Heat-diffusion maximal operators for Laguerre semigroups with negative parameters *

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Dedicated to the memory of Miguel de Guzmán

Abstract

We prove L^p boundedness for the maximal operator of the heat semigroup associated to the Laguerre functions, $\{\mathcal{L}_k^\alpha\}_k$, when the parameter α is greater than -1. Namely, the maximal operator is of strong type (p,p) if $p>1$ and $\frac{2}{2+\alpha}<p<\frac{2}{-\alpha}$, when $-1<\alpha<0$. If $\alpha\geq 0$ there is strong type for $1<p\leq\infty$. The behavior at the end points is studied in detail.

1 Introduction

The Laguerre polynomials $L_k^\alpha(y)$ are given by

$$e^{-y}y^\alpha L_k^\alpha(y)=\frac{1}{k!}\frac{d}{dy^k}\left(e^{-y}y^{k+\alpha}\right),$$

where y is positive. We assume that $\alpha>-1$. The Laguerre polynomials $\{L_k^\alpha(y)\}_{k=0}^\infty$ form a orthogonal system with respect to the measure $e^{-y}y^\alpha dy$. More precisely,

$$\int_0^\infty L_k^\alpha(y)\,L_j^\alpha(y)\,e^{-y}y^\alpha dy=\frac{\Gamma(k+\alpha+1)}{\Gamma(k+1)}\delta_{kj},$$

thus, the Laguerre functions $\mathcal{L}_k^\alpha(y)$ defined by

$$\mathcal{L}_k^\alpha(y)=\left(\frac{\Gamma(k+1)}{\Gamma(k+\alpha+1)}\right)^{1/2}e^{-y/2}y^{\alpha/2}L_k^\alpha(y),$$

for a given α, form an orthonormal set with respect to the Lebesgue measure. Standard references for Laguerre functions and polynomials are [1], [8] and [9].

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We define the heat diffusion kernel $W^{\alpha}(t,y,z)$ for $\alpha>-1$, $t>0$, $y>0$ and $z>0$ as

$$W^{\alpha}(t,y,z)=\sum_{n=0}^{\infty}e^{-t(n+(\alpha+1)/2)}\mathcal{L}_n^{\alpha}(y)\,\mathcal{L}_n^{\alpha}(z)\,, \tag{1}$$

and the heat diffusion integral $W^{\alpha}f(t,y)$ as

$$W^{\alpha}f(t,y)=\int_0^{\infty}W^{\alpha}(t,y,z)\,f(z)\,dz.$$

The heat diffusion integral $W_{\alpha}f(t,y)$ satisfies the semigroup property

$$W^{\alpha}f(t+s,y)=\int_0^{\infty}W^{\alpha}(t,y,z)\,W^{\alpha}f(s,z)\,dz.$$

The maximal operator $W^{\alpha,*}f(t)$ associated to the heat diffusion integral $W_{\alpha}f(t,y)$ is given by $W^{\alpha,*}f(y)=sup_{t>0}\left|W^{\alpha}f(t,y)\right|$.

We define the fractional maximal function $M_{\theta}f(x)$ for $0\le\theta<1$ as

$$M_{\theta}f(x)=\sup_{h>0}\frac{1}{(2h)^{1-\theta}}\int_{|y|\le h}|f(x-y)|\,dy.$$

If $\theta=0$, $M_0f(x)$ is the Hardy-Littlewood maximal function. It is well known that if y^{δ} is a weight with $-1<\delta<p-1$, then M_0f is of strong type (p,p) for $p>1$ and of weak type $(1,1)$ if $p=1$ with the measure $y^{\delta}dy$. As references see [4], [6].

The purpose of this paper is to study the action of the maximal operator $W^{\alpha,*}f$ just defined on the spaces $L^p((0,\infty),dy)$. For $\alpha\ge0$ the positive results we give here were obtained by K. Stempak in [7] using techniques similar to those of B. Muckenhoupt, see [3]. The situation for $-1<\alpha<0$ is rather different since, for instance, the functions $\mathcal{L}_k^{\alpha}(y)$ do not belong to every $L^p((0,\infty),dy)$. This indicates that the general theory of semigroup, see [5], does not apply.

2 Statement of the results

Let N_{α} denote the interval

$$N_{\alpha}=\begin{cases}(2/(2+\alpha),2/(-\alpha)) & ,\ \text{if } -1<\alpha<0,\\ \text{and} & \\ (1,\infty] & ,\ \text{if } \alpha\ge0.\end{cases}$$

With this notation, we have the following result:

Theorem 1 (Strong type) *Let $-1<\alpha<\infty$. The maximal operator $W^{\alpha,*}f(y)$ satisfies*

$$\int_0^{\infty}W^{\alpha,*}f(y)^p\,dy\le C_{\alpha,p}\int_0^{\infty}|f(y)|^p\,dy,$$

with a constant $C_{\alpha,p}$ depending only on α, provided that $p\in N_{\alpha}$. That is to say

(2)
a) If $-1<\alpha\le0$, then $2/(2+\alpha)<p<2/(-\alpha)$, and
b) If $\alpha\ge0$, then $1<p\le\infty$.

The following theorem gives the behaviour of $W^{\alpha,*}f$ at the end points of N_α.

Theorem 2 (End points) *Let $-1<\alpha$. At the end points of N_α we have*

(a) *If $-1<\alpha<0$, the upper end point of N_α is equal to $2/(-\alpha)$. The operator $W^{\alpha,*}f$ is of weak type and not of strong type $(2/(-\alpha),2/(-\alpha))$.*

(b) *If $\alpha\geq 0$, the upper end point of N_α is equal to ∞. The operator $W^{\alpha,*}f$ is of strong type (∞,∞).*

(c) *If $-1<\alpha<0$, the lower end point of N_α is equal to $2/(2+\alpha)$. The operator $W^{\alpha,*}f$ is of restricted weak type and not of weak type $(2/(2+\alpha),2/(2+\alpha))$.*

(d) *If $\alpha\geq 0$, the lower end point of N_α is equal to 1. The operator $W^{\alpha,*}f$ is of weak type and not of strong type $(1,1)$.*

3 Lemmas

Throughout this paper we shall assume that $f(x)$ is a non-negative function. The constants will not have the same value in each occurrence.

