Free algebras in varieties of BL-algebras generated by a BL_n-chain

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Abstract

Free algebras with an arbitrary number of free generators in varieties of BL-algebras generated by one BL-chain which is an ordinal sum of a finite MV-chain $\mathbf{L}_n$ and a generalized BL-chain $\mathbf{B}$ are described in terms of weak boolean products of BL-algebras that are ordinal sums of subalgebras of $\mathbf{L}_n$ and free algebras in the variety of basic hoops generated by $\mathbf{B}$. The boolean products are taken over the Stone spaces of the boolean subalgebras of idempotents of free algebras in the variety of MV-algebras generated by $\mathbf{L}_n$.

Keywords: BL-algebras, Residuated lattices, Hoops, Moisil algebras, Free algebras, Boolean products.

Introduction

Basic Fuzzy Logic (BL for short) was introduced by Hájek (see [21] and the references given there) to formalize fuzzy logics in which the conjunction is interpreted by a continuous t-norm on the real segment $[0, 1]$ and the implication by its corresponding adjoint. He also introduced BL-algebras as the algebraic counterpart of these logics. BL-algebras form a variety (or equational class) of residuated lattices [21]. More precisely, they can be characterized as *bounded basic hoops* [7, 2]. Subvarieties of the variety of BL-algebras are in correspondence with axiomatic extensions of BL. Important examples of subvarieties of BL-algebras are MV-algebras (that correspond to Łukasiewicz many-valued logics, see [18]), linear Heyting algebras (that correspond to the superintuitionistic logic characterized by the axiom $(P \Rightarrow Q) \vee (Q \Rightarrow P)$, see [26] for a historical account about this logic), PL-algebras (that correspond to the logic determined by the t-norm given by the ordinary product on [0,1], see [14]), and also boolean algebras (that correspond to classical logic).

Since the propositions under BL equivalence form a free BL-algebra, descriptions of free algebras in terms of functions give concrete representations of these propositions. Such descriptions are known for some subvarieties of BL-algebras. The best known example is the representation of classical propositions by boolean functions. Free MV-algebras have been described in terms of continuous piecewise linear functions by McNaughton [24] (see also [18]). Finitely generated free linear Heyting algebras were described by Horn [22], and a description of finitely generated free PL-algebras is given in [14]. Linear Heyting algebras and PL-algebras are examples of varieties of BL-algebras satisfying the *boolean retraction property*. Free algebras in these varieties were completely described in [16].

In [10] the first author described the finitely generated free algebras in the varieties of BL-algebras generated by a single BL-chain which is an ordinal sum of a finite MV-chain $\mathbf{L}_n$ and a generalized BL-chain $\mathbf{B}$. We call these chains BL_n-chains. The aim of this paper is to extend the results of [10] considering the case of infinitely many free generators. The results of [10] were heavely based on the fact that the boolean subalgebras of finitely generated algebras in the varieties generated by BL_n-chains are finite. Therefore the methods of [10] can not be applied to the general case.

As a preliminary step we characterize the boolean algebra of idempotent elements of a free algebra in $\mathcal{MV}_n$, the variety of MV-algebras generated by the finite MV-chain $\mathbf{L}_n$. It is the free boolean algebra over a poset which is the cardinal sum of chains of length $n-1$. In the proof of this result a central role is played by the Moisil algebra reducts of algebras in $\mathcal{MV}_n$.

Free algebras in varieties of BL-algebras generated by a single BL_n-chain $\mathbf{L}_n \uplus \mathbf{B}$ are then described in terms of weak boolean products of BL-algebras that are ordinal sums of subalgebras of $\mathbf{L}_n$ and free algebras in the variety of basic hoops generated by $\mathbf{B}$. The boolean products are taken over the Stone spaces of the boolean algebras of idempotent elements of free algebras in $\mathcal{MV}_n$. An important intermediate step is the characterization of the variety of generalized BL-algebras generated by $\mathbf{B}$ (Corollary 3.5).

The paper is organized as follows: in the first section we recall, for further reference, some basic notions on BL-algebras and on the varieties $\mathcal{MV}_n$. We also recall some facts about the representation of free algebras in varieties of BL-algebras as weak boolean products.The only new result is given in Theorem 1.5. In Section 2, after giving the necessary background on Moisil algebra reducts of algebras in $\mathcal{MV}_n$, we characterize the boolean algebras of idempotent elements of free algebras in $\mathcal{MV}_n$. These results are used in Section 3 to give the mentioned description of free algebras in the varieties of BL-algebras generated by a BL_n-chain. Finally in Section 4 we give some examples and we compare our results with those of [16] and [10].

1 Preliminaries

1.1 BL-algebras: Basic Notions

A **hoop** [7] is an algebra $\mathbf{A} = (A, *, \to, \top)$ of type $(2,2,0)$, such that $(A, *, \top)$ is a commutative monoid and for all $x, y, z \in A$:

1. $x \to x = \top$,

2. $x * (x \to y) = y * (y \to x)$,

3. $x \to (y \to z) = (x * y) \to z$.

A **basic hoop** [2] or a **generalized BL-algebra** [17], is a hoop that satisfies the equation:

$$(((x \to y) \to z) * ((y \to x) \to z)) \to z = \top. \tag{1}$$

It is shown in [2] that generalized BL-algebras can be characterized as algebras $\mathbf{A} = (A, \wedge, \vee, *, \to, \top)$ of type $(2, 2, 2, 2, 0)$ such that:

1. $(A, *, \top)$, is an commutative monoid,
2. $\mathbf{L}(\mathbf{A}) := (A, \wedge, \vee, \top)$, is a lattice with greatest element $\top$,
3. $x \to x = \top$,
4. $x \to (y \to z) = (x * y) \to z$,
5. $x \wedge y = x * (x \to y)$,
6. $(x \to y) \vee (y \to x) = \top$.

A **BL-algebra** or **bounded basic hoop** is a bounded generalized BL-algebra, that is, it is an algebra $\mathbf{A} = (A, \wedge, \vee, *, \to, \bot, \top)$ of type $(2, 2, 2, 2, 0, 0)$ such that $(A, \wedge, \vee, *, \to, \top)$ is a generalized BL-algebra, and $\bot$ is the lower bound of $\mathbf{L}(\mathbf{A})$. In this case, we define the unary operation $\neg$ by the equation:

$$\neg x = x \to \bot.$$

The BL-algebra $\mathbf{A}$ with only one element, that is $\bot = \top$, is called the **trivial BL-algebra**. The varieties of BL-algebras and of generalized BL-algebras will be denoted by $\mathcal{BL}$ and $\mathcal{GBL}$, respectively.

In every generalized BL-algebra $\mathbf{A}$ we denote by $\leq$ the (partial) order defined on A by the lattice $\mathbf{L}(\mathbf{A})$, i.e. for $a, b \in A$, $a \leq b$ iff $a = a \wedge b$ iff $b = a \vee b$. This order is called the **natural order** of $\mathbf{A}$. When this natural order is total (i.e., for each $a, b, \in A$, $a \leq b$ or $b \leq a$), we say that $\mathbf{A}$ is a **generalized BL-chain** (**BL-chain** in case $\mathbf{A}$ is a BL-algebra). The following theorem makes obvious the importance of BL-chains and can be easily derived from [21, Lemma 2.3.16].

Theorem 1.1 *Each BL-algebra is a subdirect product of BL-chains.*

In every BL-algebra $\mathbf{A}$ we define a binary operation:

$$x \oplus y = \neg(\neg x * \neg y).$$

For each positive integer k, the operations x^k and $k \cdot x$ are inductively defined as follows:

- $x^1 = x$ and $x^{k+1} = x^k * x$,
- $1 \cdot x = x$ and $(k + 1) \cdot x = (k \cdot x) \oplus x$.

MV-algebras, the algebras of Łukasiewicz infinite-valued logic, form a subvariety of $\mathcal{BL}$, which is characterized by the equation:

$$\neg\neg x = x$$

(see [21]). The variety of MV-algebras is denoted by $\mathcal{MV}$. Totally ordered MV-algebras are called **MV-chains**. For each BL-algebra $\mathbf{A}$, the set

$$MV(\mathbf{A}) := \{x \in A : \neg\neg x = x\}$$

is the universe of a subalgebra $\mathbf{MV}(\mathbf{A})$ of $\mathbf{A}$ which is an MV-algebra (see [17]).

A **PL-algebra** is a BL-algebra that satisfies the following two axioms:

1. $(\neg\neg z * ((x * z) \to (y * z))) \to (x \to y) = \top$,
2. $x \wedge \neg x = \bot$.

PL-algebras correspond to **product fuzzy logic**, see [14] and [21].

It follows from Theorem 1.1 that for each BL-algebra $\mathbf{A}$ the lattice $\mathbf{L}(\mathbf{A})$ is distributive. The complemented elements of $L(\mathbf{A})$ form a subalgebra $\mathbf{B}(\mathbf{A})$ of $\mathbf{A}$ which is a boolean algebra. Elements of $B(\mathbf{A})$ are called **boolean elements** of $\mathbf{A}$.

1.2 Implicative filters

Definition 1.2 *An* **implicative filter** *of a BL-algebra* $\mathbf{A}$ *is a subset* $F \subseteq A$ *satisfying the following conditions:*

1. $\top \in F$,
2. *If* $x \in F$ *and* $x \to y \in F$*, then* $y \in F$.

