GRASSMANNIANS OF A FINITE ALGEBRA IN THE STRONG OPERATOR TOPOLOGY *

Esteban Andruchow and Lázaro Recht

Instituto de Ciencias, UNGS, Los Polvorines, Argentina
and
Depto. de Matemáticas P Y A, USB, Caracas, Venezuela

Abstract

If $\mathcal{M}$ is a type II$_1$ von Neumann algebra with a faithful trace τ, we consider the set $\mathcal{P}$ of selfadjoint projections of $\mathcal{M}$ as a subset of the Hilbert space $\mathcal{H} = L^2(\mathcal{M}, \tau)$. We prove that though it is not a differentiable submanifold, the geodesics of the natural Levi-Civita connection given by the trace have minimal length. More precisely: the curves of the form $\gamma(t) = e^{itx}pe^{-itx}$ with $x^* = x$, $pxp = (1-p)x(1-p) = 0$ have minimal length when measured in the Hilbert space norm of $\mathcal{H}$, provided that the *operator* norm $\|x\|$ is less or equal than $\pi/2$. Moreover, any two projections which are unitary equivalent are joined by at least one such minimal geodesic, and only unitary equivalent projections can be joined by a smooth curve. Finally, we prove that these geodesics have also minimal length if one measures them with the Schatten k-norms of τ, $\|x\|_k = \tau((x^*x)^{k/2})^{1/k}$, for all $k \in \mathbb{R}$, $k \geq 0$. We also characterize curves of unitaries which have minimal length with these k-norms.

Keywords: curves of projections, curves of unitaries.

1 Preliminaries

Let $\mathcal{M}$ be type II$_1$ von Neumann algebra with faithful normal trace τ. Denote by $\mathcal{H} = L^2(\mathcal{M}, \tau)$ the Hilbert space obtained by completion of $\mathcal{M}$ with the inner product given by τ: $< m', m >= \tau(m^*m')$, with norm $\|m\|_2 = \tau(m^*m)^{1/2}$. Denote by $\mathcal{P}$ the set of selfadjoint projections of $\mathcal{M}$, and by $U_{\mathcal{M}}$ the unitary group of $\mathcal{M}$. This note studies the geometric structure of $\mathcal{P}$ in the topology induced by τ. That is, we regard $\mathcal{P}$ as a subset of $\mathcal{H}$ (identifying $\mathcal{M}$ with its image in the standard representation in $\mathcal{H}$). It is well known that this topology coincides in $\mathcal{P}$ with the strong and weak operator topologies. We consider the following problems

1. Local (smooth?) structure of $\mathcal{P}$.

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2. Existence and uniqueness of short curves (geodesics) joining elements of $\mathcal{P}$. To measure lengths of (piecewise smooth) curves we use the metric of $\mathcal{H}$:

$$length(\gamma) = \int_0^1 \|\dot{\gamma}\|_2 dt,$$

where γ is parametrized in the interval $[0, 1]$.

Concerning the local structure of $\mathcal{P}$, first we prove that a curve in $\mathcal{P}$ which is differentiable (as a map on $\mathcal{H}$) has constant trace, i.e. remains inside a unitary orbit in $\mathcal{P}$. If $p \in \mathcal{P}$, denote by $\mathcal{P}_p = \{upu^* : u \in U_{\mathcal{M}}\}$. We show that $\mathcal{P}_p$ is not a differentiable submanifold of $\mathcal{H}$. Nevertheless, we introduce a riemannian metric and its corresponding Levi-Civita connection in $\mathcal{P}$. Namely, the trace inner product given by τ in every tangent space. This connection looks formally identical to the reductive connection of $\mathcal{P}$ in the norm topology (see [4]).

With respect to the second question, we prove that the geodesics of the Levi-Civita connection are short curves for the metric. More precisely: two given points p, q of $\mathcal{P}$, lying in the same unitary orbit, are joined by a minimizing geodesic, which is a curve of the form

$$\gamma(t) = e^{tx} p e^{-tx}, \quad t \in [0, 1]$$

with $x^* = -x$ codiagonal with respect to p and $\|x\| \le \pi/2$. If $\|x\| < \pi/2$ (a fact equivalent to $\|p - q\| < 1$), then the geodesic is unique.

Moreover, we prove that these curves have also minimal length, if one measures them using the k-norm, $\|x\|_k = \tau((x^*x)^{k/2})^{1/k}$, for $k \in \mathbb{R}$, $k \ge 2$. This strenghtens a result in [5], where minimality was proved among so called "non wandering curves", and for k an even integer. Non wandering curves are curves of the form $e^{ix(t)} p e^{-ix(t)}$, were $x(t)$ is a piecewise (norm) smooth curve of selfadjoint elements with $\|x(t)\| < \pi/2$. To illustrate this difference, our result here shows that γ remains minimal also among *strongly* differentiable curves with unbounded velocity vectors(=selfadjoint operators affiliated to $\mathcal{M}$).

In order to prove this, we use a stronger fact. Namely, that curves of the form $\alpha(t) = e^{itx}u$ with u unitary and x selfadjoint have minimal length among (piecewise C^1) curves of unitaries when measured with the k-norms, provided that $\|x\| \le \pi$. For $k = 2$ this was proved in [1]. For $k > 2$ it is proved here.

The contents of the paper are as follows. In section 2 we establish basic facts and notation. In section 3 we introduce de Levi-Civita connection of $\mathcal{P}$, and prove that any two points in a unitary orbit in $\mathcal{P}$ are joined by a minimizing geodesic (in the 2-norm). In section 4 we prove that $\mathcal{P} \subset \mathcal{H}$ is not a differentiable submanifold, but that the geodesic distance and the 2-metric of $\mathcal{H}$ are equivalent in $\mathcal{P}$. In section 5 we prove the minimality result for curves of unitaries in the k-norms. As a consequence, we obtain the minimality result in $\mathcal{P}$ for these norms.