Lemma 1 *Given β, $0\leq\beta<1$, there exists a constant C_β such that for every $y>0$*

$$\begin{aligned} y^{-\beta/2}M_\beta\left(f(z)\,z^{-\beta/2}\right)(y) \;\leq\; & C_\beta\left\{y^{\beta/2}\frac{1}{y}\int_0^y f(z)\,z^{-\beta/2}dz\right. \\ & \left.+\sup\nolimits_{h\geq y/2}\left(\frac{y^{-\beta/2}}{(2h)^{1-\beta}}\int_y^{y+h} f(z)\,z^{-\beta/2}dz\right)+M_0f(y)\right\}, \end{aligned} \tag{3}$$

and

$$\sup\nolimits_{h\geq y/2}\left(y^{-\beta/2}\frac{1}{(2h)^{1-\beta}}\int_y^{y+h} f(z)\,z^{-\beta/2}dz\right)\leq C_\beta y^{-\beta/2}M\left(f(z)\,z^{\beta/2}\right)(y). \tag{4}$$

Proof. If $y\leq 2h$, we have

$$\begin{aligned} \frac{1}{(2h)^{1-\beta}}\int_{y-h}^{y+h} f(z)\,z^{-\beta/2}dz \;\leq\; & \frac{1}{(2h)^{1-\beta}}\int_0^y f(z)\,z^{-\beta/2}dz+\frac{1}{(2h)^{1-\beta}}\int_y^{y+h} f(z)\,z^{-\beta/2}dz \\ \leq\; & \frac{1}{y^{1-\beta}}\int_0^y f(z)\,z^{-\beta/2}dz+\sup\nolimits_{h\geq y/2}\left(\frac{1}{(2h)^{1-\beta}}\int_y^{y+h} f(z)\,z^{-\beta/2}dz\right). \end{aligned}$$

On the other hand, if $y\geq 2h$, then $h\leq y-h$ and we get

$$\begin{aligned} \frac{1}{(2h)^{1-\beta}}\int_{y-h}^{y+h} f(z)\,z^{-\beta/2}dz \;\leq\; & \frac{2\cdot h^{1-\beta/2}}{(2h)^{1-\beta}}\frac{1}{2h}\int_{y-h}^{y+h} f(z)\,dz \\ \leq\; & 2^{\beta/2}(2h)^{\beta/2}M_0f(y)\leq 2^{\beta/2}y^{\beta/2}M_0f(y). \end{aligned}$$

Thus,

$$\frac{y^{-\beta/2}}{(2h)^{1-\beta}}\int_{y-h}^{y+h} f(z)\, z^{-\beta/2}dz \leq y^{\beta/2}\frac{1}{y}\int_0^y f(z)\, z^{-\beta/2}dz$$
$$+sup_{y\leq 2h}\left(\frac{y^{-\beta/2}}{(2h)^{1-\beta}}\int_y^{y+h} f(z)\, z^{-\beta/2}dz\right)+2^{\beta/2}M_0 f(y).$$

which proves (3).

Now, let $y \leq 2h$ and choose k_0 as the unique integer satisfying $2^{k_0}y < y+h \leq 2^{k_0+1}y$, then

$$\begin{aligned}
\frac{1}{(2h)^{1-\beta}}\int_y^{y+h} f(z)\, z^{-\beta/2}dz &\leq \frac{1}{(2h)^{1-\beta}}\sum_{k=0}^{k_0}\int_{2^k y}^{2^{k+1}y} f(z)\frac{z^{\beta/2}}{z^{\beta}}dz\\
&\leq \frac{y^{-\beta}}{(2h)^{1-\beta}}\sum_{k=0}^{k_0}2^{-\beta k}\int_{2^k y}^{2^{k+1}y} f(z)\, z^{\beta/2}dy\\
&\leq 2\frac{y^{1-\beta}}{(2h)^{1-\beta}}\sum_{k=0}^{k_0}2^{(1-\beta)k}\frac{1}{2^{k+1}y}\int_0^{2^{k+1}y} f(z)\, z^{\beta/2}dy\\
&\leq c_\beta 2^{(1-\beta)k_0}\frac{y^{1-\beta}}{(2h)^{1-\beta}}M_0\left(f(z)\, z^{\beta/2}\right)(y)\\
&\leq c_\beta\left(\frac{y+h}{y}\right)^{1-\beta}\frac{y^{1-\beta}}{(2h)^{1-\beta}}M_0\left(f(z)\, z^{\beta/2}\right)(y)\\
&\leq c_\beta\left(\frac{y+h}{2h}\right)^{1-\beta}M_0\left(f(z)\, z^{\beta/2}\right)(y) \leq c_\beta M_0\left(f(z)\, z^{\beta/2}\right)(y),
\end{aligned}$$

thus, we get

$$\sup_{h\geq y/2}\left(\frac{y^{-\beta/2}}{(2h)^{1-\beta}}\int_y^{y+h} f(z)\, z^{-\beta/2}dz\right) \leq c_\beta y^{-\beta/2}M_0\left(f(z)\, z^{\beta/2}\right)(y),$$

ending the proof of the lemma. ■

Lemma 2 *Let* $-1 < \alpha$. *We have the following estimates for the heat diffusion integral* $W^\alpha f(t,y)$:

(a) If $-1 < \alpha \leq 0$, *we denote* $\beta = -\alpha$. *Then*

(5) $$W^\alpha f(t,y) \leq C_\alpha\left\{e^{-t/4}M_0 f(y) + e^{-t(1-\beta)/4}y^{-\beta/2}M_\beta\left(z^{-\beta/2}f(z)\right)(y)\right\}.$$