An implicative filter is called **proper** provided that $F \neq A$. If W is a subset of a BL-algebra $\mathbf{A}$, the implicative filter generated by W will be denoted by $\langle W \rangle$. If U is a filter of the boolean subalgebra $\mathbf{B}(\mathbf{A})$, then the implicative filter $\langle U \rangle$ is called **Stone filter of A**. An implicative filter F of a BL-algebra $\mathbf{A}$ is called **maximal** iff it is proper and no proper implicative filter of $\mathbf{A}$ strictly contains F.

Implicative filters characterize congruences in BL-algebras. Indeed, if F is an implicative filter of a BL-algebra $\mathbf{A}$ it is well known (see [21, Lemma 2.3.14]), that the binary relation $\equiv_F$ on A defined by:

$$x \equiv_F y \quad \text{iff} \quad x \to y \in F \text{ and } y \to x \in F$$

is a congruence of $\mathbf{A}$. Moreover, $F = \{x \in A : x \equiv_F \top\}$. Conversely, if $\equiv$ is a congruence relation on A, then $\{x \in A : x \equiv \top\}$ is an implicative filter, and $x \equiv y$ iff $x \to y \equiv \top$ and $y \to x \equiv \top$. Therefore, the correspondence $F \mapsto \equiv_F$ is a bijection from the set of implicative filters of $\mathbf{A}$ onto the set of congruences of $\mathbf{A}$.

Lemma 1.3 *(see [16]) Let* $\mathbf{A}$ *be a BL-algebra, and let* F *be a filter of* $\mathbf{B}(\mathbf{A})$*. Then*

$$(\equiv_F) = \{(a, b) \in A \times A : a \wedge c = b \wedge c \text{ for some } c \in F\}$$

is a congruence relation on $\mathbf{A}$ *that coincides with the congruence relation given by the implicative filter* $\langle F \rangle$ *generated by* F.

1.3 MV_n-algebras

For $n \geq 2$, we define:

$$L_n = \{\frac{0}{n-1}, \frac{1}{n-1}, \frac{2}{n-1}, \ldots, \frac{n-1}{n-1}\}.$$

The set L_n equipped with the operations $x * y = \max(0, x + y - 1)$, $x \to y = \min(1, 1 - x + y)$, and with $\bot = 0$ defines a finite MV-algebra which shall be denoted by $\mathbf{L}_n$. Clearly $B(\mathbf{L}_n) = \{0, 1\}$.

A BL-algebra $\mathbf{A}$ is said to be **simple** provided it is nontrivial and the only proper implicative filter of $\mathbf{A}$ is the singleton $\{\top\}$. In [18] it is proved that $\mathbf{L}_n$ is a simple MV-algebra for each integer n.

We shall denote by $\mathcal{MV}_n$ the subvariety of $\mathcal{MV}$ generated by $\mathbf{L}_n$. The elements of $\mathcal{MV}_n$ are called **MV$_n$-algebras**. A finite MV-chain $\mathbf{L}_m$ belongs to $\mathcal{MV}_n$ iff $m-1$ is a divisor of $n-1$. Therefore it is not hard to corroborate that every MV$_n$-algebra is a subdirect product of a family of algebras $(\mathbf{L}_{m_i}, i \in I)$ where $m_i - 1$ divides $n-1$ for each $i \in I$.

It can be deduced from [18, Corollary 8.2.4 and Theorem 8.5.1] that $\mathcal{MV}_n$ is the proper subvariety of $\mathcal{MV}$ characterized by the following equations:

- (α_n) $\quad x^{(n-1)} = x^n$,

  and if $n \geq 4$, for every integer $p = 2, \ldots, n-2$ that does not divide $n-1$:

- (β_n) $\quad (p \cdot x^{p-1})^n = n \cdot x^p$.

If $\mathbf{A}$ is an MV$_n$-algebra, it is not hard to verify that for each $x \in A \setminus \{\top\}$, $x^n = \bot$ and for each $y \in A \setminus \{\bot\}$, $n.y = \top$.

1.4 Ordinal sum and decomposition of BL-chains

Let $\mathbf{R} = (R, *_{\mathbf{R}}, \to_{\mathbf{R}}, \top)$ and $\mathbf{S} = (S, *_{\mathbf{S}}, \to_{\mathbf{S}}, \top)$ be two hoops such that $R \cap S = \{\top\}$. Following [7] we can define the **ordinal sum** $\mathbf{R} \uplus \mathbf{S}$ of these two hoops as the hoop given by $(R \cup S, *, \to, \top)$ where the operations (*,→) are defined as follows:

$$x * y = \begin{cases} x *_{\mathbf{R}} y & \text{if } x, y \in R, \\ x *_{\mathbf{S}} y & \text{if } x, y \in S, \\ x & \text{if } x \in R \setminus \{\top\} \text{ and } y \in S, \\ y & \text{if } y \in R \setminus \{\top\} \text{ and } x \in S. \end{cases}$$

$$x \to y = \begin{cases} \top & \text{if } x \in R \setminus \{\top\},\ y \in S, \\ x \to_{\mathbf{R}} y & \text{if } x, y \in R, \\ x \to_{\mathbf{S}} y & \text{if } x, y \in S, \\ y & \text{if } y \in R \setminus \{\top\} \text{ and } x \in S. \end{cases}$$

If $R \cap S \neq \{\top\}$, $\mathbf{R}$ and $\mathbf{S}$ can be replaced by isomorphic copies whose intersection is $\{\top\}$, thus their ordinal sum can be defined. Observe that when $\mathbf{R}$ is a generalized BL-chain and $\mathbf{S}$ is a generalized BL-algebra, the hoop resulting from their ordinal sum satisfies equation (1). Thus $\mathbf{R} \uplus \mathbf{S}$ is a generalized BL-algebra. Moreover, if $\mathbf{R}$ is a BL-chain, then $\mathbf{R} \uplus \mathbf{S}$ is a BL-algebra, where $\bot = \bot_{\mathbf{R}}$. In this case it is obvious that the chain $\mathbf{R} \uplus \mathbf{S}$ is subdirectly irreducible if and only if $\mathbf{S}$ is subdirectly irreducible. Notice also that for any generalized BL-algebra $\mathbf{S}$, $\mathbf{L_2} \uplus \mathbf{S}$ is the BL-algebra that arises from adjoining a bottom element to $\mathbf{S}$.

Given a BL-algebra $\mathbf{A}$ we can consider the set

$$D(\mathbf{A}) := \{x \in \mathbf{A} : \neg x = \bot\}.$$

It is shown in [17], that $\mathbf{D}(\mathbf{A}) = (D(\mathbf{A}), \wedge, \vee, *, \to, \top)$ is a generalized BL-algebra.

Theorem 1.4 *(see [10]) For each BL-chain* $\mathbf{A}$ *we have that*

$$\mathbf{A} \cong \mathbf{MV}(\mathbf{A}) \uplus \mathbf{D}(\mathbf{A}).$$

Theorem 1.5 *Let* $\mathbf{A}$ *be a BL-algebra such that* $\mathbf{MV}(\mathbf{A}) \cong \mathbf{L}_n$ *for some integer* n*. Then*

$$\mathbf{A} \cong \mathbf{MV}(\mathbf{A}) \uplus \mathbf{D}(\mathbf{A}) \cong \mathbf{L}_n \uplus \mathbf{D}(\mathbf{A}).$$

Proof: From Theorem 1.1, we can think of each non trivial BL-algebra $\mathbf{A}$ as a subdirect product of a family $(\mathbf{A}_i, i \in I)$ of non trivial BL-chains, that is, there exists an embedding

$$e : \mathbf{A} \to \prod_{i \in I} \mathbf{A}_i,$$

such that $\pi_i(e(\mathbf{A})) = \mathbf{A}_i$ for each $i \in I$, where π_i denotes each projection. We shall identify $\mathbf{A}$ with $e(\mathbf{A})$. Then each element of A is a tuple $\mathbf{x}$ and coordinate i is $x_i \in A_i$. With this notation we have that for each $\mathbf{x} \in A$, $\pi_i(\mathbf{x}) = x_i$. We will proof the following items:

1. *For each* $i \in I$, $\mathbf{MV}(\mathbf{A}_i)$ *is isomorphic to* $\mathbf{L}_n$.

   Since for each $i \in I$, π_i is a homomorphism and $\pi_i(MV(\mathbf{A})) \subseteq A_i$, we have that $\pi_i(MV(\mathbf{A})) \subseteq MV(\mathbf{A}_i)$. Then $\pi_i(\mathbf{MV}(\mathbf{A}))$ is a subalgebra of $\mathbf{MV}(\mathbf{A}_i)$. On the other hand, given $i \in I$, let $x_i \in MV(\mathbf{A}_i)$. Then $\neg\neg x_i = x_i$ and there exists an element $\mathbf{x} \in A$ such that $\pi_i(\mathbf{x}) = x_i$. Taking $\mathbf{y} = \neg\neg\mathbf{x} \in MV(\mathbf{A})$ we have that $\pi_i(\mathbf{y}) = x_i$ and $x_i \in \pi_i(MV(\mathbf{A}))$. Hence $MV(\mathbf{A}_i) \subseteq \pi_i(MV(\mathbf{A}))$.

   In conclusion $\mathbf{MV}(\mathbf{A}_i) = \pi_i(\mathbf{MV}(\mathbf{A})) = \pi_i(\mathbf{L}_n) = \mathbf{L}_n$, because $\mathbf{L}_n$ is simple.

2. *If* $\mathbf{x} \in A$*, then* $\mathbf{x} \in MV(\mathbf{A}) \cup D(\mathbf{A})$.

