Decomposition of Selfadjoint Projections in Krein Spaces

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Abstract

Given a Hilbert space $(\mathcal{H}, \langle\ ,\ \rangle)$ and a bounded selfadjoint operator B consider the sesquilinear form over $\mathcal{H}$ induced by B,

$$\langle\, x, y\,\rangle_B = \langle\, Bx, y\,\rangle\,, \quad x, y \in \mathcal{H}.$$

A bounded operator T is B-selfadjoint if it is selfadjoint respect to this sesquilinear form. We study the set $\mathcal{P}(B, \mathcal{S})$ of B-selfadjoint projections with range $\mathcal{S}$, where $\mathcal{S}$ is a closed subspace of $\mathcal{H}$. We state several conditions which characterize the existence of B-selfadjoint projections with a given range; among them certain decompositions of $\mathcal{H}$, $R(|B|)$ and $R(|B|^{1/2})$. We also show that every B-selfadjoint projection can be factorized as the product of a B-contractive, a B-expansive and a B-isometric projection. Finally two different formulas for B-selfadjoint projections are given.

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1 Introduction

Consider a Hilbert space $(\mathcal{H}, \langle \ , \ \rangle)$ and $L(\mathcal{H})$ the algebra of bounded linear operators on $\mathcal{H}$. Every selfadjoint operator $B \in L(\mathcal{H})$ induces an indefinite sesquilinear form, given by

$$\langle x, y \rangle_B = \langle Bx, y \rangle, \qquad x, y \in \mathcal{H}.$$

Given a selfadjoint operator $B \in L(\mathcal{H})$ and a closed subspace $\mathcal{S}$ of $\mathcal{H}$ we study the existence of projections $Q \in L(\mathcal{H})$ with range $\mathcal{S}$ which are selfadjoint with respect to the form $\langle \ , \ \rangle_B$. It is easy to see that Q is B-selfadjoint if and only if $BQ = Q^*B$, so that we are interested in studying in which cases the set

$$\mathcal{P}(B, \mathcal{S}) = \{Q \in L(\mathcal{H}) \ : \ Q^2 = Q, \ R(Q) = \mathcal{S}, \ BQ = Q^*B\}$$

is not empty. If $\mathcal{P}(B, \mathcal{S}) \neq \varnothing$ we say that the pair $(B, \mathcal{S})$ is *compatible*.

Observe that if B is positive and invertible then there exists a unique B-selfadjoint projection onto $\mathcal{S}$, because in this case $\langle \ , \ \rangle_B$ is an inner product equivalent to $\langle \ , \ \rangle$. In [4], [5], [6], many conditions for the existence of these projections have been given when B is (semidefinite) positive. For instance, it was proven that if $\mathcal{S}$ has finite dimension then there always exists a B-selfadjoint projection onto $\mathcal{S}$. However, these facts do not hold in the general case, even if $\mathcal{H}$ has finite dimension (see Example 3.1).

Most differences between the positive and the selfadjoint case are related to the indefinite metric space structure of $(\mathcal{H}, \langle \ , \ \rangle_B)$. When B is a symmetry, i.e. $B = B^* = B^{-1}$, $(\mathcal{H}, \langle \ , \ \rangle_B)$ is a Krein space. An exposition of the properties of these spaces can be found in the books by T. Ya. Azizov and I. S. Iokhvidov [2], J. Bognár [3] and in the lecture notes by T. Ando [1], where, in particular, the problem of the existence of projections is studied in detail; see also [10], [11], [15], [16], [17], [19] and [20].

It is well known that a projection Q in a Hilbert space $(\mathcal{H}, \langle \ , \ \rangle)$ is orthogonal if and only if it is a contraction. This fact can be generalized to B-selfadjoint projections when B is (semidefinite) positive, in this case, a projection Q is B-selfadjoint if and only if Q is a B-contraction, see [4]. However, this is no longer true when B is selfadjoint. S. Hassi and K. Nordström [14] proved that, given a selfadjoint operator B, then a projection Q is a B-contraction if and only if Q is B-selfadjoint and the nullspace of Q is B-nonnegative. They also proved that, if B is also invertible, a B-selfadjoint projection can be factorized in terms of a B-contraction and a B-expansion.

Although some of the results stated in this paper for selfadjoint operators are similar to the corresponding results for (semidefinite) positive operators (see [4], [5], [6]) the techniques to prove them are quite different and are related to some of the ideas that usually appear in Krein spaces problems.

The contents of the paper are as follows: in section 2 the basic notation is introduced together with some known results which are used later, mainly in three different topics: (bounded) operator factorization, Krein spaces, and, angles between (closed) subspaces in a Hilbert space.

Section 3 is devoted to state the background needed to study the main problem of this paper: the existence of B-selfadjoint projections with a prescribed range $\mathcal{S}$ for a selfadjoint operator $B \in L(\mathcal{H})$. The differences between the selfadjoint and the positive case naturally arise here. We also introduce the notion of the B-Gram operator associated to a closed subspace $\mathcal{S}$, which is closely related to compatibility of $(B, \mathcal{S})$.

We begin section 4 by giving necessary conditions involving the positive part (or modulus) of the polar decomposition of B. Then, we show that compatibility is equivalent to certain decompositions of operator ranges. For example we show that $(B, \mathcal{S})$ is compatible if and only if

$$R(|B|^{1/2}) = \mathcal{M} \cap R(|B|^{1/2}) \dotplus \mathcal{M}^{[\perp]} \cap R(|B|^{1/2}),$$

where $\mathcal{M} = \overline{|B|^{1/2}(\mathcal{S})}$, $\mathcal{M}^{[\perp]} = J^{-1}(\mathcal{M}^{\perp})$ and J is the unitary part in the polar decomposition of B. We also give equivalent conditions to compatibility in terms of angles between subspaces and we study in detail the particular case in which B has closed range.

In section 5, following the ideas of S. Hassi and K. Nordström, we study possible decompositions of a B-selfadjoint projection Q. First of all, we write Q as a sum of a B-positive, a B-negative and a B-neutral projection, which is unique under a few additional conditions. Then, we extend Hassi-Nordström's factorization for an arbitrary selfadjoint operator B: every B-selfadjoint projection Q admits a factorization $Q = Q_0Q_1Q_2$, where Q_0, Q_1 and Q_2 are commuting projections such that Q_0 is B-isometric, Q_1 is B-contractive and Q_2 is B-expansive.

Section 6 is devoted to present two different formulas for B-selfadjoint projections which resemble (and generalize) those obtained in [5].

2 Preliminaries

In what follows $\mathcal{H}$ denotes a Hilbert space with inner product $\langle\ ,\ \rangle$ and $L(\mathcal{H})$ is the algebra of bounded linear operators on $\mathcal{H}$. Let $L(\mathcal{H})^s$ be the (real) subspace of selfadjoint operators in $L(\mathcal{H})$ and $L(\mathcal{H})^+$ the cone of (semidefinite) positive operators in $L(\mathcal{H})$; $GL(\mathcal{H})$ is the group of invertible operators in $L(\mathcal{H})$, $GL(\mathcal{H})^s = GL(\mathcal{H}) \cap L(\mathcal{H})^s$ and $GL(\mathcal{H})^+ = GL(\mathcal{H}) \cap L(\mathcal{H})^+$. $\mathcal{Q}$ denotes the set of oblique projections in $L(\mathcal{H})$, i.e., $\mathcal{Q} = \{Q \in L(\mathcal{H}) : Q^2 = Q\}$. Given $T \in L(\mathcal{H})$, $R(T)$ denotes the range of T and $N(T)$ its nullspace.

Given two subspaces $\mathcal{S}$ and $\mathcal{T}$ of $\mathcal{H}$, denote by $\mathcal{S} \dotplus \mathcal{T}$ the direct sum of $\mathcal{S}$ and $\mathcal{T}$, $\mathcal{S} \oplus \mathcal{T}$ the orthogonal sum of them and $\mathcal{S} \ominus \mathcal{T} = \mathcal{S} \cap (\mathcal{S} \cap \mathcal{T})^{\perp}$. If $\mathcal{H} = \mathcal{S} \dotplus \mathcal{T}$, the oblique projection onto $\mathcal{S}$ along $\mathcal{T}$, $P_{\mathcal{S}//\mathcal{T}}$, is the projection with $R(P_{\mathcal{S}//\mathcal{T}}) = \mathcal{S}$ and $N(P_{\mathcal{S}//\mathcal{T}}) = \mathcal{T}$. In particular, $P_{\mathcal{S}} = P_{\mathcal{S}//\mathcal{S}^{\perp}}$ is the orthogonal projection onto $\mathcal{S}$.

Given $T \in L(\mathcal{H})$ and a fixed closed subspace $\mathcal{S}$ of $\mathcal{H}$, the operator T can be represented as a 2×2 matrix according to the decomposition $\mathcal{H} = \mathcal{S} \oplus \mathcal{S}^\perp$. More precisely, if $P = P_{\mathcal{S}}$, T can be represented as

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where $a = PTP|_{\mathcal{S}}$, $b = PT(I-P)|_{\mathcal{S}^\perp}$, $c = (I-P)TP|_{\mathcal{S}}$ and $d = (I-P)T(I-P)|_{\mathcal{S}^\perp}$. In particular, $P = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$. Observe that every oblique projector Q onto $\mathcal{S}$ has the form $Q = \begin{pmatrix} I & x \\ 0 & 0 \end{pmatrix}$.

The following result due to R. G. Douglas [9], characterizes operator range inclusions. It is frequently used along the paper.

Theorem 2.1. *Given Hilbert spaces $\mathcal{H}$, $\mathcal{K}_1$, $\mathcal{K}_2$ and operators $A \in L(\mathcal{K}_1, \mathcal{H})$ and $B \in L(\mathcal{K}_2, \mathcal{H})$, the following conditions are equivalent:*

1. *the equation $AX = B$ has a solution in $L(\mathcal{K}_2, \mathcal{K}_1)$;*
2. *$R(B) \subseteq R(A)$;*
3. *there exists $\lambda > 0$ such that $BB^* \leq \lambda AA^*$.*

In this case, there exists a unique $D \in L(\mathcal{K}_2, \mathcal{K}_1)$ such that $AD = B$ and $R(D) \subseteq \overline{R(A^*)}$; moreover, $N(D) = N(B)$ and $\|D\| = \inf\{\lambda > 0 : BB^* \leq \lambda AA^*\}$. The operator D is called the reduced solution *of $AX = B$.*

In what follows we give some basic results on Krein spaces, see the book by T. Ya. Azizov and I. S. Iokhvidov [2] for the proofs of the results below.

A *Krein space* (or a J-space) is a triple $(\mathcal{H}, \langle\, .,. \,\rangle, J)$ such that $(\mathcal{H}, \langle\, .,. \,\rangle)$ is a Hilbert space and $J \in L(\mathcal{H})$ is a symmetry which defines an indefinite metric (a J-metric) on $\mathcal{H}$ by

$$[\, x, y \,] := \langle\, Jx, y \,\rangle, \qquad x, y \in \mathcal{H}.$$

A vector $x \in \mathcal{H}$ is *J-positive, J-negative or J-neutral* according to the sign of $[\, x, x \,]$, that is, if $[\, x, x \,] > 0$, $[\, x, x \,] < 0$ or $[\, x, x \,] = 0$ respectively. A subspace $\mathcal{S}$ of $\mathcal{H}$ is *J-nonnegative* if $\mathcal{S} \subseteq \{x \in \mathcal{H} : [\, x, x \,] \geq 0\}$. J-positive, J-neutral, J-nonpositive and J-negative subspaces are defined analogously. $\mathcal{S}$ is *indefinite* if $\mathcal{S}$ contains J-positive and J-negative vectors.

Consider $x, y \in \mathcal{H}$, then x, y are *J-orthogonal* vectors, $x [\perp] \, y$, if $[\, x, y \,] = 0$. Given $\mathcal{S} \subset \mathcal{H}$, the set

$$\mathcal{S}^{[\perp]} = \{x \in \mathcal{H} : [\, x, y \,] = 0 \text{ for every } y \in \mathcal{S}\}$$

is the *J-orthogonal complement* of $\mathcal{S}$.