2 Unitary orbits

In the norm topology, the connected components of the set $\mathcal{P}$ of projections of a von Neumann algebra are precisely the unitary orbits. For example, if $\mathcal{M} = \mathcal{B}(H)$, they are parametrized by the rank and corank of the projections. Since algebras of type II_1 have continuous dimension, it is natural that nonequivalent projections can be joined by a strong operator continuous curve. For example, let $\mathcal{M} = L^\infty(0, 1) \otimes \mathcal{M}_0$ with $\mathcal{M}_0$ also of type II_1 and trace τ_0. Put τ in $\mathcal{M}$ given by $\tau(f \otimes x) = (\int_0^1 f(s)ds)\tau_0(x)$. Then $\delta(t) = \chi_{(0,t)} \otimes 1_{\mathcal{M}_0}$ is a continuous curve of projections joining 0 and 1. Note that δ, regarded as a curve in $\mathcal{H}$, with the 2-norm $\| \ \|_2$, is non differentiable. We show now that if a curve of projections is differentiable (as a curve in $\mathcal{H}$), then it lies inside a unitary orbit.

Let us fix some notation. If $\xi \in \mathcal{H}$ and $a \in m$, we denote by $a\xi$ (resp. ξa) the vector $L_a\xi$ (resp. $R_a\xi$), where L_a (resp. R_a) is the representation of $\mathcal{M}$ as left (resp. right) multiplication operator in $\mathcal{H} = L^2(\mathcal{M}, \tau)$. As is usual, denote by J the completion of the semilinear isometric operator $\mathcal{M} \ni x \mapsto x^* \in \mathcal{M}$. Note that the trace τ can be interpreted as $\tau(a) =< a, 1 >$, where 1 here is regarded as a (ciclic) vector in $\mathcal{H}$. Then τ can be extended to a continuous functional in $\mathcal{H}$, which we still denote by τ. Note that $\tau(a\xi) = \tau(\xi a)$. If $\mathcal{M}$ is not a factor, let Tr be the center valued trace. Since this map is normal, it can be extended to $\mathcal{H}$, and we denote also by Tr its extension.

Proposition 2.1 *Let γ be a curve in $\mathcal{P}$, which is differentiable as a curve in $\mathcal{H}$. Then γ stays within a unitary orbit.*

Proof. Note that since the product is differentiable, the fact $\gamma(t)^2 = \gamma(t)$ implies that

$$\dot{\gamma}\gamma + \gamma\dot{\gamma} = \dot{\gamma}, \tag{2.1}$$

and therefore, multiplying by γ on both sides, $\gamma\dot{\gamma}\gamma = 0$. Then $Tr(\gamma\dot{\gamma}\gamma) = Tr(\gamma\dot{\gamma}) = Tr(\dot{\gamma}\gamma) = 0$. Taking trace in (2.1), one obtains $Tr(\dot{\gamma}) = 0$, i.e. $Tr(\gamma)$ is constant. □

As in the proof above, the velocity vectors of curves in $\mathcal{P}$ at p are vectors ξ satisfying $\xi p + p\xi = \xi$ and $J\xi = \xi$, the latter because $\mathcal{P} \subset \mathcal{M}_h$. We define

$$(T\mathcal{P})_p = \{\xi \in \mathcal{H} : J\xi = \xi \text{ and } \xi p + p\xi = \xi\}. \tag{2.2}$$

Then $(T\mathcal{P})_p = \{\xi \in \mathcal{H} : J\xi = \xi, (p + JpJ)\xi = \xi\}$. Note that $(T\mathcal{P})_p$ is closed in $\mathcal{H}$.

If $\xi \in \mathcal{H}$, denote by L_ξ the closure of the densely defined linear (possibly unbounded, affiliated to $\mathcal{M}$) operator given by $L_\xi x = \xi x$, if $x \in \mathcal{M} \subset \mathcal{H}$. If $J\xi = \xi$, then L_ξ is selfadjoint [11].

Proposition 2.2 *Any element $\xi \in (T\mathcal{P})_p$ is the velocity of a C^1 curve $\alpha(t) \in \mathcal{P}$: $\alpha(0) = p$ and $\dot{\alpha}(0) = \xi$. If $\xi = x \in \mathcal{M}$, the curve α can be chosen C^∞ .*

Proof. Let $\nu = i(p\xi - \xi p)$. Then clearly $J\nu = \nu$. Put $\alpha(t) = e^{itL_\nu} p e^{-itL_\nu}$. This curve is continuous and takes values in $\mathcal{P}$. Moreover, it can be differentiated in $\mathcal{H}$,

$$\dot{\alpha}(t) = i\nu\alpha(t) - i\alpha(t)\nu,$$

which is continuous in $\mathcal{H}$. At $t = 0$ one gets

$$\dot{\alpha}(0) = i\nu p - ip\nu = (\xi p - p\xi)p - p(\xi p - p\xi) = \xi p + p\xi = \xi,$$

because $\xi \in (T\mathcal{P})_p$ implies that $p\xi p = 0$. Note that if $\xi = x \in \mathcal{M}$, then α is C^∞ (the obstruction to further regularity of α is that the powers of ν may lie outside $L^2(\mathcal{M}, \tau)$). □

Let us finish this section by stating these basic facts. Certainly they are well known.

Proposition 2.3 *Both $\mathcal{P}$ and the unitary orbit $\mathcal{P}_p = \{upu^* : u \in U_\mathcal{M}\}$ are closed in $\mathcal{H}$.*

Proof. Let p_n be a sequence in $\mathcal{P}$ converging to ξ in $\mathcal{H}$. Then, for any $\eta \in \mathcal{H}$, $p_n\eta$ is a Cauchy sequence in $\mathcal{H}$. Indeed, if $\eta = x \in \mathcal{M} \subset \mathcal{H}$,

$$\|p_n x - p_k x\|_2^2 = \tau(x^*(p_n - p_k)^2 x) = \tau((p_n - p_k)x^*x(p_n - p_k)) \le \|x\|^2\tau((p_n - p_k)^2) = \|x\|^2\|p_n - p_k\|_2^2.$$

In general, there exists $x \in \mathcal{M}$ such that $\|x - \eta\|_2 < \epsilon/2$. And therefore

$$\|(p_n - p_k)\eta\|_2 \le \|(p_n - p_k)x\|_2 + \|(p_n - p_k)(\eta - x)\|_2 \le \|(p_n - p_k)x\|_2 + 2\|\eta - x\|_2 < \|(p_n - p_k)x\|_2 + \epsilon.$$

Therefore p_n converges strongly to a linear operator in $\mathcal{H}$, which is bounded by the uniform boundedness principle, i.e. $p_n \to a \in \mathcal{M}$. By strong continuity of the product and the adjoint ($\mathcal{M}$ is finite), clearly $a^2 = a^* = a$.