   Let $\mathbf{x} \in A$ and let $\mathbf{y} = n.(\neg\mathbf{x})$. If $x_i \in L_n \setminus \{\top\}$, then $\neg x_i \in L_n \setminus \{\bot\}$. From equation (α_n) we obtain that $y_i = n.(\neg x_i) = \top$. On the other hand if $\neg x_i = \bot$, then $y_i = n.(\neg x_i) = \bot$. Now let $\mathbf{z} = (\neg\neg\mathbf{x})^n$. If $x_i \in L_n \setminus \{\top\}$, then $z_i = \bot$, but if $\neg\neg x_i = \top$, then $z_i = \top$.

   Suppose there exists $\mathbf{x} \in A$ such that $\mathbf{x} \notin MV(\mathbf{A})$ and $\mathbf{x} \notin D(\mathbf{A})$. It follows from Theorem 1.4 that for each $i \in I$, $\mathbf{A}_i = \mathbf{MV}(\mathbf{A}_i) \uplus \mathbf{D}(\mathbf{A}_i)$, then there exist $i, j \in I$, such that $x_i \in MV(\mathbf{A}_i) \setminus \{\top\} = L_n \setminus \{\top\}$ and $x_j \in D(\mathbf{A}_j) \setminus \{\top\}$.

   Let $\mathbf{y} = n.(\neg\mathbf{x})$. Then $y_i = \top$, $y_j = \bot$ and $y_k \in \{\bot, \top\}$ for each $k \in I \setminus \{i, j\}$. Now let $\mathbf{z} = (\neg\neg\mathbf{x})^n$. We have that $z_j = \top$, $z_i = \bot$ and $z_k \in \{\bot, \top\}$ for each $k \in I \setminus \{i, j\}$. It follows that $\mathbf{y}$ and $\mathbf{z}$ are elements in the chain $MV(\mathbf{A}) = L_n$ which are not comparable, which is a contradiction.

3. *If* $\mathbf{x} \in MV(\mathbf{A}) \setminus \{\top\}$ *and* $\mathbf{y} \in D(\mathbf{A})$*, then* $\mathbf{x} < \mathbf{y}$.

   The statement is clear if $x_i \in MV(\mathbf{A}_i) \setminus \{\top\}$ for every $i \in I$ or if $y_i = \top$ for each $i \in I$. Otherwise, suppose $x_i = \top$ for some $i \in I$. Since $\mathbf{x} \neq \top$ there must exists $j \in I$ such that $x_j \neq \top$. If $y_i = \top$ for each $i \in I$ such that $x_i = \top$, then $\mathbf{x} < \mathbf{y}$. If not, let $\mathbf{z} = \mathbf{x} \wedge \mathbf{y}$. Since operations are coordinatewise, $z_j \in MV(\mathbf{A}_j) \setminus \{\top\}$ and $z_i \in D(\mathbf{A}_i) \setminus \{\top\}$, for some $i \in I$. Hence $\mathbf{z} \notin MV(\mathbf{A})$ and $\mathbf{z} \notin D(\mathbf{A})$ contradicting the previous item.

4. *If* $\mathbf{x} \in MV(\mathbf{A}) \setminus \{\top\}$ *and* $\mathbf{y} \in D(\mathbf{A})$*, then* $\mathbf{y} \to \mathbf{x} = \mathbf{x}$ *and* $\mathbf{y} * \mathbf{x} = \mathbf{x}$.

   Since $\neg\mathbf{y} = \bot$ we have that

$$\mathbf{y} \to \mathbf{x} = \mathbf{y} \to \neg\neg\mathbf{x} = \mathbf{y} \to (\neg\mathbf{x} \to \bot) = \neg\mathbf{x} \to (\mathbf{y} \to \bot) =$$

$$= \neg\mathbf{x} \to \bot = \neg\neg\mathbf{x} = \mathbf{x},$$

   and

$$\mathbf{x} = \mathbf{y} \wedge \mathbf{x} = \mathbf{y} * (\mathbf{y} \to \mathbf{x}) = \mathbf{y} * \mathbf{x}.$$

From the previous items it follows that $\mathbf{A} \cong \mathbf{MV}(\mathbf{A}) \uplus \mathbf{D}(\mathbf{A}) = \mathbf{L}_n \uplus \mathbf{D}(\mathbf{A})$.■

1.5 Free algebras in varieties of BL-algebras generated by a BL$_n$-chain.

Recall that an algebra $\mathbf{A}$ in a variety $\mathcal{K}$ is said to be **free over a set** Y if and only if for every algebra $\mathbf{C}$ in $\mathcal{K}$ and every function $f : Y \to \mathbf{C}$, f can be uniquely extended to a homomorphism of $\mathbf{A}$ into $\mathbf{C}$. Given a variety $\mathcal{K}$ of algebras, we denote by $\mathbf{Free}_{\mathcal{K}}(X)$ the free algebra in $\mathcal{K}$ over X. As mentioned in the introduction, we define a **BL$_n$-chain** as a BL-chain which is an ordinal sum of the MV-chain $\mathbf{L}_n$ and a generalized BL-chain. Once we fixed the generalized BL-chain $\mathbf{B}$, we are going to study the free algebra $\mathbf{Free}_{\mathcal{V}}(X)$ where $\mathcal{V}$ is the variety of BL-algebras generated by the BL$_n$-chain

$$\mathbf{T}_n := \mathbf{L}_n \uplus \mathbf{B}.$$

Notice that $\mathbf{MV}(\mathbf{T}_n) \cong \mathbf{L}_n$ and if $x \notin MV(\mathbf{T}_n) \setminus \{\top\}$, then $x \in D(\mathbf{T}_n) = B$.

Recall that a **weak boolean product** of a family $(A_y, y \in Y)$ of algebras over a boolean space Y is a subdirect product $\mathbf{A}$ of the given family such that the following conditions hold:

- if $a, b \in A$, then $[a = b] = \{y \in Y : a_y = b_y\}$ is open,
- if $a, b \in A$ and Z is a clopen in X, then $a|_Z \cup b|_{X \setminus Z} \in A$.

Since the variety $\mathcal{BL}$ is congruence distributive, it has the Boolean Factor Congruence property. Therefore each nontrivial BL-algebra can be represented as a weak boolean product of directly indecomposable BL-algebras (see [5] and [20]). The explicit representation of each BL-algebra as a weak boolean product of directly indecomposable algebras is given in [16] by the following lemma:

Lemma 1.6 *Let* $\mathbf{A}$ *be a BL-algebra and let* $Sp\ \mathbf{B}(\mathbf{A})$ *be the boolean space of ultrafilters of the boolean algebra* $\mathbf{B}(\mathbf{A})$*. The correspondence:*

$$a \mapsto (a/\langle U\rangle)_{U \in Sp\ \mathbf{B}(\mathbf{A})}$$

gives an isomorphism of $\mathbf{A}$ *onto the weak boolean product of the family*

$$(\mathbf{A}/\langle U\rangle) : U \in Sp\ \mathbf{B}(\mathbf{A})$$

over the boolean space $Sp\ \mathbf{B}(\mathbf{A})$*. This representation is called* **the Pierce representation**. *Any other representation of* $\mathbf{A}$ *as a weak boolean product of a family of directly indecomposable algebras is equivalent to the Pierce representation.*

Therefore, to describe $\mathbf{Free}_{\mathcal{V}}(X)$ we need to describe $\mathbf{B}(\mathbf{Free}_{\mathcal{V}}(X))$ and the quotients $\mathbf{Free}_{\mathcal{V}}(X)/\langle U\rangle$ for each $U \in Sp\ \mathbf{B}(\mathbf{Free}_{\mathcal{V}}(X))$.

In the next section we will obtain a characterization of the boolean algebra $\mathbf{B}(\mathbf{Free}_{\mathcal{V}}(X))$. Once this aim is achieved, we shall consider the quotients $\mathbf{Free}_{\mathcal{V}}(X)/\langle U\rangle$.

2 $\mathbf{B}(\mathbf{Free}_{\mathcal{V}}(X))$

The next two results can be found in [17].

Theorem 2.1 *For each BL-algebra* $\mathbf{A}$, $\mathbf{B}(\mathbf{A}) \cong \mathbf{B}(\mathbf{MV}(\mathbf{A}))$.

Theorem 2.2 *For each variety* $\mathcal{K}$ *of BL-algebras and each set* X *one has that:*

$$\mathbf{MV}(\mathbf{Free}_{\mathcal{K}}(X)) \cong \mathbf{Free}_{\mathcal{MV}\cap\mathcal{K}}(\neg\neg X).$$

Theorem 2.3 $\mathcal{V} \cap \mathcal{MV}$ *is the variety* $\mathcal{MV}_n$.

Proof: Since $\mathbf{L}_n \cong \mathbf{MV}(\mathbf{T_n})$ is in $\mathcal{V} \cap \mathcal{MV}$, we have that $\mathcal{MV}_n \subseteq \mathcal{V} \cap \mathcal{MV}$. On the other hand, let $\mathbf{A}$ be an MV-algebra in $\mathcal{V} \cap \mathcal{MV}$. Suppose $\mathbf{A}$ is not in $\mathcal{MV}_n$. Then there exists an equation $e(x_1, \ldots, x_p) = \top$ that is satisfied by $\mathbf{L}_n$ and is not satisfied by $\mathbf{A}$, that is, there exist elements $a_1, \ldots, a_p$ in A such that $e(a_1, \ldots, a_p) \neq \top$. Since $(\neg\neg b_1, \ldots, \neg\neg b_p)$ is in $(L_n)^p$, for each tuple $(b_1, \ldots, b_p)$ in $(T_n)^p$, the equation $e'(x_1, \ldots, x_p) = e(\neg\neg x_1, \ldots, \neg\neg x_p) = \top$ is satisfied in $\mathcal{V}$. Since $\mathbf{A} \in \mathcal{V} \cap \mathcal{MV}$, it follows that $\top = e'(a_1, \ldots, a_p) = e(\neg\neg a_1, \ldots, \neg\neg a_p) = e(a_1, \ldots, a_p) \neq \top$, a contradiction. Hence $\mathcal{MV}_n = \mathcal{V} \cap \mathcal{MV}$. ■

From these results we obtain:

Theorem 2.4

$$\mathbf{B}(\mathbf{Free}_{\mathcal{V}}(X)) \cong \mathbf{B}(\mathbf{Free}_{\mathcal{MV}_n}(\neg\neg X)).$$

2.1 n-valued Moisil algebras

Boolean elements of $\mathbf{Free}_{\mathcal{MV}_n}(\neg\neg X)$ depend on some operators that can be defined on each MV$_n$-algebra. Such operators provide each MV$_n$-algebra with an n-valued Moisil algebra structure, in the sense of the following definition.

