THE INTERCHANGE BETWEEN THE PRODUCT AND THE CONVOLUTION OF THE n-DIMENSIONAL DISTRIBUTIONAL HANKEL TRANSFORMS

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ABSTRACT. In this Note, we prove several formulas about the interchange between the product and the convolution of the n - dimensional Hankel transforms. In fact we obtain the following formulas:

1) $\mathcal{H}\{\delta^{(\ell)}(u(x))\}\cdot\mathcal{H}\{\delta^{(k)}(u(x))\}=\mathcal{H}\{\delta^{(\ell)}(u(x))*\delta^{(k)}(u(x))\}$, (cfr. (II,13))

2) $\mathcal{H}\Big\{\delta^{(k)}\big(u(|x|^2)\big)^{\frac{n-2}{2}+k}*\delta^{(\ell)}\big(u(|x|^2)\big)^{\frac{n-2}{2}+\ell}\Big\}=D\Big\{\big(u(|x|^2)\big)^{\frac{n-2}{2}+k}*\big(u(|x|^2)\big)^{\frac{n-2}{2}+\ell}\Big\}$,

(cfr. (II,20))

where D is the constant given by (II,18).

3) $\mathcal{H}\Big\{\big(u(|x|^2)\big)^{\frac{n-2}{2}+k}\cdot\big(u(|x|^2)\big)^{\frac{n-2}{2}+\ell}\Big\}=C\,\mathcal{H}\Big\{\big(u(|x|^2)\big)^{\frac{n-2}{2}+k}\Big\}*\mathcal{H}\Big\{\big(u(|x|^2)\big)^{\frac{n-2}{2}+\ell}\Big\}$,

cfr. (II,15))

where C is the constant given by (II,16).

4) $\mathcal{H}\Big\{\big(u(|x|^2)\big)^{\frac{n-2}{2}+k}*\big(u(|x|^2)\big)^{\frac{n-2}{2}+\ell}\Big\}=D\,\mathcal{H}\Big\{\big(u(|x|^2)\big)^{\frac{n-2}{2}+k}\Big\}\cdot\mathcal{H}\Big\{\big(u(|x|^2)\big)^{\frac{n-2}{2}+\ell}\Big\}$,

(cfr. (II,17))

where D is the constant given by (II,18).

5) $\delta^{(k)}\big(u(|x|^2)\big)*\delta^{(\ell)}\big(u(|x|^2)\big)=D\;\delta^{\left(k+\ell+\frac{n-2}{2}\right)}\big(u(|x|^2)\big)$, (cfr. (II,21))

here D is the constant given by (II,18).

6) $\mathcal{H}\{\delta^{(\ell)}(P)*\delta^{(k)}(P)\}=\mathcal{H}\{\delta^{(\ell)}(P)\}\cdot\mathcal{H}\{\delta^{(k)}(P)\}$. (cfr. (III, 7)).

I. Introduction.

We begin with some definitions. Let $x=(x_1,x_2,\dots,x_n)$ be a point of the n-dimensional Euclidean space $I\!R^n$. Consider a non-degenerate quadratic form in n variables of the form

$$P=P(x)=x_1^2+\cdots+x_p^2-x_{p+1}^2-\cdots-x_{p+q}^2\ , \tag{I,1}$$

where $n=p+q$.

We define the two following distributions, as follows

$$P_+^\lambda=\begin{cases}P^\lambda & \text{if } P>0,\\ 0 & \text{if } P\le 0;\end{cases} \tag{I,2}$$

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and

$$P_-^\lambda = \begin{cases} 0 & \text{if } P > 0, \\ (-P)^\lambda & \text{if } P \leq 0. \end{cases} \tag{I, 3}$$

