C}) \text{ unitary}\}) = \{M_\alpha \in \mathcal{D} : \ \alpha \prec \beta\}, \tag{1}$$

where M_α is the diagonal matrix with the entries of α in the main diagonal.

This combination of the two results, commonly known as Schur-Horn theorem, has played a significant role in many contexts of matrix analysis: although simple, vector majorization expresses a natural and deep relation among vectors, and as such it has been a useful tool both in pure and applied mathematics. We refer to the books [5, 14] and the introductions of [6, 16] for more on this.

During the last 25 years, several extensions of majorization have been proposed by, among others, Ando [1] (to selfadjoint matrices), Kamei [13] (to selfadjoint operators in a II_1 factor), Hiai [10, 11] (to normal operators in a von Neumann algebra), and Neumann [16] (to vectors in $\ell^\infty(\mathbb{N})$). With these generalizations at hand, it is natural to ask about extensions of the Schur-Horn theorem.

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In [16], Neumann developed his extension of majorization with the goal of using it to prove a Schur-Horn type theorem in $\mathcal{B}(\mathcal{H})$ in the vein of previous works in convexity (see the introduction in [16] for details and bibliography). Other versions of the Schur-Horn theorem have been considered in [3] and [6]. It is interesting to note that the motivation in [16] comes from geometry, in [3] comes from the study of frames on Hilbert spaces, while in [6] it is of an operator theoretic nature.

In [6] Arveson and Kadison proposed the study of a Schur-Horn type theorem in the context of II_1 factors, which are for such purpose the most natural generalization of full matrix algebras. They proved a Schur type theorem for II_1 factors and they posed as a problem a converse of this result, i.e. a Horn type theorem. In this note we prove a Schur-Horn type theorem that is inspired by Arveson-Kadison's conjecture (Theorem 3.4).

2. Preliminaries

Throughout the paper $\mathcal{M}$ denotes a II_1 factor with normalized faithful normal trace τ. We denote by $\mathcal{M}^{\rm sa}$, $\mathcal{M}^+$, $\mathcal{U}_{\mathcal{M}}$, the sets of selfadjoint, positive, and unitary elements of $\mathcal{M}$, and by $\mathcal{Z}(\mathcal{M})$ the center of $\mathcal{M}$. Given $a \in \mathcal{M}^{sa}$ we denote its spectral measure by p^a. The characteristic function of the set Δ is denoted by 1_Δ. For $n \in \mathbb{N}$, the algebra of $n \times n$ matrices over $\mathbb{C}$ is denoted by $M_n(\mathbb{C})$, and its unitary group by $\mathcal{U}_n$. By dt we denote integration with respect to Lebesgue measure. To simplify terminology, we will refer to non-decreasing functions simply as "increasing"; similarly, "decreasing" will be used instead of "non-increasing".

Besides the usual operator norm in $\mathcal{M}$, we consider the 1-norm induced by the trace, $\|x\|_1 = \tau(|x|)$. As we will be always dealing with bounded sets in a II_1 factor, we can profit from the fact that the topology induced by $\|\cdot\|_1$ agrees with the σ-strong operator topology. Because of this we will express our results in terms of σ-strong closures although our computations are based on estimates for $\|\cdot\|_1$. For $X \subset \mathcal{M}$, we shall denote by $\overline{X}$ and $\overline{X}^{\sigma\text{-sot}}$ the respective closures in the norm and in the σ-strong operator topology.

2.1. **Spectral scale and spectral preorders.** The *spectral scale* [17] of $a \in \mathcal{M}^{sa}$ is defined as

$$\lambda_a(t) = \min\{s \in \mathbb{R} : \ \tau(p^a(s,\infty)) \le t\}, \quad t \in [0,1).$$

The function $\lambda_a : [0,1) \to [0, \|a\|]$ is decreasing and right-continuous. The map $a \mapsto \lambda_a$ is continuous with respect to both $\|\cdot\|$ and $\|\cdot\|_1$, since [17]

$$\|\lambda_a - \lambda_b\|_\infty \le \|a-b\|, \quad \|\lambda_a - \lambda_b\|_1 \le \|a-b\|_1 \quad a, b \in \mathcal{M}^{sa}, \tag{2}$$

where the norms on the left are those of $L^\infty([0,1], dt)$ and $L^1([0,1], dt)$ respectively.