Definition 2.5 *For each integer $n \geq 2$, an* **n-valued Moisil algebra** *([8] and [11]) or* **n-valued Łukasiewicz algebra** *([4], [12] and [13]) is an algebra $\mathbf{A} = (A, \wedge, \vee, \neg, \sigma_1^n, \ldots, \sigma_{n-1}^n, 0, 1)$ of type $(2, 2, 1, \ldots, 1, 0, 0)$ such that $(A, \wedge, \vee, 0, 1)$ is a distributive lattice with unit 1 and zero 0, and $\neg, \sigma_1^n, \ldots, \sigma_{n-1}^n$ are unary operators defined on A that satisfy the following conditions:*

1. $\neg\neg x = x$,
2. $\neg(x \vee y) = \neg x \wedge \neg y$,
3. $\sigma_i^n(x \vee y) = \sigma_i^n x \vee \sigma_i^n y$,
4. $\sigma_i^n x \vee \neg\sigma_i^n x = 1$,
5. $\sigma_i^n \sigma_j^n x = \sigma_j^n x$, *for* $i, j = 1, 2, \ldots n-1$,
6. $\sigma_i^n(\neg x) = \neg(\sigma_{n-i}^n x)$,
7. $\sigma_i^n x \vee \sigma_{i+1}^n x = \sigma_{i+1}^n x$, *for* $i = 1, 2, \ldots, n-2$,
8. $x \vee \sigma_{n-1}^n x = \sigma_{n-1}^n x$,
9. $(x \wedge \neg\sigma_i^n x \wedge \sigma_{i+1}^n y) \vee y = y$, *for* $i = 1, 2, \ldots, n-2$.

Properties and examples of n-valued Moisil algebras can be found in [4] and in [8]. The variety of n-valued Moisil algebras will be denoted $\mathcal{M}_n$. An important property of n-valued Moisil algebras is the following:

We also have that:

Theorem 2.6 *(see [11]) Let $\mathbf{A}$ be in $\mathcal{M}_n$. Then $x \in B(\mathbf{A})$ if and only if $\sigma_{n-1}^n(x) = x$. Furthermore,*

$$\sigma_{n-1}^n(x) = \min\{b \in B(\mathbf{A}) : x \leq b\} \text{ and } \sigma_1^n(x) = \max\{a \in B(\mathbf{A}) : a \leq x\}.$$

Definition 2.7 *For each integer $n \geq 2$,* **Post algebra of order** *n is a system*

$$\mathbf{A} = (A, \wedge, \vee, \neg, \sigma_1^n, \ldots, \sigma_{n-1}^n, e_1, \ldots, e_{n-1}, 0, 1)$$

such that $(A, \wedge, \vee, \neg, \sigma_1^n, \ldots, \sigma_{n-1}^n, 0, 1)$ is an n-valued Moisil algebra and $e_1, \ldots, e_{n-1}$ are constants that satisfy the following equations:

$$\sigma_i^n(e_j) = \begin{cases} 0 & \text{if } i + j < n; \\ 1 & \text{if } i + j \geq n. \end{cases}$$

For every $n \geq 2$ we can define one-variable terms $\sigma_1^n(x), \ldots, \sigma_{n-1}^n(x)$ in the language $(\neg, \rightarrow, \top)$ such that evaluated on the algebras $\mathbf{L}_n$ give:

$$\sigma_i^n(\frac{j}{(n-1)}) = \begin{cases} 1 & \text{if } i + j \geq n, \\ 0 & \text{if } i + j < n, \end{cases}$$

for $i = 1, \ldots, n-1$ (see [13] or [25]) . It is easy to check that

$$\mathbf{M}(\mathbf{L}_n) = (L_n, \wedge, \vee, \neg, \sigma_1^n, \ldots, \sigma_{n-1}^n, 0, 1)$$

is a n-valued Moisil algebra. Since these algebras are defined by equations and $\mathbf{L}_n$ generates the variety $\mathcal{MV}_n$, we have that each $\mathbf{A} \in \mathcal{MV}_n$ admits a structure of an n-valued Moisil algebra, denoted by $\mathbf{M}(\mathbf{A})$. The chain $\mathbf{M}(\mathbf{L}_n)$ plays a very important role in the structure of n-valued Moisil algebras, since each n-valued Moisil algebra is a subdirect product of subalgebras of $\mathbf{M}(\mathbf{L}_n)$ (see [4] or [12]). If we add to the structure $\mathbf{M}(\mathbf{L}_n)$ the constants

$$e_i = \frac{i}{n-1},$$

for $i = 1, \ldots, n-1$, then $\mathbf{PT}(\mathbf{L}_n) = (L_n, \wedge, \vee, \neg, \sigma_1^n, \ldots, \sigma_{n-1}^n, e_1, \ldots, e_{n-1}, 0, 1)$ is a Post algebra.

Not every n-valued Moisil algebra has a structure of MV$_n$-algebra (see [23]). For example, a subalgebra of $\mathbf{M}(\mathbf{L}_n)$ may not be a subalgebra of $\mathbf{L}_n$ as MV$_n$-algebra. That is the case of the set

$$C = \{\frac{0}{4}, \frac{1}{4}, \frac{3}{4}, \frac{4}{4}\}$$

which is the universe of a subalgebra of $\mathbf{M}(\mathbf{L_5})$, but not the universe of a subalgebra of $\mathbf{L_5}$. On the other hand, every Post algebra has a structure of MV$_n$-algebra (see [25, Theorem 10]).

The next example will play an important role in what follows:

Example 2.8 Let $\mathbf{C} = (C, \wedge, \vee, \neg, 0, 1)$ be a boolean algebra. We define

$$C^{[n]} := \{\mathbf{z} = (z_1, \ldots, z_{n-1}) \in C^{n-1} : z_1 \leq z_2 \leq \ldots \leq z_{n-1}\}$$

For each $\mathbf{z} = (z_1, \ldots, z_{n-1}) \in C^{[n]}$ we define:

$$\neg_n \mathbf{z} = (\neg z_{n-1}, \ldots \neg z_1),$$

$$\mathbf{0} = (0, \ldots, 0),$$

$$\mathbf{1} = (1, \ldots, 1),$$

$$\sigma_i^n(\mathbf{z}) = (z_i, z_i, \ldots, z_i) \text{ for } i = 1, \ldots, n-1.$$

With $\wedge$ and $\vee$ defined coordinatewise, $\mathbf{C}^{[n]} = (C^{[n]}, \wedge, \vee, \neg_n, \sigma_1^n, \ldots, \sigma_{n-1}^n, \mathbf{0}, \mathbf{1})$ is an n-valued Moisil algebra (see [8, Chapter 3, Example 1.10]). If we define $\mathbf{e_j} = (e_{j,1}, \ldots, e_{j,n-1})$ by

$$e_{j,i} = \begin{cases} 0 & \text{if } i < j, \\ 1 & \text{if } i \geq j, \end{cases}$$

then $\mathbf{C}^{[n]} = (C^{[n]}, \wedge, \vee, \neg_n, \sigma_1^n, \ldots, \sigma_{n-1}^n, \mathbf{e_1}, \ldots, \mathbf{e_{n-1}}, \mathbf{0}, \mathbf{1})$ is a Post algebra. Consequently, $\mathbf{C}^{[n]}$ has a structure of MV$_n$-algebra.

It is easy to see that for each MV$_n$-algebra $\mathbf{A}$,

$$\mathbf{B}(\mathbf{A}) = \mathbf{B}(\mathbf{M}(\mathbf{A})).$$

We need to show that the boolean elements of the MV$_n$-algebra generated by a set G coincide with the boolean elements of the n-valued Moisil algebra generated by the same set. In order to prove this result it is convenient to consider the following operators on each n-valued Moisil algebra $\mathbf{A}$: for each $i = 0, \ldots, n-1$

$$J_i(x) = \sigma^n_{n-i}(x) \wedge \neg\sigma^n_{n-i-1}(x),$$

where $\sigma^n_0(x) = 0$ and $\sigma^n_n(x) = 1$. Notice that in $\mathbf{M}(\mathbf{L}_n)$ we have:

$$J_i(\frac{j}{(n-1)}) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Lemma 2.9 *Let* $\mathbf{A}$ *be an* MV_n*-algebra, and let* $G \subset A$*. If* $< G >_{\mathcal{MV}_n}$ *is the subalgebra of* $\mathbf{A}$ *generated by the set* G *and* $< G >_{\mathcal{M}_n}$ *is the subalgebra of* $\mathbf{M}(\mathbf{A})$ *generated by* G*, then*

$$\mathbf{B}(< G >_{\mathcal{MV}_n}) = \mathbf{B}(< G >_{\mathcal{M}_n}).$$

Proof: Since $< G >_{\mathcal{M}_n}$ is always a subalgebra of $\mathbf{M}(< G >_{\mathcal{MV}_n})$, we have that $\mathbf{B}(< G >_{\mathcal{M}_n})$ is a subalgebra of $\mathbf{B}(< G >_{\mathcal{MV}_n})$.