Let $u_n \in U_{\mathcal{M}}$ such that $u_n p u_n^*$ is convergent in $\mathcal{H}$, by the above argument, to a projection $q \in \mathcal{P}$. Since the center valued trace Tr of $\mathcal{M}$ is strongly continuous, $Tr(p) = Tr(u_n p u_n^*) \to Tr(q)$, and therefore p and q are unitarily equivalent. □

3 Levi-Civita connection of the trace

We shall treat $\mathcal{P}$ as if it were a submanifold of $\mathcal{H}$ (which we shall see later that it is not). We endow each tangent space with the ambient metric (inner product) given by the trace τ: if $\xi, \eta \in (T\mathcal{P})_p$, $<\xi, \eta>_p = <\xi, \eta>$, which is real valued because $J\xi = \xi$ and $J\eta = \eta$. Since we consider $\mathcal{P}$ with the flat euclidean metric at each point, the Levi-Civita connection consists of projecting orthogonally onto $T\mathcal{P}$ the usual derivative in $\mathcal{H}$ of a field. That is, if $X(t)$ is a C^1, $\mathcal{H}$ valued map, which is tangent along a curve γ in $\mathcal{P}$, i.e. $X(t) \in (T\mathcal{P})_{\gamma(t)}$, then

$$\frac{DX}{dt} = \Pi_\gamma(\dot{X}).$$

The projection $\Pi_p : \mathcal{H} \to (T\mathcal{P})_p$ can be computed explicitely:

$$\Pi_p(\nu) = \frac{1}{2}(1-p)(\nu + J\nu)p + \frac{1}{2}p(\nu + J\nu)(1-p).$$

Indeed, note that if $\nu \in (T\mathcal{P})_p$, $J\nu = \nu$ and then $\Pi_p(\nu) = \nu$, and clearly Π_p takes values in $(T\mathcal{P})_p$. It remains to verify that Π_p is orthogonal, which is straightforward. In particular, a curve γ is a geodesic if it satisfies

$$0 = \Pi_\gamma(\ddot{\gamma}) = (1-\gamma)\ddot{\gamma}\gamma + \gamma\ddot{\gamma}(1-\gamma). \tag{3.3}$$

If $\xi \in \mathcal{H}$ with $J\xi = \xi$ and $p\xi p = 0 = (1-p)\xi(1-p)$ then $\gamma(t) = e^{itL_\xi} p e^{-itL_\xi}$ is a C^2 geodesic of $\mathcal{P}$, provided that $\xi^2 \in L^2(\mathcal{M}, \tau)$. This is to ensure that $\ddot{\gamma}$ remains in $L^2(\mathcal{M}, \tau)$. Indeed,

$$\ddot{\gamma} = e^{itL_\xi}(-\xi^2 p + 2\xi p \xi - p\xi^2)e^{-itL_\xi},$$

and therefore (3.3) yields

$$e^{itL_\xi}\{p(-\xi^2 p + 2\xi p\xi - p\xi^2)(1-p) + (1-p)(-\xi^2 p + 2\xi p \xi - p\xi^2)p\}e^{-itL_\xi}$$
$$= e^{itL_\xi}\{2p\xi p\xi(1-p) - p\xi^2(1-p) - (1-p)\xi^2 p + 2(1-p)\xi p\xi p\}e^{-itL_\xi}.$$

Note that $p\xi p = 0$ and ξ^2 commutes with p, so that the expression above equals zero.

If $\alpha(t)$, $t \in [0,1]$ is a C^1 curve in $\mathcal{P}$, we measure its length as follows:

$$length(\alpha) = \int_0^1 \|\dot{\alpha}\|_2 dt.$$

and define the rectifiable metric accordingly

$$d(p,q) = inf\{length(\alpha) : \alpha \text{ is piecewise smooth and joins } p \text{ and } q\}, \quad p, q \in \mathcal{P}. \tag{3.4}$$

Note that in fact the metric d is finite if and only if p and q are unitarily equivalent. Let us state the following result, which is a consequence of Th. 3.3 in [5]:

Theorem 3.1 (Th.3.3, [5]) *If γ is a C^2 curve in $\mathcal{P}_p$, which achieves the distance between its endpoints, then it is a geodesic of the above connection.*

Theorem 3.3 of [5] was proved in more generality, for the metric induced by the k-norms (k even), but for the class of smooth (=C^∞) curves. However, the proof remains valid for C^2 curves, with unbounded (outside $\mathcal{M}$) velocity.

There is an alternate description for $\mathcal{P}$, as (selfadjoint) symmetries or reflections. Namely

$$\mathcal{E} = \{\epsilon \in \mathcal{M} : \epsilon^2 = 1, \epsilon^* = \epsilon\}.$$

One passes from projections to symmetries by means of the affine, one to one map $p \mapsto \epsilon = 2p - 1$. The Riemanian metric and the Levi-Civita connection can be translated to $\mathcal{E}$, the above map is an isometric isomorphism with a correction factor 2. Geodesics have a particular nice description in this setting. The velocity vector ξ of a geodesic γ of $\mathcal{P}$ starting at p, is codiagonal with respect to p, and therefore anticommutes with $\epsilon = 2p - 1$: $\xi\epsilon = -\epsilon\xi$. Therefore the corresponding geodesic $\rho = 2\gamma - 1$ of $\mathcal{E}$ is of the form

$$\rho(t) = e^{2itL_\xi}\epsilon = \epsilon e^{-2itL_\xi}. \tag{3.5}$$

Theorem 3.2 *Let $p, q \in \mathcal{P}$ such that $\|p - q\| < 1$. Then there exists a unique C^2 curve $\gamma(t)$, $t \in [0, 1]$, with $\gamma(0) = p$ and $\gamma(1) = q$, such that*

$$length(\gamma) = d(p, q).$$

Moreover, this curve is a C^∞ geodesic.