Let $\mathcal{S}$ be a subspace of $\mathcal{H}$. The *isotropic part of* $\mathcal{S}$ is the subspace $\mathcal{S}^0 = \mathcal{S} \cap \mathcal{S}^{[\perp]}$. If $\mathcal{S}^0 = \{0\}$, $\mathcal{S}$ is *J-non-degenerated.*

Proposition 2.2. *If a subspace $\mathcal{S}$ of $\mathcal{H}$ admits a direct sum decomposition $\mathcal{S} = \mathcal{S}^+ \dotplus \mathcal{S}^-$, where $\mathcal{S}^+$ is a J-positive subspace and $\mathcal{S}^-$ is a J-negative subspace, then $\mathcal{S}$ is J-non degenerated.*

Proposition 2.3. *Let $\mathcal{S}$ be a subspace of a Krein space $(\mathcal{H}, \langle\ ,\ \rangle, J)$. Then $\mathcal{S}^{[\perp]}$ is closed and*

$$\mathcal{S}^{[\perp]} = J(\mathcal{S}^{\perp}), \qquad \mathcal{S}^{\perp} = J(\mathcal{S}^{[\perp]}).$$

Furthermore, $\mathcal{S}^{[\perp]} = \overline{\mathcal{S}}^{[\perp]}$ and $\mathcal{S}^{[\perp][\perp]} = \overline{\mathcal{S}}$. In particular, if $\mathcal{S}$ is a closed subspace, $\mathcal{S}^0 = (\mathcal{S}^{[\perp]})^0$.

If $(\mathcal{H}, \langle\ ,\ \rangle, J)$ is a Krein space and $\mathcal{S}$ is a closed subspace of $\mathcal{H}$ the subspace $\mathcal{S} + \mathcal{S}^{[\perp]}$ is not necessarily $\mathcal{H}$, in fact:

$$\mathcal{H} = \overline{(\mathcal{S} + \mathcal{S}^{[\perp]})} \oplus J(\mathcal{S}^0).$$

Therefore, $\mathcal{H} = \overline{(\mathcal{S} + \mathcal{S}^{[\perp]})}$ if and only if $\mathcal{S}$ is J-non-degenerated.

Definition. A subspace $\mathcal{S}$ of a Krein space $(\mathcal{H}, \langle\ ,\ \rangle, J)$ is *projectively complete* if $\mathcal{H} = \mathcal{S} + \mathcal{S}^{[\perp]}$.

It is well known that if $\mathcal{S}$ is a projectively complete subspace of $(\mathcal{H}, \langle\ ,\ \rangle, J)$ then $\mathcal{S}$ is closed and J-non degenerated. Then, there exists a unique projection Q with range $\mathcal{S}$ and nullspace $\mathcal{S}^{[\perp]}$. This projection is J-ortogonal, i.e. $JQ = Q^*J$. Observe that if $\dim \mathcal{S} < \infty$ and $\mathcal{S}$ is J-non-degenerated then $\mathcal{S}$ is projectively complete.

The bounded operator $G_{\mathcal{S}} = P_{\mathcal{S}} J|_{\mathcal{S}} : \mathcal{S} \to \mathcal{S}$ is the *Gram operator of* $\mathcal{S}$. The existence of a J-orthogonal projection onto $\mathcal{S}$ is characterized in the next theorem:

Theorem 2.4. *Let $(\mathcal{H}, \langle\ ,\ \rangle, J)$ be a Krein space and $\mathcal{S}$ be a closed subspace of $\mathcal{H}$. The following conditions are equivalent:*

1. *$\mathcal{S}$ is projectively complete.*
2. *$G_{\mathcal{S}}$ is invertible.*
3. *$\mathcal{S}$ is a Krein space (with the induced metric).*

To end this section recall the following definitions of angle between subspaces. Given two closed subspaces $\mathcal{S}$ and $\mathcal{T}$ of $\mathcal{H}$, the cosine of the *Friedrichs angle* between them is defined by

$$c(\mathcal{S}, \mathcal{T}) = \sup \{ |\langle x, y \rangle| \,:\, x \in \mathcal{S} \ominus \mathcal{T},\ \|x\| \le 1,\ \ y \in \mathcal{T} \ominus \mathcal{S},\ \|y\| \le 1 \}.$$

The following conditions are equivalent (see [8], [18]):

1. $c(\mathcal{S},\mathcal{T})<1$,
2. $\mathcal{S}+\mathcal{T}$ is closed,
3. $c(\mathcal{S}^{\perp},\mathcal{T}^{\perp})<1$,
4. $P_{\mathcal{S}^{\perp}}(\mathcal{T})$ is closed.

The *minimal angle* between $\mathcal{S}$ and $\mathcal{T}$ is the angle whose cosine is defined by

$$c_0(\mathcal{S},\mathcal{T})=\sup\{|\langle x,y\rangle| \,:\, x\in\mathcal{S},\|x\|\leq 1,\ y\in\mathcal{T},\|y\|\leq 1\}\,.$$

Observe that $c(\mathcal{S},\mathcal{T})\leq c_0(\mathcal{S},\mathcal{T})$ and $c(\mathcal{S},\mathcal{T})=c_0(\mathcal{S},\mathcal{T})$ when $\mathcal{S}\cap\mathcal{T}=\{0\}$.

3 Definitions and Basic Properties

Every $B\in L(\mathcal{H})^s$ induces a sesquilinear form in $\mathcal{H}\times\mathcal{H}$ given by

$$\langle x,y\rangle_B=\langle Bx,y\rangle\,,\qquad x,y\in\mathcal{H}.$$

If $\mathcal{S}$ is a closed subspace of $\mathcal{H}$ and $B\in L(\mathcal{H})^s$, the B-orthogonal complement of $\mathcal{S}$ is given by

$$\mathcal{S}^{\perp_B}:=\{x\in\mathcal{H}\,:\,\langle Bx,s\rangle=0 \text{ for every } s\in\mathcal{S}\}.$$

It holds that $\mathcal{S}^{\perp_B}=B^{-1}(\mathcal{S}^{\perp})=B(\mathcal{S})^{\perp}$.

A vector $x\in\mathcal{H}$ is *B-positive* if $\langle x,x\rangle_B>0$. A subspace $\mathcal{S}$ of $\mathcal{H}$ is B-positive if every non-trivial $x\in\mathcal{S}$ is a B-positive vector. B-nonnegative, B-neutral, B-negative and B-nonpositive vectors (and subspaces) are defined analogously.

An operator $T\in L(\mathcal{H})$ is *B-selfadjoint* if $\langle Tx,y\rangle_B=\langle x,Ty\rangle_B$ for $x,y\in\mathcal{H}$. It is easy to see that this condition is equivalent to the equality $BT=T^*B$. $T\in L(\mathcal{H})$ is *B-positive* if $\langle Tx,x\rangle_B\geq 0$ for every $x\in\mathcal{H}$, i.e. BT is a (semidefinite) positive operator. B-neutral and B-negative operators are defined in a similar way.

Definition. Let $B\in L(\mathcal{H})^s$ and $\mathcal{S}$ be a closed subspace of $\mathcal{H}$. The pair $(B,\mathcal{S})$ is *compatible* if there exists a B-selfadjoint projection with range $\mathcal{S}$, i.e. if the set

$$\mathcal{P}(B,\mathcal{S})=\{Q\in\mathcal{Q}\,:\,R(Q)=\mathcal{S},\ BQ=Q^*B\}$$

is not empty.

In [14, p. 404] (see also [4, Lemma 3.2]) it was stated that a projection Q is B-selfadjoint if and only if its nullspace satisfies the inclusion $N(Q)\subseteq R(Q)^{\perp_B}$. Then it follows that $(B,\mathcal{S})$ is compatible if and only if

$$\mathcal{H}=\mathcal{S}+B^{-1}(\mathcal{S}^{\perp}).$$

If $A \in GL(\mathcal{H})^+$ then $(A, \mathcal{S})$ is compatible for every closed subspace $\mathcal{S}$ of $\mathcal{H}$. Also, if $A \in L(\mathcal{H})^+$ and $\mathcal{S}$ is finite dimensional then $\mathcal{P}(A, \mathcal{S}) \neq \varnothing$, see [4]. However, the following example shows that there exist pairs $(B, \mathcal{S})$ with $B \in GL(\mathcal{H})^s$ such that $\mathcal{P}(B, \mathcal{S}) = \varnothing$, even for finite-dimensional $\mathcal{H}$.

Example 3.1. Let $\mathcal{H} = \mathbb{C}^2$, consider the subspace $\mathcal{S} = \{(x, y) \in \mathbb{C}^2 : y = -x\}$ and

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in GL(\mathbb{C}^2)^s.$$

Then $(J, \mathcal{S})$ *is not compatible*: It is easy to see that $J^{-1}(\mathcal{S}^\perp) = \mathcal{S}$ so that $\mathcal{S} + J^{-1}(\mathcal{S}^\perp) = \mathcal{S} \neq \mathbb{C}^2$. Moreover, the subspace $\mathcal{S}$ is J-degenerated, in fact $\mathcal{S}^0 = \mathcal{S}$. (In Section 4, given a symmetry $J \in L(\mathcal{H})$, we construct a J-non-degenerated subspace $\mathcal{T}$ of $(\mathcal{H}, \langle\, , \,\rangle, J)$ such that $(J, \mathcal{T})$ is not compatible.)

Given a compatible pair $(B, \mathcal{S})$, define $\mathcal{N} = \mathcal{S} \cap B(\mathcal{S})^\perp$. Since $\mathcal{H} = \mathcal{S} \dotplus (B(\mathcal{S})^\perp \ominus \mathcal{N})$, consider the oblique projection

$$P_{B,\mathcal{S}} := P_{\mathcal{S}//B(\mathcal{S})^\perp \ominus \mathcal{N}}.$$

Observe that $P_{B,\mathcal{S}} \in \mathcal{P}(B, \mathcal{S})$ because $R(P_{B,\mathcal{S}}) = \mathcal{S}$ and $N(P_{B,\mathcal{S}}) \subseteq B(\mathcal{S})^\perp$. In fact, the set $\mathcal{P}(B, \mathcal{S})$ is an affine manifold that can be parametrized as

$$\mathcal{P}(B, \mathcal{S}) = P_{B,\mathcal{S}} + L(\mathcal{S}^\perp, \mathcal{N}),$$

where $L(\mathcal{S}^\perp, \mathcal{N})$ is viewed as a subspace of $L(\mathcal{H})$; and $\mathcal{P}(B, \mathcal{S})$ is a singleton if and only if $\mathcal{N} = \{0\}$. See [4] for a proof of these facts.

It is easy to prove that, if $A \in L(\mathcal{H})^+$ and $\mathcal{S}$ is a closed subspace of $\mathcal{H}$, then $\mathcal{N} = \mathcal{S} \cap N(A)$. In general, this equality does not hold for a selfadjoint operator, for instance the pair $(J, \mathcal{S})$ of Example 3.1 satisfies $\mathcal{N} = \mathcal{S}$ and $\mathcal{S} \cap N(J) = \{0\}$. However,

Proposition 3.2. *Consider $B \in L(\mathcal{H})^s$ and $\mathcal{S}$ a closed subspace of $\mathcal{H}$ such that $(B, \mathcal{S})$ is compatible. Then $\mathcal{N} = \mathcal{S} \cap N(B)$.*

Proof. The inclusion $\mathcal{S} \cap N(B) \subseteq \mathcal{N}$ always holds. Suppose that $(B, \mathcal{S})$ is compatible and consider $Q \in \mathcal{P}(B, \mathcal{S})$. If $x \in \mathcal{N}$ and $y \in \mathcal{H}$ then,

$$\langle Bx, y \rangle = \langle BQx, y \rangle = \langle Q^* Bx, y \rangle = \langle Bx, Qy \rangle = 0$$

because $Bx \in \mathcal{S}^\perp$. Thus, $Bx = 0$ i.e. $x \in \mathcal{S} \cap N(B)$. □

Remark 3.3. Let $B \in L(\mathcal{H})^s$ and $\mathcal{S}$ be a closed subspace of $\mathcal{H}$.

1. *If $(B, \mathcal{S})$ is compatible then $(B, \mathcal{S} \ominus \mathcal{N})$ is compatible*: if $(B, \mathcal{S})$ is compatible then $\mathcal{H} = \mathcal{S} + B(\mathcal{S})^\perp = \mathcal{S} \ominus \mathcal{N} + B(\mathcal{S})^\perp \subseteq \mathcal{S} \ominus \mathcal{N} + B(\mathcal{S} \ominus \mathcal{N})^\perp$, so $(B, \mathcal{S} \ominus \mathcal{N})$ is compatible.