We will see that $B(< G >_{\mathcal{MV}_n}) \subseteq B(< G >_{\mathcal{M}_n})$. The case $G = \emptyset$ is clear. Suppose that G is a finite set of cardinality $p \geq 1$. Since MV$_n$-algebras are locally finite (see [9, Chapter II, Theorem 10.16]), we obtain that $< G >_{\mathcal{MV}_n}$ is a finite MV$_n$-algebra. Since finite MV$_n$-algebras are direct product of simple algebras, there exists a finite $k \geq 1$ such that

$$< G >_{\mathcal{MV}_n} = \prod_{i=1}^{k} \mathbf{L}_{m_i},$$

where each $m_i - 1$ divides $n-1$, for each $i = 1, \ldots, k$. If $k = 1$, then $< G >_{\mathcal{M}_n}$ and $< G >_{\mathcal{MV}_n}$ are finite chains whose only boolean elements are their extremes. Otherwise, we can think of the elements of $< G >_{\mathcal{MV}_n}$ as k-tuples, i.e., if $\mathbf{x} \in < G >_{\mathcal{MV}_n}$, then $\mathbf{x} = (x_1, \ldots, x_k)$. We shall denote by $\mathbf{1}^j$ the k-tuple given by

$$(\mathbf{1}^j)_i = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

It is clear that for each $j = 1, \ldots, k$, $\mathbf{1}^j$ is in $< G >_{\mathcal{MV}_n}$. From this it follows that for every pair $i \neq j$, $i, j \in \{1, \ldots, k\}$, there exists an element $\mathbf{x} \in G$ such that $x_j \neq x_i$. Indeed, suppose on the contrary that there exist $i, j \leq k$ such that $x_i = x_j$, for every $\mathbf{x} \in G$. Then for every $\mathbf{z} \in < G >_{\mathcal{MV}_n}$ we would have $z_j = z_i$ contradicting the fact that $\mathbf{1}^i$ is in $< G >_{\mathcal{MV}_n}$.

To see that every boolean element in $< G >_{\mathcal{MV}_n}$ is also in $< G >_{\mathcal{M}_n}$ it is enough to prove that $\mathbf{1}^j$ is in $< G >_{\mathcal{M}_n}$ for every $j = 1, \ldots, k$. For a fixed j, for each $i \neq j$, $i = 1, \ldots k$, we choose $\mathbf{x}^i \in G$ such that $x^i_j \neq x^i_i$. Let j_i be the numerator of $x^i_j \in L_n$. It is not hard to verify that

$$\mathbf{1}^j = \bigwedge_{i=1, i\neq j}^{k} J_{j_i}(\mathbf{x}^i).$$

Therefore $\mathbf{1}^j \in < G >_{\mathcal{M}_n}$ and $B(< G >_{\mathcal{MV}_n}) \subseteq B(< G >_{\mathcal{M}_n})$.

If G is not finite, let $\mathbf{y}$ be a boolean element in $< G >_{\mathcal{MV}_n}$. Hence, there exists a finite subset $G_{\mathbf{y}}$ of G such that $\mathbf{y}$ belongs to the subalgebra of $< G >_{\mathcal{MV}_n}$ generated by $G_{\mathbf{y}}$. Therefore, since $\mathbf{y}$ is boolean, $\mathbf{y}$ belongs to the subalgebra of $< G >_{\mathcal{M}_n}$ generated by $G_{\mathbf{y}}$, and we conclude that

$$B(< G >_{\mathcal{MV}_n}) \subseteq B(< G >_{\mathcal{M}_n})$$

for all sets G. ■

Given an algebra $\mathbf{A}$ in a variety $\mathcal{K}$, a subalgebra $\mathbf{S}$ of $\mathbf{A}$, and an element $x \in A$, we shall denote by $< \mathbf{S}, x >_{\mathcal{K}}$ the subalgebra of $\mathbf{A}$ generated by the set $S \cup \{x\}$ in $\mathcal{K}$.

Lemma 2.10 *Let $\mathbf{C}$ be in $\mathcal{M}_n$ and $x \in C$. Let $\mathbf{S}$ be a subalgebra of $\mathbf{C}$ such that $\sigma_i^n(x)$ belongs to $B(\mathbf{S})$ for each $i = 1, \ldots, n-1$. Then*

$$\mathbf{B}(< \mathbf{S}, x >_{\mathcal{M}_n}) = \mathbf{B}(\mathbf{S}).$$

Proof: Clearly $\mathbf{B}(\mathbf{S})$ is a subalgebra of $\mathbf{B}(< \mathbf{S}, x >_{\mathcal{M}_n})$, then it is left to check that $B(< \mathbf{S}, x >_{\mathcal{M}_n}) \subseteq B(\mathbf{S})$. To achieve such aim, we shall study the form of the elements in $< \mathbf{S}, x >_{\mathcal{M}_n}$. We define for each $s \in S$

- $\alpha(s) = s \wedge x$,
- $\beta(s) = s \wedge \neg x$,
- $\gamma_i(s) = s \wedge \sigma_i^n(x)$, for $i = 1, \ldots n-1$,
- $\delta_i(s) = s \wedge \neg\sigma_i^n(x)$, for $i = 1, \ldots n-1$.

Notice that for all $s \in S$ we have that $\gamma_i(s)$ and $\delta_i(s)$ are in S for $i = 1, \ldots, n-1$. Let

$$M := \{y = \bigvee_{j=1}^{k_y} \bigwedge_{i=1}^{p_j} f_i(s_i) : f_i \in \{\alpha, \beta, \gamma_1, \delta_1, \ldots \gamma_{n-1}, \delta_{n-1}\} \text{ and } s_i \in S\}.$$

We shall see that $< \mathbf{S}, x >_{\mathcal{M}_n} = \mathbf{M} = (M, \wedge, \vee, \neg, \sigma_1^n, \ldots, \sigma_{n-1}^n, 0, 1)$. Indeed, for all $s \in S$, $s = \gamma_1(s) \vee \delta_1(s)$, then $S \subseteq M$. Besides, $x \in M$ because $x = \alpha(1)$. Lastly, it is easy to see that M is closed under the operations of n-valued Moisil algebra, thus $< \mathbf{S}, x >_{\mathcal{M}_n}$ is a subalgebra of $\mathbf{M}$. From the definition of M, it is obvious that $M \subseteq < \mathbf{S}, x >_{\mathcal{M}_n}$, and the equality follows.

Now let $z \in B(< \mathbf{S}, x >_{\mathcal{M}_n})$. By Theorem 2.6, $\sigma_{n-1}^n(z) = z$ and $z = \bigvee_{j=1}^{k_z} \bigwedge_{i=1}^{p_j} f_i(s_i)$ with $f_i \in \{\alpha, \beta, \gamma_1, \delta_1, \ldots \gamma_{n-1}, \delta_{n-1}\}$ and $s_i \in S$. Then we have:

$$z = \sigma_{n-1}^n(z) = \sigma_{n-1}^n(\bigvee_{j=1}^{k_z} \bigwedge_{i=1}^{p_j} f_i(s_i)) = \bigvee_{j=1}^{k_z} \bigwedge_{i=1}^{p_j} \sigma_{n-1}^n(f_i(s_i)),$$

is in $B(\mathbf{S})$ because $\sigma_{n-1}^n(f_i(s_i)) = \gamma_k(\sigma_{n-1}^n(s_i))$ or $\sigma_{n-1}^n(f_i(s_i)) = \delta_k(\sigma_{n-1}^n(s_i))$, for some $k = 1, \ldots, n-1$. ■

Theorem 2.11 *Let $\mathbf{C}$ be an MV_n-algebra and $x \in \mathbf{C}$. Let $\mathbf{S}$ be a subalgebra of $\mathbf{C}$ such that $\sigma_i^n(x)$ belongs to $B(\mathbf{S})$ for each $i = 1, \ldots, n-1$. Then*

$$\mathbf{B}(< \mathbf{S}, x >_{\mathcal{MV}_n}) = \mathbf{B}(\mathbf{S}).$$

Proof: By Lemmas 2.9 and 2.10 we obtain:

$$\mathbf{B}(<\mathbf{S}, x>_{\mathcal{MV}_n}) = \mathbf{B}(<\mathbf{S}, x>_{\mathcal{M}_n}) = \mathbf{B}(\mathbf{S}).$$

■

2.2 Boolean elements in $\mathbf{Free}_{\mathcal{MV}_n}(Z)$

Recall that a boolean algebra $\mathbf{B}$ is said to be **free over a poset** Y if for each boolean algebra $\mathbf{C}$ and for each non-decreasing function $f : Y \to \mathbf{C}$, f can be uniquely extended to a homomorphism from $\mathbf{B}$ into $\mathbf{C}$.

Theorem 2.12 $\mathbf{B}(\mathbf{Free}_{\mathcal{MV}_n}(Z))$ *is the free boolean algebra over the poset* $Z' := \{\sigma_i^n(z) : z \in Z, i = 1, \ldots, n-1\}$.

