Weighted generalized inverses, oblique projections and least squares problems *

G. Corach and A. Maestripieri

Gustavo Corach (corresponding author)
Depto. de Matemática, FI-UBA,
Paseo Colón 850
1063 - Buenos Aires, Argentina
and
IAM-CONICET,
Saavedra 15,
1083 - Buenos Aires, Argentina.
e-mail: gcorach@fi.uba.ar

Alejandra Maestripieri
Instituto de Ciencias, Universidad Nacional de General Sarmiento,
1613 - Los Polvorines, Argentina
and
IAM-CONICET.
e-mail: amaestri@ungs.edu.ar

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Abstract

A generalization with singular weights of Moore-Penrose generalized inverses of closed range operators in Hilbert spaces is studied using the notion of compatibility of subspaces and positive operators.

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1 Introduction

Given a matrix $B \in \mathbb{C}^{m\times n}$, the Moore-Penrose generalized inverse of B is the unique matrix $C \in \mathbb{C}^{n\times m}$ which satisfies the system

$$BXB = B,\ XBX = X,\ (BX)^* = BX,\ (XB)^* = XB.$$

Thus, BC is the orthogonal projection onto the column space $R(B)$ of B and CB is the orthogonal projection onto the column space $R(B^*)$. In many applications, it appears to be necessary to change the scalar products in the spaces of input and output vectors. More precisely, given $B \in \mathbb{C}^{m\times n}$ and $A_1 \in \mathbb{C}^{n\times n}$, $A_2 \in \mathbb{C}^{m\times m}$ which are positive definite, the system to be solved is

$$BXB = B,\ XBX = X,\ (A_2BX)^* = A_2BX,\ (A_1XB)^* = A_1XB. \tag{$*$}$$

Again, there exists a unique solution $C' \in \mathbb{C}^{n\times m}$, BC' (resp. $C'B$) is the orthogonal projection onto $R(B)$ (resp. $R(B^*)$) with respect to the scalar product on $\mathbb{C}^m$ (resp. $\mathbb{C}^n$) defined by A_2 (resp. A_1). In some applications a singular version of the problem needs to be solved. Thus, A_1 and A_2 are supposed to be positive semidefinite. In this case, solutions of $(*)$ always exist but they are infinitely many. Among them, there exists a unique solution of minimal Euclidean norm. In other applications in which very large numbers of variables are involved, it can be desirable to solve system $(*)$ for bounded linear operators between Hilbert spaces. It should be noticed that, in such cases, the first two conditions of $(*)$ force B to have a closed range and any solution will have, also, a closed range. In this case, the existence of a solution is not guaranteed. The goal of this paper is the complete solution of the following problems. Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces, $B : \mathcal{H} \to \mathcal{K}$ a bounded linear operator with closed range and $A_1 : \mathcal{H} \to \mathcal{H}$, $A_2 : \mathcal{K} \to \mathcal{K}$ positive semidefinite bounded linear operators. Consider the seminorm $\|\cdot\|_{A_1}$, (resp. $\|\cdot\|_{A_2}$) on $\mathcal{H}$ (resp. on $\mathcal{K}$) defined by $\|x\|_{A_1} = \langle A_1x, x\rangle^{1/2}$, for $x \in \mathcal{H}$ (respectively $\|x\|_{A_2} = \langle A_2x, x\rangle^{1/2}$, for $x \in \mathcal{K}$).

Problem I
Find necessary and sufficient conditions for the existence of solutions of system $(*)$.

Problem II
Find all solutions of system $(*)$, in case there exists one.

Problem III
Find necessary and sufficient conditions for the existence of $u_0 \in \mathcal{H}$ such that $\|y - Bu_0\|_{A_2} \le \|y - Bx\|_{A_2}$ for every $x \in \mathcal{H}$ and $\|u_0\|_{A_1} \le \|u\|_{A_1}$, for every $u \in \mathcal{H}$ such that $\|y - Bu\|_{A_2} \le \|y - Bx\|_{A_2}$, for every $x \in \mathcal{H}$.

Problem IV
In case there exists an u_0 as above, find all of them and, among them, find one of minimal Euclidean norm.

It should be mentioned that, if the weights A_1 and A_2 are supposed to be invertible, then existence and uniqueness of solutions of system $(*)$ follow immediately from the analogous results on Moore-Penrose generalized inverses, changing the inner products of $\mathcal{H}$ and $\mathcal{K}$ (or

$\mathbb{C}^n$ and $\mathbb{C}^m$ in the finite dimensional case). In this case, Problems I to IV have a unique solution. The reader is referred to the complete survey by Nashed and Votruba [22], section 4.5, and to the more modern treatment by Nashed [21], with emphasis in Banach and Hilbert space operators.

Before the description of the main results of the paper let us give a look to the history of the subject.

Historical notes.