(b) If $0 \leq \alpha$, *then,*

(6) $$W^\alpha f(t,y) \leq C_\alpha e^{-t/4}M_0 f(y).$$

Let us consider the generating function for the Laguerre polynomials

$$\sum_0^\infty \frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)} L_n^\alpha(y) L_n^\alpha(z) r^n = \frac{1}{1-r} e^{-r\frac{z+y}{1-r}} (ryz)^{-\alpha/2} I_\alpha\left(2\frac{(ryz)^{1/2}}{1-r}\right),$$

where $0 \le r < 1$, and $I_\alpha(x) = e^{-i\alpha\pi/2} J_\alpha(ix)$, the modified Bessel function, see [1], p.189 (20)], [8] and [9] Then, let $Q_\alpha(y,z,r)$ be the function

$$\begin{aligned} Q_\alpha(y,z,r) &= \sum\nolimits_0^\infty \frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)} e^{-y/2} y^{\alpha/2} L_n^\alpha(y) e^{-z/2} z^{\alpha/2} L_n^\alpha(z) r^{n+(\alpha+1)/2} \\ &= \sum\nolimits_0^\infty \mathcal{L}_n^\alpha(y) \mathcal{L}_n^\alpha(z) r^{n+(\alpha+1)/2} = \frac{r^{1/2}}{1-r} e^{-(z+y)/2} e^{-r(z+y)/(1-r)} I_\alpha\left(2\frac{(ryz)^{1/2}}{1-r}\right). \end{aligned}$$

Taking into account (1), this shows that $Q_\alpha(y,z,e^{-t}) = W^\alpha(t,y,z)$.

If $e^{-t} = \left(\frac{1-s}{1+s}\right)^2$, then $0 < s \le 1$ holds if and only if $0 < t \le \infty$. Thus, if we define

$$R_\alpha(y,z,s) = Q_\alpha\left(y, z, \left(\frac{1-s}{1+s}\right)^2\right),$$

we get that $R_\alpha\left(y, z, \frac{1-e^{-t/2}}{1+e^{-t/2}}\right) = W^\alpha(t,y,z)$. Moreover,

$$R_\alpha(y,z,s) = \frac{1}{2}\frac{1-s^2}{2s} e^{-\frac{1}{4}\left(s+\frac{1}{s}\right)\left(y^{1/2}-z^{1/2}\right)^2} e^{-\frac{1}{2}\left(s+\frac{1}{s}\right)(yz)^{1/2}} I_\alpha\left(\frac{1-s^2}{2s}(yz)^{1/2}\right).$$

Thus,

$$W^\alpha f(t,y) = \int_0^\infty W^\alpha(t,y,z) f(z)\,dz = \int_0^\infty R_\alpha(y,z,s) f(z)\,dz$$

for $s = \left(1-e^{-t/2}\right)/\left(1+e^{-t/2}\right)$.

We shall need in the sequel the following estimations for $I_\alpha(x)$: Let $\alpha > -1$, there exist two constants c_α and C_α such that

$$(7) \qquad \begin{array}{lll} (1) \text{ If } 0 \le x \le 1, & \text{then} & c_\alpha x^\alpha \le I_\alpha(x) \le C_\alpha x^\alpha. \\ (2) \text{ If } x \ge 1, & \text{then} & c_\alpha \frac{1}{x^{1/2}} e^x \le I_\alpha(x) \le C_\alpha \frac{1}{x^{1/2}} e^x. \end{array}$$

See, [1], p.5, (12) and [1], p.86, (5).

Let $D_s = \left\{x : \left(\frac{1-s^2}{2s}\right)^2 x \ge 1\right\}$. Then, the function $\chi_{D_s}(yz)$ is equal to 1 if and only if $\left(\frac{1-s^2}{2s}\right)^2 yz \ge 1$. By (7), $\chi_{D_s}(yz) R_\alpha(y,z,s)$ is bounded by constant times

$$(8) \qquad \frac{1}{2}\frac{1-s^2}{2s} e^{-\frac{1}{4}\left(s+\frac{1}{s}\right)\left(z^{1/2}-y^{1/2}\right)^2 - \frac{1}{2}\left(s+\frac{1}{s}\right)(zy)^{1/2}} \chi_{D_s}(yz) \frac{e^{\left(\frac{1-s^2}{2s}\right)(zy)^{1/2}}}{\left(\frac{1-s^2}{2s}\right)^{1/2} (zy)^{1/4}}.$$

Since $-\frac{1}{2}\left(s+\frac{1}{s}\right)+\left(\frac{1-s^2}{2s}\right)=-s$, we get that (8) is equal to

$$\frac{1}{2}\frac{1-s^2}{2s}e^{-\frac{1}{4}\left(s+\frac{1}{s}\right)\left(z^{1/2}-y^{1/2}\right)^2-s(zy)^{1/2}}\chi_{D_s}(yz)\frac{1}{\left(\frac{1-s^2}{2s}\right)^{1/2}(zy)^{1/4}}$$

$$\leq \frac{1}{2}\frac{1-s^2}{2s}e^{-\frac{1}{4s}\left(z^{1/2}-y^{1/2}\right)^2}\chi_{D_s}(yz)\frac{1}{\left(\frac{1-s^2}{2s}\right)^{1/2}(zy)^{1/4}}. \tag{9}$$

Analogously, by (7), $\chi_{D_s^c}(yz)\,R_\alpha(y,z,s)$ is bounded by constant times

$$\frac{1}{2}\frac{1-s^2}{2s}e^{-\frac{1}{4s}\left(z^{1/2}-y^{1/2}\right)^2}\chi_{D_s^c}(yz)\left(\frac{1-s^2}{2s}(yz)^{1/2}\right)^\alpha. \tag{10}$$

We denote

$$\begin{aligned} H_{\alpha,1}(s,y) &= \int_0^\infty \chi_{D_s}(yz)\,R_\alpha(y,z,s)\,f(z)\,dz \\ \text{and}\qquad H_{\alpha,2}(s,y) &= \int_0^\infty \chi_{D_s^c}(yz)\,R_\alpha(y,z,s)\,f(z)\,dz. \end{aligned}$$