We say that a is *submajorized* by b, written $a \prec_w b$, if

$$\int_0^s \lambda_a(t)\ dt \le \int_0^s \lambda_b(t)\ dt, \quad \text{for every } s \in [0,1).$$

If in addition $\tau(a) = \tau(b)$ then we say that a is *majorized* by b, written $a \prec b$.

These preorders play an important role in many papers (among them we mention [8, 10, 11, 13]), and they arise naturally in several contexts in operator theory and operator algebras: some recent examples closely related to our work are the study of Young's type [9] and Jensen's type inequalities [2, 6, 7, 10].

Theorem 2.1 ([10]). *Let $a, b \in \mathcal{M}^{sa}$. Then $a \prec b$ (resp. $a \prec_w b$) if and only if $\tau(f(a)) \leq \tau(f(b))$ for every convex (resp. increasing convex) function $f : J \to \mathbb{R}$, where J is an open interval such that $\sigma(a), \sigma(b) \subseteq J$.*

If $\mathcal{N} \subset \mathcal{M}$ is a von Neumann subalgebra and $b \in \mathcal{M}^{sa}$, we denote by $\Omega_{\mathcal{N}}(b)$ the set of elements in $\mathcal{N}^{sa}$ that are majorized by b, i.e.

$$\Omega_{\mathcal{N}}(b) = \{a \in \mathcal{N}^{sa} : \ a \prec b\}.$$

The unitary orbit of $a \in \mathcal{M}^{sa}$ is the set $\mathcal{U}_{\mathcal{M}}(a) = \{u^*au : u \in \mathcal{U}_{\mathcal{M}}\}$.

Proposition 2.2. *Let $\mathcal{N} \subset \mathcal{M}$ be a von Neumann subalgebra and let $E_{\mathcal{N}}$ be the trace preserving conditional expectation onto $\mathcal{N}$. Then, for any $b \in \mathcal{M}^{sa}$,*

(i) $E_{\mathcal{N}}(b) \prec b$.
(ii) $\|E_{\mathcal{N}}(b)\|_1 \leq \|b\|_1$.
(iii) $\overline{E_{\mathcal{N}}(\mathcal{U}_{\mathcal{M}}(b))}^{\sigma\text{-}sot} \subset \Omega_{\mathcal{N}}(b)$.

Proof. (i) The map $E_{\mathcal{N}}$ is doubly stochastic (i.e. trace preserving, unital, and positive), so this follows from [10, Theorem 4.7] (see also [6, Theorem 7.2]).

(ii) Consider the convex function $f(x) = |x|$. Since $E_{\mathcal{N}}(b) \prec b$, using Theorem 2.1 we get

$$\|E_{\mathcal{N}}(b)\|_1 = \tau(f(E_{\mathcal{N}}(b))) \leq \tau(f(b)) = \|b\|_1.$$

(iii) By (i) and the fact that $ubu^* \prec b$ for every $u \in \mathcal{U}_{\mathcal{M}}$, we just have to prove that the set $\Omega_{\mathcal{N}}(b)$ is $\|\cdot\|_1$-closed. So, let $(a_n)_{n\in\mathbb{N}} \subset \Omega_{\mathcal{N}}(b)$ be such that $\lim_{n\to\infty} \|a_n - a\|_1 = 0$ for some $a \in \mathcal{N}$. Then, necessarily, $a \in \mathcal{N}^{sa}$. By (2),

$$\int_0^s \lambda_a(t)\ dt = \lim_{n\to\infty} \int_0^s \lambda_{a_n}(t)\ dt \leq \int_0^s \lambda_b(t)\ dt.$$

Also, $\tau(a) = \lim_n \tau(a_n) = \tau(b)$, so $a \prec b$. □

The following result seems to be well-known, but we have not been able to find a reference. Thus we give a sketch of a proof.