The first appearance of weighted generalized inverses of matrices is due to Greville [16] who used them in problems involving least squares fitting of curves and surfaces. As it happens with every natural useful notion, many results on generalized inverses have been discovered once and again by mathematicians, statisticians and egineers. Thus, Chipman [6] reintroduced the notion for linear regression problems. Also Goldman and Zelen [15], Watson [32], Zyskind [34] and Rao and Mitra [25], [26], [24], found applications to statistics. Milne [18] introduced a version of "oblique pseudoinverse" for matrices and Ward, Boullion and Lewis [29] proved that Milne's oblique pseudoinverses can be thought as weighted generalized inverses with invertible weights. In a later paper [30] they extended some results to singular weights. In fact, in the papers mentioned above the weights are represented by positive definite matrices and Ward, Boullion and Lewis relaxed the hypothesis on the weights. Ward [31] found a limit formula for weighted generalized inverses. Some related results with a different approach have been obtained by Rao and Mitra [25], [24], [19] and Morley [20].

In 1980, Eldén [13] published a complete treatment of the existence of optimal weighted generalized inverses for singular weights in finite dimensional spaces. The present paper can be seen as an extension of Eldén's approach to infinite dimensional Hilbert spaces. For recent results on this subject the reader is referred to the papers by Sun and Wei [28], Stanimirović and Stanković [27] and Djordjević, Stanimirović and Wei [11]. For applications to parallel computing, image processing and many algorithmical results which use weighted generalized inverses with singular weights, the reader is referred to the papers by Censor, Gordon and and Gordon [4], [5] and Censor and Elfving, [2], [3]. The papers by Nashed and Votruba [22] and Nashed [21], and the books by Rao and Mitra [24] and Ben-Israel and Greville [1] are excellent references, which contain many results on weighted generalized inverses.

The contents of the paper are the following. Section 2 contains all results on the notion of compatibility of a closed subspace of a Hilbert space $\mathcal{H}$ and a positive bounded operator A acting on $\mathcal{H}$. Section 3 is devoted to solve Problems I and II in terms of compatibility. In Section 4 we solve Problems III and IV and show an application of our techniques by proving a result by Morley [20] on an infinite dimension regression model with singular covariance.

2 Preliminaries

Throughout, $\mathcal{H}, \mathcal{K}, \mathcal{G}$ denote Hilbert spaces, $L(\mathcal{H}, \mathcal{K})$ is the space of bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$, $L(\mathcal{H})$ is the algebra $L(\mathcal{H}, \mathcal{H})$ and $L(\mathcal{H})^+$ denotes the cone of positive semidefinite operators. For any $C \in L(\mathcal{H}, \mathcal{K})$ the image or range (resp. the nullspace) of C is denoted by $R(C)$ (resp. $N(C)$). $CR(\mathcal{H}, \mathcal{K})$ is the subset of $L(\mathcal{H}, \mathcal{K})$ of all operators with closed range. For any $B \in CR(\mathcal{H}, \mathcal{K})$ the Moore-Penrose inverse of B is the operator $B^\dagger \in CR(\mathcal{K}, \mathcal{H})$ such that $B^\dagger Bx = x$ for every $x \in N(B)^\perp$ and $B^\dagger y = 0$ for every $y \in R(B)^\perp$. $B^\dagger$ is characterized by the properties $BB^\dagger B = B$, $B^\dagger BB^\dagger = B^\dagger$, $(BB^\dagger)^* = BB^\dagger$, $(B^\dagger B)^* = B^\dagger B$. If $\mathcal{H}$ is decomposed as a direct sum of closed subspaces $\mathcal{H} = \mathcal{M} \oplus \mathcal{N}$, the projection onto $\mathcal{M}$ with nullspace $\mathcal{N}$ is denoted by $P_{\mathcal{M}\|\mathcal{N}}$. In particular, given a closed subspace $\mathcal{M}$ of $\mathcal{H}$, $P_{\mathcal{M}}$ denotes the projection $P_{\mathcal{M}\|\mathcal{M}^\perp}$. Denote $\mathcal{Q} = \{Q \in L(\mathcal{H}) : Q^2 = Q\}$. The Moore-Penrose inverse $B^\dagger$ of $B \in CR(\mathcal{H}, \mathcal{K})$ is determined by the properties $BB^\dagger = P_{R(B)}$ and $B^\dagger B = P_{N(B)^\perp}$ or, equivalently, by:

(i) $\|BB^\dagger y - y\| \le \|Bx - y\|$ for every $x \in \mathcal{H}$;

(ii) $\|B^\dagger y\| \le \|z\|$ for every $z \in \mathcal{H}$ such that $\|Bz - y\| \le \|Bx - y\|$ for every $x \in \mathcal{H}$.

An operator $A \in L(\mathcal{H})^+$ and a closed subspace $\mathcal{S}$ of $\mathcal{H}$ form a compatible pair $(A, \mathcal{S})$ if there exists a projection $Q \in L(\mathcal{H})$ such that $R(Q) = \mathcal{S}$ and $AQ = Q^*A$. The last condition means that Q is A-Hermitian in the sense that $< Qx, x' >_A = < x, Qx' >_A$, for every $x, x' \in \mathcal{H}$, where $< x, x' >_A = < Ax, x' >$ defines a semi inner product on $\mathcal{H}$, which is an inner product only if $N(A) = \{0\}$. There is also a seminorm defined by A, namely $\|x\|_A = < Ax, x >^{1/2}$ for $x \in \mathcal{H}$.