Let $y>0$, $z>0$ and $s>0$. For every integer k we define

$$B_k(y)=\left\{z: 2^k s^{1/2}<\left|z^{1/2}-y^{1/2}\right|\leq 2^{k+1}s^{1/2}\right\},$$

and let k_0 be an integer to be fixed later, then

$$\begin{aligned} H_{\alpha,1}(s,y) &\leq C_\alpha \sum_{-\infty}^{k_0}\frac{1-s^2}{2s}e^{-2^{2k}/4}\int_{B_k(y)}\chi_{D_s}(yz)\frac{f(z)dz}{\left(\frac{1-s^2}{2s}\right)^{1/2}(zy)^{1/4}} \\ &\quad + C_\alpha\sum_{k_0+1}^{+\infty}\frac{1-s^2}{2s}e^{-2^{2k}/4}\int_{B_k(y)}\chi_{D_s}(yz)\frac{f(z)dz}{\left(\frac{1-s^2}{2s}\right)^{1/2}(zy)^{1/4}} \\ &= H_{\alpha,11}(s,y)+H_{\alpha,12}(s,y). \end{aligned} \tag{11}$$

For k_0 and $B_k(y)$ having the same meaning as before, by (10), we have that $H_{\alpha,2}(s,y)$ is bounded by a constant times

$$\begin{aligned} &\sum_{-\infty}^{k_0}\frac{1-s^2}{2s}e^{-2^{2k}/4}\int_{B_k(y)}\chi_{D_s^c}(yz)\left(\frac{1-s^2}{2s}(yz)^{1/2}\right)^\alpha f(z)\,dz \\ &+ \sum_{k_0+1}^{+\infty}\frac{1-s^2}{2s}e^{-2^{2k}/4}\int_{B_k(y)}\chi_{D_s^c}(yz)\left(\frac{1-s^2}{2s}(yz)^{1/2}\right)^\alpha f(z)\,dz \\ &= H_{\alpha,21}(s,y)+H_{\alpha,22}(s,y). \end{aligned} \tag{12}$$

Let us fixed k_0 as the unique integer satisfying $2^{k_0+2}s^{1/2}<y^{1/2}\leq 2^{k_0+3}s^{1/2}$. Since $\left|z^{1/2}-y^{1/2}\right|\leq 2^{k+1}s^{1/2}$, if $k\leq k_0$ and $z\in B_k(y)$ then, we get

$$y/4\leq\left(y^{1/2}-2^{k+1}s^{1/2}\right)^2\leq z\leq\left(y^{1/2}+2^{k+1}s^{1/2}\right)^2\leq 2y. \tag{13}$$

If $k \geq k_0$ and $z \in B_k(y)$, since $\left|z^{1/2} - y^{1/2}\right| \leq 2^{k+1} s^{1/2}$, we get

$$0 \leq z \leq 100.2^{2k} s \text{ and } 0 \leq y \leq 100.2^{2k} s. \tag{14}$$

Proof of Lemma 2. We are going to estimate $H_{\alpha,11}(s,y)$, $H_{\alpha,12}(s,y)$, $H_{\alpha,21}(s,y)$, and $H_{\alpha,22}(s,y)$ for $\alpha > -1$.

Estimate of $H_{\alpha,11}(s,y)$. By (11) and (13), $H_{\alpha,11}(s,y)$ is less than or equal to a constant times the sum for $k \leq k_0$ of the terms

$$\left(\frac{1-s^2}{2s}\right)^{1/2} e^{-2^{2k}/4} y^{-1/2} \int_{\left(y^{1/2}-2^{k+1}s^{1/2}\right)^2}^{\left(y^{1/2}+2^{k+1}s^{1/2}\right)^2} f(z)\, dz. \tag{15}$$

Since $\left(y^{1/2} + 2^{k+1}s^{1/2}\right)^2 - \left(y^{1/2} - 2^{k+1}s^{1/2}\right)^2 = 4y^{1/2}2^{k+1}s^{1/2}$, we have that (15) is bounded by a constant times

$$\left(1-s^2\right)^{1/2} e^{-2^{2k}/4} 2^k \frac{1}{4y^{1/2}2^{k+1}s^{1/2}} \int_{\left(y^{1/2}-2^{k+1}s^{1/2}\right)^2}^{\left(y^{1/2}+2^{k+1}s^{1/2}\right)^2} f(z)\, dz. \tag{16}$$

Then, by (13) and (16), , we get

$$H_{\alpha,11}(s,y) \leq C_\alpha \left(1-s^2\right)^{1/2} M_0 f(y). \tag{17}$$

holds for $\alpha > -1$.

Estimate of $H_{\alpha,12}(s,y)$. By (11), we have that $H_{\alpha,12}(s,y)$ is bounded by a constant times the sum for $k > k_0$ of the terms

$$\frac{1-s^2}{2s} e^{-2^{2k}/4} \int\limits_{B_k(y)} \chi_{D_s}(yz) \frac{f(z)\, dz}{\left(\frac{1-s^2}{2s}\right)^{1/2} (zy)^{1/4}}. \tag{18}$$

The condition $\chi_{D_s}(yz) = 1$ is equivalent to $z \geq \frac{1}{y}\left(\frac{2s}{1-s^2}\right)^2$, and by (14), $y \leq 1002^{2k}s$. Hence,(18) is bounded by

$$\begin{aligned}
\frac{1-s^2}{2s} e^{-2^{2k}/4} \int_0^{1002^{2k}s} \chi_{D_s}(yz) \frac{f(z)\, dz}{\left(\frac{1-s^2}{2s}\right)^{1/2} (zy)^{1/4}} &\leq \left(\frac{1-s^2}{2s}\right) e^{-2^{2k}/4} \int_0^{1002^{2k}s} f(z)\, dz \\
&\leq C\left(1-s^2\right) e^{-2^{2k}/4} 2^{2k} \frac{1}{1002^{2k}s} \int_0^{1002^{2k}s} f(z)\, dz \\
&\leq C\left(1-s^2\right) e^{-2^{2k}/4} 2^{2k} M_0 f(y).
\end{aligned}$$

Then, we get that $H_{\alpha,12}(s,y)$ is bounded by a constant times by

$$\left(1-s^2\right) M_0 f(y), \tag{19}$$

for $\alpha > -1$.