Proof. It is well known([3],[8],[4]) that $\|p - q\| < 1$ is equivalent to the existence of $x \in \mathcal{M}$ such that $x^* = x$, $\|x\| < \pi/2$, x is codiagonal with respect to p and $e^{ix}pe^{-ix} = q$. Then $\gamma(t) = e^{itx}pe^{-itx}$ is a C^∞ geodesic of $\mathcal{P}$ with $\gamma(0) = p$ and $\gamma(1) = q$. Let us prove that it has minimal length. Denote by $\rho(t) = 2\gamma(t) - 1$ the corresponding geodesic of $\mathcal{E}$. Note that the velocity vector of ρ satisfies $\|2x\| < \pi$. In [1] it was proven that a curve of the form $e^{itz}u$ with u unitary and z selfadjoint such that $\|z\| < \pi$, is the shortest curve of unitaries of $\mathcal{M}$ joining u and $e^z u$, when the length is measured with the $\| \ \|_2$ norm. It follows that if α is a curve in $\mathcal{P}$ joining p and q, the curve $2\alpha - 1$ is longer that ρ. Therefore

$$length(\gamma) = \frac{1}{2}length(\rho) \leq \frac{1}{2}length(2\alpha - 1) = length(\alpha).$$

Clearly this curve γ is unique. □

In other words, two projections such that $\|p - q\| < 1$, are joined by a unique C^∞ geodesic of the $\| \ \|_2$ metric, which is minimal with respect to this metric. This strenghtens partially a result in [5], where it was shown that these geodesics are minimal for all the k-norms, but only among what they call there "non wandering curves". In section 5 we show that this results also holds for the k-norms, with real k.

It remains to examine the case $\|p - q\| = 1$. The following result settles this case. We use ideas and facts from [12] and [3].

Theorem 3.3 *If $\|p - q\| = 1$, then there exists a C^∞ minimizing geodesic in $\mathcal{P}_p$ joining p and q, with length at most $\pi/2$.*

Proof. In Prop. 5 of [3] it is shown that if p and q are projections in a von Neumann algebra, with $\|p-q\| = 1$ and $rank(p \wedge (1-q)) = rank(q \wedge (1-p))$ then there exists a smooth curve of projections in $\mathcal{H}$ joining p and q, of length $\pi/2$. The length is measured with respect to the rectifiable metric with the usual norm. First note that in a II_1 von Neumann algebra, $rank(p\wedge(1-q)) = rank(q\wedge(1-p))$. This fact is probably well known ([12]). We include a proof. Let Tr be the center valued trace of $\mathcal{M}$. Recall that $p\wedge(1-q)$ is the strong limit of the powers $(p(1-q))^n$. Since Tr is strongly continuous, to prove the claim it suffices to show that for all $n\geq 1$,

$$Tr((p-pq)^n) = Tr((q-qp)^n).$$

In the span of $(p-pq)^n$, the monomial

$$p^{i_1}(pq)^{j_1}p^{i_2}(pq)^{j_2}\dots(pq)^{j_k}p^{j_{k+1}} = (pq)^{j_1+\dots+j_k}p^{j_{k+1}},$$

can be paired up with the corresponding monomial in the span of $(q-qp)^n$,

$$q^{i_1}(qp)^{j_1}q^{i_2}(qp)^{j_2}\dots(qp)^{j_k}q^{j_{k+1}} = (qp)^{j_1+\dots+j_k}q^{j_{k+1}}.$$

Clearly, both monomials have the same trace. In Prop. 5 of [3], Brown uses a canonic representation ρ of the C*-algebra generated by p and q to find a curve joining them. This representation sends p and q to (respectively)

$$\rho(p) = 0\oplus\begin{pmatrix}1&0\\0&0\end{pmatrix}\oplus\begin{pmatrix}1&0\\0&0\end{pmatrix}\oplus 1$$

and

$$\rho(q) = 0\oplus\begin{pmatrix}0&0\\0&1\end{pmatrix}\oplus\begin{pmatrix}\cos^2\varphi&\cos\varphi\sin\varphi\\\cos\varphi\sin\varphi&\sin^2\varphi\end{pmatrix}\oplus 1,$$

where φ is a measurable function on certain measure space X, with $0\leq\varphi(x)\leq\pi/2$. A straightforward computation shows that

$$exp\left(\begin{pmatrix}0&-\varphi\\\varphi&0\end{pmatrix}\right)\begin{pmatrix}1&0\\0&0\end{pmatrix}exp\left(\begin{pmatrix}0&\varphi\\-\varphi&0\end{pmatrix}\right) = \begin{pmatrix}\cos^2\varphi&\cos\varphi\sin\varphi\\\cos\varphi\sin\varphi&\sin^2\varphi\end{pmatrix}$$

(here exp denotes the usual exponential). In particular, also

$$exp\left(\begin{pmatrix}0&-\pi/2\\\pi/2&0\end{pmatrix}\right)\begin{pmatrix}1&0\\0&0\end{pmatrix}exp\left(\begin{pmatrix}0&\pi/2\\-\pi/2&0\end{pmatrix}\right) = \begin{pmatrix}0&0\\0&1\end{pmatrix}.$$