Denote $P(A, \mathcal{S}) = \{Q \in \mathcal{Q} : R(Q) = \mathcal{S} \text{ and } AQ = Q^*A\}$, i.e., $P(A, \mathcal{S})$ is the set of A-Hermitian projections with fixed range $\mathcal{S}$. The set $P(A, \mathcal{S})$ can be empty (if $(A, \mathcal{S})$ is not compatible), or have one element (for example, if A is positive definite) or have infinite elements. It is easy to see that if $\mathcal{S}$ is finite dimensional (and, a fortiori, if $\mathcal{H}$ is finite dimensional), then every pair $A, \mathcal{S}$ is compatible [9]. The compatibility of a given pair $(A, \mathcal{S})$ has been characterized in terms of angles between subspaces and decompositions of the ranges of A and $A^{1/2}$. It has also been proven that the compatibility of $(A, \mathcal{S})$ is equivalent to the existence of a solution of the equation $PAPX = PA(I - P)$, where $P = P_{\mathcal{S}}$. (See [8], [9] for details). This kind of equations can be studied applying Douglas theorem:

Theorem 2.1. *Given Hilbert spaces $\mathcal{H}$, $\mathcal{K}$, $\mathcal{G}$ and operators $A \in L(\mathcal{H}, \mathcal{G})$, $B \in L(\mathcal{K}, \mathcal{G})$ then the following conditions are equivalent:*

i) the equation $AX = B$ has a solution in $L(\mathcal{K}, \mathcal{H})$;

ii) $R(B) \subseteq R(A)$;

iii) there exists $\lambda > 0$ such that $BB^* \le \lambda AA^*$. In this case, there exists a unique $D \in L(\mathcal{K}, \mathcal{H})$ such that $AD = B$ and $R(D) \subseteq \overline{R(A^*)}$; moreover, $\|D\|^2 = \inf\{\lambda > 0 : BB^* \le \lambda AA^*\}$. We shall say that D is the **reduced solution** *of $AX = B$.*

The reader is referred to [12] and [14] for the proof of Douglas theorem and related results.

Suppose that $(A, \mathcal{S})$ is compatible and consider the reduced solution D of the equation $PAPX = PA(I - P)$. Define $P_{A,\mathcal{S}} = P + D$, or, in terms of the matrix representation induced by P, $P_{A,\mathcal{S}} = \begin{pmatrix} 1 & d \\ 0 & 0 \end{pmatrix}$, where P is identified with the identity in $L(\mathcal{S})$, D with the operator $d = D|_{\mathcal{S}^\perp} \in L(\mathcal{S}^\perp, \mathcal{S})$. The next theorem characterizes the set $P(A, \mathcal{S})$:

Theorem 2.2. *Let $A \in L(\mathcal{H})^+$ and $\mathcal{S}$ a closed subspace of $\mathcal{H}$ such that $(A, \mathcal{S})$ is compatible. Then $P_{A,\mathcal{S}} \in P(A, \mathcal{S})$ and it is the projection onto $\mathcal{S}$ with nullspace $A^{-1}(\mathcal{S}^\perp) \ominus (N(A) \cap \mathcal{S})$. The set $P(A, \mathcal{S})$ is an affine manifold and it can be parametrized as*

$$P(A, \mathcal{S}) = P_{A,\mathcal{S}} + L(\mathcal{S}^\perp, N(A) \cap \mathcal{S}),$$

where $L(\mathcal{S}^\perp, N(A)\cap\mathcal{S})$ is viewed as a subspace of $L(\mathcal{H})$.

Given $T \in L(\mathcal{H},\mathcal{K})$, a closed subspace $\mathcal{S}$ of $\mathcal{H}$, and an element $y \in \mathcal{H}$, an *abstract spline* or a $(T,\mathcal{S})$-*spline interpolant* to y is any element of the set

$$spl(T,\mathcal{S},y) = \{x \in y+\mathcal{S} : \|Tx\| \leq \|T(y+s)\| \text{ for all } s \in \mathcal{S}\}.$$

It holds that $spl(T,\mathcal{S},y) = (y+\mathcal{S}) \cap A(\mathcal{S})^\perp$ where $A = T^*T$.

The following theorem relates the existence of splines to compatibility:

Theorem 2.3. *Let $T \in L(\mathcal{H},\mathcal{K})$ and $\mathcal{S}$ a closed subspace of $\mathcal{H}$. If $A = T^*T$, then:*

a) $spl(T,\mathcal{S},y)$ is not empty for every $y \in \mathcal{H}$ if and only if the pair $(A,\mathcal{S})$ is compatible.

b) If $(A,\mathcal{S})$ is compatible and $y \in \mathcal{H}\setminus\mathcal{S}$ then $spl(T,\mathcal{S},y) = \{(I-Q)y : Q \in P(A,\mathcal{S})\}$. Furthermore, $(I-P_{A,\mathcal{S}})y$ is the unique vector in $spl(T,\mathcal{S},y)$ with minimal norm.

See [10] for the proofs of these assertions.