Estimate of $H_{\alpha,21}(s,y)$, for the case $-1<\alpha<0$. Let $\beta=-\alpha$. By (13) we have that $H_{\alpha,21}(s,y)$ is bounded by a constant times the sum for $k\le k_0$ of the terms

$$\left(\frac{1-s^2}{2s}\right)e^{-2^{2k}/4}\int_{B_k(y)}\chi_{D_s^c}(zy)\left(\left(\frac{1-s^2}{2s}\right)(zy)^{1/2}\right)^{-\beta}f(z)\,dz$$

$$(20)\quad \le\left(\frac{1-s^2}{2s}\right)^{1-\beta}e^{-2^{2k}/4}y^{-\beta/2}\left(\frac{4y^{1/2}2^{k+1}s^{1/2}}{4y^{1/2}2^{k+1}s^{1/2}}\right)^{1-\beta}\int_{(y^{1/2}-2^{k+1}s^{1/2})^2}^{(y^{1/2}+2^{k+1}s^{1/2})^2}\chi_{D_s^c}(zy)\,z^{-\beta/2}f(y)\,dz.$$

If $\left(\frac{1-s^2}{2s}\right)(yy/4)^{1/2}>1$, then for $z\ge y/4$ it turns out that $\left(\frac{1-s^2}{2s}\right)(yz)^{1/2}>1$, thus $\chi_{D_s^c}(zy)=0$ and the integral above is equal to zero. Therefore, we can assume that $\left(\frac{1-s^2}{2s}\right)(yy/4)^{1/2}\le 1$, which implies that

$$(21)\qquad \left(\frac{1-s^2}{2s}\right)y\le 2.$$

Then, by (21), the expression (20), is smaller than or equal to a constant times

$$\left(1-s^2\right)^{\frac{1-\beta}{2}}e^{-2^{2k}/4}2^{k(1-\beta)}y^{-\beta/2}\frac{1}{\left(4y^{1/2}2^{k+1}s^{1/2}\right)^{1-\beta}}\int_{\left(y^{1/2}-2^{k+1}s^{1/2}\right)^2}^{\left(y^{1/2}+2^{k+1}s^{1/2}\right)^2}z^{-\beta/2}f(z)\,dz$$
$$\le\ \left(1-s^2\right)^{\frac{1-\beta}{2}}e^{-2^{2k}/4}2^{k(1-\beta)}y^{-\beta/2}M_\beta\left(z^{-\beta/2}f(z)\right)(y).$$

Hence, for $-1<\alpha<0$, $H_{\alpha,21}(s,y)$ is bounded by a constant times

$$(22)\qquad \left(1-s^2\right)^{\frac{1-\beta}{2}}y^{-\beta/2}M_\beta\left(z^{-\beta/2}f(z)\right)(y).$$

Estimate of $H_{\alpha,21}(s,y)$ for the case $\alpha\ge 0$. By (13), we have that $H_{\alpha,21}(s,y)$ bounded by a constant times the sum for $k\le k_0$ of the terms

$$\frac{1-s^2}{2s}e^{-2^{2k}/4}\int_{B_k(y)}\chi_{D_s^c}(yz)\left(\frac{1-s^2}{2s}(yz)^{1/2}\right)^{\alpha}f(z)\,dz$$
$$(23)\qquad \le\ C\left(\frac{1-s^2}{2s}\right)^{1+\alpha}e^{-2^{2k}/4}y^{\alpha}\int_{\left(y^{1/2}-2^{k+1}s^{1/2}\right)^2}^{\left(y^{1/2}+2^{k+1}s^{1/2}\right)^2}\chi_{D_s^c}(yz)\,f(z).$$

Then, by (21), we get that (23) is bounded by a constant times

$$\left(1-s^2\right)^{1/2}e^{-2^{2k}/4}2^k\frac{1}{4y^{1/2}2^{k+1}s^{1/2}}\int_{\left(y^{1/2}-2^{k+1}s^{1/2}\right)^2}^{\left(y^{1/2}+2^{k+1}s^{1/2}\right)^2}f(z)\,dz$$
$$\le\ C\left(1-s^2\right)^{1/2}e^{-2^{2k}/4}2^kM_0f(y).$$

Thus, summing up for $k\le k_0$, we get that

$$(24)\qquad H_{\alpha,21}(s,y)\le C_\alpha\left(1-s^2\right)^{1/2}M_0f(y)$$

holds for the case $\alpha \geq 0$.

Estimate of $H_{\alpha,22}(s,y)$, for the case $-1 < \alpha \leq 0$. Let $\beta = -\alpha$. By (13) we have that $H_{\alpha,22}(s,y)$ is bounded by a constant times the sum for $k > k_0$ of the terms

$$\begin{aligned} &\left(\frac{1-s^2}{2s}\right) e^{-2^{2k}/4} \int_{B_k(y)} \chi_{D_s^c}(zy) \left(\left(\frac{1-s^2}{2s}\right)(zy)^{1/2}\right)^{-\beta} f(z)\, dz \\ \leq\ & \left(\frac{1-s^2}{2s}\right)^{1-\beta} e^{-2^{2k}/4} y^{-\beta/2} \left(\frac{1002^{2k}s}{1002^{2k}s}\right)^{1-\beta} \int_0^{1002^{2k}s} \chi_{D_s^c}(zy)\, z^{-\beta/2} f(y)\, dz. \end{aligned}$$

By (14) the former expression is smaller than or equal to a constant times

$$\left(1-s^2\right)^{1-\beta} e^{-2^{2k}/4} 2^{k(1-\beta)} y^{-\beta/2} M_\beta\left(z^{-\beta/2} f(z)\right)(y).$$

Hence, for $-1 < \alpha \leq 0$, $H_{\alpha,22}(s,y)$ is bounded by a constant times

$$\left(1-s^2\right)^{1-\beta} y^{-\beta/2} M_\beta\left(z^{-\beta/2} f(z)\right)(y). \tag{25}$$