Therefore, if we denote

$$z = 0\oplus\begin{pmatrix}0&-\pi/2\\\pi/2&0\end{pmatrix}\oplus\begin{pmatrix}0&-\varphi\\\varphi&0\end{pmatrix}\oplus 0,$$

then

$$e^z\rho(p)e^{-z} = \rho(q).$$

Note that z is antihermitian and lies in the C*-algebra generated by $\rho(p)$ and $\rho(q)$. Let x be an element in the C*-algebra generated by p and q (in $\mathcal{M}$) such that $\rho(ix) = z$. Then x is selfadjoint, is codiagonal with respect to p (i.e. $x\in(T\mathcal{P})_p$) and verifies

$$e^{ix}pe^{-ix} = q.$$

Note also that $\|x\| = \|z\| = \pi/2$. In particular, $\delta(t) = e^{itx}pe^{-itx}$ is a geodesic $\mathcal{P}_p$ with $\delta(0) = p$ and $\delta(1) = q$. To finish the proof, it remains to prove that this geodesic δ is minimizing for the 2-metric. Suppose that γ joins p and q and $length(\delta) - length(\gamma) = r > 0$. For $t_r < 1$, let δ_r be the restriction of δ to the interval $[0, t_r]$. Note that, by (3.2), δ_r is a minimizing geodesic. Adjust t_r such that the length of the curve $e^{itx}pe^{-itx}$ with $t\in[t_r,1]$, is less than $r/2$. Then this curve, followed by γ, form a piecewise C^1 curve which joins p and $e^{it_rx}pe^{-it_rx}$. Moreover, it measures $length(\gamma) + r/2 < length(\delta_r)$, which is a contradiction. □

In this case, the geodesic δ may not be unique.

Corollary 3.4 *The exponential map*

$$exp_p : (T\mathcal{P})_p \to \mathcal{P}_p, exp_p(x) = e^{ix}pe^{-ix}$$

is surjective.

The above corollary is of course also valid in the norm topology context of [4], if the algebra is (a von Neumann algebra) of type II_1. There it was shown that exp_p fills the set $\{q \in \mathcal{P} : \|q-p\| < 1\}$.

Corollary 3.5 *The geodesic radius of $\mathcal{P}_p$ in the 2-metric is $\pi/2$.*

Proof. $length(\delta) = \|xp - px\|_2 \leq \|xp - px\| = \|x\| \leq \pi/2$. □

Remark 3.6 *The curvature tensor of this connection is given by* [4]

$$R_p(x, y, z) = [[x, y], z],\ \ x, y, z \in (T\mathcal{P})_p,$$

where $[\ ,\ ]$ is the usual commutant of operators. The sectional curvature of $\mathcal{P}$ is non negative:

$$< R_p(x, y)y, x >= 2(\tau(x^2y^2) - \tau(xyxy)).$$

Note that $\tau(xyxy) =< xy, yx >\leq< xy, xy >^{1/2}< yx, yx >^{1/2}= \tau(x^2y^2)$ by the Cauchy-Schwarz inequality. In particular, the sectional curvature vanishes if and only equality occurs, which implies that there exists $\lambda \geq 0$ such that $xy = \lambda yx$ (or $yx = \lambda xy$, which is dealt analogously). Since x and y are selfadjoint, this implies that $yx = \lambda xy = \lambda^2 yx$. Then either $yx = xy = 0$ or $\lambda = 1$. Therefore vanishing of the sectional curvature implies commutation of x and y, the generators of the corresponding plane in $(T\mathcal{P})_p$.

4 $\mathcal{P}$ is not a submanifold of $\mathcal{H}$

Here we prove that $\mathcal{P}$ (or rather $\mathcal{P}_p$) is not a submanifold of $\mathcal{H} = L^2(\mathcal{M}, \tau)$, a fact which makes more relevant the minimality results of the previous section. As a consequence we obtain that the mapping $\xi \mapsto e^{i\xi}pe^{-i\xi}$, for $\xi \in \mathcal{H}$ with $J\xi = \xi$, is non differentiable. We also prove that, though $\mathcal{P}_p$ is not a differentable manifold, the geodesic distance and the norm $\|\ \|_2$ are equivalent metrics in $\mathcal{P}_p$.

Theorem 4.1 *Let $p \in \mathcal{P}$, denote by $\mathcal{P}_p$ the unitary orbit of p. Then $\mathcal{P}_p \subset \mathcal{H}$ is not a differentiable submanifold.*

Proof. Suppose that $\mathcal{P}_p \subset \mathcal{H}$ is a differentiable submanifold. Then the tangent space at p is the set denoted earlier by $(T\mathcal{P})_p$. The trace inner product provides each tangent space with a complete inner product. Therefore $\mathcal{P}_p$ is a Riemannian manifold. In particular, there is a normal neighbourhood for p, i.e. a ball in the $\|\ \|_2$ metric of $T(\mathcal{P})_p$ where the exponential map

$$exp_p : (T\mathcal{P})_p \to \mathcal{P}_p$$

is a diffeomorphism. By (3.1), the exponential map is given by

$$exp_p(z) = e^{iz}pe^{-iz}, z \in (T\mathcal{P})_p.$$

Suppose r is the radius of such a normal ball around the origin in $(T\mathcal{P})_p$ and fix $r > \delta > 0$. Let q_n and q'_n be projections in $\mathcal{M}$ such that

$$q_n \leq p,\ q'_n \leq 1 - p,\ q_n \sim q'_n \text{ and } \tau(q_n) = \tau(q'_n) = \delta^2/n^2.$$

Let v_n be partial isometries in $\mathcal{M}$ implementing the equivalence between q_n and q_n': $v_n^*v_n = q_n$ and $v_nv_n^* = q_n'$. Note that $\tau(v_n) = 0$. Put $a_n = \frac{n}{\sqrt{2}}(v_n + v_n^*)$. Then $a_n^* = a_n$ and $a_n^2 = n^2/2(v_n^*v_n + v_nv_n^*) = n^2/2(q_n + q_n')$, because $v_n^2 = 0$. Then

$$\|a_n\|_2 = \tau(a_n^2)^{1/2} = (n^2\tau(q_n))^{1/2} = \delta.$$

Compute the powers a_n^k: for k even,

$$(v_n + v_n^*)^k = q_n + q_n',$$

for k odd

$$(v_n + v_n^*)^k = v_n + v_n^*.$$

Then

$$e^{ia_n} = 1 + (q_n + q_n') \sum_{k>0 \text{ even}} (\frac{i\delta n}{\sqrt{2}})^k \frac{1}{k!} + (v_n + v_n^*) \sum_{k \text{ odd}} (\frac{i\delta n}{\sqrt{2}})^k \frac{1}{k!}$$