3 Weighted generalized inverses

Theorem 3.1. *Given $B \in CR(\mathcal{H},\mathcal{K})$, $A_1 \in L(\mathcal{H})^+$ and $A_2 \in L(\mathcal{K})^+$ there exists $C \in L(\mathcal{K},\mathcal{H})$ such that*

$$BCB = B, CBC = C, A_1CB = B^*C^*A_1, A_2BC = C^*B^*A_2 \tag{1}$$

if and only if $(A_1, N(B))$ and $(A_2, R(B))$ are compatible pairs.

Proof. Suppose that $C \in L(\mathcal{K},\mathcal{H})$ satisfies (1). Notice that C has closed range: in fact the projection $P = BC$ on $\mathcal{K}$ has the same range as B and the projection $Q = CB$ has the same range as C; of course, Q is a bounded linear projection and, therefore, its range is closed. It follows easily that Q (resp. P) and B (resp. C) have the same nullspace. Observe also that the third and fourth conditions of (1) say that Q is A_1-Hermitian and P is A_2-Hermitian. Then $I-Q$ is also A_1-Hermitian and $R(I-Q) = N(Q) = N(B)$, which proves that $I-Q \in P(A_1, N(B))$. Analogously, $P \in P(A_2, R(B))$. This shows that $(A_1, N(B))$ and $(A_2, R(B))$ are compatible pairs.

Conversely, suppose there exist $Q' \in P(A_1, N(B))$ and $P \in P(A_2, R(B))$. Then $Q = I - Q'$ is A_1-Hermitian and $N(Q) = R(Q') = N(B)$. Consider the decomposition $\mathcal{K} = R(B) \oplus N(P)$ and define $C : \mathcal{K} \to \mathcal{H}$ by $C(Bx+z) = Qx$, for $x \in \mathcal{H}$, $z \in N(P)$. C is well defined because $N(B) = N(Q)$. It is also easy to check that C is a linear operator, with $R(C) = R(Q)$ and $N(C) = N(P)$; C is also bounded, because $B|_{R(Q)} : R(Q) \to R(B)$ is an isomorphism by the closed graph theorem and $C|_{R(B)} = (B|_{R(Q)})^{-1}$. This also implies $BCB = B$. It remains to prove the other conditions of (1). On one side, it holds $CBC(Bx+z) = CBx = C(Bx+z)$ for every $x \in \mathcal{H}$ and $z \in N(P)$. On the other side, $CB = Q$ is A_1-Hermitian and $BC = P$ is A_2-Hermitian. □

From now on, $GI(B, A_1, A_2)$ denotes the set of all bounded linear solutions of $(*)$:

$$GI(B,A_1,A_2) = \{C \in CR(\mathcal{K},\mathcal{H}) : BCB = B, CBC = C, A_1CB = B^*C^*A_1, A_2BC = C^*B^*A_2\}.$$

The proof of the theorem above and the characterization of the set of generalized Hermitian projections of a given range described in section 2, provide the following parametrization of $GI(B, A_1, A_2)$:

Proposition 3.2. *The set $GI(B, A_1, A_2)$ is parametrized by the vector space*

$$L(N(B)^{\perp}, N(A_1) \cap N(B)) \times L(R(B)^{\perp}, N(A_2) \cap R(B)).$$

Proof. The proof of the theorem above shows that the construction of a bounded linear solution of $(*)$, if there exists any, is based in the choice of two projections, namely, $I - Q \in P(A_1, N(B))$ and $P \in P(A_2, R(B))$. It is not difficult to prove that different choices provide different solutions of $(*)$. On the other hand, following the notations and results of section 2, $P(A_1, N(B))$ is in bijection with $L(N(B)^{\perp}, N(A_1) \cap N(B))$ and $P(A_2, R(B))$ is in bijection with $L(R(B)^{\perp}, N(A_2) \cap R(B))$. With these comments, the result follows straightforward. $\square$

The parametrization just obtained is quite indirect. The following results of this section are devoted to find more explicit parametrizations of $GI(B, A_1, A_2)$.

The first goal is to generalize Douglas theorem in order to get convenient solutions of Douglas-type equations.

Theorem 3.3. *Let $\mathcal{H}, \mathcal{K}$ and $\mathcal{G}$ be Hilbert spaces. Given $A \in L(\mathcal{H}, \mathcal{G})$ and $B \in L(\mathcal{K}, \mathcal{G})$ such that $R(B) \subseteq R(A)$, for every closed subspace $\mathcal{M}$ of $\mathcal{H}$ such that $\mathcal{H} = N(A) \oplus \mathcal{M}$ there exists a unique solution $C \in L(\mathcal{K}, \mathcal{H})$ of the operator equation $AX = B$ such that $R(C) \subseteq \mathcal{M}$. The nullspace of C coincides with that of B.*

Proof. Consider the reduced solution $C' \in L(\mathcal{K}, \mathcal{H})$ of $AX = B$ and define $C = P_{\mathcal{M}\|N(A)}C'$. Obviously, $R(C) \subseteq \mathcal{M}$. Observe that $AC = AP_{\mathcal{M}\|N(A)}C' = AC' = B$ because $A(I - P_{\mathcal{M}\|N(A)}) = AP_{N(A)\|\mathcal{M}} = 0$. Therefore, $AC = B$, which proves the existence statement.