Proof: Let $\mathbf{S}$ be the subalgebra of $\mathbf{B}(\mathbf{Free}_{\mathcal{MV}_n}(Z))$ generated by Z'. Let $\mathbf{C}$ be a boolean algebra and let $f : Z' \to \mathbf{C}$ be a non-decreasing function. The monotonicity of f implies that the prescription

$$f'(z) = (f(\sigma_1^n(z)), \ldots, f(\sigma_{n-1}^n(z)))$$

defines a function $f' : Z \to \mathbf{C}^{[n]}$. Since $\mathbf{C}^{[n]} \in \mathcal{MV}_n$, there is a unique homomorphism $h' : \mathbf{Free}_{\mathcal{MV}_n}(Z) \to \mathbf{C}^{[n]}$ such that $h'(z) = f'(z)$ for every $z \in Z$. Let $\pi : \mathbf{C}^{[n]} \to \mathbf{C}$ be the projection over the first coordinate. The composition $\pi o h'$ restricted to $\mathbf{S}$ is a homomorphism $h : \mathbf{S} \to \mathbf{C}$, and for $y = \sigma_j^n(z) \in Z'$ we have:

$$h(y) = \pi(h'(\sigma_j^n(z))) = \pi(\sigma_j^n(h'(z))) = \pi(\sigma_j^n(f'(z))) =$$

$$\pi(\sigma_j^n(f(\sigma_1^n(z)), \ldots, f(\sigma_{n-1}^n(z)))) = \pi(f(\sigma_j^n(z)), \ldots, f(\sigma_j^n(z))) = f(\sigma_j^n(z)) = f(y).$$

Hence $\mathbf{S}$ is the free boolean algebra over the poset Z'. But since $\sigma_j^n(z)$ is in $\mathbf{S}$ for all $z \in Z$ and $j = 1, \ldots n-1$, Theorem 2.11 asserts that

$$\mathbf{S} = \mathbf{B}(\mathbf{S}) = \mathbf{B}(<\mathbf{S}, z>_{\mathcal{MV}_n})$$

for every $z \in Z$. From the fact that $\mathbf{S}$ is a subalgebra of $\mathbf{B}(\mathbf{Free}_{\mathcal{MV}_n}(Z))$ we obtain:

$$\mathbf{S} = \mathbf{B}(<\mathbf{S}, Z>_{\mathcal{MV}_n}) = \mathbf{B}(\mathbf{Free}_{\mathcal{MV}_n}(Z))$$

that is, $\mathbf{B}(\mathbf{Free}_{\mathcal{MV}_n}(Z))$ is the free boolean algebra generated by the poset Z'.■

From Theorem 2.4 we obtain:

Corollary 2.13 $\mathbf{B}(\mathbf{Free}_{\mathcal{V}}(X))$ *is the free boolean algebra generated by the poset* $Y := \{\sigma_i^n(\neg\neg x) : x \in X, i = 1, \ldots, n-1\}$.

Remark 2.14 Notice that if $n = 2$, i.e, the variety considered $\mathcal{V}$ is generated by a BL_2-chain, then $\sigma_1^2(x) = x$ for each $x \in X$. Therefore, in this case, $Y = \{\neg\neg x : x \in X\}$, and the cardinality of Y equals the cardinality of X. It follows that $\mathbf{B}(\mathbf{Free}_{\mathcal{V}}(X))$ is the free boolean algebra over the set Y.

3 $\mathbf{Free}_{\mathcal{V}}(X)/\langle U\rangle$

Following the program established at the end of section 2, our next aim is to describe $\mathbf{Free}_{\mathcal{V}}(X)/\langle U\rangle$ for each ultrafilter U in the free boolean algebra generated by $Y = \{\sigma_i^n(\neg\neg x) : x \in X, i = 1, \ldots, n-1\}$, where $\langle U\rangle$ is the BL-filter generated by the boolean filter U.

The plan is to prove that $\mathbf{MV}(\mathbf{Free}_{\mathcal{V}}(X)/\langle U\rangle)$ is a subalgebra of $\mathbf{L}_n$ and then, using Theorem 1.5, decompose each quotient $\mathbf{Free}_{\mathcal{V}}(X)/\langle U\rangle$ into an ordinal sum. To accomplish this we need the following results:

Theorem 3.1 *Let $\mathbf{A}$ be a BL-algebra and $U \in Sp\ \mathbf{B}(\mathbf{A})$. Then*

$$\mathbf{MV}(\mathbf{A}/\langle U\rangle) \cong \mathbf{MV}(\mathbf{A})/(\langle U\rangle \cap \mathbf{MV}(\mathbf{A})).$$

Proof: Let $V =: \langle U\rangle \cap \mathbf{MV}(\mathbf{A})$ and let $f : \mathbf{MV}(\mathbf{A})/V \to \mathbf{MV}(\mathbf{A}/\langle U\rangle)$ be given by

$$f(a/V) = a/\langle U\rangle,$$

for each $a \in MV(\mathbf{A})$. It is easy to see that f is a homomorphism into $\mathbf{MV}(\mathbf{A}/\langle U\rangle)$. Besides, we have that:

1. *f is injective*

   Let $a/\langle U\rangle = b/\langle U\rangle$, with $a, b \in MV(\mathbf{A})$. From Lemma 1.3 we know that there exists $u \in U$ such that $a \wedge u = b \wedge u$. Since $U \subseteq MV(\mathbf{A})$, then $u \in V$. From the fact that u is boolean (see [16, Lemma 2.2]), we have that: $a * u = a \wedge u = b \wedge u \leq b$, thus $u \leq a \to b$ and similarly $u \leq b \to a$. Then $a \to b$ and $b \to a$ are in V and this means that $a/V = b/V$.

2. *f is surjective*

   Let $a/\langle U\rangle \in MV(\mathbf{A}/\langle U\rangle)$. Then

   $$a/\langle U\rangle = \neg\neg(a/\langle U\rangle) = \neg\neg a/\langle U\rangle,$$

   and since $\neg\neg a \in MV(\mathbf{A})$ we obtain that $f(\neg\neg a/V) = a/\langle U\rangle$.

■

By Theorem 2.4, if $U \in Sp\ \mathbf{B}(\mathbf{Free}_{\mathcal{V}}(X))$, then U is an ultrafilter in $\mathbf{B}(\mathbf{Free}_{\mathcal{MV}_n}(\neg\neg X))$. Moreover,

$$\langle U\rangle \cap \mathbf{MV}(\mathbf{Free}_{\mathcal{V}}(X)) = \langle U\rangle \cap \mathbf{Free}_{\mathcal{MV}_n}(\neg\neg X)$$

is the Stone ultrafilter of $\mathbf{Free}_{\mathcal{MV}_n}(\neg\neg X)$ generated by U. From [18, Chapter 6.3], we have that $\langle U\rangle \cap \mathbf{Free}_{\mathcal{MV}_n}(\neg\neg X)$ is a maximal filter of $\mathbf{Free}_{\mathcal{MV}_n}(\neg\neg X)$. It follows from [18, Corollary 3.5.4] that the MV-algebra

$$\mathbf{MV}(\mathbf{Free}_{\mathcal{V}}(X))/(\langle U\rangle \cap \mathbf{MV}(\mathbf{Free}_{\mathcal{V}}(X)))$$

is an MV-chain in $\mathcal{MV}_n$, thus from Theorem 3.1 we have:

Theorem 3.2

$$\mathbf{MV}(\mathbf{Free}_{\mathcal{V}}(X)/\langle U\rangle) \cong \mathbf{L}_s$$

with $s-1$ dividing $n-1$.

From Theorems 1.5 and 3.2 we obtain:

Theorem 3.3 *For each $U \in Sp\ \mathbf{B}(\mathbf{Free}_{\mathcal{V}}(X))$ we have that*

$$\mathbf{Free}_{\mathcal{V}}(X)/\langle U\rangle \cong \mathbf{L}_s \uplus \mathbf{D}(\mathbf{Free}_{\mathcal{V}}(X)/\langle U\rangle)$$

for some $s-1$ dividing $n-1$.

In order to obtain a complete description of $\mathbf{Free}_{\mathcal{V}}(X)$ there is only left to find a description of $\mathbf{D}(\mathbf{Free}_{\mathcal{V}}(X)/\langle U\rangle)$ for each $U \in Sp\ \mathbf{B}(\mathbf{Free}_{\mathcal{V}}(X))$. This last description will depend on the characterization of the variety $\mathcal{W}$ of generalized BL-algebras generated by the generalized BL-chain $\mathbf{B}$. Therefore we shall firstly consider such variety.

3.1 The subvariety of $\mathcal{GBL}$ generated by $\mathbf{B}$.

We recall that $\mathcal{V}$ is the variety of BL-algebras generated by the BL-chain $\mathbf{T}_n = \mathbf{L}_n \uplus \mathbf{B}$. Let $\mathcal{W}$ be the variety of generalized BL-algebras generated by the chain $\mathbf{B}$.

Let $\{e_i, i \in I\}$ be the set of equations that define $\mathcal{MV}_n$ as a subvariety of $\mathcal{BL}$, and $\{d_j, j \in J\}$ be the set of equations that define $\mathcal{W}$ as a subvariety of $\mathcal{GBL}$. For each $i \in I$, let e'_i be the equation that results from substituting $\neg\neg x$ for each variable x in e_i, and for each $j \in J$, let d'_j the equation that results from substituting $\neg\neg x \to x$ for each variable x in the equation d_j. Let $\mathcal{V}'$ the variety of BL-algebras characterized by the equations of BL-algebras plus the equations $\{e'_i, i \in I\} \cup \{d'_j, j \in J\}$.

Theorem 3.4 $\mathcal{V}' \subseteq \mathcal{V}$.