2. If $A \in L(\mathcal{H})^+$ then $\mathcal{N} = \mathcal{S} \cap N(A)$ and $\mathcal{S} + A(\mathcal{S})^\perp = \mathcal{S} \ominus \mathcal{N} + A(\mathcal{S} \ominus \mathcal{N})^\perp$. Therefore, $(A, \mathcal{S})$ is compatible if and only if $(A, \mathcal{S} \ominus \mathcal{N})$ is compatible. But, *if $B \in L(\mathcal{H})^s$, the compatibility of $(B, \mathcal{S} \ominus \mathcal{N})$ does not imply the compatibility of* $(B, \mathcal{S})$ as the following example shows: let $\{e_1, e_2, e_3, e_4\}$ be the canonical basis of $\mathbb{R}^4$,
$$B = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix} \in GL(\mathbb{R}^4)^s$$
and $\mathcal{S} = \langle e_1, e_2, e_3 \rangle$, the subspace generated by e_1, e_2 and e_3. Then, $\mathcal{N} = \langle e_3 \rangle$ and $\mathcal{S} \ominus \mathcal{N} = \langle e_1, e_2 \rangle$, the pair $(B, \mathcal{S} \ominus \mathcal{N})$ is compatible but $(B, \mathcal{S})$ is not compatible.

3.1 The B-Gram operator of a subspace

Definition. Given $B \in L(\mathcal{H})^s$ and a closed subspace $\mathcal{S}$ of $\mathcal{H}$, the *B-Gram operator of $\mathcal{S}$* induced by B is
$$G_{B,\mathcal{S}} = PBP,$$
where P is the orthogonal projection onto $\mathcal{S}$. It is easy to see that $N(G_{B,\mathcal{S}}) = \mathcal{S}^\perp \dot{+} \mathcal{N}$ and $\overline{R(G_{B,\mathcal{S}})} = \mathcal{S} \ominus \mathcal{N}$. In [4, Proposition 3.3] it was proven that, $(B, \mathcal{S})$ is compatible if and only if $R(PB) \subseteq R(PBP)$, or equivalently, equation
$$G_{B,\mathcal{S}} X = PB \tag{3.1}$$
admits a bounded solution.

Remark 3.4. Let $(B, \mathcal{S})$ be a compatible pair and let $\mathcal{T} = \mathcal{S} \ominus \mathcal{N}$. Then,

1. $G_{B,\mathcal{S}} = G_{B,\mathcal{T}}$.

   In fact, from Proposition 3.2 $\mathcal{N} \subseteq N(B)$ so that $B(\mathcal{S}) = B(\mathcal{T})$, or $BP = BP_\mathcal{T}$. Therefore, $G_{B,\mathcal{S}} = PBP = P_\mathcal{T} B P_\mathcal{T} = G_{B,\mathcal{T}}$.

2. $\mathcal{T} \cap B(\mathcal{T})^\perp = \{0\}$ and the restriction of the B-Gram operator $G_{B,\mathcal{T}}$ to the subspace $\mathcal{T}$, $G = P_\mathcal{T} B|_\mathcal{T} : \mathcal{T} \to \mathcal{T}$ is injective.

   Indeed, since $B(\mathcal{T}) = B(\mathcal{S})$, $\mathcal{T} \cap B(\mathcal{T})^\perp = \mathcal{T} \cap B(\mathcal{S})^\perp = \mathcal{S} \cap \mathcal{N}^\perp \cap B(\mathcal{S})^\perp = \mathcal{N} \cap \mathcal{N}^\perp = \{0\}$. Furthermore, given $x \in \mathcal{T}$, $x \in N(G)$ if and only if $Bx \in \mathcal{T}^\perp$, or equivalently, $x \in \mathcal{T} \cap B^{-1}(\mathcal{T}^\perp) = \{0\}$. Therefore, G is injective.

If $\mathcal{S}$ is a J-definite subspace of a Krein space $(\mathcal{H}, \langle\ ,\ \rangle, J)$ then it is well known that there exists a J-selfadjoint projection onto $\mathcal{S}$ if and only if $\mathcal{S}$ is uniformly J-definite i.e. there exists $\alpha > 0$ such that $[\,x, x\,] \geq \alpha \|x\|^2$ for every $x \in \mathcal{S}$, see [2, Corollary 7.17].

If $B \in GL(\mathcal{H})^s$ and $\mathcal{S}$ is a B-definite subspace of $\mathcal{H}$ then it is easy to see that $(B, \mathcal{S})$ is compatible if and only if $\mathcal{S}$ is uniformly B-definite, i.e. there exists $\alpha > 0$ such that $G_{B,\mathcal{S}} \geq \alpha P_{\mathcal{S}}$ (see Proposition 3.7). However, only one of the implications remains true for an arbitrary $B \in L(\mathcal{H})^s$. The following results are stated for B-positive subspaces, the reader can deduce the analogue for B-negative subspaces.

Proposition 3.5. *If $\mathcal{S}$ is uniformly B-positive then $(B, \mathcal{S})$ is compatible.*

Proof. Suppose that $\mathcal{S}$ is uniformly B-positive. Then $R(G_{B,\mathcal{S}})$ is closed. Furthermore, $\mathcal{N} = \{0\}$. In fact, if $x \in \mathcal{N}$ then $0 = \langle Bx, x \rangle = \langle G_{B,\mathcal{S}} x, x \rangle \geq \alpha \|x\|^2$. Thus, $R(G_{B,\mathcal{S}}) = \mathcal{S}$ and, by Douglas' theorem, equation (3.1) has a bounded solution, i.e. $(B, \mathcal{S})$ is compatible. □

Remark 3.6. Given a closed B-positive subspace $\mathcal{S}$ of $\mathcal{H}$, the compatibility of $(B, \mathcal{S})$ does not imply that $\mathcal{S}$ is uniformly B-positive. Indeed, given $A \in L(\mathcal{H})^+$ injective with $R(A) \neq \mathcal{H}$, consider the Hilbert space $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}$, $\mathcal{S} = \mathcal{H} \oplus \{0\}$ and $B \in L(\mathcal{K})^+$ represented by

$$B = \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}$$

in the decomposition induced by $\mathcal{S}$. Then, $B(\mathcal{S})^\perp = (R(A) \oplus \{0\})^\perp = \{0\} \oplus \mathcal{H}$ so $\mathcal{S} + B(\mathcal{S})^\perp = \mathcal{K}$, i.e. $(B, \mathcal{S})$ is compatible. On the other hand, its clear that $\mathcal{S}$ is not uniformly B-positive because $R(A)$ is not closed.

Proposition 3.7. *Suppose that $B \in L(\mathcal{H})^s$ has closed range and $\mathcal{S}$ is a B-positive subspace. If $(B, \mathcal{S})$ is compatible then $\mathcal{S}$ is uniformly B-positive.*

Proof. If $\mathcal{S}$ is B-positive then $G_{B,\mathcal{S}} \in L(\mathcal{H})^+$ and $\mathcal{N} = \{0\}$. Also, $B(\mathcal{S})$ is closed: if $B(s_n) \to y$ then $y \in R(B)$ because $R(B)$ is closed. So that $y = Bx$ for $x \in \mathcal{H}$ and $Bs_n = BQs_n = Q^*Bs_n \to Q^*Bx = BQx$. Therefore, $y = BQx \in B(\mathcal{S})$. Furthermore, from $\mathcal{H} = \mathcal{S} + B(\mathcal{S})^\perp$ it follows that $\mathcal{S}^\perp + B(\mathcal{S})$ is closed, or equivalently $R(G_{B,\mathcal{S}}) = P_{\mathcal{S}}(B(\mathcal{S}))$ is closed (see the preliminaries on angles between subspaces). Then, there exists $\alpha > 0$ such that $G_{B,\mathcal{S}} \geq \alpha P_{\mathcal{S}}$ i.e. $\mathcal{S}$ is uniformly B-positive. □

4 Necessary and Sufficient Conditions for Compatibility

It is well known that every $T \in L(\mathcal{H})$ has a (unique) polar decomposition $T = UA$, where $A = (T^*T)^{1/2} \in L(\mathcal{H})^+$ and U is a partial isometry from $N(T)^\perp$ onto $\overline{R(T)}$ with nullspace $N(T)$. If $B \in L(\mathcal{H})^s$ and $B = UA$, it is easy to see that $U = U^*$ so that A and U commute. In this case we can replace the partial isometry U by an unitary operator J, satisfying $B = JA$ and $Jx = x$ for every $x \in N(T)$. Observe that J is a *symmetry* i.e. $J = J^* = J^{-1}$, and $AJ = JA$. Therefore, $R(B) = R(A)$, $N(B) = N(A)$ and both subspaces are invariant under J.

Let $B \in L(\mathcal{H})^s$ with the polar decomposition $B = JA$, where $A \in L(\mathcal{H})^+$ and $J = J^* = J^{-1}$. Observe that $(B, \mathcal{S})$ is compatible if and only if $\mathcal{H} = \mathcal{S} + A^{-1}(\mathcal{S}^{[\perp]})$. In particular, if $\mathcal{S}$ is invariant by J then, $(B, \mathcal{S})$ is compatible if and only if $(A, \mathcal{S})$ is compatible.

The next proposition shows some conditions, involving the positive part of B, which are necessary for compatibility.

Proposition 4.1. *Consider $B \in L(\mathcal{H})^s$ with the polar decomposition $B = JA$, where $A \in L(\mathcal{H})^+$ and $J = J^{-1} = J^*$. Consider the following conditions:*

1. *The pair $(B, \mathcal{S})$ is compatible,*
2. *$A(\mathcal{S})$ is closed in $R(A)$ i.e. $\overline{A(\mathcal{S})} \cap R(A) = A(\mathcal{S})$,*
3. *$A^{1/2}(\mathcal{S})$ is closed in $R(A^{1/2})$,*
4. *$\mathcal{S} + N(A)$ is closed.*
5. *$P_{\overline{R(B)}}(\mathcal{S})$ is closed.*

Then, 1. $\Rightarrow$ 2. $\Rightarrow$ 3. $\Rightarrow$ 4. $\Leftrightarrow$ 5.

Proof. 1. $\Rightarrow$ 2. : Assume that $(B, \mathcal{S})$ is compatible and let $Q \in \mathcal{P}(B, \mathcal{S})$. Then, with an argument similar to the one used in Proposition 3.7 it follows that $B(\mathcal{S})$ is closed in $R(B)$. But $B(\mathcal{S})$ is closed in $R(B)$ if and only if $A(\mathcal{S})$ is closed in $R(A)$: if $B(\mathcal{S})$ is closed in $R(B)$ then $B(\mathcal{S}) = \overline{B(\mathcal{S})} \cap R(B)$, or $J(A(\mathcal{S})) = \overline{JA(\mathcal{S})} \cap J(R(A)) = J(\overline{A(\mathcal{S})} \cap R(A))$. Then $A(\mathcal{S}) = \overline{A(\mathcal{S})} \cap R(A)$ i.e. $A(\mathcal{S})$ is closed in $R(A)$. The converse is similar.
2. $\Rightarrow$ 3. $\Rightarrow$ 4. $\Leftrightarrow$ 5. follow from Proposition 3.4 of [6]. $\square$

Proposition 4.2. *Let $B \in L(\mathcal{H})^s$ and consider a closed subspace $\mathcal{S}$ of $\mathcal{H}$. Then, $(B, \mathcal{S})$ is compatible if and only if $P_{\overline{R(B)}}(\mathcal{S})$ is closed and $\left(B, P_{\overline{R(B)}}(\mathcal{S})\right)$ is compatible.*

Proof. The proof is similar to that of Corollary 3.5 of [6]. $\square$

Remark 4.3. Let $B \in L(\mathcal{H})^s$ and consider $B_R = B|_{\overline{R(B)}} \in L(\overline{R(B)})$. Then, $(B, \mathcal{S})$ is compatible if and only if $P_{\overline{R(B)}}(\mathcal{S})$ is closed and $(B_R, P_{\overline{R(B)}}(\mathcal{S}))$ is compatible.

In fact, applying Proposition 4.2 we can assume that $\mathcal{S} \subseteq \overline{R(B)}$. Then, the proof is straightforward, observing that $B^{-1}(\mathcal{S}^{\perp}) = B_R^{-1}(\overline{R(B)} \ominus \mathcal{S}) + N(B)$ and that $\overline{R(B)} \ominus \mathcal{S}$ is the orthogonal complement of $\mathcal{S}$ in $\overline{R(B)}$.