Suppose that $D \in L(\mathcal{K}, \mathcal{H})$ satisfies $AD = B$ and $R(D) \subseteq \mathcal{M}$. Then, $A(D - C) = 0$ so that $R(D - C) \subseteq N(A)$. But $R(D - C) \subseteq \mathcal{M}$ and, therefore, $R(D - C) \subseteq N(A) \cap \mathcal{M} = \{0\}$. This shows that $D = C$.

The last assertion follows easily: $N(C) \subseteq N(B)$ because $AC = B$; conversely, if $Bx = 0$ then $Cx \in N(A) \cap R(C) \subseteq N(A) \cap \mathcal{M} = \{0\}$, which shows that $N(B) \subseteq N(C)$. $\square$

Given $B \in CR(\mathcal{H}, \mathcal{K})$ let us denote $B\{1\} = \{C \in L(\mathcal{K}, \mathcal{H}) : BCB = B\}$ and $B\{1, 2\} = \{C \in L(\mathcal{K}, \mathcal{H}) : BCB = B \text{ and } CBC = C\}$. Following the notations of Ben Israel and Greville [1], we call any $C \in B\{1\}$ an $\{1\}$-inverse of B and any $C \in B\{1, 2\}$ an $\{1, 2\}$-inverse of B.

Corollary 3.4. *Consider $B \in CR(\mathcal{H}, \mathcal{K})$ and projections $Q \in L(\mathcal{H})$, $P \in L(\mathcal{K})$ such that $N(Q) = N(B)$ and $R(P) = R(B)$. Then there exists a unique solution $C \in L(\mathcal{K}, \mathcal{H})$ of*

$$BX = P, \; R(X) = R(Q). \tag{2}$$

It holds $C \in B\{1, 2\}$ and $N(C) = N(P)$.

Proof. Observe the decompositions $\mathcal{H} = N(Q) \oplus R(Q) = N(B) \oplus R(Q)$ and the inclusion $R(P) \subseteq R(B)$. By Theorem 3.3, there exists a unique $C \in L(\mathcal{H}, \mathcal{K})$ such that $BC = P$ and $R(C) \subseteq R(Q)$, and C satisfies also $N(C) = N(P)$. It remains to prove that $C \in B\{1,2\}$ and $R(Q) \subseteq R(C)$.

Since $R(P) = R(B)$ it follows that $PB = B$ so that $BCB = PB = B$; also $C(I - P) = 0$ because $N(C) = N(P) = R(I - P)$; therefore, $CBC = CP = C$ and this proves that $C \in B\{1,2\}$. In order to prove the inclusion $R(Q) \subseteq R(C)$, observe first that $N(B) \subseteq N(CB) \subseteq N(CBC) = N(B)$, so that $N(CB) = N(B) = N(Q)$. Then, CB and Q are bounded linear projections with the same nullspace and $R(CB) \subseteq R(C) \subseteq R(Q)$ and, therefore, $CB = Q$ and, a fortiori, $R(C) = R(Q)$. □

Observe first that any solution C of

$$BX = P, \ N(X) = N(P) \tag{2'}$$

satisfies $BCB = B$ because $BC = P$ and $PB = B$; similarly, $C(I - P) = 0$ because $N(C) = N(P)$ and then, $CBC = CP = C$. Thus $C \in B\{1,2\}$. By the generalization of Douglas theorem, there exists a unique solution $C \in L(\mathcal{K}, \mathcal{H})$ of

$$BX = P, \ N(X) = N(P), \ R(X) \subseteq R(Q). \tag{3}$$

By the first remark, it holds $CBC = C$, so that $(CB)^2 = CB$. Moreover, $N(C) = N(BC) = N(P)$ and $R(CB) \subseteq R(C) \subseteq R(Q)$. Thus, CB is a projection with $N(CB) = N(B) = N(Q)$ and $R(B) \subseteq R(Q)$. By the first remark, it holds $CBC = C$, so that $(CB)^2 = CB$. Moreover, $N(C) = N(BC) = N(P)$ and $R(CB) \subseteq R(C) \subseteq R(Q)$. Thus, CB is a projection with $N(CB) = N(B) = N(Q)$ and $R(B) \subseteq R(Q)$. By elementary theory of projections, it holds $CB = Q$ and, a fortiori, $R(Q) = R(CB) \subseteq R(C) \subseteq R(Q)$, so that $R(C) = R(Q)$ and this proves that C is a solution of (2). Uniqueness of solutions of (2) follows from that of (3).

Notation: In what follows, $B^{\dagger}_{P,Q}$ denotes the unique solution of (2). It follows from the proof that $B^{\dagger}_{P,Q}$ is the unique operator in $L(\mathcal{K}, \mathcal{H})$ such that

$$BB^{\dagger}_{P,Q} = P \text{ and } B^{\dagger}_{P,Q}B = Q. \tag{4}$$

Of course, $B^{\dagger}_{P,Q}$ has closed range, namely $R(Q)$.