Proposition 2.3. *Let $a \in \mathcal{A}^{sa}$, where $\mathcal{A} \subset \mathcal{M}$ is a diffuse von Neumann subalgebra. Then there exists a spectral resolution $\{e(t)\}_{t\in[0,1]} \subset \mathcal{A}$ with $\tau(e(t)) = t$ for every $t \in [0,1]$, and such that*

$$a = \int_0^1 \lambda_a(t)\, de(t).$$

Proof. Since $\tau(1) < \infty$, it is enough to show the result for $a \geq 0$. By [15, Theorem 3.2] there exists $a' \in \mathcal{A}$ with $g_{a'}(s) = \tau(p^{a'}(-\infty, s])$ continuous in $\mathbb{R}$, and an increasing left-continuous function f such that $p^a(-\infty, s] = p^{a'}(-\infty, f(s)]$. Although the original statement in [15] involves a masa, only the fact that the algebra is diffuse is needed for its proof.

Let $g_{a'}^{\dagger}(t) = \min\{s : \ g_{a'}(s) \geq t\}$, and let $q(t) = p^{a'}(-\infty, g_{a'}^{\dagger}(t)]$. Since $\tau(q(t)) = g_{a'}(g_{a'}^{\dagger}(t)) = t$ for every $t \in [0,1]$, it follows that $\{q(t)\}_{t\in[0,1]}$ is a continuous spectral resolution. Moreover, $q(g_{a'}(t)) = p^{a'}(-\infty, t]$, so $p^a(-\infty, t] = q(g_{a'}(f(t)))$. As $g_{a'} \circ f$ is increasing and right-continuous, by [15, Theorem 4.4] there exists an increasing and left-continuous function h_a such that $a = \int h_a(t)\, dq(t)$. Define $e(t) = 1 - q(1-t)$, and $h(t) = h_a(1-t)$. Then $\tau(e(t)) = t$, $a = \int h(t)\, de(t)$. As h is decreasing and right-continuous, it can be seen that $h = \lambda_a$. □

A spectral resolution $\{e(t)\}_{t\in[0,1]}$ as in Proposition 2.3 is called a *complete flag* for a.

3. A Schur-Horn Theorem for II_1 factors

For each $n\in\mathbb{N}$, $k\in 1,\ldots,2^n$, let $\{I_k^{(n)}\}_{k=1}^{2^n}$ be the partition of $[0,1]$ associated to the points $\{h\,2^{-n}:\ h=0,\ldots,2^n\}$.

Definition 3.1. *For each $n\in\mathbb{N}$ and every $f\in L^1([0,1])$, let*

$$E_n(f)=\sum_{i=1}^{2^n}\left(2^n\int_{I_i^{(n)}}f\right)\,1_{I_i^{(n)}}.$$

It is clear that each operator E_n is a linear contraction for both $\|\cdot\|_1$ and $\|\cdot\|_\infty$.

Given the flag $\{e(t)\}_t$, we write $e([t_0,t_1])$ for $e(t_1)-e(t_0)$. Note that, since we consider $e(t)$ diffuse, $e([t_0,t_1])=e((t_0,t_1))=e([t_0,t_1))=e((t_0,t_1])$.

Lemma 3.2. *Let $\{e(t)\}_{t\in[0,1]}$, $\{I_i^{(n)}\}_{i=1}^{2^n}$, $n\in\mathbb{N}$ and $\{E_n\}_{n\in\mathbb{N}}$ as above. Then, for each $a\in\mathcal{M}^{sa}$,*

$$\lim_{n\to\infty}\|\,a-\int_0^1 E_n(\lambda_a)(t)\,de(t)\,\|_1=0. \tag{3}$$

Proof. By continuity of the trace, we only need to check that

$$\|\lambda_a-E_n(\lambda_a)\|_1\underset{n}{\longrightarrow}0 \tag{4}$$

in $L^1([0,1])$. Consider first a continuous function g. By uniform continuity, $\|g-E_n(g)\|_1\to 0$. Since continuous functions are dense in $L^1([0,1])$ and because the operators E_n are $\|\cdot\|_1$-contractive for every $n\in\mathbb{N}$, a standard $\varepsilon/3$ argument proves (4) for any integrable function. □

Recall that $\mathcal{D}$ denotes the diagonal masa in $M_n(\mathbb{C})$, and that for $\alpha\in\mathbb{R}^n$ we denote by M_α the matrix with the entries of α in the diagonal and zero off-diagonal. The projection $E_{\mathcal{D}}$ of $M_n(\mathbb{C})$ onto $\mathcal{D}$ is then given by $E_{\mathcal{D}}(A)=M_{\mathrm{diag}(A)}$, where $\mathrm{diag}(A)\in\mathbb{R}^n$ is the main diagonal of A. We use $\{e_{ij}\}$ to denote the canonical system of matrix units in $M_n(\mathbb{C})$.