4.1 Compatibility and Decomposition of Ranges

The next proposition relates the compatibility of the pair $(B, \mathcal{S})$ to certain decompositions of $R(A)$ and $R(A^{1/2})$.

Proposition 4.4. *Given $B \in L(\mathcal{H})^s$ consider its polar decomposition $B = JA$, with $A \in L(\mathcal{H})^+$ and $J = J^* = J^{-1}$. The following conditions are equivalent:*

1. *The pair $(B, \mathcal{S})$ is compatible;*
2. *$R(A) = A(\mathcal{S}) \dot{+} \mathcal{S}^{[\perp]} \cap R(A)$;*
3. *$R(A^{1/2}) = A^{1/2}(\mathcal{S}) \dot{+} A^{1/2}(\mathcal{S})^{[\perp]} \cap R(A^{1/2})$.*

Proof. 1. $\Leftrightarrow$ 2.: If $(B, \mathcal{S})$ is compatible then $\mathcal{H} = \mathcal{S} + B^{-1}(\mathcal{S}^\perp)$. Applying A to both sides of this equality it follows that $R(A) = A(\mathcal{S}) + \mathcal{S}^{[\perp]} \cap R(A)$. But it is a direct sum because $A(\mathcal{S}) \cap S^{[\perp]} = J(B(\mathcal{S}) \cap S^\perp) \subseteq J(\overline{B(\mathcal{S})} \cap S^\perp) = J(\mathcal{H}^\perp) = \{0\}$. The converse is similar.
1. $\Leftrightarrow$ 3.: If $(B, \mathcal{S})$ is compatible then, applying $A^{1/2}$ as in 1. $\Leftrightarrow$ 2., $R(A^{1/2}) = A^{1/2}(\mathcal{S}) + A^{1/2}(\mathcal{S})^{[\perp]} \cap R(A^{1/2})$. Furthermore,

$$A^{1/2}(\mathcal{S}) \cap A^{1/2}(\mathcal{S})^{[\perp]} = A^{-1/2}(A(\mathcal{S}) \cap \mathcal{S}^{[\perp]}) \cap R(A^{1/2}) = N(A^{1/2}) \cap R(A^{1/2}) = \{0\}.$$

The converse is similar. □

Corollary 4.5. *Let $B \in L(\mathcal{H})^s$ and define $\mathcal{M} = \overline{A^{1/2}(\mathcal{S})}$. Then, $(B, \mathcal{S})$ is compatible if and only if $A^{1/2}(\mathcal{S})$ is closed in $R(A^{1/2})$ and*

$$R(A^{1/2}) = \mathcal{M} \cap R(A^{1/2}) \dot{+} \mathcal{M}^{[\perp]} \cap R(A^{1/2}).$$

Proof. It is immediate from Propositions 4.1 and 4.4. □

Corollary 4.6. *If $(B, \mathcal{S})$ is compatible then $\mathcal{M} = \overline{A^{1/2}(\mathcal{S})}$ is a J-non degenerated subspace of $(\mathcal{H}, \langle\ ,\ \rangle, J)$.*

Proof. By the above corollary,

$$\mathcal{H} \;=\; \overline{R(A^{1/2})} \oplus N(A^{1/2}) \subseteq \overline{\mathcal{M} + \mathcal{M}^{[\perp]}} + N(A^{1/2}) = \overline{\mathcal{M} + \mathcal{M}^{[\perp]}},$$

because $N(A^{1/2}) \subseteq \mathcal{M}^{[\perp]}$. Then, $\mathcal{M}$ is J-non degenerated. □

4.2 Compatibility and angles between subspaces

Compatibility can also be given in terms of angle conditions between certain subspaces. Look at the Preliminaries for the definitions and properties of the minimal angle.

Theorem 4.7. *Let $B \in L(\mathcal{H})^s$ and $\mathcal{S}$ be a closed subspace of $\mathcal{H}$. Then, $(B, \mathcal{S})$ is compatible if and only if $c_0(\mathcal{S}^\perp, \overline{B(\mathcal{S})}) < 1$.*

Proof. The proof given in [5, Theorem 2.15] remains valid for $B \in L(\mathcal{H})^s$. □

Proposition 4.8. *Let $B \in L(\mathcal{H})^s$ be injective and $\mathcal{S}$ be a closed subspace of $\mathcal{H}$. Then, $(B, \mathcal{S})$ is compatible if and only if $c_0(\mathcal{S}, B(\mathcal{S})^\perp) < 1$.*

Proof. If $(B, \mathcal{S})$ is compatible then $\mathcal{S} + B(\mathcal{S})^\perp = \mathcal{H}$, therefore $\mathcal{S} + B(\mathcal{S})^\perp$ is closed, or equivalently $c(\mathcal{S}, B(\mathcal{S})^\perp) < 1$. Furthermore, by Proposition 3.2, $\mathcal{S} \cap B(\mathcal{S})^\perp = \mathcal{S} \cap N(B) = \{0\}$. Thus $c_0(\mathcal{S}, B(\mathcal{S})^\perp) = c(\mathcal{S}, B(\mathcal{S})^\perp) < 1$.

Conversely, if $c_0(\mathcal{S}, B(\mathcal{S})^\perp) < 1$ then $\mathcal{S} + B(\mathcal{S})^\perp$ is closed and $\mathcal{S} \cap B^{-1}(\mathcal{S}^\perp) = \{0\}$. Since B is injective, $B(\mathcal{S} \cap B^{-1}(\mathcal{S}^\perp)) = B(\mathcal{S}) \cap B(B^{-1}(\mathcal{S}^\perp)) = B(\mathcal{S}) \cap (\mathcal{S}^\perp \cap R(B)) = B(\mathcal{S}) \cap \mathcal{S}^\perp$. Therefore $B(\mathcal{S}) \cap S^\perp = \{0\}$ and $\mathcal{H} = (S^\perp \cap B(\mathcal{S}))^\perp = \overline{\mathcal{S} + B(\mathcal{S})^\perp} = \mathcal{S} + B(\mathcal{S})^\perp$. □

Corollary 4.9. *Let $B \in L(\mathcal{H})^s$ and consider a closed subspace $\mathcal{S}$ of $\mathcal{H}$. Then, $(B, \mathcal{S})$ is compatible if and only if $P_{\overline{R(B)}}(\mathcal{S})$ is closed and $c_0(P_{\overline{R(B)}}(\mathcal{S}), B(\mathcal{S})^\perp) < 1$.*

Proof. By Remark 4.3, $(B, \mathcal{S})$ is compatible if and only if $\mathcal{S}' = P_{\overline{R(B)}}(\mathcal{S})$ is closed and the pair $(B_R, \mathcal{S}')$ is compatible in $\overline{R(B)}$. Since $B_R \in L(\overline{R(B)})^s$ is injective, applying Proposition 4.8 we have that

$$c_0(\mathcal{S}', B(\mathcal{S}')^{\perp_R}) = \sup\big\{ |\langle x, y \rangle| \, : \, x \in \mathcal{S}', \|x\| \leq 1, \ y \in B(\mathcal{S}')^{\perp_R}, \|y\| \leq 1 \big\} < 1,$$

where the Hilbert space considered in the angle condition is $\overline{R(B)}$ with the usual norm and, if $\mathcal{M}$ is a subspace of $\overline{R(B)}$, $\mathcal{M}^{\perp_R} = \overline{R(B)} \ominus \mathcal{M}$.

Since $B = BP_{\overline{R(B)}}$ we have that $B(\mathcal{S}') = B(\mathcal{S}) \subset R(B)$ therefore $B(\mathcal{S})^\perp = B(\mathcal{S})^{\perp_R} \oplus N(B)$. Then, $c_0(\mathcal{S}', B(\mathcal{S})^\perp) \leq c_0(\mathcal{S}', B(\mathcal{S}')^{\perp_R}) < 1$. □

The following example is based on one appeared on Halmos' book [13, pages 28-29]. It proves that, given a symmetry J, a J-non-degenerated subspace is not necessarily compatible.

Example 4.10. Let $\mathcal{H}$ be a separable infinite-dimensional Hilbert space and $\mathcal{S}$ be a closed subspace of $\mathcal{H}$ such that $\dim \mathcal{S} = \dim \mathcal{S}^\perp$. Consider the symmetry $J = 2P_{\mathcal{S}} - I \in L(\mathcal{H})$.

Given orthonormal bases $\{a_n\}_{n\in\mathbb{N}}$ of $\mathcal{S}$ and $\{b_n\}_{n\in\mathbb{N}}$ of $\mathcal{S}^\perp$, consider the orthonormal families $\{f_n\}_{n\in\mathbb{N}}$ and $\{h_n\}_{n\in\mathbb{N}}$, where $f_n = \frac{1}{\sqrt{2}}(a_n + b_n)$ and $h_n = \frac{1}{\sqrt{2}}(a_n - b_n)$.

Finally, define $g_n = (\cos\frac{1}{n})f_n + (\sin\frac{1}{n})h_n$ and consider $\mathcal{T} = \langle \{g_n\}_{n\in\mathbb{N}} \rangle$, the closed subspace generated by $\{g_n\}_{n\in\mathbb{N}}$. Then, $(J, \mathcal{T})$ is not compatible: in fact,

$Jg_n = (\cos\frac{1}{n})h_n + (\sin\frac{1}{n})f_n$ is an orthonormal basis of $J(\mathcal{T})$ and $\{u_n\}_{n\in\mathbb{N}}$, with $u_n = (\sin\frac{1}{n})h_n - (\cos\frac{1}{n})f_n$, is an orthonormal basis of $J(\mathcal{T})^\perp$. Observe that

$$\begin{aligned}\langle g_n, u_n \rangle &= \left\langle (\cos\tfrac{1}{n})f_n + (\sin\tfrac{1}{n})h_n, (\sin\tfrac{1}{n})h_n - (\cos\tfrac{1}{n})f_n \right\rangle = \\ &= \left\langle \cos\tfrac{1}{n})f_n, -(\cos\tfrac{1}{n})f_n \right\rangle + \left\langle (\sin\tfrac{1}{n})h_n, (\sin\tfrac{1}{n})h_n \right\rangle = \\ &= -\cos\tfrac{2}{n},\end{aligned}$$

and $c_0(\mathcal{T}, J(\mathcal{T})^\perp) \geq |\langle g_n, u_n \rangle| = \cos\frac{2}{n} \xrightarrow[n\to+\infty]{} 1$, i.e. $c_0(\mathcal{T}, J(\mathcal{T})^\perp) = 1$. Then, by Proposition 4.8, $(J, \mathcal{T})$ is not compatible.

To complete the example it remains to prove that $\mathcal{T}$ is J-non degenerated. Suppose that $v \in \mathcal{T} \cap J(\mathcal{T})^\perp$, $v = \sum_{n=1}^\infty \alpha_n g_n = \sum_{n=1}^\infty \beta_n u_n$, with $\sum_{n=1}^\infty |\alpha_n|^2 < \infty$, $\sum_{n=1}^\infty |\beta_n|^2 < \infty$. Then,

$$\sum_{n=1}^\infty \alpha_n(\cos\tfrac{1}{n})f_n + \sum_{n=1}^\infty \alpha_n(\sin\tfrac{1}{n})h_n = v = \sum_{n=1}^\infty \beta_n(\sin\tfrac{1}{n})h_n - \sum_{n=1}^\infty \beta_n(\cos\tfrac{1}{n})f_n,$$

which implies that $\alpha_n = \beta_n = 0$ for every $n \in \mathbb{N}$, i.e. $v = 0$. Therefore, $\mathcal{T}$ is J-non-degenerated.

4.3 B-selfadjoint projections: the closed range case

Throughout this subsection $B \in L(\mathcal{H})^s$ has closed range, $B = JA$ is its polar decomposition with $A \in L(\mathcal{H})^+$ and $J = J^* = J^{-1}$, $\mathcal{S}$ is a closed subspace of $\mathcal{H}$, $P = P_\mathcal{S}$ is the orthogonal projection onto $\mathcal{S}$ and $\mathcal{M} = A^{1/2}(\mathcal{S})$. Observe that $\mathcal{M}$ is closed if and only if $\mathcal{S} + N(B)$ is closed. Furthermore, Theorem 6.2 of [4] states that, if $A \in L(\mathcal{H})^+$ has closed range, then $(A, \mathcal{S})$ is compatible if and only if $A^{1/2}(\mathcal{S})$ is closed. The next proposition generalizes this result.