$\mathcal{H}$ denotes the distributional Hankel transform. Let $\phi(t)$ be defined in $I\!R^+ : \{t, t > 0\}$. By the Hankel transform of the function $\phi(t)$ we mean the function $g(s)$, $0 \leq s < \infty$, defined by the formula

$$g(s) = \mathcal{H}\{\phi(t)\} = \int_0^\infty \phi(t) J_\nu(xt)\sqrt{xt}dt \ , \tag{I, 4}$$

or, equivalently,

$$g(s) = (\mathcal{H}\{\phi(t)\}) = \frac{1}{2}\int_0^\infty \phi(t) t^{\frac{n-2}{2}} R_{\frac{n-2}{2}}(\sqrt{st})dt \ , \tag{I, 5}$$

where

$$R_m(x) = \frac{J_m(x)}{x^m} \ , \tag{I, 6}$$

and $J_m(x)$ is the well-known Bessel function defined by the formula

$$J_m(x) = \sum_{\nu=0}^{\infty} \frac{(-1)^\nu \left(\frac{x}{2}\right)^{m+2\nu}}{\nu!\,\Gamma(m+\nu+1)} \ . \tag{I, 7}$$

It is well known (cfr.[5], p. 240) that if $\phi(t)$ satisfies adequate conditions, for example if $\phi(t)$ belongs to $S_{I\!R^+}$, the following formula is valid:

$$\phi(t) = (\mathcal{H}\{g(s)\}) = \frac{1}{2}\int_0^\infty g(s) s^{\frac{n-2}{2}} R_{\frac{n-2}{2}}(\sqrt{st})ds \ . \tag{I, 8}$$

Let $S_{I\!R^+}$ designate the space of functions $f \in S$ defined in the positive half line $I\!R^+ = \{t, t > 0\}$. By $S'_{I\!R^+}$ we designate the dual of $S_{I\!R^+}$.

Let $U(t) \in S'_{I\!R^+}$. The Hankel transform of $U(t)$ will be, by definition, the distribution $V(s) \in S'_{I\!R^+}$, defined by the formula

$$\langle \mathcal{H}\{U(t), \phi(s)\}\rangle = \langle U(t), (\mathcal{H}\{\phi(s)\})\rangle \ , \tag{I, 9}$$

for every $\phi \in S_{I\!R^+}$.

There are other definitions of the Hankel transform of distributions (cfr. [6]). We use the definition which appears in [7], p. 64, especially, Theorem 26, p. 72. In fact, we have that

$$\widetilde{\mathcal{H}(\widetilde{T})} = \{T\}^\wedge \ , \tag{II, 10}$$

here $\widetilde{T}$ is the image of T belongs to $S'^{\natural}_{I\!R^n}$ in S', defined by the formula

$$\langle \widetilde{T}, \phi(t)\rangle = \langle T, \phi(r^2)\rangle \ , \tag{I, 11}$$

for every $\phi \in S_{I\!R^+}$.

We designate $S^{\natural}_{I\!R^n}$ the family of functions $f(x)$ belongs to $S_{I\!R^n}$ and, further, invariable by rotations. Moreover, $S'^{\natural}_{I\!R^n}$ designates the dual of $S^{\natural}_{I\!R^n}$.

Following strictly the definitions of [1], we shall define the k-th derivative of Dirac delta in $u(x_1, x_2, \ldots, x_n)$.

Let ϕ_t denote a distribution of one variable t. Let $u \in C^\infty(I\!R^n)$ be such that $(n-1)$-dimensional manifold $u(x_1, x_2, \ldots, x_n) = 0$ has no critical point. By $\phi_{u(x)}$ (cfr. [8], page 102) we designate the distribution defined on $I\!R^n$ by

$$\langle \phi_{u(x)}, \varphi(x) \rangle = \langle \phi_t, \psi(t) \rangle \ , \qquad (I,12)$$

where

$$\psi(t) = \int_{u(x)=t} \varphi(x) w_u(x, dx) \ , \qquad (I,13)$$

and $\varphi \in C_0^\infty(I\!R^n)$ is the set of infinitely differentiable functions with compact support and w_u is a $(n-1)$-dimensional exterior differential form on u defined as follows:

$$du \wedge dw = dx_1 \wedge \ldots \wedge dx_n \ , \qquad (I,14)$$

and the orientation of the manifold $u(x) = t$ is such that $w_u(x, dx) > 0$.