As a corollary, we get another parametrization of $GI(B, A_1, A_2)$:

Corollary 3.5. *Suppose that $(A_1, N(B))$ and $(A_2, R(B))$ are compatible pairs. Then*

$$GI(B, A_1, A_2) = \{B^{\dagger}_{P,I-Q} : P \in P(A_2, R(B)), Q \in P(A_1, N(B))\}.$$

Proof. Let $Q \in P(A_1, N(B))$, $P \in P(A_2, R(B))$. Then $B^{\dagger}_{P,I-Q}$ satisfies the equivalent of (4): $BB^{\dagger}_{P,I-Q} = P$, $B^{\dagger}_{P,I-Q}B = I - Q$. The fact that P (resp. $I - Q$) is A_2 (resp. A_1)-Hermitian, together with the identities $BB^{\dagger}_{P,I-Q}B = B$ and $B^{\dagger}_{P,I-Q}BB^{\dagger}_{P,I-Q} = B^{\dagger}_{P,I-Q}$, prove that $B^{\dagger}_{P,I-Q}$ belongs to $GI(B, A_1, A_2)$.

Conversely, if $C \in GI(B, A_1, A_2)$ then C satisfies $(*)$. Then $P = BC \in P(A_2, R(B))$ and if $Q = CB$ then $I - Q \in P(A_1, N(B))$. Thus, C is the unique solution $B^{\dagger}_{P,I-Q}$ of (2). □

The next result gives a better way of constructing $B^\dagger_{P,Q}$ in terms of $B\{1\}$. As a corollary we shall get a simpler parametrization of $GI(B, A_1, A_2)$.

Proposition 3.6. *Given $B \in CR(\mathcal{H}, \mathcal{K})$ and projections $Q \in L(\mathcal{H})$ and $P \in L(\mathcal{K})$ such that $N(Q) = N(B)$ and $R(P) = R(B)$ it holds $B^\dagger_{P,Q} = QB^{(1)}P$ for any $B^{(1)} \in B\{1\}$.*

Proof. Take any $B^{(1)} \in B\{1\}$ and let $\mathcal{M} = R(B^{(1)}B)$. Then $\mathcal{H} = \mathcal{M} \oplus N(B)$ because $N(B) = N(B^{(1)}B)$, and $B^{(1)}B$ is a projection onto $\mathcal{M}$. Define $C = QB^{(1)}P$. Straightforward computations show that $N(C) = N(P)$. Let us prove that $R(C) = R(Q)$: observe that $R(Q) = Q\mathcal{M}$; then $R(C) = Q(R(B'P)) = Q(R(B'B)) = Q\mathcal{M}$, because P and B have the same range. Finally, the identity $BQ = B$, due to the fact that $N(Q) = N(B)$, implies $BC = BQB^{(1)}P = BB^{(1)}P = P$, because $BB^{(1)}$ is a projection onto $R(B) = R(P)$. Thus, C satisfies (2) and, by Proposition 3.4 it follows that $B^\dagger_{P,Q} = QB^{(1)}P$, as claimed. □

The last result of this section gives a more explicit parametrization of $GI(B, A_1, A_2)$. The fact that we use the Moore-Penrose inverse of B instead of an arbitrary choice of a $\{1\}$-inverse of B is not relevant.

Theorem 3.7. *If $(A_1, N(B))$ and $(A_2, R(B))$ are compatible pairs then*

$$GI(B, A_1, A_2) = \{(I - Q)B^\dagger P : Q \in P(A_1, N(B)), P \in P(A_2, R(B))\}$$

where $B^\dagger$ is the Moore-Penrose inverse of B.

Proof. It follows by combining the last proposition with Corollary 3.5. □

Remark 3.8. Milne [18] defined what he called the *oblique pseudoinverse* of an operator acting between finite dimensional Hilbert spaces. Let $B \in L(\mathcal{V}, \mathcal{W})$ and let $\mathcal{K} \subseteq \mathcal{V}$, $\mathcal{L} \subseteq \mathcal{W}$ be two subspaces such that $\mathcal{V} = \mathcal{K} \oplus N(B)$ and $\mathcal{W} = R(B) \oplus \mathcal{L}$. The oblique pseudoinverse of B with respect to the subspaces $\mathcal{K}$ and $\mathcal{L}$ is defined as the unique $B^\dagger_{\mathcal{K},\mathcal{L}} \in L(\mathcal{W}, \mathcal{V})$ satisfying $B^\dagger_{\mathcal{K},\mathcal{L}}Bv = v$ for every $v \in \mathcal{K}$ and $B^\dagger_{\mathcal{K},\mathcal{L}}w = 0$ for every $w \in \mathcal{L}$. Milne's definition and results have trivial extensions to closed range operators between infinite dimensional Hilbert spaces. If $Q \in L(\mathcal{V})$ is the projection onto $\mathcal{K}$ with nullspace $N(B)$ and $P \in L(\mathcal{W})$ is the projection onto $R(B)$ with nullspace $\mathcal{L}$, then it can easily be shown that $B^\dagger_{\mathcal{K},\mathcal{L}}$ satisfies (2) so that $B^\dagger_{\mathcal{K},\mathcal{L}} = B^\dagger_{P,Q}$. It should be remarked that Milne proved that $B^\dagger_{\mathcal{K},\mathcal{L}} = QB^{(1)}P$ for any $B^{(1)} \in B\{1\}$. An algebraic treatment of the properties of $B^\dagger_{P,Q}$ can be found in the survey by Nashed and Votruba [22].