Proposition 4.11. *Under the above conditions, $(B, \mathcal{S})$ is compatible if and only if $\mathcal{M}$ is closed and $(J, \mathcal{M})$ is compatible.*

Proof. The compatibility of $(B, \mathcal{S})$ implies that $\mathcal{M}$ is closed in $R(A^{1/2})$. Then $\mathcal{M}$ is closed because $R(A^{1/2}) = R(B)$ is closed. By Proposition 4.4 $R(A^{1/2}) = \mathcal{M} + \mathcal{M}^{[\perp]} \cap R(A^{1/2})$ so $\mathcal{H} = R(A^{1/2}) + N(A^{1/2}) = \mathcal{M} + \mathcal{M}^{[\perp]}$, because $\mathcal{M}^{[\perp]} = \mathcal{M}^{[\perp]} \cap R(A^{1/2}) \dot{+} N(A^{1/2})$. The converse is similar.

□

Corollary 4.12. *Let $B \in GL(\mathcal{H})^s$. Then, $(B, \mathcal{S})$ is compatible if and only if $(J, \mathcal{M})$ is compatible.*

If $A \in L(\mathcal{H})^+$ has closed range, then $(B, \mathcal{S})$ is compatible if and only if $R(G_{A,\mathcal{S}})$ is closed (see [4, Theorem 6.2]). In the selfadjoint case we have the following:

Proposition 4.13. *Under the above conditions,* $(B, \mathcal{S})$ *is compatible if and only if* $\mathcal{N} = \mathcal{S} \cap N(B)$ *and* $G_{B,\mathcal{S}}$ *has closed range.*

Proof. Recall that, by Remark 3.4, if $(B, \mathcal{S})$ is compatible and $\mathcal{T} = \mathcal{S} \ominus \mathcal{N}$ then $(B, \mathcal{T})$ is compatible, $G_{B,\mathcal{S}} = G_{B,\mathcal{T}}$ and the restriction of the B-Gram operator $G_{B,\mathcal{T}}$ to the subspace $\mathcal{T}$, $G = P_{\mathcal{T}} B|_{\mathcal{T}} : \mathcal{T} \to \mathcal{T}$ is injective.

By Proposition 4.1 $B(\mathcal{T})$ is closed and from $\mathcal{H} = \mathcal{T} + B(\mathcal{T})^{\perp}$ it follows that $\mathcal{T}^{\perp} + B(\mathcal{T})$ is closed, or equivalently $R(G) = P_{\mathcal{T}}(B(\mathcal{T}))$ is closed (see the Preliminaries on angles between subspaces).

Conversely, if $R(G_{B,\mathcal{S}})$ is closed then $R(G_{B,\mathcal{S}}) = \mathcal{S} \ominus \mathcal{N}$ and $G_{B,\mathcal{S}}{}^{\dagger} \in L(\mathcal{H})$. Consider the operator $Q = P_{\mathcal{S}} G_{B,\mathcal{S}}{}^{\dagger} P_{\mathcal{S}} B \in L(\mathcal{H})$. Then Q is an oblique projection such that $BQ = Q^* B$ and $R(Q) = \mathcal{S} \ominus \mathcal{N}$. Indeed, $R(Q) \subseteq \mathcal{S}$ but, if $x \in \mathcal{N}$ then $Qx = 0$. On the other hand, if $x \in \mathcal{S} \ominus \mathcal{N}$, $Qx = P_{\mathcal{S}} P_{R(G_{B,\mathcal{S}})} x = P_{\mathcal{S}} x = x$. Therefore, $R(Q) = \mathcal{S} \ominus \mathcal{N}$. Observe that $Q' = Q + P_{\mathcal{N}}$ is an oblique projection with $R(Q') = \mathcal{S}$ and $BQ' = (Q')^* B$ (because $P_{\mathcal{N}} B = B P_{\mathcal{N}} = 0$). Thus, $Q' \in \mathcal{P}(B, \mathcal{S})$ i.e. $(B, \mathcal{S})$ is compatible. □

The next theorem gives a complete characterization of the compatibility of a pair $(B, \mathcal{S})$ in terms of the subspace $\mathcal{M}$, when $B \in L(\mathcal{H})^s$ is a closed range operator.

Theorem 4.14. *Let* $B \in L(\mathcal{H})^s$ *be a closed range operator. The following conditions are equivalent:*

1. $(B, \mathcal{S})$ *is compatible.*
2. $P_{\mathcal{M}//\mathcal{M}^{[\perp]}} \in L(\mathcal{H})$.
3. $\mathcal{M}$ *is closed and* $c_0(\mathcal{M}, \mathcal{M}^{[\perp]}) < 1$.
4. $R(G_{\mathcal{M}}) = \mathcal{M}$.
5. $\mathcal{M}$ *is a Krein space (with the induced metric given by* J*).*

Proof. 1. $\leftrightarrow$ 2.: By Proposition 4.11, $(B, \mathcal{S})$ is compatible if and only if $\mathcal{M}$ is closed and $\mathcal{H} = \mathcal{M} + J^{-1}(\mathcal{M}^{\perp}) = \mathcal{M} + \mathcal{M}^{[\perp]}$. Since J is invertible, $\mathcal{M} \cap \mathcal{M}^{[\perp]} = \{0\}$ and therefore $P_{\mathcal{M}//\mathcal{M}^{[\perp]}} \in L(\mathcal{H})$.
2. $\leftrightarrow$ 3.: If $P_{\mathcal{M}//\mathcal{M}^{[\perp]}} \in L(\mathcal{H})$ then $\mathcal{M}$ is closed, $\mathcal{M} \cap \mathcal{M}^{[\perp]} = \{0\}$ and $\mathcal{H} = \mathcal{M} + \mathcal{M}^{[\perp]}$. Therefore $c_0(\mathcal{M}, \mathcal{M}^{[\perp]}) = c(\mathcal{M}, \mathcal{M}^{[\perp]}) < 1$. The converse is similar.
2. $\leftrightarrow$ 4.: Recall that $N(G_{\mathcal{M}}) = \mathcal{M}^{\perp} \dotplus \mathcal{M} \cap \mathcal{M}^{[\perp]}$. If $(B, \mathcal{S})$ is compatible then, by Proposition 4.13, $\mathcal{M} \cap \mathcal{M}^{[\perp]} = \{0\}$ and $R(G_{\mathcal{M}})$ is closed. Then $R(G_{\mathcal{M}}) = N(G_{\mathcal{M}})^{\perp} = \mathcal{M}$. Conversely, if $R(G_{\mathcal{M}}) = \mathcal{M}$ then $N(G_{\mathcal{M}}) = \mathcal{M}^{\perp}$ i.e. $\mathcal{M} \cap \mathcal{M}^{[\perp]} = \{0\}$. Again, by Proposition 4.13, $(J, \mathcal{M})$ is compatible i.e. $P_{\mathcal{M}//\mathcal{M}^{[\perp]}} \in L(\mathcal{H})$.
4. $\to$ 5. and 5. $\to$ 2.: See T. Ya. Azizov's book [2, Theorem 7.16]. □

5 Decompositions of B-selfadjoint projections

In [5, Proposition 3.5] it was shown that, if $(B,\mathcal{S})$ is compatible and $Q \in \mathcal{P}(B,\mathcal{S})$ then

$$Q = P_{B,\mathcal{S}\ominus\mathcal{N}} + P_{\mathcal{N}}Q = P_{B,\mathcal{S}\ominus\mathcal{N}} + P_{\mathcal{N}//(\mathcal{S}\ominus\mathcal{N}+N(Q))}.$$

Observe that $P_{\mathcal{N}//(\mathcal{S}\ominus\mathcal{N}+N(Q))}$ is B-neutral because $\mathcal{N} \subseteq N(B)$ (see Proposition 3.2). The following theorem proves that $P_{B,\mathcal{S}\ominus\mathcal{N}}$ is the sum of a B-positive and a B-negative projection.

Theorem 5.1. *Let $B \in L(\mathcal{H})^s$ and $\mathcal{S}$ be a closed subspace of $\mathcal{H}$ such that $(B,\mathcal{S})$ is compatible. Then, every $Q \in \mathcal{P}(B,\mathcal{S})$ admits a factorization $Q = Q_0 + Q_1 + Q_2$, where Q_0 is B-neutral, Q_1 is B-positive and Q_2 is B-negative.*

Proof. Let $Q \in \mathcal{P}(B,\mathcal{S})$ and consider the subspaces $\mathcal{T} = \mathcal{S} \ominus \mathcal{N}$ and $\mathcal{T}_0 = \mathcal{N}$. Recall that $(B,\mathcal{T})$ is compatible, $G_{B,\mathcal{T}} = G_{B,\mathcal{S}}$ and $G = G_{B,\mathcal{T}}|_{\mathcal{T}}$ is injective (see Remark 3.4). So, if $G = U|G|$ is the polar decomposition of G then U is a symmetry on $\mathcal{T}$. Extend U by $U = 0$ on $\mathcal{T}^{\perp}$. Then there exist closed subspaces $\mathcal{T}_1$ and $\mathcal{T}_2$ such that $\mathcal{T} = \mathcal{T}_1 \oplus \mathcal{T}_2$ and $U = P_{\mathcal{T}_1} - P_{\mathcal{T}_2}$. Since $\mathcal{T}_i \subseteq \mathcal{T} = \mathcal{S} \cap \mathcal{T}_0^{\perp}$ it is clear that for $i = 0,1,2$,

$$QP_{\mathcal{T}_i} = P_{\mathcal{T}_i} \quad \text{and} \quad P_{\mathcal{T}_i}P_{\mathcal{T}_j} = 0 \quad \text{if} \quad i \neq j. \tag{5.1}$$

Consider $Q_i = P_{\mathcal{T}_i}Q$, $i = 0,1,2$. Then, from Eq. (5.1) it follows easily that $Q_i \in \mathcal{Q}$ for $i,j = 0,1,2$ and $Q_iQ_j = 0$ if $i \neq j$.

Let us prove that Q_i is B-selfadjoint: first of all, observe that $BQ_0 = 0 = Q_0^*B$ because $\mathcal{T}_0 \subset N(B)$, and therefore Q_0 is B-neutral. If $i = 1,2$,

$$\begin{aligned} BQ_i &= BP_{\mathcal{T}_i}Q = BQP_{\mathcal{T}_i}Q = Q^*BP_{\mathcal{T}_i}Q = Q^*(P_{\mathcal{T}_0} + P_{\mathcal{T}})BP_{\mathcal{T}_i}Q = \\ &= Q^*P_{\mathcal{T}}BP_{\mathcal{T}_i}Q = Q^*G_{B,\mathcal{T}}P_{\mathcal{T}_i}Q = Q^*P_{\mathcal{T}_i}G_{B,\mathcal{T}}Q = Q^*P_{\mathcal{T}_i}BQ = \\ &= Q^*P_{\mathcal{T}_i}Q^*B = Q^*P_{\mathcal{T}_i}B = Q_i^*B \end{aligned}$$

because $G_{B,\mathcal{T}}P_{\mathcal{T}_i} = P_{\mathcal{T}_i}G_{B,\mathcal{T}}$. Furthermore, Q_1 is B-positive: given $x \in \mathcal{H}$,

$$\langle BQ_1x, x \rangle = \langle BQ_1x, Q_1x \rangle = \langle G_{B,\mathcal{T}}Q_1x, Q_1x \rangle = \langle |G|Q_1x, Q_1x \rangle > 0.$$

Analogously, Q_2 is B-negative and $Q_0 + Q_1 + Q_2 = (P_{\mathcal{T}_0} + P_{\mathcal{T}_1} + P_{\mathcal{T}_2})Q = (P_{\mathcal{N}} + P_{\mathcal{S}\ominus\mathcal{N}})Q = Q$. □

It follows from the proof of Theorem 5.1 that the constructed projections Q_i also satisfy:

(i) $Q_iQ_j = 0$ if $i \neq j$,

(ii) $R(Q_i) \perp R(Q_j)$ for $i \neq j$, and

(iii) $\langle Bx, x \rangle > 0$ for every $x \in R(Q_1)$ and $\langle Bx, x \rangle < 0$ for every $x \in R(Q_2)$.