Estimate of $H_{\alpha,22}(s,y)$ for $\alpha \geq 0$. By (12) and (14), $H_{\alpha,22}(s,y)$ is bounded by a constant times the sum for $k > k_0$ of the terms

$$\begin{aligned} &\frac{1-s^2}{2s} e^{-2^{2k}/4} \int_{B_k(y)} \chi_{D_s^c}(yz) \left(\frac{1-s^2}{2s}(yz)^{1/2}\right)^{\alpha} f(z)\, dz \\ \leq\ & \left(\frac{1-s^2}{2s}\right)^{1+\alpha} e^{-2^{2k}/4} y^{\alpha/2} \int_0^{1002^{2k}s} z^{\alpha/2} f(z)\, dz \\ \leq\ & \left(\frac{1-s^2}{2s}\right)^{1+\alpha} e^{-2^{2k}/4} y^{\alpha/2} \left(1002^{2k}s\right)^{\alpha/2} \int_0^{1002^{2k}s} f(z)\, dz. \end{aligned} \tag{26}$$

Since $y \leq 1002^{2k}s$, we obtain that (26) is bounded by a constant times

$$\begin{aligned} &\left(1-s^2\right)^{1+\alpha} e^{-2^{2k}/4} 2^{2(1+\alpha)k} \frac{1}{1002^{2k}s} \int_0^{1002^{2k}s} f(z)\, dz \\ \leq\ & \left(1-s^2\right)^{1+\alpha} e^{-2^{2k}/4} 2^{2(1+\alpha)k} M_0 f(y). \end{aligned}$$

Thus, $H_{\alpha,22}(s,y)$ is bounded by a constant times

$$\left(1-s^2\right)^{1+\alpha} M_0 f(y), \tag{27}$$

when $\alpha \geq 0$.

A simple computation show that $1 - s^2 \leq 4e^{-t/2}$.Then the estimates (17), (19), (22) and (25) obtained for $-1 < \alpha \leq 0$, show that (5) holds, and the estimates (17),(19), (24) and (27) obtained for $\alpha \geq 0$, show that (6) holds. ■

4 Proof of the results

Proof of Theorem 1 Let us consider the case $-1 < \alpha < 0$, and let $\beta = -\alpha$. By part (a) of Lemma 4, see (5), we have

$$W^{\alpha} f(t, y) \leq C_{\alpha} \left\{ e^{-t/4} M_0 f(y) + e^{-t(1-\beta)/4} y^{-\beta/2} M_{\beta}\left(z^{-\beta/2} f(z)\right)(y) \right\},$$

and from Lemma 3, (3) and (4), we have

$$y^{-\beta/2} M_{\beta}\left(f(z)\, z^{-\beta/2}\right)(y) \tag{28}$$

$$\leq C_{\beta} \left\{ y^{\beta/2} \frac{1}{y} \int_0^y f(z)\, z^{-\beta/2} dz + y^{-\beta/2} M_0\left(f(z)\, z^{\beta/2}\right)(y) + M_0 f(y) \right\}, \tag{29}$$

thus, $W^{\alpha,*} f(y)$ is smaller than or equal to a constant times

$$y^{\beta/2} M_0\left(f(z)\, z^{-\beta/2}\right)(y) + y^{-\beta/2} M_0\left(f(z)\, z^{\beta/2}\right)(y) + M_0 f(y).$$

The hypothesis that $2/(2+\alpha) < p < 2/(-\alpha)$ and $p > 1$ for $-1 < \alpha < 0$ is equivalent to $-1 < -p\beta/2 < p\beta/2 < p-1$ and $p > 1$. This shows that the weights $y^{-p\beta/2}$ and $y^{p\beta/2}$ belong to the Muckenhoupt class A_p and therefore

$$\begin{aligned}
\int_0^{\infty} \left(y^{\beta/2} M_0\left(f(z)\, z^{-\beta/2}\right)(y)\right)^p dy &\leq C_{\alpha,p} \int_0^{\infty} f(y)^p\, dy, \\
\int_0^{\infty} \left(y^{-\beta/2} M_0\left(f(z)\, z^{\beta/2}\right)(y)\right)^p dy &\leq C_{\alpha,p} \int_0^{\infty} f(y)^p\, dy, \\
\text{and} \qquad \int_0^{\infty} M_0 f(y)^p\, dy &\leq C_p \int_0^{\infty} f(y)^p\, dy,
\end{aligned}$$

which proves the theorem for $-1 < \alpha < 0$.

Let us consider the case $\alpha \geq 0$. By Lemma 2 part (b), inequality (6),

$$W^{\alpha} f(s, y) \leq C_{\alpha} e^{-t/4} M_0 f(y),$$

thus $W^{\alpha,*} f(y) \leq C_{\alpha} M_0 f(y)$, and

$$\int_0^{\infty} \left(W^{\alpha,*} f(y)\right)^p dy \leq C_{\alpha} \int_0^{\infty} M_0 (f(y))^p\, dy \leq C_{\alpha,p} \int_0^{\infty} f(y)^p\, dy,$$

which proves the theorem for $\alpha \geq 0$. ■

Remark 1 *A second proof of Theorem 1 can be given by interpolation of the results obtained for the end points of N_{α} in Theorem 2. For the results on interpolation needed, see [2], page 59, Theorem (3.4.6).*

Proof of Theorem 2.

Proof of part (a). If $-1<\alpha<0$ it follows that $2/(-\alpha)>1$, and the upper end point of N_α is equal to $2/(-\alpha)$. For a fixed s, $0<s<1$, consider the points y and z satisfying $\left(\frac{1-s^2}{2s}\right)y\le 1$ and $\left(\frac{1-s^2}{2s}\right)z\le 1$, then, by (7), we obtain

$$R_\alpha(y,z,s)\ge C_\alpha\frac{1}{2}\frac{1-s^2}{2s}e^{-\frac{1+s^2}{1-s^2}}\left(\frac{1-s^2}{2s}(yz)^{1/2}\right)^\alpha\ge C_{\alpha,s}y^{\alpha/2}z^{\alpha/2}\ .$$