4 Least squares formulation

The great impact that Moore-Penrose inverses have in science is due to the fact that they solve a least squares problems, namely, $B^\dagger c$ is the unique vector in $\mathcal{H}$ with minimal norm among those which minimize $\|Bx - c\|$. We generalize this result for the weighted case, i.e., if we consider weights A_1 and A_2 on $\mathcal{H}$ and $\mathcal{K}$, respectively.

Definition 4.1. *Given $B \in L(\mathcal{H}, \mathcal{K})$, $A_1 \in L(\mathcal{H})^+$, $A_2 \in L(\mathcal{K})^+$ and $y \in \mathcal{K}$, an element $u \in \mathcal{H}$ is said to be an A_2-least squares solution (hereafter, $A_2 - LSS$) of the equation*

$$Bx = y \tag{5}$$

if $\|Bu - y\|_{A_2} \leq \|Bx - y\|_{A_2}$ for every $x \in \mathcal{H}$.

An element $u_0 \in \mathcal{H}$ is said to be an A_1A_2-least squares solution $(hereafter A_1A_2 - LSS)$ of (5) if u_0 is an A_2-LSS of (5) and $\|u_0\|_{A_1} \leq \|u\|_{A_1}$ for every u which is an A_2-LSS of (5).

Lemma 4.2. *Given $B \in CR(\mathcal{H}, \mathcal{K})$ and $A \in L(\mathcal{K})^+$ there exists an $A-LSS$ $u \in \mathcal{H}$ of the equation $Bx = y$ for every $y \in \mathcal{K}$ if and only if the pair $(A, R(B))$ is compatible.*

Proof. Observe that $u \in \mathcal{H}$ is an $A-$LSS of $Bx = y$ if and only if $u \in spl(A^{1/2}, R(B), y)$. Then $Bx = y$ admits an $A-$LSS for every $y \in \mathcal{K}$ if and only if $spl(A^{1/2}, R(B), y)$ is not empty, for every $y \in \mathcal{K}$, which, by item a) of Theorem 2.3, is equivalent to the compability of $(A, \mathcal{S})$. □

Remark 4.3. Given $B \in CR(\mathcal{H}, \mathcal{K})$, $T \in L(\mathcal{K}, \mathcal{G})$ and $y \in \mathcal{K}$, it follows from the preliminaries that $u_0 \in \mathcal{H}$ is an A-LSS of $Bx = y$ (where $A = T^*T \in L(\mathcal{K})^+$) if and only if $y - Bu_0 \in spl(T, R(B), y)$. Therefore, by the characterization of splines in the preliminaries section, given $y \in \mathcal{K}$ there exists an A-LSS u_0 of $Bx = y$ if and only if $y \in Bu_0 + R(AB)^{\perp}$.

The next result determines all A-LSS of (5) if $(A, R(B))$ is compatible.

Proposition 4.4. *Given $B \in CR(\mathcal{H}, \mathcal{K})$, $A \in L(\mathcal{K})^+$ such that $(A, R(B))$ is compatible, and $y \in \mathcal{K} \setminus R(B)$, then $u \in \mathcal{H}$ is an A-LSS of (5) if and only if there exists $P \in P(A, R(B))$ such that $Bu = Py$.*

Proof. If $y \in \mathcal{K} \setminus R(B)$ then by Theorem 2.3 it holds $spl(A^{1/2}, R(B), y) = \{(I - P)y : P \in P(A, R(B))\}$ so, by the last remark, u is an A-LSS of (5) if and only if $y - Bu \in spl(A^{1/2}, R(B), y)$ and the result follows. □

Observe that if $y \in R(B)$ then every element $u \in B^{-1}(\{y\})$ (i.e., every solution of $Bx = y$) is trivially an A-LSS solution of (5). In fact, in this case, $u \in \mathcal{H}$ is an A-LSS of (5) if and only if $y - Bu \in N(A)$.

Remark 4.5. If $N(A) \cap R(B) = \{0\}$ then $u \in \mathcal{H}$ is an A-LSS of (5) if and only if $B^*A(Bx - y) = 0$. In fact, $P(A, R(B))$ consists of a single element $P = P_{A,R(B)}$ whose nullspace is $A^{-1}(R(B)^{\perp}) = R(AB)^{\perp}$. Straightforward computations prove the statement.

If $y \in \mathcal{K} \setminus R(B)$ then, by Proposition 4.4, the set of all A-LSS of (5) is given by $\bigcup\{B^{-1}\{Py\} : P \in P(A, R(B))\}$ and, for a fixed P, $B^{-1}\{Py\} = x_0 + N(B)$, where x_0 is the unique element of $B^{-1}\{Py\} \cap N(B)^{\perp}$. Notice that $x_0 = P_{N(B)^{\perp}} x_0 = B^{\dagger}Bx_0 = B^{\dagger}Py$.