Proposition 5.2 shows that, under conditions $(i)-(iii)$, the above decomposition is unique. Recall that $\mathcal{T}_1$ and $\mathcal{T}_2$ are the eigenspaces of U corresponding to $\lambda = 1$ and $\lambda = -1$ respectively.

Proposition 5.2. *Let $Q \in \mathcal{P}(B, \mathcal{S})$ and suppose that there exist projections Q_0, Q_1 and Q_2, such that Q_0 is B-neutral, Q_1 is B-positive, Q_2 is B-negative, $Q = Q_0 + Q_1 + Q_2$ and they satisfy conditions $(i)-(iii)$. Then,*

$$Q_0 = P_{\mathcal{N}//N(Q)\dot{+}\mathcal{S}\ominus\mathcal{N}} \quad \text{and} \quad Q_i = P_{B,\mathcal{T}_i} \quad \text{for } i = 1, 2.$$

Proof. If $\mathcal{V}_i = R(Q_i)$ then $\mathcal{S} = \mathcal{V}_0 \oplus \mathcal{V}_1 \oplus \mathcal{V}_2$: the sum is direct because, by (ii), $\mathcal{V}_i \perp \mathcal{V}_j$ if $i \neq j$ and $\mathcal{S} \subseteq \mathcal{V}_0 \oplus \mathcal{V}_1 \oplus \mathcal{V}_2$. Conversely, if $x \in \mathcal{V}_0 \oplus \mathcal{V}_1 \oplus \mathcal{V}_2$ then $x = v_0 + v_1 + v_2$ with $v_i \in \mathcal{V}_i$, but, applying (i), $Q_i(v_j) = 0$ if $i \neq j$ and $Q(v_i) = v_i$, so that $x = Qx \in \mathcal{S}$.

Also, $\mathcal{V}_1 \oplus \mathcal{V}_2 = \mathcal{S} \ominus \mathcal{N}$: since $BQ_0 = 0$ it follows that $\mathcal{V}_0 \subseteq \mathcal{S} \cap N(B) = \mathcal{T}_0$. Then, taking orthogonal complement in $\mathcal{S}$, $\mathcal{T}_1 \oplus \mathcal{T}_2 \subseteq \mathcal{V}_1 \oplus \mathcal{V}_2$ so that $\mathcal{V}_1 \oplus \mathcal{V}_2 = \mathcal{T}_1 \oplus \mathcal{T}_2 \oplus \mathcal{U}$ and $\mathcal{U} \subseteq \mathcal{T}_0$. But $\mathcal{U} = \{0\}$ because, if $u \in \mathcal{U}$, $u \neq 0$ then $u = v_1 + v_2$ with $v_i \in \mathcal{V}_i$, $v_1 \neq 0$ (or $v_2 \neq 0$) and $Bu = 0$. Therefore, $Bv_1 = -Bv_2$ and, by condition (iii), $0 < \langle Bv_1, v_1 \rangle = -\langle Bv_2, v_1 \rangle = -\langle v_2, Bv_1 \rangle = \langle Bv_2, v_2 \rangle < 0$, which is absurd. Then, $\mathcal{V}_1 \oplus \mathcal{V}_2 = \mathcal{T}_1 \oplus \mathcal{T}_2 = \mathcal{S} \ominus \mathcal{N}$ and $\mathcal{V}_0 = \mathcal{T}_0 = \mathcal{N}$.

For $i = 0, 1, 2$, Q_i is B-selfadjoint and $Q_i P_{\mathcal{V}_i} = P_{\mathcal{V}_i}$. Then, $P_{\mathcal{V}_i} B P_{\mathcal{V}_j} = P_{\mathcal{V}_j} B P_{\mathcal{V}_i} = 0$ for $i \neq j$. In fact,

$$P_{\mathcal{V}_i} B P_{\mathcal{V}_j} = P_{\mathcal{V}_i} B Q_j P_{\mathcal{V}_j} = P_{\mathcal{V}_i} Q_j^* B P_{\mathcal{V}_j} = (Q_j P_{\mathcal{V}_i})^* B P_{\mathcal{V}_j} = 0$$

since $Q_j P_{\mathcal{V}_i} = (Q_j Q_i) P_{\mathcal{V}_i} = 0$ for $i \neq j$. Therefore, $G_{B,\mathcal{V}_0} = 0$ and $G_{B,\mathcal{S}} = G_{B,\mathcal{T}_1} + G_{B,\mathcal{T}_2}$. By condition (iii), $G_{B,\mathcal{V}_1}|_{\mathcal{V}_1} \in L(\mathcal{V}_1)^+$ and $-G_{B,\mathcal{V}_2}|_{\mathcal{V}_2} \in L(\mathcal{V}_2)^+$ are injective. Furthermore, since $P_{\mathcal{V}_i} P_{\mathcal{V}_j} = P_{\mathcal{V}_j} P_{\mathcal{V}_i} = 0$,

$$G_{B,\mathcal{S}} = G_{B,\mathcal{V}_1} + G_{B,\mathcal{V}_2} = (G_{B,\mathcal{V}_1} - G_{B,\mathcal{V}_2})(P_{\mathcal{V}_0} + P_{\mathcal{V}_1} - P_{\mathcal{V}_2}),$$

$A = G_{B,\mathcal{V}_1} - G_{B,\mathcal{V}_2} \in L(\mathcal{H})^+$ (note that $A|_{\mathcal{V}_1 \oplus \mathcal{V}_2} \in L(\mathcal{V}_1 \oplus \mathcal{V}_2)^+$ is injective) and, if $U = P_{\mathcal{V}_0} + P_{\mathcal{V}_1} - P_{\mathcal{V}_2}$, then $U = U^*$ and $U^2 = P_{\mathcal{S}}$. If $A_0 = A|_{\mathcal{S}}$ then $G_{B,\mathcal{S}}|_{\mathcal{S}} = A_0 U|_{\mathcal{S}}$ where $A_0 \in L(\mathcal{S})^+$ and $U|_{\mathcal{S}} \in L(\mathcal{S})$ is a symmetry. By uniqueness of the polar decomposition, U is the unitary part of $G_{B,\mathcal{S}} = G_{B,\mathcal{S}\ominus\mathcal{N}}$, $\mathcal{V}_1 = \mathcal{T}_1$ and $\mathcal{V}_2 = \mathcal{T}_2$. Hence, if $i = 1, 2$, $Q_i \in \mathcal{P}(B, \mathcal{T}_i)$ and, since $\mathcal{T}_i \cap N(B) = \{0\}$, $Q_i = P_{B,\mathcal{T}_i}$. Therefore, $Q_0 = Q - P_{B,\mathcal{T}_1} - P_{B,\mathcal{T}_2}$; multiplying both sides of the last equality by $P_{\mathcal{N}}$ it follows that $Q_0 = P_{\mathcal{N}} Q = P_{\mathcal{N}//N(Q)\dot{+}\mathcal{S}\ominus\mathcal{N}}$. □

Given a (bounded) projection Q on $(\mathcal{H}, \langle\ ,\ \rangle)$, it is well known that Q is orthogonal if and only if Q is a contraction, i.e. $\|Q\| \leq 1$. More generally, given $B \in L(\mathcal{H})^s$ consider the indefinite metric space $(\mathcal{H}, \langle\ ,\ \rangle_B)$. An operator $C \in L(\mathcal{H})$ is a *B-contraction* if $\langle Cx, Cx \rangle_B \leq \langle x, x \rangle_B$. It is easy to see that C is a B-contraction if and only if $C^* B C \leq B$. In [4, Lemma 3.2] it was shown that, if $A \in L(\mathcal{H})^+$ and $Q \in \mathcal{Q}$, then Q is A-selfadjoint if and only if Q is an A-contraction.

For a selfadjoint operator B, S. Hassi and K. Nordström proved the following result, which characterizes those B-selfadjoint projections which are B-contractive (see [14, Proposition 5]).

Proposition 5.3. *Let $B \in L(\mathcal{H})^s$. If $Q \in \mathcal{Q}$ then Q is a B-contraction if and only if Q is B-selfadjoint and $N(Q)$ is B-nonnegative.*

Remark 5.4. Consider $B \in L(\mathcal{H})^s$ and $Q \in \mathcal{Q}$. Then,

1. *Q is B-contractive if and only if $I-Q$ is B-positive.* Indeed, if Q is B-contractive then Q is B-selfadjoint i.e. $BQ = Q^*B$. So, $B(I-Q) = B - BQ = B - Q^*BQ \geq 0$. Conversely, if $I-Q$ is B-positive then $I-Q$ and Q are B-selfadjoint projections. But $B \geq BQ = Q^*BQ$ i.e. Q is B-contractive.

2. An operator $C \in L(\mathcal{H})$ is a *B-expansion* if $\langle Cx, Cx \rangle_B \geq \langle x, x \rangle_B$ (i.e. $C^*BC \geq C$) and C is a *B-isometry* if $\langle Cx, Cx \rangle_B = \langle x, x \rangle_B$ (i.e. $C^*BC = C$). It is easy to see that $Q \in \mathcal{Q}$ is B-expansive (respectively B-isometric) if and only if $I-Q$ is B-negative (respectively B-neutral).

Hassi and Nordström [14, Theorem 2] also proved that, if $B \in GL(\mathcal{H})^s$, every B-selfadjoint projection Q can be represented as the product of two commuting B-selfadjoint projections Q_1 and Q_2 such that Q_1 is B-contractive and Q_2 is B-expansive. The following corollary shows that their result also holds for not necessarily invertible selfadjoint operators.

Corollary 5.5. *Every B-selfadjoint projection Q admits a factorization $Q = Q_0Q_1Q_2$, where Q_0, Q_1 and Q_2 are commuting projections such that Q_0 is B-isometric, Q_1 is B-contractive and Q_2 is B-expansive.*

Proof. If Q is B-selfadjoint then $I - Q \in \mathcal{P}(B, N(Q))$ and, by Theorem 5.1,

$$I - Q = E_0 + E_1 + E_2,$$

where E_0 is B-neutral, E_1 is B-positive, E_2 is B-negative and they satisfy $E_iE_j = 0$ if $i \neq j$. If $Q_i = I - E_i$ then $Q_iQ_j = Q_jQ_i$ and

$$Q = I - (E_0 + E_1 + E_2) = (I - E_0)(I - E_1)(I - E_2) = Q_0Q_1Q_2.$$

Furthermore, Q_0 is B-isometric, Q_1 is B-contractive and Q_2 is B-expansive (see Remark 5.4). □

Observe that Proposition 5.2 also states the uniqueness of the above factorization if E_0, E_1 and E_2 satisfy conditions $(i) - (iii)$.

We end this section with a description of the so called canonical decompositions of a Krein space. Given a Krein space $(\mathcal{H}, \langle\ ,\ \rangle, J)$ (or $(\mathcal{H}, [\ ,\ ])$), a *canonical decomposition of* $\mathcal{H}$ is a decomposition of $\mathcal{H}$ as a direct sum

$$\mathcal{H} = \mathcal{S} \dotplus \mathcal{S}^{[\perp]},$$

where $\mathcal{S}$ is a closed subspaces of $\mathcal{H}$ such that $\mathcal{S}$ is J-positive and $\mathcal{S}^{[\perp]}$ is J-negative. Observe that $\mathcal{H} = \mathcal{S} \dot{+} \mathcal{S}^{[\perp]}$ is a canonical decomposition if and only if $P_{\mathcal{S}//\mathcal{S}^{[\perp]}}$ is B-expansive and $I - P_{\mathcal{S}//\mathcal{S}^{[\perp]}}$ is B-contractive. Each canonical decomposition defines a *reflection* $K_{\mathcal{S}}$ (i.e. a bounded invertible operator which coincides with its inverse), by means of $K_{\mathcal{S}} = 2P_{\mathcal{S}//\mathcal{S}^{[\perp]}} - I$ (see [2]).