Then

$$\tau(e^{ia_n}) = 1 + 2(\cos(n/\sqrt{2}) - 1)\tau(q_n) = 1 + 2(\cos(n/\sqrt{2}) - 1)\frac{\delta^2}{n^2}.$$

Then $\tau(e^{ia_n}) \to 1$. Analogously $\tau(e^{-ia_n}) \to 1$. It follows that

$$\|e^{\pm ia_n} - 1\|_2^2 = 2 - \tau(e^{ia_n}) - \tau(e^{-ia_n}) \to 0.$$

The elements a_n lie in $(T\mathcal{P})_p$. Indeed, this is a consequence of $pv_np = (1-p)v_n(1-p) = 0$. This leads to a contradiction. On one hand, a_n form a sequence in a normal ball in $(T\mathcal{P})_p$ which do not converge to zero, because $\|a_n\|_2 = \delta > 0$. On the other hand, $e^{\pm ia_n} \to 1$ strongly, a fact which implies that

$$exp_p(a_n) \to p,$$

strongly, which contradicts the fact that exp_p is a (local) homeomorphism. □

Denote by $\mathcal{H}_h = \{\xi \in \mathcal{H} : J\xi = \xi\}$, i.e. the completion of the real subspace of selfadjoint elements of $\mathcal{M}$ in the 2-norm. In [2] it was proven that the exponential map

$$\mathcal{H}_h \to U_{\mathcal{M}} \subset \mathcal{H},\ \xi \mapsto e^{iL_\xi} \tag{4.6}$$

is continuous, weakly C^1 but non (strongly) differentiable. The following is a related result.

Corollary 4.2 *If $p \in \mathcal{P}$, the map*

$$exp_p : (T\mathcal{P})_p \to \mathcal{P}_p \subset \mathcal{H},\ exp_p(\xi) = e^{iL_\xi}pe^{-iL_\xi}$$

is continuous but non differentiable. Moreover, the restriction of the map (4.6)

$$(T\mathcal{P})_p \to U_{\mathcal{M}},\ \xi \mapsto e^{iL_\xi}$$

is non differentiable.

Proof. If exp_p were differentiable, it would provide a local chart for $\mathcal{P}_p \subset \mathcal{H}$ near p. Translating this chart via the unitary action would endow $\mathcal{P}$ with an atlas, a fact which contradicts the result above. If the second map $(T\mathcal{P})_p \ni \xi \mapsto e^{iL_\xi}$ were differentiable, then exp_p, which can be described in terms of products of this map, would be differentiable. Note that the product of differentiable maps on $\mathcal{H}$, which are uniformly bounded in the *usual* norm of $\mathcal{M}$, is also differentiable. □

In view of (3.2) and (3.3), for each pair of equivalent projections p, q, there exists $x \in (T\mathcal{P})_p$ with $\|x\| \leq \pi/2$ such that $q = e^{ix}pe^{-ix}$ and the correspondig geodesic is minimal. Then the geodesic distance equals

$$d(p,q) = \|ixp - ipx\|_2 = (2\tau(x^2p))^{1/2} = \tau(x^2)^{1/2}.$$

Proposition 4.3 *The metric d and the norm $\| \ \|_2$ are equivalent in $\mathcal{P}_p$.*

Proof. With the above notations, one has

$$\|p-q\|_2^2 = \|p - e^{ix}pe^{-ix}\|_2^2 = 2\tau(p) - 2\tau(pe^{ix}pe^{-ix}p).$$

Note that since $x = xp + px$, x^2 commutes with p. Then $pe^{\pm ix}p = p\cos(x)p$. Also note that since x is selfadjoint with $\|x\| \leq \pi/2$,

$$\frac{4}{\pi}x^2 \leq 1 - \cos(x) \leq \frac{1}{2}x^2,$$

and therefore

$$\frac{4}{\pi^2}px^2p \leq p - p\cos(x)p \leq \frac{1}{2}px^2p. \tag{4.7}$$

On the other hand, $\tau\big(p-(p\cos(x)p)^2\big) = \tau\big((p-p\cos(x)p)(p+p\cos(x)p)\big)$, and $p \leq p+p\cos(x)p \leq 2p$. Then, using that $p - p\cos(x)p \geq 0$,

$$p - p\cos(x)p \leq (p-p\cos(x)p)^{1/2}(p+p\cos(x)p)(p-p\cos(x)p)^{1/2} \leq 2(p-p\cos(x)p).$$

Taking traces one obtains

$$2\tau(p-p\cos(x)p) \leq \|p-q\|_2^2 \leq 4\tau(p-p\cos(x)p).$$

Combining this with the elementary estimate (4.7), one gets

$$\frac{8}{\pi^2}\tau(px^2p) \leq \|p-q\|_2^2 \leq 2\tau(px^2p),$$

i.e.

$$\frac{2}{\pi}d(p,q) \leq \|p-q\|_2 \leq d(p,q).$$

□

5 k-norms

In this section we study the minimality problem of geodesics in $\mathcal{P}$ measured in the k-norms, for $k \in \mathbb{R}$, $k > 2$. To do this we study first short curves of unitaries in these norms. Minimality of geodesics in $\mathcal{P}$ will follow with arguments similar as in (3.2) and (3.3). As is standard notation [11], for $x \in \mathcal{M}$, let

$$\|x\|_k = \tau((x^*x)^{k/2})^{1/k},$$

and denote by $\mathcal{L}^k = L^k(\mathcal{M}, \tau)$ the completion of $\mathcal{M}$ with this norm $\| \ \|_k$. Fix $r > 0$ and let