On the other hand (cfr. [9], p. 230, form. (6)), we have

$$\Big(\delta^{(k)}(G(x_1, \ldots, x_n)), \varphi(x_1, \ldots, x_n)\Big) = (-1)^k \int_{G(x)=0} w_k(\varphi) \ , \qquad (I,15)$$

$k = 0, 1, \ldots$; where $x = (x_1, \ldots, x_n)$, $G(x_1, \ldots, x_n)$ is such an infinitely differentiable function that

$$\operatorname{grad} G = \left(\frac{\partial G}{\partial x_1}, \ldots, \frac{\partial G}{\partial x_n}\right) \neq 0 \ , \qquad (I,16)$$

$$w_k(\varphi) = \frac{\partial^k}{\partial u_1^k} \left\{ D\binom{x}{u} \varphi, (u_1, \ldots, u_n) \right\} du_1 \ldots du_n \ , \qquad (I,17)$$

$$w_0 = \varphi . w \ , \qquad (I,18)$$

$$\begin{aligned} u_1 &= G(x_1, \ldots, x_n), \\ u_2 &= x_2, \\ &\vdots \\ u_n &= x_n, \end{aligned} \qquad (I,19)$$

and

$$D\binom{x}{u} = \left[D\binom{u}{x}^{-1} \right]^{-1} = \frac{1}{\frac{\partial G}{\partial x_1}} \ , \qquad (I,20)$$

with

$$\frac{\partial G}{\partial x_1} > 0 \ . \tag{I,21}$$

Otherwise, from [9], form. (8),

$$\delta^{(k)}\langle (G(x)), \varphi\rangle = (-1)^k \int_G f_{u_1}^{(k)}(0, u_2, \ldots, u_n) du_2, \ldots, du_n \ , \tag{I,22}$$

where

$$f(u_1, u_2, \ldots, u_n) = \varphi_1(u_1, \ldots, u_n) D\binom{x}{u} \ , \tag{I,23}$$

$$\varphi_1(u_1, u_2, \ldots, u_n) = \varphi(x_1, x_2, \ldots, x_n) \ , \tag{I,24}$$

and $D\binom{x}{u}$ is defined by (I,20).

II. "$\mathcal{H}\left\{\delta^{(\ell)}(u(x)) * \delta^{(k)}(u(x))\right\} = \mathcal{H}\left\{\delta^{(\ell)}(u(x))\right\} \cdot \mathcal{H}\left\{\delta^{(k)}(u(x))\right\}$".

In this paragraph we shall prove the following formula

$$\mathcal{H}\left\{\delta^{(\ell)}(u(x)) * \delta^{(k)}(u(x))\right\} = \mathcal{H}\left\{\delta^{(\ell)}(u(x))\right\} \cdot \mathcal{H}\left\{\delta^{(k)}(u(x))\right\} \ , \tag{II,1}$$

which expresses the interchange between the product and the convolution of the Hankel transform of the p-derivative Dirac-delta of $u(x)$. Here, $x = (x_1, x_2, \ldots, x_n)$, $|x|^2 = x_1^2 + \cdots + x_n^2$ and $u(x) = u(|x|^2)$.

From formula (32), p. 5 of [1], we have

$$\mathcal{H}\left\{\delta^{(\ell)}(u(x))\right\} = \frac{1}{2^{2\ell+\frac{n}{2}}\Gamma\left(\frac{n}{2}+\ell\right)} (u(y))^{\frac{n-2}{2}+\ell} \ . \tag{II,2}$$

From (II,2), we obtain

$$\mathcal{H}\left\{\delta^{(\ell)}(u(x))\right\} \cdot \mathcal{H}\left\{\delta^{(k)}(u(x))\right\} = \frac{(u(y))^{\frac{n-2}{2}+k+\frac{n-2}{2}+\ell}}{2^{2k+\frac{n}{2}}\Gamma\left(\frac{n}{2}+k\right) 2^{2\ell+\frac{n}{2}}\Gamma\left(\frac{n}{2}+\ell\right)} \ . \tag{II,3}$$