Lemma 5.6. *Let $B \in L(\mathcal{H})^s$ and $Q \in \mathcal{Q}$ such that $R(Q)$ is B-nonnegative and $N(Q)$ is B-nonpositive. Then, Q is B-selfadjoint if and only if the reflection $2Q - I$ is B-positive.*

Proof. If Q is a B-selfadjoint projection then $BQ = Q^*BQ$, so that if $K = 2Q - I$

$$\langle Kx, x \rangle_B = \langle Qx, Qx \rangle_B - \langle (I - Q)x, (I - Q)x \rangle_B \geq 0$$

because $R(Q)$ is B-nonnegative and $N(Q)$ is B-nonpositive. Thus K is B-positive. Conversely, if K is B-positive then K is B-selfadjoint. Therefore $Q = \frac{I+K}{2}$ is B-selfadjoint. □

Consider the set of reflections $\mathcal{K} = \{K \in GL(\mathcal{H}) : K^{-1} = K\}$ and the set of symmetries $\mathcal{J} = \{J \in GL(\mathcal{H})^s : J^{-1} = J\} \subset \mathcal{K}$. If $K \in \mathcal{K}$ and $K = J_K|K|$ is the polar decomposition of K then $J_K \in \mathcal{J}$ (see [7, Proposition 3.1]). Then, the map

$$\pi : \mathcal{K} \to \mathcal{J}, \quad \pi(K) = J_K,$$

is well defined and continuous. Furthermore, the set of canonical decompositions of $(\mathcal{H}, \langle \ , \ \rangle, J)$ can be parametrized by the reflections in the fibre $\pi^{-1}(\{J\})$. Observe that in [7], Propositions 3.1 and 3.2, the formulation is quite different but the contents are equivalent to those of our Proposition 5.7.

Proposition 5.7. *Let $(\mathcal{H}, \langle \ , \ \rangle, J)$ be a Krein space. Then $\mathcal{H} = \mathcal{S} \dot{+} \mathcal{S}^{[\perp]}$ is a canonical decomposition of $\mathcal{H}$ if and only if $K_{\mathcal{S}} \in \pi^{-1}(\{J\})$.*

Proof. Suppose that $\mathcal{H} = \mathcal{S} \dot{+} \mathcal{S}^{[\perp]}$, or equivalently, $P_{\mathcal{S}//\mathcal{S}^{[\perp]}}$ is J-selfadjoint, $\mathcal{S}$ is J-positive and $\mathcal{S}^{[\perp]}$ is J-negative. By Lemma 5.6 $K_{\mathcal{S}}$ is a J-positive operator i.e. $A := JK \in L(\mathcal{H})^+$. Then, $K = JA$ and $A^*A = K^*J^2K = K^*K$ i.e. $A = |K|$ and J is the unitary part in the polar decomposition of K. Therefore, $K_{\mathcal{S}} \in \pi^{-1}(\{J\})$.

Conversely, if $K \in \pi^{-1}(\{J\})$ then K is J-positive. By Lemma 5.6 the projection $Q = \frac{I+K}{2}$ is J-selfadjoint and its is easy to see that $\mathcal{S} := R(Q)$ is J-positive and $\mathcal{S}^{[\perp]} = N(Q)$ is J-negative. Therefore, $\mathcal{H} = \mathcal{S} \dot{+} \mathcal{S}^{[\perp]}$ is a canonical decomposition of $\mathcal{H}$. □

6 Formulas for $P_{B,\mathcal{S}}$

Let $B \in GL(\mathcal{H})^s$ and consider its polar decomposition $B = JA$. Then,

$$\langle x, y \rangle_B = \langle Bx, y \rangle = \langle AJx, y \rangle = \langle Jx, y \rangle_A, \qquad x, y \in \mathcal{H}.$$

Since $A \in GL(\mathcal{H})^+$, it follows that $\langle\ ,\ \rangle_A$ is an inner product equivalent to $\langle\ ,\ \rangle$ and therefore $(\mathcal{H}, \langle\ ,\ \rangle_A, J)$ is a Krein space. Observe that Corollary 4.12 says that the compatibility of $(B, \mathcal{S})$ is equivalent to the existence of a (unique) J-orthogonal projection in the Krein space $(\mathcal{H}, \langle\ ,\ \rangle_A, J)$ with range $\mathcal{M} = A^{1/2}(\mathcal{S})$.

Proposition 6.1. *Let $B \in GL(\mathcal{H})^s$ and let $\mathcal{S}$ be a closed subspace of $\mathcal{H}$. If $(B, \mathcal{S})$ is compatible, then $P_{B,\mathcal{S}} = A^{-1/2} P_{\mathcal{M}//\mathcal{M}^{[\perp]}} A^{1/2}$.*

Proof. If $(B, \mathcal{S})$ is compatible then, by Theorem 4.14, $P_{\mathcal{M}//\mathcal{M}^{[\perp]}} \in L(\mathcal{H})$ and $\mathcal{P}(B, \mathcal{S})$ is a singleton. Consider

$$Q = A^{-1/2} P_{\mathcal{M}//\mathcal{M}^{[\perp]}} A^{1/2}.$$

Q is a projection such that $R(Q) = \mathcal{S}$ and $BQ = Q^*B$. Then, $Q = P_{B,\mathcal{S}}$. □

The first part of this section is devoted to generalize the above formula $P_{B,\mathcal{S}} = A^{-1/2} P_{\mathcal{M}//\mathcal{M}^{[\perp]}} A^{1/2}$ (obtained for $B \in GL(\mathcal{H})^s$) to an arbitrary selfadjoint operator.

Given an operator $B \in L(\mathcal{H})^s$, consider its polar decomposition $B = JA$, with $A \in L(\mathcal{H})^+$, $J = J^* = J^{-1}$. If $\mathcal{S}$ is a closed subspace of $\mathcal{H}$ let $P = P_{\mathcal{S}}$ be the orthogonal projection onto $\mathcal{S}$ and define the closed subspace $\mathcal{M} = \overline{A^{1/2}(\mathcal{S})}$.

Recall that if $(B, \mathcal{S})$ is compatible then $\mathcal{M}$ is a J-non degenerated subspace of $(\mathcal{H}, \langle\ ,\ \rangle, J)$. Therefore, the projection $P_{\mathcal{M}//\mathcal{M}^{[\perp]}}$ has dense domain $(\mathcal{M} \dot{+} \mathcal{M}^{[\perp]})$ but it can be an unbounded operator. However, $R(A^{1/2}) \subseteq \mathcal{M} \dot{+} \mathcal{M}^{[\perp]}$ and the product $P_{\mathcal{M}//\mathcal{M}^{[\perp]}} A^{1/2}$ remains bounded as shown in the following Proposition.

Proposition 6.2. *Let $(B, \mathcal{S})$ be compatible and consider $T = P_{\mathcal{M}//\mathcal{M}^{[\perp]}} A^{1/2}$. Then, T is well defined and bounded.*

Proof. If $(B, \mathcal{S})$ is compatible then, by Corollary 4.5,

$$R(A^{1/2}) = \mathcal{M} \cap R(A^{1/2}) \dot{+} \mathcal{M}^{[\perp]} \cap R(A^{1/2}) \subseteq \mathcal{M} + \mathcal{M}^{[\perp]}.$$

Therefore, T is well defined because $\mathrm{Dom}(P_{\mathcal{M}//\mathcal{M}^{[\perp]}}) = \mathcal{M} + \mathcal{M}^{[\perp]}$. Let $Q \in \mathcal{P}(B, \mathcal{S})$. For every $x \in \mathcal{H}$,

$$Tx = TQx + T(I - Q)x.$$

Since $Qx \in \mathcal{S}$, $TQx = P_{\mathcal{M}//\mathcal{M}^{[\perp]}} A^{1/2} Qx = A^{1/2} Qx$ and $T(I - Q)x = 0$ because $A^{1/2}(I - Q)x \in A^{1/2} B^{-1}(\mathcal{S}^\perp) = A^{1/2}(\mathcal{S})^{[\perp]} \cap R(A^{1/2}) \subseteq \mathcal{M}^{[\perp]}$. Therefore,

$$P_{\mathcal{M}//\mathcal{M}^{[\perp]}} A^{1/2} = T = A^{1/2} Q \in L(\mathcal{H}). \tag{6.1}$$

□

Corollary 6.3. *$(B, \mathcal{S})$ is compatible if and only if $R(A^{1/2}) \subseteq \mathcal{M} \dot{+} \mathcal{M}^{[\perp]}$ and*

$$R(P_{\mathcal{M}//\mathcal{M}^{[\perp]}} A^{1/2}) \subseteq R(A^{1/2} P).$$

Proof. If $(B, \mathcal{S})$ is compatible then, by Corollary 4.5, $R(A^{1/2}) \subseteq \mathcal{M} \dotplus \mathcal{M}^{[\perp]}$ and, by Eq. (6.1), $P_{\mathcal{M}//\mathcal{M}^{[\perp]}} A^{1/2} = A^{1/2}Q$ with $Q \in \mathcal{P}(B, \mathcal{S})$. Therefore $R(P_{\mathcal{M}//\mathcal{M}^{[\perp]}} A^{1/2}) \subseteq R(A^{1/2}Q) = R(A^{1/2}P)$.

Conversely, suppose that $R(A^{1/2}) \subseteq \mathcal{M} \dotplus \mathcal{M}^{[\perp]}$ and $R(P_{\mathcal{M}//\mathcal{M}^{[\perp]}} A^{1/2}) \subseteq R(A^{1/2}P)$. Let $y \in R(A^{1/2})$ and consider the vectors $y_1 = P_{\mathcal{M}//\mathcal{M}^{[\perp]}} y$ and $y_2 = (I - P_{\mathcal{M}//\mathcal{M}^{[\perp]}})y$. Note that $y_1 \in R(P_{\mathcal{M}//\mathcal{M}^{[\perp]}} A^{1/2}) \subseteq R(A^{1/2}P) = A^{1/2}(\mathcal{S})$. Then, it is clear that $y_2 = y - y_1 \in R(A^{1/2})$ and, $y_2 \in N(P_{\mathcal{M}//\mathcal{M}^{[\perp]}}) = \mathcal{M}^{[\perp]}$. Thus, $y_2 \in A^{1/2}(\mathcal{S})^{[\perp]} \cap R(A^{1/2})$ and the decomposition $R(A^{1/2}) = A^{1/2}(\mathcal{S}) \dotplus A^{1/2}(\mathcal{S})^{[\perp]} \cap R(A^{1/2})$ is proved. By Proposition 4.4, $(B, \mathcal{S})$ is compatible. □

Observe that the above corollary is a generalization of the result stated for positive operators in Proposition 2.14 of [5]: If $A \in L(\mathcal{H})^+$, then $(A, \mathcal{S})$ is compatible if and only if $R(P_{\mathcal{M}} A^{1/2}) \subseteq R(A^{1/2}P)$.

Corollary 6.4. *Let $B = JA$ be the polar decomposition of $B \in L(\mathcal{H})^s$ with $A \in L(\mathcal{H})^+$ and $J = J^* = J^{-1}$. Given a closed subspace $\mathcal{S}$ of $\mathcal{H}$, consider the subspace $\mathcal{M} = \overline{A^{1/2}(\mathcal{S})}$. Then, $(B, \mathcal{S})$ is compatible if and only if the following conditions holds:*

1. $P_{\mathcal{M}//\mathcal{M}^{[\perp]}} A^{1/2} \in L(\mathcal{H})$,
2. $A^{1/2}(\mathcal{S})$ *is closed in* $R(A^{1/2})$,
3. $R(P_{\mathcal{M}//\mathcal{M}^{[\perp]}} A^{1/2}) \subseteq R(A^{1/2})$.

Proof. Suppose that $(B, \mathcal{S})$ is compatible. Then, condition 1. is a consequence of Proposition 6.2. Furthermore, by Corollary 4.5, $A^{1/2}(\mathcal{S})$ is closed in $R(A^{1/2})$ and

$$R(A^{1/2}) = A^{1/2}(\mathcal{S}) + \mathcal{M}^{[\perp]} \cap R(A^{1/2}).$$

Therefore, $R(P_{\mathcal{M}//\mathcal{M}^{[\perp]}} A^{1/2}) = A^{1/2}(\mathcal{S}) \subseteq R(A^{1/2})$.

Conversely, suppose that conditions 1., 2. and 3. are fulfilled. If $P_{\mathcal{M}//\mathcal{M}^{[\perp]}} A^{1/2} \in L(\mathcal{H})$ then $(I - P_{\mathcal{M}//\mathcal{M}^{[\perp]}})A^{1/2}$ is also bounded and, for every $x \in \mathcal{H}$,

$$A^{1/2}x = P_{\mathcal{M}//\mathcal{M}^{[\perp]}} A^{1/2}x + (I - P_{\mathcal{M}//\mathcal{M}^{[\perp]}})A^{1/2}x.$$

Observe that $P_{\mathcal{M}//\mathcal{M}^{[\perp]}} A^{1/2}x \in R(P_{\mathcal{M}//\mathcal{M}^{[\perp]}} A^{1/2}) \subseteq \mathcal{M} \cap R(A^{1/2})$, so

$$(I - P_{\mathcal{M}//\mathcal{M}^{[\perp]}})A^{1/2}x \in \mathcal{M}^{[\perp]} \cap R(A^{1/2}).$$

Then, $R(A^{1/2}) = \mathcal{M} \cap R(A^{1/2}) + \mathcal{M}^{[\perp]} \cap R(A^{1/2})$ and, by Corollary 4.5, $(B, \mathcal{S})$ is compatible.