Thus, denoting $a=\frac{2s}{1-s^2}$, we get that

$$W^\alpha\left(\chi_{(0,a)}\right)(s,y)\ge C_{\alpha,s}y^{\alpha/2}\int_0^a z^{\alpha/2}dz=C_{\alpha,s}y^{\alpha/2},$$

holds for every $0\le y\le a$. Since

$$\int_0^a\left(y^{\alpha/2}\right)^{2/(-\alpha)}dy=\int_0^a y^{-1}dy=\infty,$$

it follows that the operator $W^{\alpha,*}f$ is not of strong type $(2/(-\alpha),2/(-\alpha))$. However, the operator $W^{\alpha,*}f$ is of weak type. In fact, by (3) in Lemma 1 and (5 in Lemma 2, it will be enough to show that the three terms of

$$y^{\beta/2}M_0\left(f(z)z^{-\beta/2}\right)(y)+sup_{y\le 2h}\left(\frac{y^{-\beta/2}}{(2h)^{1-\beta}}\int_y^{y+h}f(z)z^{-\beta/2}dz\right)+M_0f(y)$$

satisfy the weak type condition. Since $-1<\alpha<0$ implies that $-1<1<2/(-\alpha)-1$, then the weight y belongs to $A_{2/(-\alpha)}$. Thus, the first term satisfies

$$\begin{aligned}\int_0^\infty\left(y^{\beta/2}M_0\left(f(z)z^{-\beta/2}\right)(y)\right)^{2/(-\alpha)}dy&=\int_0^\infty\left(M_0\left(f(z)z^{-\beta/2}\right)(y)\right)^{2/(-\alpha)}ydy\\ &\le\ C_\alpha\int_0^\infty\left(f(y)y^{-\beta/2}\right)^{2/(-\alpha)}ydy=C_\alpha\int_0^\infty f(y)^{2/(-\alpha)}dy.\end{aligned}$$

This shows that the first term is of strong type and therefore of weak type $(2/(-\alpha),2/(-\alpha))$. The third term is obviously of strong and weak type $(2/(-\alpha),\ 2/(-\alpha))$. For the second term if we denote $2/(-\alpha)$ by p, then $p'=2/(2+\alpha)$. By Hölder's inequality, we obtain

$$\frac{1}{(2h)^{1-\beta}}\int_y^{y+h}f(z)z^{-\beta/2}dz\le\frac{1}{(2h)^{1-\beta}}\|f\|_{L^p(y,y+h)}\left\|z^{\alpha/2}\right\|_{L^{p'}(y,y+h)}.$$

In order to estimate $\left\|z^{\alpha/2}\right\|_{L^{p'}(y,y+h)}$ we observe that

$(\alpha/2)\,2/(2+\alpha)=2(1+\alpha)/(2+\alpha)-1>-1$ and $(2(1+\alpha)/(2+\alpha))(2+\alpha)/2=1+\alpha$

hold. Then $\left\|z^{\alpha/2}\right\|_{L^{p'}(y,y+h)}\le C_\alpha(y+h)^{1+\alpha}$. Thus, since $y\le 2h$, we get

$$\frac{1}{(2h)^{1+\alpha}}\int_y^{y+h}f(z)z^{\alpha/2}dz\ \le C_\alpha\left(\tfrac{y+h}{h}\right)^{1+\alpha}\|f\|_{L^p(y,y+h)}\le C_\alpha\|f\|_{L^p(0,\infty)}.$$

Multiplying by $y^{-\beta/2}$ and taking the supremum in $h \geq y/2$, we obtain

$$\sup_{h\geq y/2}\left(y^{-\beta/2}\frac{1}{(2h)^{1-\beta}}\int_y^{y+h} f(z)\, z^{-\beta/2}dz\right) \leq C_\alpha y^{-\beta/2}\left\|f\right\|_{L^p(0,\infty)}.$$

From this inequality the weak type (p,p) for $p = 2/(-\alpha)$ follows readily.

Proof of part (b). If $\alpha \geq 0$, the upper end point of N_α is equal to ∞ and, by (6), we have $W^{\alpha,*}f(y) \leq C_\alpha M_0 f(y)$. Therefore, the operator $W^{\alpha,*}f$ is of strong type (∞,∞).

Proof of part (c). If the lower end point of N_α is greater than 1, then it coincides with $2/(2+\alpha)$. This implies that $\alpha < 0$. If for a given $a = \frac{2s}{1-s^2}$, as before, the integral $\int_0^a f(z)\, z^{\alpha/2}dz$ is finite for every $f(z) \in L^{2/(2+\alpha)}(0,a)$, then since $(2/(2+\alpha))' = 2/(-\alpha)$, by uniform boundedness, it follows that $z^{\alpha/2} \in L^{2/(-\alpha)}(0,a)$. This is a contradiction since $z^{(\alpha/2)2/(-\alpha)} = z^{-1}$. Then, there exists $f \in L^{2/(2+\alpha)}(0,a)$ such that $\int_0^a f(z)\, z^{\alpha/2}dz = \infty$. This shows that for the given f, $\int_0^a R(s,y,z) f(z)\, dz \geq C_{\alpha,s} y^{\alpha/2}\int_0^a z^{\alpha/2} f(z)\, dz = \infty$, showing that $W^{\alpha,*}f(y) = \infty$ for every $y \leq a$. This is telling us that the operator $W^{\alpha,*}$ cannot be of weak type at the lower end point $2/(2+\alpha) > 1$. Let $-1 < \alpha < 0$ and $\beta = -\alpha$. By (5), (3) and (4), we have

$$(30)\quad W^{\alpha,*}f(y) \;\leq\; C_\alpha\left\{M_0 f(y) + y^{\beta/2}\frac{1}{y}\int_0^y f(z)\, z^{-\beta/2}dz + y^{-\beta/2}M_0\left(f(z)\, z^{\beta/2}\right)(y)\right\}.$$

It is easy to see that the inequalities

$$1 < 2/(2-\beta) \quad \text{and} \quad -1 < -(\beta/2)\, 2/(2-\beta) < 2/(2-\beta) - 1$$

hold. These inequalities imply that the weight $y^{-(\beta/2)2/(2-\beta)}$ belong to $A_{2/(2-\beta).}$ Therefore, the operators

$$M_0 f(y) \qquad \text{and} \qquad y^{-\beta/2}M_0\left(f(z)\, z^{\beta/2}\right)(y)$$

are of strong type $(2/(2-\beta), 2/(2-\beta))$. We have not considered yet the second term of (30). Let E be a measurable set contained in $(0,\infty)$. Then,

$$\begin{aligned}\int_0^y \chi_E(z)\, z^{-\beta/2}dz &\leq \int_0^\infty \chi_E(z)\, z^{-\beta/2}dz \leq \int_0^{|E|} z^{-\beta/2}dz = \frac{1}{1-\beta/2}\left|E\right|^{1-\beta/2} \\ &= C_\beta\left(\int_0^\infty \chi_E(z)\, dz\right)^{(2-\beta)/2}.\end{aligned}$$

In consequence,

$$y^{\beta/2}\frac{1}{y}\int_0^y \chi_E(z)\, z^{-\beta/2}dz \leq C_\beta\; y^{\beta/2}\frac{1}{y}\left(\int_0^\infty \chi_E(z)\, dz\right)^{(2-\beta)/2}.$$

From this inequality the restricted weak type $\left(\frac{2}{2-\beta}, \frac{2}{2-\beta}\right)$ of the second term (30) is readily obtained.