$$\mathcal{S}_r^k = \{x \in \mathcal{L}^k : \|x\|_k = r\}$$

be the sphere of radius r in $\mathcal{L}^k$. Let us transcribe Jensen's trace inequality for C*-algebras by Hansen and Pedersen ([7], Th. 2.7): if $f(t)$ is a convex continuous real function, defined on an interval I and and $\mathcal{A}$ is a C*-algebra with trace tr, then the inequality

$$tr\big(f(\sum_{i=1}^n b_i^* a_i b_i)\big) \leq tr\big(\sum_{i=1}^n b_i^* f(a_i) b_i\big) \tag{5.8}$$

is valid for every n-tuple $(a_1,\dots,a_n)$ of selfadjoint elements in $\mathcal{A}$ with spectra contained in I and every n-tuple $(b_1,\dots,b_n)$ in $\mathcal{A}$ with $\sum_{i=1}^n b_i^* b_i = 1$. The following inequality is a trivial consequence of (5.8) in its simplest form: a is a selfadjoint element in a C^*-algebra with trace tr, then $tr(f(a)) \leq f(tr(a))$ for every convex continuous real function defined in the spectrum of a. We state it as a lemma for it will be useful below.

Lemma 5.1 *Let $a \in \mathcal{M}$ be positive and $p \in \mathcal{P}$. Then, if $r \in \mathbb{R}$, $r \geq 1$*

$$\tau(pap)^r \leq \tau(p)^{r-1}\tau((pap)^r).$$

Proof. If $p = 0$ the result is trivial. Suppose $\tau(p) \neq 0$. Consider the algebra $p\mathcal{M}p$, with unit p and normalized trace $tr(pxp) = \frac{\tau(pxp)}{\tau(p)}$. Then by Jensen's trace inequality for the map $f(t) = t^r$,

$$\frac{\tau(pap)^r}{\tau(p)^r} \leq \frac{\tau((pap)^r)}{\tau(p)},$$

which is the desired inequality. □

Using this inequality we obtain a minimality result for curves in spheres of the k-norms, for $k > 2$. If $\mu(t)$ is a curve of unitaries in $\mathcal{M}$, and p is a projection with $\tau(p) = r^k$, then $\mu(t)p$ is a curve in $\mathcal{S}_r^k$: $\|\mu p\|_k = \tau((p\mu^*\mu p)^{k/2})^{1/k} = r$.

Lemma 5.2 *Let $\mu(t)$ be a smooth curve of unitaries in $\mathcal{M}$, such that $\mu(0)p = p$ and $\mu(1)p = e^{i\alpha}p$ with $-\pi < \alpha < \pi$. Then the curve μp of $\mathcal{S}_r^k$, measured with the $k-norm$, is longer than the curve $\epsilon(t) = e^{it\alpha}p$.*

Proof. The length of μp is (in the k-norm) measured by

$$\int_0^1 \|\dot\mu(t)p\|_k dt = \int_0^1 \tau((p\dot\mu(t)^*\dot\mu(t)p)^{k/2})^{1/k} dt.$$

by the inequality in the above lemma,

$$length_k(\mu p) \geq \tau(p)^{\frac{k/2-1}{k}} \int_0^1 \tau(p\dot\mu(t)^*\dot\mu(t)p)^{1/2} dt.$$

This last integral measures the length of the curve μp in the 2-sphere $\mathcal{S}_r^2$ of radius $r^{1/2}$ in the Hilbert space $L^2(\mathcal{M},\tau)$. It is well known that the curves $\epsilon(t) = e^{it\alpha}p$ are minimizing geodesics of these spheres, provided that $|\alpha| r^{1/2} < \pi$, which holds because $r = \tau(p)^{1/k} < 1$. It follows that

$$\int_0^1 \tau(p\dot\mu(t)^*\dot\mu(t)p)^{1/2} \geq length_2(\epsilon) = |\alpha|\tau(p)^{1/2}.$$

Then

$$length_k(\mu p) \geq |\alpha|\tau(p)^{\frac{k/2-1}{k}}\tau(p)^{1/2} = |\alpha|\tau(p)^{1/k} = length_k(\epsilon).$$

□

Lemma 5.3 *Let $x = \sum_{i=1}^n \alpha_i p_i$ with $\sum_{i=1}^n p_i = 1$ and $-\pi < \alpha_i < \pi$ (i.e. $\|x\| < \pi$). Then the curve $\delta(t) = e^{itx}$, $t \in [0,1]$ is the shortest curve in $U_{\mathcal{M}}$ joining its endpoints, when measured with the k-norm.*