□

Recall that D is the reduced solution of $(A^{1/2}P)X = P_{\mathcal{M}//\mathcal{M}^{[\perp]}}A^{1/2}$ if and only if $R(D) \subseteq \overline{R((A^{1/2}P)^*)} \subseteq R(P) = \mathcal{S}$. Then,

$$A^{1/2}D = P_{\mathcal{M}//\mathcal{M}^{[\perp]}}A^{1/2}.$$

Furthermore, if $A^{1/2}$ is injective, then $A^{-1/2} : R(A^{1/2}) \to \mathcal{H}$ is densely defined. Therefore,

Proposition 6.5. *Let $B \in L(\mathcal{H})^s$ be injective and suppose that $(B, \mathcal{S})$ is compatible. Then,*

$$P_{B,\mathcal{S}} = A^{-1/2}P_{\mathcal{M}//\mathcal{M}^{[\perp]}}A^{1/2}.$$

Proof. If $(B, \mathcal{S})$ is compatible then $\mathcal{P}(B, \mathcal{S}) = \{P_{B,\mathcal{S}}\}$ because B is injective, see Proposition 3.2. Denote by $Q = A^{-1/2}P_{\mathcal{M}//\mathcal{M}^{[\perp]}}A^{1/2}$. It is well defined since $P_{\mathcal{M}//\mathcal{M}^{[\perp]}}A^{1/2}$ is well defined and bounded (see Proposition 6.2) and, by Corollary 6.3,

$$R(P_{\mathcal{M}//\mathcal{M}^{[\perp]}}A^{1/2}) \subseteq R(A^{1/2}).$$

Recall that, by Eq. (6.1), $P_{\mathcal{M}//\mathcal{M}^{[\perp]}}A^{1/2} = A^{1/2}P_{B,\mathcal{S}}$. Then,

$$Q = A^{-1/2}A^{1/2}P_{B,\mathcal{S}} = P_{B,\mathcal{S}}.$$

□

We generalize this formula for a not necessarily injective $B \in L(\mathcal{H})^s$. Given the matrix decomposition induced by $\overline{R(B)}$,

$$B = \begin{pmatrix} B_R & 0 \\ 0 & 0 \end{pmatrix}$$

with $B_R = B|_{\overline{R(B)}} \in L(\overline{R(B)})^s$ injective, we define

$$B^\sharp = \begin{pmatrix} B_R^{-1} & 0 \\ 0 & 0 \end{pmatrix}$$

where $B_R^{-1} : \overline{R(B)} \to \overline{R(B)}$ is densely defined. Observe that $B^\sharp$ is a linear, densely defined operator. If $R(B)$ is closed then $B^\dagger = B^\sharp P_{R(B)}$, where $B^\dagger$ stands for the Moore-Penrose pseudoinverse of B.

Proposition 6.6. *Consider $B \in L(\mathcal{H})^s$ such that $(B, \mathcal{S})$ is compatible.*

1. If $\mathcal{S} \subseteq \overline{R(B)}$ then

$$P_{B,\mathcal{S}} = (A^{1/2})^\sharp P_{\mathcal{M}//\mathcal{M}^{[\perp]}}A^{1/2}. \tag{6.2}$$

2. If $\mathcal{N}=\{0\}$ then

$$P_{B,\mathcal{S}}=(P_{\overline{R(B)}}P)^{\dagger}P_{B,P_{\overline{R(B)}}(\mathcal{S})}=(P_{\overline{R(B)}}P)^{\dagger}(A^{1/2})^{\sharp}P_{\mathcal{M}//\mathcal{M}^{[\perp]}}A^{1/2}. \tag{6.3}$$

Proof. It is similar to the proof of Proposition 3.4 of [5]. □

As it was stated in Section 3, $(B,\mathcal{S})$ is compatible if and only if equation

$$G_{B,\mathcal{S}}X=PB \tag{6.4}$$

admits a bounded solution. Furthermore, since the reduced solution D of equation (6.4) satisfies $R(D)\subseteq\overline{R(G_{B,\mathcal{S}})}=\overline{R(G_{B,\mathcal{S}\ominus\mathcal{N}})}=\mathcal{S}\ominus\mathcal{N}$, it is not hard to prove that $D=P_{B,\mathcal{S}\ominus\mathcal{N}}$ and

$$P_{B,\mathcal{S}}=P_{B,\mathcal{S}\ominus\mathcal{N}}+P_{\mathcal{N}}. \tag{6.5}$$

Proposition 6.7. *If the pair $(B,\mathcal{S})$ is compatible, then the reduced solution Q of the equation*

$$(PBP)X=PB \tag{6.6}$$

coincides with the reduced solution of

$$(A^{1/2}P)X=P_{\mathcal{M}//\mathcal{M}^{[\perp]}}A^{1/2}. \tag{6.7}$$

Proof. By Proposition 6.2, $P_{\mathcal{M}//\mathcal{M}^{[\perp]}}A^{1/2}$ is a well defined bounded operator. Then, Eq. (6.7) is well defined and has a bounded solution (see Corollary 6.3).

By the discussion following Eq. (6.4), the projection $Q=P_{B,\mathcal{S}\ominus\mathcal{N}}$ is the reduced solution of Eq. (6.6), but it is also the reduced solution of Eq. (6.7): If $z\in\mathcal{S}\ominus\mathcal{N}$ then $(A^{1/2}P)Qz=A^{1/2}z=P_{\mathcal{M}//\mathcal{M}^{[\perp]}}A^{1/2}z$. On the other hand, if $z\in B(\mathcal{S}\ominus\mathcal{N})^{\perp}=B(\mathcal{S})^{\perp}$ then $A^{1/2}z\in\mathcal{M}^{[\perp]}$. So $P_{\mathcal{M}//\mathcal{M}^{[\perp]}}A^{1/2}z=0=(A^{1/2}P)Qz$. Therefore Q is a solution of (6.7). Moreover, Q is the reduced solution since

$$R(Q)=\mathcal{S}\ominus\mathcal{N}=(\mathcal{S}^{\perp}+\mathcal{N})^{\perp}=(\mathcal{S}^{\perp}+\mathcal{S}\cap N(B))^{\perp}=N(A^{1/2}P)^{\perp}=\overline{R((A^{1/2}P)^{*})},$$

and the proof is complete. □

Suppose that $B\in L(\mathcal{H})^{s}$ is a closed range operator. If $(B,\mathcal{S})$ is compatible and $\mathcal{N}=\{0\}$ then the proof of Proposition 4.13 suggests the following formula for $P_{B,\mathcal{S}}$:

$$P_{B,\mathcal{S}}=P\,G_{B,\mathcal{S}}{}^{\dagger}\,PB=P(PBP)^{\dagger}PB. \tag{6.8}$$

Furthermore, if $\mathcal{N}\neq\{0\}$ we have that $P=P_{\mathcal{S}\ominus\mathcal{N}}+P_{\mathcal{N}}$. Then, by Eq. (6.8) and Remark 3.4 we get

$$P_{B,\mathcal{S}}=P_{\mathcal{N}}+P(PBP)^{\dagger}PB \tag{6.9}$$

In what follows we are going to show that, in the above formulas, we can replace the orthogonal projection P for an arbitrary bounded operator $C\in L(\mathcal{H})$ with $R(C)=\mathcal{S}$, for an arbitrary selfadjoint operator B.

Proposition 6.8. *Let $B \in L(\mathcal{H})^s$ and consider $\mathcal{S}$ a closed subspace of $\mathcal{H}$. Then, $(B, \mathcal{S})$ is compatible if and only if $R(C^*BC) = R(C^*B)$ for every $C \in L(\mathcal{H})$ with $R(C) = \mathcal{S}$.*

Proof. The sufficiency follows from Equation (6.4) taking $C = P_{\mathcal{S}}$, the orthogonal projection onto $\mathcal{S}$.

Conversely, fix $C \in L(\mathcal{H})$ with $R(C) = \mathcal{S}$ and observe that $C = PC$. Since $R(PBP) = R(PB)$,

$$R(C^*BC) = C^*(PB(\mathcal{S})) = C^*(R(PBP)) = C^*(R(PB)) = R(C^*B).$$

□

Proposition 6.9. *Let $B \in L(\mathcal{H})^s$ and $\mathcal{S}$ be a closed subspace of $\mathcal{H}$ such that $(B, \mathcal{S})$ is compatible and consider $C \in L(\mathcal{H})$ with $R(C) = \mathcal{S}$. Then $P_{B,\mathcal{S}\ominus\mathcal{N}} = CD$, where D is the reduced solution of the equation*

$$(C^*BC)X = C^*B. \tag{6.10}$$

Proof. First of all note that we can suppose that $\mathcal{N} = \mathcal{S} \cap B(\mathcal{S})^{\perp} = \{0\}$. Indeed, since $(B, \mathcal{S})$ is compatible, $\mathcal{N} = \mathcal{S} \cap N(B)$ and, for a fixed $C \in L(\mathcal{H})$ with $R(C) = \mathcal{S}$, $BC = BP_{\mathcal{S}}C = B(P_{\mathcal{S}\ominus\mathcal{N}} + P_{\mathcal{N}})C = BP_{\mathcal{S}\ominus\mathcal{N}}C$. Then, if $C_1 = P_{\mathcal{S}\ominus\mathcal{N}}C$ we have that $R(C_1) = \mathcal{S} \ominus \mathcal{N}$, $BC_1 = BC$ and

$$C_1^*BC_1 = C_1^*BC = (BC_1)^*C = (BC)^*C = C^*BC.$$

So, the equation $(C^*BC)X = C^*B$ can be rewritten as $(C_1^*BC_1)X = C_1^*B$, where $R(C_1) = \mathcal{S} \ominus \mathcal{N}$ and $\mathcal{N}_1 = (\mathcal{S} \ominus \mathcal{N}) \cap B(\mathcal{S} \ominus \mathcal{N})^{\perp} = (\mathcal{S} \ominus \mathcal{N}) \cap B(\mathcal{S})^{\perp} = \{0\}$.

Therefore, fix $C \in L(\mathcal{H})$ with $R(C) = \mathcal{S}$ and suppose that $\mathcal{N} = \{0\}$. If $D \in L(\mathcal{H})$ is the reduced solution of Eq. (6.10), then $R(D) \subseteq \overline{R(C^*BC)}$ and $N(D) = N(C^*B) = B(\mathcal{S})^{\perp}$. Furthermore,

$$(C^*BC)DCD = C^*BCD = C^*B$$

and $R(DCD) \subseteq R(D) \subseteq \overline{R(C^*BC)}$. Therefore, by uniqueness of the reduced solution, $DCD = D$. Then, $CD \in \mathcal{Q}$ and $N(CD) = N(D) = B(\mathcal{S})^{\perp}$.

It is clear that $R(CD) \subseteq R(C) = \mathcal{S}$ but, since $\mathcal{H} = \mathcal{S} \dotplus B(\mathcal{S})^{\perp}$ and $N(CD) = B(\mathcal{S})^{\perp}$, we have that $R(CD) = \mathcal{S}$. □

Suppose that $(B, \mathcal{S})$ is compatible. If D is the reduced solution of Eq. (6.10) then $D = P_{\overline{R(C^*BC)}}D = (C^*BC)^{\sharp}C^*B$ because $R(D) \subseteq \overline{R(C^*BC)}$. Therefore,

$$P_{B,\mathcal{S}\ominus\mathcal{N}} = CD = C(C^*BC)^{\sharp}C^*B$$

and, by Equation (6.5),

$$P_{B,\mathcal{S}} = P_{\mathcal{N}} + C(C^*BC)^{\sharp}C^*B. \tag{6.11}$$

This formula for $P_{B,\mathcal{S}}$ generalizes the one given in Proposition 3.6 of [5] when B is a positive operator with closed range.

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