Lemma 3.3. *Let $\mathcal{N}\subset\mathcal{M}$ be a von Neumann subalgebra, and assume $E_{\mathcal{N}}$ denotes the unique trace preserving conditional expectation onto $\mathcal{N}$. Let $\{p_i\}_{i=1}^n\subset\mathcal{Z}(\mathcal{N})$ be a set of mutually orthogonal equivalent projections such that $\sum_{i=1}^n p_i=I$. Then there exists a unital *-monomorphism $\pi: M_n(\mathbb{C})\to\mathcal{M}$ satisfying*

$$\pi(e_{ii})=p_i,\quad 1\le i\le n, \tag{5}$$

$$E_{\mathcal{N}}(\pi(A))=\pi(E_{\mathcal{D}}(A)),\quad A\in M_n(\mathbb{C}). \tag{6}$$

Proof. Since the projections p_i are equivalent in $\mathcal{M}$, for each i there exists a partial isometry v_{i1} such that $v_{i1}v_{i1}^*=p_i$ and $v_{i1}^*v_{i1}=p_1$. Let $v_{11}=p_1$, $v_{1i}=v_{i1}^*$ for $2\le i\le n$ and $v_{ij}=v_{i1}v_{1j}$ for $1\le i,j\le n$. In this way we get the standard associated system of matrix units $\{v_{ij},\ 1\le i,\,j\le n\}$ in $\mathcal{M}$. Define $\pi: M_n(\mathbb{C})\to\mathcal{M}$ by $\pi(A)=\sum_{i,\,j=1}^n a_{ij}\,v_{ij}$.

The matrix unit relations imply that π is a *-monomorphism and it is clear that (5) is also satisfied. Moreover,

$$E_{\mathcal{N}}(v_{ij})=E_{\mathcal{N}}(p_i\,v_{ij}\,p_j)=p_i\,E_{\mathcal{N}}(v_{ij})\,p_j=\delta_{ij}\,p_i,$$

since $p_i\, p_j = \delta_{ij}\, p_i$, $E_{\mathcal{N}}(v_{ii}) = E_{\mathcal{N}}(p_i) = p_i$, and $p_i \in \mathcal{Z}(\mathcal{N})$. Finally, we check (6):

$$E_{\mathcal{N}}(\pi(A)) = \sum_{i,j} a_{ij} E_{\mathcal{N}}(v_{ij}) = \sum_i a_{ii} p_i = \pi(E_{\mathcal{D}}(A)). \qquad \square$$

Next we state and prove our version of the Schur-Horn theorem for II_1 factors. Note the formal analogy with (1).

Theorem 3.4. *Let $\mathcal{A} \subset \mathcal{M}$ be a diffuse abelian von Neumann subalgebra and let $b \in \mathcal{M}^{sa}$. Then*

$$\overline{E_{\mathcal{A}}(\mathcal{U}_{\mathcal{M}}(b))}^{\sigma\text{-}sot} = \Omega_{\mathcal{A}}(b). \tag{7}$$

Proof. By Proposition 2.2, we only need to prove $\overline{E_{\mathcal{A}}(\mathcal{U}_{\mathcal{M}}(b))}^{\sigma\text{-sot}} \supset \Omega_{\mathcal{A}}(b)$.

So let $a \in \mathcal{A}^{sa}$ with $a \prec b$. By Proposition 2.3,

$$a = \int_0^1 \lambda_a(t)\, de(t), \quad b = \int_0^1 \lambda_b(t)\, df(t)$$

where $\{e(t)\}$ and $\{f(t)\}$ are complete flags with $\tau(e(t)) = \tau(f(t)) = t$, $e(t) \in \mathcal{A}$, $t \in [0,1]$.