Proof of part (d). Let us show that if the lower end point of N_α is equal to 1, then the operator $W^{\alpha,*}f$ cannot be of strong type $(1,1)$. By (7), we have

$$\chi_{D_s}(yz)\,R_\alpha(y,z,s)\geq c_\alpha\left(\frac{1-s^2}{2s}\right)^{1/2}e^{-\frac{1}{4s}\left(y^{1/2}-z^{1/2}\right)^2}e^{-\frac{s}{4}\left(y^{1/2}-z^{1/2}\right)^2}e^{-s(yz)^{1/2}}\chi_{D_s}(yz)\,\frac{1}{(yz)^{1/4}}.$$

Take $0<\varepsilon\leq 1$. Let us assume that $1<z\leq 1+\varepsilon,\ 1+2\varepsilon\leq y\leq 2$, and $s=(y-1)^2/4$. Then it follows that $s\leq 1/4$, $\left(\frac{1-s^2}{2s}\right)\geq 1$, and $\left(\frac{1-s^2}{2s}\right)^{1/2}(yz)^{1/4}\geq 1$. Thus $\chi_{D_s}(yz)=1$, and since

$$\begin{aligned}\frac{1}{4s}\left(y^{1/2}-z^{1/2}\right)^2 &= \frac{\left(y^{1/2}-z^{1/2}\right)^2}{(y-1)^2}\leq\left(\frac{(y-z)}{2(y-1)}\right)^2\leq\left(\frac{1}{2}+\frac{|1-z|}{2(y-1)}\right)^2\\ &\leq \left(\frac{1}{2}+\frac{\varepsilon}{4\varepsilon}\right)^2\leq 1,\end{aligned}$$

we get that $R_\alpha(y,z,s)\geq\frac{C_\alpha}{(y-1)}$ holds. Then

$$W^{\alpha,*}\left(\chi_{(1,1+\varepsilon)}\right)(y)\geq\frac{C_\alpha}{(y-1)}\int_0^\infty\chi_{(1,1+\varepsilon)}(z)\,dz=C_\alpha\frac{\varepsilon}{(y-1)},$$

holds for $1+2\varepsilon\leq y\leq 2$. If the operator $W^{\alpha,*}$ were of strong type $(1,1)$ we would have that

$$\int_0^\infty W^{\alpha,*}\left(\chi_{(1,1+\varepsilon)}\right)(y)\,dy\leq A_\alpha\int_0^\infty\chi_{(1,1+\varepsilon)}(y)\,dy=A_\alpha\varepsilon \tag{31}$$

holds for a finite constant A_α depending on α only. On the other hand, we have

$$\int_{1+2\varepsilon}^2 W^{\alpha,*}\left(\chi_{(1,1+\varepsilon)}\right)(y)\,dy\geq C_\alpha\int_{1+2\varepsilon}^2\frac{\varepsilon}{y-1}dy=C_\alpha\varepsilon\log\left(\frac{1}{2\varepsilon}\right). \tag{32}$$

In consequence, from (31) and (32), it follows that

$$C_\alpha\varepsilon\log\left(\frac{1}{2\varepsilon}\right)\leq A_\alpha\varepsilon \text{ or also } C_{\alpha,\varepsilon}\log\left(\frac{1}{2\varepsilon}\right)\leq A_\alpha.$$

This is a contradiction since the left hand side of the last inequality above tends to ∞ when ε tends to 0, proving that there is no strong type $(1,1)$ for the operator $W^{\alpha,*}f$. However as we are going to prove, the operator is of weak type $(1,1)$. Since $2/(2+\alpha)\leq 1$, it follows that $\alpha\geq 0$. Then, by Lemma 2, (6), we have that $W^{\alpha,*}f(y)\leq C_\alpha M_0 f(y)$ which implies the weak type $(1,1)$ for the operator $W^{\alpha,*}f(y)$. ■

References

[1] Erdélyi, Arthur; Magnus, Wilhelm; Oberhettinger, Fritz; Tricomi, Francesco G. Higher transcendental functions. Vol. II. Based, in part, on notes left by Harry Bateman. McGraw-Hill Book Company, Inc., New York-Toronto-London, 1953.

[2] M. de Guzmán, Real Variable Methods in Fourier Analysis, North-Holland Mathematics Studies 46. North-Holland Publishing Company, 1981.

[3] B. Muckenhoupt, *Mean convergence of Hermite and Laguerre expansions,* Trans. Amer. Math. Soc. **139** (1969), 231-242.

[4] B. Muckenhoupt, *Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Soc., **165**, 1972, 207-226.

[5] E. M. Stein, Topics in Harmonic Analysis Related to the Littlewood-Paley Theory, Annals of Mathematical Studies 63. Princeton University Press, 1970.

[6] E. M. Stein, Singular Integrals and Differentiability Properties of Functions, Princeton Mathematical Series 30. Princeton University Press, 1970.

[7] K. Stempak, *Heat-diffusion and Poisson integrals for Laguerre expansions,* Tohoku Math. J. **46** (1994) , 83-104.

[8] G. Szego, Orthogonal Polynomials, Amer. Math. Soc. Colloquium Publication XXIII, American Mathematical Society,1939.

[9] S. Thangavelu, Lectures on Hermite and Laguerre Expansions, Mathematical Notes 24. Princeton University Press, 1993.

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