Proof: Let $\mathbf{A}$ be a subdirectly irreducible BL-algebra in $\mathcal{V}'$. From Theorem 1.1, $\mathbf{A}$ is a BL-chain, and by Theorem 1.4, $\mathbf{A} = \mathbf{MV}(\mathbf{A}) \uplus \mathbf{D}(\mathbf{A})$. Since for each $x \in MV(\mathbf{A})$, $\neg\neg x = x$, $\mathbf{MV}(\mathbf{A})$ satisfies equations $\{e_i, i \in I\}$. Then $\mathbf{MV}(\mathbf{A})$ is a chain in $\mathcal{MV}_n$, i. e., $\mathbf{MV}(\mathbf{A}) \cong \mathbf{L}_s$, with $s-1$ dividing $n-1$. Moreover, since for each $x \in D(\mathbf{A})$, $\neg\neg x \to x = x$, $\mathbf{D}(\mathbf{A})$ satisfies equations $\{d_j, j \in J\}$. Hence $\mathbf{D}(\mathbf{A}) = \mathbf{C}$ is a generalized BL-chain in $\mathcal{W}$. Since $\mathbf{A}$ is subdirectly irreducible, $\mathbf{C}$ is also subdirectly irreducible, and since $\mathcal{GBL}$ is a congruence distributive variety, we can apply Jónsson's Lemma (see [9]) to conclude that $\mathbf{C} \in \mathbf{HSP}_u(\mathbf{B})$. Hence there is a set $J \neq \emptyset$ and an ultrafilter U over J such that $\mathbf{C}$ is a homomorphic image of a subalgebra of $\mathbf{B}^J/U$. From the proof of [1, Proposition 3.3] it follows that $(\mathbf{L}_n \uplus \mathbf{B})^J/U = \mathbf{L}_n^J/U \uplus \mathbf{B}^J/U$, and since $\mathbf{L}_n$ is finite, $\mathbf{L}_n^J/U \cong \mathbf{L}_n$. Now it is easy to see that $\mathbf{A} = \mathbf{L}_s \uplus \mathbf{C} \in \mathbf{HSP}_u(\mathbf{L}_n \uplus \mathbf{B}) \subseteq \mathcal{V}$. ■

The next corollary states the main result of this section.

Corollary 3.5 *The variety $\mathcal{W}$ of generalized BL-algebras generated by $\mathbf{B}$ consist on the generalized BL-algebras $\mathbf{C}$ such that $\mathbf{L}_n \uplus \mathbf{C}$ belongs to $\mathcal{V}$.*

Proof: Given $\mathbf{C} \in \mathcal{W}$, $\mathbf{L}_n \uplus \mathbf{C} \in \mathcal{V}' \subseteq \mathcal{V}$. On the other hand, if $\mathbf{C}$ is a generalized BL-algebra such that $\mathbf{L}_n \uplus \mathbf{C} \in \mathcal{V}$, then the elements of C satisfy equations d'_j for each $j \in J$ and since $\neg\neg x \to x = x$ for each $x \in C$, the elements of C satisfy equations d_j for each $j \in J$. Hence $\mathbf{C}$ is in $\mathcal{W}$. ■

3.2 $\mathbf{D}(\mathbf{Free}_{\mathcal{V}}(X)/\langle U\rangle)$

We know that the ultrafilters of a boolean algebra are in bijective correspondence with the homomorphisms from the algebra into the two elements boolean algebra, $\mathbf{2}$. Since every upwards closed subset of the poset $Y = \{\sigma_i^n(\neg\neg x) : x \in X, i = 1, \dots, n-1\}$ is in correspondence with an increasing function from Y onto $\mathbf{2}$, and every increasing function from Y can be extended to a homomorphism from $\mathbf{B}(\mathbf{Free}_{\mathcal{V}}(X))$ into $\mathbf{2}$, the ultrafilters of $\mathbf{B}(\mathbf{Free}_{\mathcal{V}}(X))$ are in correspondence with the upwards closed subsets of Y. This is summarized in the following lemma:

Lemma 3.6 *Consider the poset $Y = \{\sigma_i^n(\neg\neg x) : x \in X, i = 1, \dots, n-1\}$. The correspondence that assigns to each upwards closed subset $S \subseteq Y$ the boolean filter U_S generated by the set $S \cup \{\neg y : y \in Y \setminus S\}$, defines a bijection from the set of upwards closed subsets of Y onto the ultrafilters of $\mathbf{B}(\mathbf{Free}_{\mathcal{V}}(X))$.*

We shall refer to each member of $Sp\ \mathbf{B}(\mathbf{Free}_{\mathcal{V}}(X))$ by U_S making explicit reference to the upwards closed subset S that correspond to it.

Lemma 3.7 *Let $\mathbf{F}_S$ be the subalgebra of the generalized BL-algebra $\mathbf{D}(\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S\rangle)$ generated by the set $X_S := \{x/\langle U_S\rangle : x \in X, \neg\neg x \in \langle U_S\rangle\}$. Then*

$$\mathbf{F}_S = \mathbf{D}(\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S\rangle).$$

Proof: $\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S\rangle$ is the BL-algebra generated by the set $Z_S = \{x/\langle U_S\rangle : x \in X\}$. From Theorem 3.3 there exists an integer m such that

$$\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S\rangle = \mathbf{L}_m \uplus \mathbf{D}(\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S\rangle).$$

Hence each element of Z_S is either in $L_m \setminus \{\top\}$ or it is in $D(\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S\rangle)$.

If $X_S = \emptyset$, then $F_S = D(\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S\rangle) = \{\top\}$. So let suppose $X_S \neq \emptyset$. Let $y \in D(\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S\rangle)$. Recalling that $\mathbf{F}_S$ is the generalized BL-algebra generated by X_S, we will check that y is in F_S. Since $y \in Free_{\mathcal{V}}(X)/\langle U_S\rangle$, y is given by a term on the elements $x/\langle U_S\rangle \in Z_S$. Making induction on the complexity of y we have:

- If y is a generator, i.e, $y = x/\langle U_S\rangle$ for some $x/\langle U_S\rangle \in Z_S$, since $y \in D(\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S\rangle)$, we have that $\top = \neg\neg y = \neg\neg(x/\langle U_S\rangle) = (\neg\neg x)/\langle U_S\rangle$. This happens only if $\neg\neg x \in X_S$.
- Suppose that for each element $z \in D(\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S\rangle)$ of complexity less than k, z can be written as a term in the variables $x/\langle U_S\rangle$ in X_S. Let $y \in D(\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S\rangle)$ be an element of complexity k. The possible cases are the following:
  1. $y = a \to b$ for some elements a, b of complexity $< k$. In this case the possibilities are:
     (a) $a \leq b$ in which case $a \to b = \top$ and y can be written as $x/\langle U_S\rangle \to x/\langle U_S\rangle$ for any $x/\langle U_S\rangle \in X_S$, thus $y \in F_S$,
     (b) $a > b$. Since $y = a \to b$ is in $D(\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S\rangle)$, the only possibility is that $a, b \in D(\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S\rangle)$ and by inductive hypothesis y is in F_S.

2. $y = a * b$ for some elements a, b of complexity $< k$. In this case necessarily $a, b \in D(\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S\rangle)$ and by inductive hypothesis y is in F_S.

Then for each $y \in D(\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S\rangle)$, y can be written as a term on the elements of X_S, therefore $y \in F_S$ and we conclude that

$$\mathbf{F}_S = \mathbf{D}(\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S\rangle).$$

■

With the notation of the previous lemma, we have:

Theorem 3.8 *For each U_S in Sp $\mathbf{B}(\mathbf{Free}_{\mathcal{V}}(X))$,*

$$\mathbf{D}(\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S\rangle) \cong \mathbf{Free}_{\mathcal{W}}(X_S).$$

Proof: From Theorem 2.6 and Lemma 3.6 we can deduced that $\neg\neg x \in \langle U_S\rangle$ iff $\sigma_1^n(\neg\neg x) \in S$ iff $\sigma_i^n(\neg\neg x) \in S$ for $i = 1, \dots, n-1$. Hence if $\neg\neg x \notin \langle U_S\rangle$ there is a j such that $\sigma_j^n(\neg\neg x) \notin S$. We define, for each $x \in X$,

$$j_x = \begin{cases} \bot & \text{if } \neg\neg x \in \langle U_S\rangle, \\ \max\{i \in \{1, \dots, n-1\} : \sigma_i^n(\neg\neg x) \notin S\} & \text{otherwise.} \end{cases}$$

Let $\mathbf{C} \in \mathcal{W}$ and let $\mathbf{C}' = \mathbf{L}_n \uplus \mathbf{C}$. From Theorem 3.5, $\mathbf{C}'$ is in $\mathcal{V}$. Given a function $f : X_S \to \mathbf{C}$, define $\hat{f} : X \to \mathbf{C}'$ by the prescriptions:

$$\hat{f}(x) = \begin{cases} f(x/\langle U_S\rangle) & \text{if } \neg\neg x \in \langle U_S\rangle, \\ \frac{n-j_x-1}{n-1} & \text{otherwise.} \end{cases}$$

There is a unique homomorphism $\hat{h} : \mathbf{Free}_{\mathcal{V}}(X) \to \mathbf{C}'$ such that $\hat{h}(x) = \hat{f}(x)$ for each $x \in X$. We have that $U_S \subseteq \hat{h}^{-1}(\{\top\})$. Indeed, if $\neg\neg x \in \langle U_S\rangle$, then $\hat{h}(\sigma_i^n(\neg\neg x)) = \sigma_i^n(\neg\neg(\hat{h}(x)) = \sigma_i^n(\neg\neg f(x/\langle U_S\rangle)) = \sigma_i^n(\top) = \top$. If $\neg\neg x \notin \langle U_S\rangle$, then

$$\hat{h}(\sigma_i^n(\neg\neg x)) = \sigma_i^n(\neg\neg\frac{n-j_x-1}{n-1}) = \sigma_i^n(\frac{n-j_x-1}{n-1}) = \begin{cases} \bot & \text{if } i \leq j_x, \\ \top & \text{otherwise.} \end{cases}$$