Taking into account (II,2), we know that

$$\delta^{(\ell)}(u(x)) = \frac{\mathcal{H}\left\{(u(y)\}^{\frac{n-2}{2}+\ell}\right\}}{2^{2\ell+\frac{n}{2}}\Gamma\left(\frac{n}{2}+\ell\right)} \ , \tag{II,4}$$

and

$$\delta^{(k)}(u(x)) = \frac{\mathcal{H}\left\{(u(y)\}^{\frac{n-2}{2}+k}\right\}}{2^{2k+\frac{n}{2}}\Gamma\left(\frac{n}{2}+k\right)} \ , \tag{II,5}$$

So, from (II,4) and (II,5), we arrive at the following formula

$$\delta^{(\ell)}(u(x)) * \delta^{(k)}(u(x)) = \frac{\mathcal{H}\left\{(u(y))^{\frac{n-2}{2}+\ell}\right\} * \mathcal{H}\left\{(u(y))^{\frac{n-2}{2}+k}\right\}}{2^{2\ell+\frac{n}{2}}\Gamma\left(\frac{n}{2}+\ell\right)2^{2k+\frac{n}{2}}\Gamma\left(\frac{n}{2}+k\right)} . \qquad (\mathrm{II}, 6)$$

Otherwise, appealing to the theorems on the equivalence between the Hankel transform and the Fourier transform for radial functions (cfr. [2] and the classic Theorem of L. Schwartz (cfr. [3], p. 268, form. (VII, 8; 4)):

$$\mathcal{F}[T \cdot U] = \mathcal{F}[T] * \mathcal{F}[U] , \qquad (\mathrm{II}, 7)$$

where $\mathcal{F}$ is the Fourier transform, $T \in O_M$ and $U \in S'$. We know that $O_M \subset S'$, so if $u(|x|^2) \in O_M$ this implies that $u(|x|^2) \in S'$ and then the formula (II,7) is valid. We remember that S designates, as always, the space of L. Schwartz (see pp. 233-237 of [3]) and O_M is the space of distributions of slow increase (cfr. [3], p. 243).

Therefore, the hypothesis for the validity of (II,7) is

$$u(|x|^2) \in O_M . \qquad (\mathrm{II}, 8)$$

Finally, we can write, taking into account the formulae (II,6), (II,7) and (II,8) and the above considerations, that

$$\delta^{(\ell)}(u(x)) * \delta^{(k)}(u(x)) = \frac{\mathcal{H}\left[\left\{(u(y))^{\frac{n-2}{2}+\ell}\right\} \cdot \left\{(u(y))^{\frac{n-2}{2}+k}\right\}\right]}{2^{2\ell+\frac{n}{2}}\Gamma\left(\frac{n}{2}+\ell\right)2^{2k+\frac{n}{2}}\Gamma\left(\frac{n}{2}+k\right)} . \qquad (\mathrm{II}, 9)$$

We know that

$$(u(x))^{\lambda} \cdot (u(x))^{\mu} = (u(x))^{\lambda+\mu} , \qquad (\mathrm{II}, 10)$$

$\lambda, \mu \in \mathbb{C}$.

We can establish (II,10) first for $\mathrm{Re}\,\lambda > 0$ and $\mathrm{Re}\,\mu > 0$, and then by analytical continuation for every $\lambda, \mu \in \mathbb{C}$; we can express (II,9) as the following formula

$$\delta^{(\ell)}(u(x)) * \delta^{(k)}(u(x)) = \frac{\mathcal{H}\left\{(u(y))^{\frac{n-2}{2}+\ell+\frac{n-2}{2}+k}\right\}}{2^{2\ell+\frac{n}{2}}\Gamma\left(\frac{n}{2}+\ell\right)2^{2k+\frac{n}{2}}\Gamma\left(\frac{n}{2}+k\right)} , \qquad (\mathrm{II}, 11)$$