Let $\{I_i^{(n)}\}_{i=1}^{2^n}$, $n \in \mathbb{N}$, be the family of partitions considered before and let $\epsilon > 0$. By Lemma 3.2 there exists $n \in \mathbb{N}$ such that

$$\left\| a - \sum_{i=1}^{2^n} \alpha_i\; p_i \right\|_1 < \epsilon, \quad \left\| b - \sum_{i=1}^{2^n} \beta_i\; q_i \right\|_1 < \epsilon, \tag{8}$$

where $\alpha_i = 2^n \int_{I_i^{(n)}} \lambda_a(t)\, dt$, $\beta_i = 2^n \int_{I_i^{(n)}} \lambda_b(t)\, dt$, $p_i = e(I_i^{(n)})$, $q_i = f(I_i^{(n)})$, $1 \leq i \leq 2^n$. Note that $\tau(p_i) = \tau(q_i) = 2^{-n}$. Let $\alpha = (\alpha_1, \ldots, \alpha_{2^n})$, $\beta = (\beta_1, \ldots, \beta_{2^n}) \in \mathbb{R}^{2^n}$. From the fact that λ_a and λ_b are decreasing, the entries of α and β are already in decreasing order. Using that $a \prec b$ IN $\mathcal{M}$, we conclude that $\alpha \prec \beta$ in $\mathbb{R}^n$.

By the classical Schur-Horn theorem (1), there exists $U \in \mathcal{U}_n(\mathbb{C})$ such that

$$E_{\mathcal{D}}(U M_\beta U^*) = M_\alpha \tag{9}$$

Consider the *-monomorphism π of Lemma 3.3 associated with the orthogonal family of projections $\{p_i\}_{i=1}^{2^n} \subset \mathcal{A}$. Let $w \in \mathcal{U}_{\mathcal{M}}$ such that $w q_i w^* = p_i$, $i = 1, \ldots, 2^n$, and put $u := \pi(U)\, w \in \mathcal{U}_{\mathcal{M}}$. By (6) and (9),

$$\begin{aligned} E_{\mathcal{A}}\left(u \left(\sum_{i=1}^{2^n} \beta_i\, q_i \right) u^* \right) &= E_{\mathcal{A}}(\pi(U M_\beta U^*)) \\ &= \pi(E_{\mathcal{D}}(U M_\beta U^*)) = \pi(M_\alpha) \\ &= \sum_{i=1}^{2^n} \alpha_i\, p_i \end{aligned}$$

Using (8), we conclude that

$$\|E_{\mathcal{A}}(ubu^*) - a\|_1 \leq \left\| E_{\mathcal{A}}(u(b - \sum_{i=1}^{2^n} \beta_i\; q_i)u^*) \right\|_1 + \left\| a - \sum_{i=1}^{2^n} \alpha_i\; p_i \right\|_1 < 2\epsilon.$$

As ϵ was arbitrary, we obtain $a \in \overline{E_{\mathcal{A}}(\mathcal{U}_{\mathcal{M}}(b))}^{\sigma\text{-sot}}$. $\square$

Corollary 3.5. *For each $b \in \mathcal{M}^+$, the set $\overline{E_{\mathcal{A}}(\mathcal{U}_{\mathcal{M}}(b))}^{\sigma\text{-}sot}$ is convex and σ-weakly compact.*

In [6], Arveson and Kadison posed the problem whether for $b \in \mathcal{M}^{sa}$, with the notations of Theorem 3.4,

$$E_{\mathcal{A}}\left(\overline{\mathcal{U}_{\mathcal{M}}(b)}\right) = \Omega_{\mathcal{A}}(b).$$

Since ([13])

$$\overline{\mathcal{U}_{\mathcal{M}}(b)} = \overline{\mathcal{U}_{\mathcal{M}}(b)}^{\sigma\text{-sot}} = \{a \in \mathcal{M}^{sa} : \ \lambda_a = \lambda_b\},$$

an affirmative answer to the Arveson-Kadison problem is equivalent to

$$E_{\mathcal{A}}\left(\overline{\mathcal{U}_{\mathcal{M}}(b)}^{\sigma\text{-sot}}\right) = \Omega_{\mathcal{A}}(b). \tag{10}$$

As a description of the set $\Omega_{\mathcal{A}}(b)$, (7) is weaker than (10), since in general

$$E_{\mathcal{A}}(\overline{\mathcal{U}_{\mathcal{M}}(b)}^{\sigma\text{-sot}}) \subset \overline{E_{\mathcal{A}}(\mathcal{U}_{\mathcal{M}}(b))}^{\sigma\text{-sot}}. \tag{11}$$

An affirmative answer to the Arveson-Kadison problem would imply equality in (11). We think it is indeed the case, although a proof of this does not emerge from our present methods.

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