Hence there is a unique homomorphism $h_1 : \mathbf{Free}_{\mathcal{V}}(X)/\langle U_S\rangle \to \mathbf{C}'$ such that $h_1(a/\langle U_S\rangle) = \hat{h}(a)$ for all $a \in \mathbf{Free}_{\mathcal{V}}(X)$. By Lemma 3.7, $\mathbf{D}(\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S\rangle)$ is the algebra generated by X_S. Then the restriction h of h_1 to $\mathbf{D}(\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S\rangle)$ is a homomorphism $h : \mathbf{D}(\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S\rangle) \to \mathbf{C}$, and for each x such that $\neg\neg x \in \langle U_S\rangle$,

$$h(x/\langle U_S\rangle) = h_1(x/\langle U_S\rangle) = \hat{h}(x) = \hat{f}(x) = f(x/\langle U_S\rangle).$$

Therefore we conclude that $\mathbf{D}(\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S\rangle) \cong \mathbf{Free}_{\mathcal{W}}(X_S)$. ■

Theorem 3.9 *The free BL-algebra $\mathbf{Free}_{\mathcal{V}}(X)$ can be represented as a weak boolean product of the family*

$$(\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S\rangle) : U_S \in Sp\ \mathbf{B}(\mathbf{Free}_{\mathcal{V}}(X))$$

where $\mathbf{B}(\mathbf{Free}_{\mathcal{V}}(X))$ is the free boolean algebra over the poset $Y = \{\sigma_i^n(\neg\neg x) : x \in X, i = 1, \dots, n-1\}$. Moreover, for each $U_S \in Sp\ \mathbf{B}(\mathbf{Free}_{\mathcal{V}}(X))$ there exists $m \geq 2$ such that $m-1$ divides $n-1$ and

$$\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S\rangle = \mathbf{L}_m \uplus \mathbf{Free}_{\mathcal{W}}(X_S)$$

where $X_S := \{x/\langle U_S\rangle : \neg\neg x \in \langle U_S\rangle\}$ and $\mathcal{W}$ is the variety of generalized BL-algebras generated by $\mathbf{B}$.

4 Examples

4.1 PL-algebras

Let $\mathbf{G}$ be a lattice-ordered abelian group (ℓ-group), and $G^- = \{x \in G : x \leq 0\}$ its negative cone. For each pair of elements $x, y \in G^-$ we define the following operators:

$$x * y = x + y \quad \text{and} \quad x \to y = 0 \wedge (y - x).$$

Then $\mathbf{G}^- = (G^-, \wedge, \vee, *, \to, 0)$ is a generalized BL-algebra. The following result can be deduced from [3] (see also [6] and [14]).

Theorem 4.1 *The following conditions are equivalent for a generalized BL-algebra* $\mathbf{A}$:

1. $\mathbf{A}$ *is a cancellative hoop,*
2. *there is an* ℓ*-group* $\mathbf{G}$ *such that* $\mathbf{A} \cong \mathbf{G}^-$,
3. $\mathbf{A}$ *is in the variety of generalized BL-algebras generated by* $\mathbf{Z}^-$, *where* $\mathbf{Z}$ *denotes the additive group of integers with the usual order.*

So let consider $\mathcal{W}$, the variety of generalized BL-algebras generated by $\mathbf{Z}^-$, that is, the variety of cancellative hoops. In [15] a description of $\mathbf{Free}_{\mathcal{W}}(X)$ is given for any set X of free generators. Therefore we can have a complete description of free algebras in varieties of BL-algebras generated by the ordinal sum

$$\mathbf{PL}_n = \mathbf{L}_n \uplus \mathbf{Z}^-.$$

Indeed, if we denote by $\mathcal{PL}_n$ the variety of BL-algebras generated by $\mathbf{PL}_n$, from Theorem 3.9 we obtain that $\mathbf{Free}_{\mathcal{PL}_n}(X)$ is a weak boolean product of algebras of the form

$$\mathbf{L}_s \uplus \mathbf{Free}_{\mathcal{W}}(X')$$

with $s-1$ dividing $n-1$ and some set X' of cardinality less or equal than X. Therefore, in the present case, the BL-algebra $\mathbf{Free}_{\mathcal{PL}_n}(X)$ can be completely described as a weak boolean product of ordinal sums of two known algebras.

From [14, Theorem 2.8] $\mathcal{PL}_2$ is the variety of PL-algebras $\mathcal{PL}$. From Remark 2.14, *Sp* $\mathbf{B}(\mathbf{Free}_{\mathcal{PL}}(X))$ is the Cantor space $\mathbf{2}^{|X|}$. From Theorem 3.9, the free PL-algebra over a set X can be describe as a weak boolean product over the Cantor space $\mathbf{2}^{|X|}$ of algebras of the form

$$\mathbf{L_2} \uplus \mathbf{Free}_{\mathcal{W}}(X')$$

for some set X' of cardinality less or equal than X.

Given a BL-algebra $\mathbf{A}$, the radical $R(\mathbf{A})$ of $\mathbf{A}$ is the intersection of all maximal implicative filters of $\mathbf{A}$. We have that $\mathbf{r}(\mathbf{A}) = (R(\mathbf{A}), *, \to, \wedge, \vee, \top)$ is a generalized BL-algebra. Let

$$\mathcal{PL}^r = \{\mathbf{R} : \mathbf{R} = \mathbf{r}(\mathbf{A}) \text{ for some } \mathbf{A} \in \mathcal{PL}\}.$$

$\mathcal{PL}^r$ is a variety of generalized BL-algebras. In [16] a description of $\mathbf{Free}_{\mathcal{PL}}(X)$ is given. From Example 4.7 and Theorem 5.7 in the mentioned paper we obtained that $\mathbf{Free}_{\mathcal{PL}}(X)$ is the weak boolean product of the family $(\mathbf{L}_2 \uplus$

$\mathbf{Free}_{\mathcal{PL}^r}(S) : S \subseteq \mathbf{2}^{|X|})$ over the Cantor space $\mathbf{2}^{|X|}$. In order to check that our description and the one given in [16] coincide there is only left to check that $\mathcal{PL}^r = \mathcal{W}$. From Corollary 3.5 we have that $\mathcal{W}$ consist on the generalized BL-algebras $\mathbf{C}$ such that $\mathbf{L}_2 \uplus \mathbf{C} \in \mathcal{PL}$.

Theorem 4.2 $\mathcal{PL}^r = \mathcal{W}$.

Proof: Let $\mathbf{C} \in \mathcal{PL}^r$. Then there exists a BL-algebra $\mathbf{A} \in \mathcal{PL}$ such that $\mathbf{r}(\mathbf{A}) = \mathbf{C}$. It is not hard to check that $\mathbf{L}_2 \uplus \mathbf{C}$ is a subalgebra of $\mathbf{A}$, thus $\mathbf{L}_2 \uplus \mathbf{C}$ is in $\mathcal{PL}$. It follows that $\mathbf{C} \in \mathcal{W}$. On the other hand, let $\mathbf{C} \in \mathcal{W}$. Then $\mathbf{L}_2 \uplus \mathbf{C}$ is in $\mathcal{PL}$, and $\mathbf{C} \in \mathcal{PL}^r$. ∎

4.2 Finitely generated free algebras

As we mentioned in the introduction, when the set of generators X is finite, let say of cardinality k, the algebra $\mathbf{Free}_{\mathcal{V}}(X)$ is described in [10] as a direct product of algebras of the form $\mathbf{L}_s \uplus \mathbf{Free}_{\mathcal{W}}(X')$, with $s-1$ that divides $n-1$ and some set X' of cardinality less or equal than the cardinality of X, where $\mathcal{W}$ is again the subvariety of $\mathcal{GBL}$ generated by $\mathbf{B}$. The method used to describe the algebras strongly relies on the fact that the boolean elements of $\mathbf{Free}_{\mathcal{V}}(X)$ form a finite boolean algebra. Indeed, $\mathbf{Free}_{\mathcal{V}}(X)$ is a direct product of n^k algebras obtained by taking the quotients by the implicative filters generated by the atoms of $\mathbf{B}(\mathbf{Free}_{\mathcal{V}}(X))$. In this case, once you know the form of the atom that generates the ultrafilter U you also know the number s such that $\mathbf{MV}((\mathbf{Free}_{\mathcal{V}}(X))/U) = \mathbf{L}_s$.

Notice that when the set X of generators is finite, of cardinality k, then $Y = \{\sigma_i^n(\neg\neg x) : x \in X, i = 1, \ldots, n-1\}$ is the cardinal sum of k chains of length $n-1$. Therefore the number of upwards closed subsets of Y is n^k. Since weak boolean products over discrete finite spaces coincide with direct products, Theorem 3.9 asserts that $\mathbf{Free}_{\mathcal{V}}(X)$ is a direct product of n^k BL-algebras of the form $\mathbf{L}_s \uplus \mathbf{Free}_{\mathcal{W}}(Y)$, with $s-1$ that divides $n-1$ and some set Y of cardinality less or equal than k.

Therefore the description given in the present paper coincides with the one in [10]. But in any case the description given in [10], based on a detailed analysis of the structure of the atoms of $\mathbf{B}(\mathbf{Free}_{\mathcal{V}}(X))$ for a finite X, is more precise because it gives the number of factors of each kind appearing in the direct product representation.

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