Appealing to the identity theorem for Hankel transforms, we have, from (II,11), that

$$\mathcal{H}\left\{\delta^{(\ell)}(u(x)) * \delta^{(k)}(u(x))\right\} = \frac{(u(y))^{\frac{n-2}{2}+\ell+\frac{n-2}{2}+k}}{2^{2\ell+\frac{n}{2}}\Gamma\left(\frac{n}{2}+\ell\right)2^{2k+\frac{n}{2}}\Gamma\left(\frac{n}{2}+k\right)} , \qquad (\mathrm{II}, 12)$$

Finally, from (II,3) and (II,12), we arrive at our thesis

$$\mathcal{H}\left\{\delta^{(\ell)}(u(x))\right\} \cdot \mathcal{H}\left\{\delta^{(k)}(u(x))\right\} = \mathcal{H}\left\{\delta^{(\ell)}(u(x)) * \delta^{(k)}(u(x))\right\} , \qquad (\mathrm{II}, 13)$$

where

$$u(x) = u(|x|^2) \ . \qquad (\mathrm{II}, 14)$$

Otherwise, if $u(|x|^2) \in O_M$, where O_M is the space of infinitely differentiable functions of slow increase, then $u(|x|^2) \in S'$.

So, taking into account the Theorem XV, form. (VII, 8; 4), p. 268 of [3], we obtain the following formula

$$\mathcal{H}\left\{\left(u\left(|x|^2\right)\right)^{\frac{n-2}{2}+k} \cdot \left(u\left(|x|^2\right)\right)^{\frac{n-2}{2}+\ell}\right\} = C\,\mathcal{H}\left\{\left(u\left(|x|^2\right)\right)^{\frac{n-2}{2}+k}\right\} * \mathcal{H}\left\{\left(u\left(|x|^2\right)\right)^{\frac{n-2}{2}+\ell}\right\} , \qquad (\mathrm{II}, 15)$$

here C is the constant given by

$$C = \frac{1}{2^{2k+\frac{n}{2}}\Gamma\left(\frac{n}{2}+k\right)2^{2\ell+\frac{n}{2}}\Gamma\left(\frac{n}{2}+\ell\right)} \ , \qquad (\mathrm{II}, 16)$$

Analogously, if $u(|x|^2) \in O'_C$, then $u(|x|^2) \in S'$, and then by Theorem XV, form. (VII, 8; 5), p. 268 of [3], we obtain the following formula

$$\mathcal{H}\left\{\left(u\left(|x|^2\right)\right)^{\frac{n-2}{2}+k} * \left(u\left(|x|^2\right)\right)^{\frac{n-2}{2}+\ell}\right\} = D\,\mathcal{H}\left\{\left(u\left(|x|^2\right)\right)^{\frac{n-2}{2}+k}\right\} \cdot \mathcal{H}\left\{\left(u\left(|x|^2\right)\right)^{\frac{n-2}{2}+\ell}\right\} , \qquad (\mathrm{II}, 17)$$

where D is the constant given by

$$D = 2^{2k+\frac{n}{2}}\Gamma\left(\frac{n}{2}+k\right)2^{2\ell+\frac{n}{2}}\Gamma\left(\frac{n}{2}+\ell\right) \ , \qquad (\mathrm{II}, 18)$$

From (II,17), (II,4) and (II,5), we have

$$\mathcal{H}\left\{\left(u\left(|x|^2\right)\right)^{\frac{n-2}{2}+k} * \left(u\left(|x|^2\right)\right)^{\frac{n-2}{2}+\ell}\right\} = D\,\delta^{(k)}\left(u(|x|^2)\right) \cdot \delta^{(\ell)}\left(u(|x|^2)\right) \ . \qquad (\mathrm{II}, 19)$$

Equivalently, from (II,19), we have the following formula

$$\mathcal{H}\left\{\delta^{(k)}\left(u\left(|x|^2\right)\right)^{\frac{n-2}{2}+k} * \delta^{(\ell)}\left(u\left(|x|^2\right)\right)^{\frac{n-2}{2}+\ell}\right\} = D\left\{\left(u\left(|x|^2\right)\right)^{\frac{n-2}{2}+k} * \left(u\left(|x|^2\right)\right)^{\frac{n-2}{2}+\ell}\right\} . \qquad (\mathrm{II}, 20)$$