Proof. Let $\tau(p_i)=r_i^k$, and put
$$\mathcal{S}_{\vec{r}}=\mathcal{S}^k_{r_1}\times\ldots\times\mathcal{S}^k_{r_n}.$$
Consider in $\mathcal{S}_{\vec{r}}$ the following norm:
$$\|(x_1,\ldots,x_n)\|=\{\sum_{i=1}^n\|x_i\|_k^k\}^{1/k}.$$
Consider the map
$$\rho:U_{\mathcal{M}}\to\mathcal{S}_{\vec{r}},\ \ \rho(u)=(up_1,\ldots,up_n).$$
Clearly it is well defined. Let us prove that it decreases the lengths of curves. Indeed, let $\mu(t)\in U_{\mathcal{M}}$ be a smooth curve of unitaries, $t\in[0,1]$. The length of μ is measured by $\int_0^1\|\dot\mu(t)^*\dot\mu(t)\|_k dt$. The length of $\rho(\mu)$ is $\int_0^1\|(\dot\mu(t)p_1,\ldots,\dot\mu(t)p_n)\|dt$. Let us show that
$$\|\dot\mu(t)^*\dot\mu(t)\|_k\ge\|(\dot\mu(t)p_1,\ldots,\dot\mu(t)p_n)\|. \tag{5.9}$$
Note that
$$\|(\dot\mu(t)p_1,\ldots,\dot\mu(t)p_n)\|=\{\sum_{i=1}^n\tau\big(p_i\dot\mu(t)^*\dot\mu(t)p_i)^{k/2}\big)\}^{1/k}.$$
This inequality (5.9) is again a consequence of Jensen's trace inequality [7] for the convex map $f(t)=t^{k/2}$ $(k\ge 2)$, putting $b_i=p_i$ and $a_i=a$ in (5.8): $\sum_{i=1}^n\tau(p_ia^{k/2}p_i)\ge\sum_{i=1}^n\tau([p_iap_i]^{k/2})$. Then
$$\|\dot\mu(t)^*\dot\mu(t)\|_k^k=\tau((\dot\mu(t)^*\dot\mu(t))^{k/2})=\sum_{i=1}^n\tau\big(p_1(\dot\mu(t)^*\dot\mu(t))^{k/2}p_i\big)\ge\sum_{i=1}^n\tau\big(p_i\dot\mu(t)^*\dot\mu(t)p_i)^{k/2}\big)$$
$$=\sum_{i=1}^n\|\dot\mu(t)p_i\|_k^k.$$
On the other hand, note that $length(\delta)=length(\rho(\delta))$. Indeed,
$$\|(\dot\delta(t)p_1,\ldots,\dot\delta(t)p_n)\|=\|(i\alpha_1e^{it\alpha_1}p_1,\ldots,\alpha_ne^{it\alpha_n}p_n)\|=\{\sum_{i=1}^n|\alpha_i|^kr_i^k)\}^{1/k}=\|\dot\delta(t)^*\dot\delta(t)\|_k.$$
We finish the proof by establishing that
$$length(\rho(\mu))\ge length(\rho(\delta))=length(\delta). \tag{5.10}$$
There is a classic Minkowski type inequality (see inequality **201** of [6]) which states that if $f_1,\ldots,f_n$ are non negative functions, then
$$\int_0^1\{\sum_{i=0}^n f_i^k(t)\}^{1/k}dt\ge\big(\sum_{i=1}^n\{\int_0^1 f_i(t)\}^k\big)^{1/k}.$$
In our case $f_i(t)=\|\dot\mu(t)p_i\|_k$:
$$\int_0^1\{\sum_{i=0}^n\|\dot\mu(t)p_i\|_k^k)\}^{1/k}dt\ge\big(\sum_{i=1}^n\{\int_0^1\|\dot\mu(t)p_i\|_kdt\}^k\big)^{1/k}\ge\{\sum_{i=1}^n|\alpha_i|^kr_i^k\}^{1/k},$$
where in the last inequality we use the previous lemma: $\int_0^1\|\dot\mu(t)\|_kdt\ge|\alpha_i|r_i$ for $i=1,\ldots,n$. □

The following result is proved analogously as Lemma 3.2 in [1].

Theorem 5.4 *Let $x \in \mathcal{M}$ be a selfadjoint element with $\|x\| \le \pi$, and $v \in U_{\mathcal{M}}$. Then the curve $\delta(t) = ve^{itx}$ has minimal length among piecewise C^1 curves of unitaries joining its endpoints, measured with the k-norm.*

Proof. There is no loss in generality if we suppose $v = 1$. Indeed, for any curve μ of unitaries, $length(\mu) = length(v^*\mu)$.

Let us first consider the case $\|x\| < \pi$. Suppose that there exists a piecewise C^1 curve of unitaries μ which is strictly shorter than δ, $\ell(\mu) < \ell(\delta) - \epsilon = \|x\|_2 - \epsilon$. The element x can be approximated in the norm topology of $\mathcal{M}$ by selfadjoint elements of $\mathcal{M}$, say z, with finite spectrum and the following conditions:

1. $\|z\| \le \|x\| < \pi$.
2. $\|x\|_k - \epsilon/2 < \|z\|_k \le \|x\|_k$.
3. $\|e^{ix} - e^{iz}\| < 2$.
4. There exists a C^∞ curve of unitaries joining e^{ix} and e^{iz} of length less than $\epsilon/2$.

The first three are clear. The fourth condition can be obtained as follows. By the third condition $e^{-ix}e^{iz} = e^{iy}$, with $y^* = y \in \mathcal{M}$. Moreover z can be adjusted so as to obtain y of arbitrarily small norm. Then the curve of unitaries $\gamma(t) = e^{ix}e^{ity}$ is C^∞, joins e^{ix} and e^{iz}, with length $\|y\|_k \le \|y\| < \epsilon/2$.

Consider now the curve μ', which is the curve μ followed by the curve $e^{ix}e^{ity}$ above. Then clearly

$$length(\mu') \le length(\mu) + \|y\|_k < length(\mu) + \epsilon/2.$$

Therefore $length(\mu') < \|x\|_k - \epsilon/2$. On the other hand, since μ' joins 1 and e^{iz}, by the lemma above, it must have length greater than or equal to $\|z\|_k$. It follows that

$$\|z\|_k \le \|x\|_k - \epsilon/2,$$

a contradiction.

If $\|x\| = \pi$, x can be approximated by selfadjoint elements z with $\|z\| < \pi$. An argument similar as the first part of this proof, shows that one cannot find a shorter curve μ of unitaries joining the same endpoints as δ. □

The following corollary is proved appealing to the inmersion of $\mathcal{P}$ as reflections inside $U_{\mathcal{M}}$, as in Theorem 3.2.

Corollary 5.5 *Let p and q be equivalent projections in $\mathcal{M}$ and let $x \in \mathcal{M}$ with $\|x\| \le \pi/2$ such that the geodesic $\delta(t) = e^{itx}pe^{-itx}$ joins them ($\delta(1) = q$). Then this geodesic has minimal length for the k-norm. If $\|p - q\| < 1$, this geodesic is unique.*

Proof. The curve $2\delta - 1$ has minimal length in $U_{\mathcal{M}}$. □

We would like to know if this minimality result holds also for for $1 \le k < 2$.

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Esteban Andruchow
Instituto de Ciencias
Universidad Nacional de Gral. Sarmiento
J. M. Gutierrez 1150
(1613) Los Polvorines
Argentina
e-mail: eandruch@ungs.edu.ar

Lázaro Recht
Departamento de Matemática P y A
Universidad Simón Bolívar
Apartado 89000
Caracas 1080A
Venezuela
e-mail: recht@usb.ve