Otherwise, from (II,6), the formula (II,15) is equivalently to the following equality

$$\delta^{(k)}\left(u\left(|x|^2\right)\right) * \delta^{(\ell)}\left(u\left(|x|^2\right)\right) = D\ \delta^{\left(k+\ell+\frac{n-2}{2}\right)}\left(u\left(|x|^2\right)\right) \ , \qquad (\mathrm{II}, 21)$$

where D is the constant given by (II,18).

III. "$\mathcal{H}\left\{\delta^{(\ell)}(P) * \delta^{(k)}(P)\right\} = \mathcal{H}\left\{\delta^{(\ell)}(P)\right\} \cdot \mathcal{H}\left\{\delta^{(k)}(P)\right\}$".

In this paragraph, we shall prove the following formula

$$\mathcal{H}\left\{\delta^{(\ell)}(P) * \delta^{(k)}(P)\right\} = \mathcal{H}\left\{\delta^{(\ell)}(P)\right\} \cdot \mathcal{H}\left\{\delta^{(k)}(P)\right\} \ , \qquad (\text{III}, 1)$$

here P is a non degenerate quadratic form in n variables of the form

$$P = P(x) = x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2 \ , \qquad (\text{III}, 2)$$

where $p+q=n$ and $x=(x_1,\ldots,x_n)$ be a point of the n-dimensional Euclidean space $I\!R^n$.

From formula (36), p. 55 of [4], we have

$$\mathcal{H}\left\{\delta^{(k)}(P)\right\} = \frac{1}{2^{2k+\frac{n}{2}}\Gamma\left(\frac{n}{2}+k\right)}\ Q^{\frac{n-2}{2}+k} \ , \qquad (\text{III}, 3)$$

where Q is given by

$$Q = Q(y) = y_1^2 + \cdots + y_p^2 - y_{p+1}^2 - \cdots - y_{p+q}^2 \ , \qquad (\text{III}, 4)$$

$p+q=n$.

Then

$$\mathcal{H}\left\{\delta^{(k)}(P)\right\} \cdot \mathcal{H}\left\{\delta^{(\ell)}(P)\right\} = \frac{Q^{\frac{n-2}{2}+k} \cdot Q^{\frac{n-2}{2}+\ell}}{2^{2k+\frac{n}{2}}\Gamma\left(\frac{n}{2}+k\right) 2^{2\ell+\frac{n}{2}}\Gamma\left(\frac{n}{2}+\ell\right)} \ . \qquad (\text{III}, 5)$$

Taking into account, the formula (I, 3; 17), p. 23 of [10], where λ, μ and $\lambda+\mu$ are positive integers and n is even, we obtain, from (III,5), the following formula

$$\mathcal{H}\left\{\delta^{(k)}(P)\right\} \cdot \mathcal{H}\left\{\delta^{(\ell)}(P)\right\} = \frac{Q^{n-2+k+\ell}}{2^{2k+\frac{n}{2}}\Gamma\left(\frac{n}{2}+k\right) 2^{2\ell+\frac{n}{2}}\Gamma\left(\frac{n}{2}+\ell\right)} \ . \qquad (\text{III}, 6)$$

Otherwise, from formula (38), p. 55 of [4], we know that

$$\mathcal{H}\left\{\delta^{(k)}(P)\right\} * \mathcal{H}\left\{\delta^{(\ell)}(P)\right\} = \frac{Q^{n-2+k+\ell}}{2^{2k+\frac{n}{2}}\Gamma\left(\frac{n}{2}+k\right) 2^{2\ell+\frac{n}{2}}\Gamma\left(\frac{n}{2}+\ell\right)} \ . \qquad (\text{III}, 7)$$

From (III,6) and (III,7) we arrive at the formula (III,1).

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