Let us study a minimizing problem in $\mathcal{H}$.

Lemma 4.6. *Consider $B \in L(\mathcal{H}, \mathcal{K})$ and $A \in L(\mathcal{H})^+$ such that $(A, N(B))$ is compatible. Then, for every non zero $x_0 \in N(B)^{\perp}$ and $u \in x_0 + N(B)$, it holds $\|u\|_A \leq \|x\|_A$ for every $x \in x_0 + N(B)$ if and only if there exists $Q \in P(A, N(B))$ such that $u = (I - Q)x_0$.*

Proof. Decompose $u = x_0 + P_{N(B)}u$. Then $\|u\|_A \leq \|x\|_A$ for every $x \in x_0 + N(B)$ if and only if $\|u\|_A \leq \|x_0 + P_{N(B)}x\|_A$ for every $x \in \mathcal{H}$ or equivalently, u is an A-LSS of the equation $P_{N(B)}x = -x_0$. Applying the last proposition to the operator $P_{N(B)}$ and the vector $x_0 \in \mathcal{H} \setminus N(B) = \mathcal{H} \setminus R(P_{N(B)})$ this is equivalent to the existence of $Q \in P(A, N(B))$ such that $P_{N(B)}u = -Qx_0$. Adding x_0 to the last equality, we get $u = (I - Q)x_0$ as claimed. □

Remark 4.7. If $x_0 = 0$ then $\|u\|_A \leq \|x\|_A$ for every $x \in N(B)$ if and only if $\|u\|_A = 0$, which means that $u \in N(A) \cap N(B)$.

We are now in position of finding all A_1A_2-LSS of $Bx = y$.

Proposition 4.8. *Let $B \in CR(\mathcal{H}, \mathcal{K})$, $A_1 \in L(\mathcal{H})^+$ and $A_2 \in L(\mathcal{K})^+$ be such that $(A_1, N(B))$ and $(A_2, R(B))$ are compatible pairs. Consider $y \in \mathcal{K} \setminus R(B)$ and $u \in \mathcal{H} \setminus N(B)$. Then u is an A_1A_2-LSS of the equation $Bx = y$ if and only if there exist $Q \in P(A_1, N(B))$ and $P \in P(A_2, R(B))$ such that $u = (I - Q)B^\dagger Py$.*

Proof. Suppose that u is an A_1A_2-LSS of $Bx = y$. In particular, u is an A_2-LSS of $Bx = y$ and, by Proposition 4.4 there exists $P \in P(A_2, R(B))$ such that $Bu = Py$. Then $x_0 = P_{N(B)^\perp}u = B^\dagger Bu = B^\dagger Py$ is non zero because $Bu \neq 0$. By the lemma above, replacing A by A_1, there exists $Q \in P(A_1, N(B))$ such that $u = (I - Q)x_0 = (I - Q)B^\dagger Py$. Conversely, suppose $u = (I - Q)B^\dagger Py$ for some $Q \in P(A_1, N(B))$ and $P \in P(A_2, R(B))$. Then $Bu = B(I - Q)B^\dagger Py = BB^\dagger Py = Py$ (the second equality holds because $BQ = 0$; the third one follows from the facts that $BB^\dagger = P_{R(B)}$ and P projects onto $R(B)$). Then, by Proposition 4.4, u is an A_2-LSS of $Bx = y$. On the other hand $u = (I-Q)B^\dagger Py = B^\dagger Py = QBPy$ is the decomposition of u according to $\mathcal{H} = N(B)^\perp \oplus N(B)$ and from the lemma above it follows that $\|u\|_A \leq \|z\|_{A_1}$ for every $z \in B^\dagger Py + N(B)$, which is the set of A_2-LSS of $Bx = y$, by the comments following Proposition 4.4. This finishes the proof. □

Theorem 4.9. *Given A_1, A_2, B and y as before consider the problem*

$$\min\{\|y - Bu\| : u \textit{ is an } A_2 - LSS \textit{ of } Bx = y\} \tag{6}$$

Then:

i) u_0 is a solution of (6) if and only if $Bu_0 = P_{A_2,R(B)}y$;

ii) u_0 is a solution of (6) and an A_1A_2-LSS of $Bx = y$ if and only if $u_0 = (I - Q)B^\dagger P_{A_2,R(B)}y$ for some $Q \in P(A_1, N(B))$;

iii) the unique minimal norm element of the set $\{(I-Q)B^\dagger P_{A_2,R(B)}y : Q \in P(A_1, N(B))\}$ is $(I - P_{A_1,N(B)})B^\dagger P_{A_2,R(B)}y$.

Proof. To prove i) observe that by Proposition 4.4 u_0 is an A_2-LSS of $Bx = y$ if and only if there exists $P \in P(A_2, R(B))$ such that $Bu = Py$; then we look for

$$\min\{\|(I - P)y\| : P \in P(A_2, R(B))\}.$$

But, by theorem 2.3 in the Preliminaries, this minimum is attained in $(I - P_{A_2,R(B)})y$ so that $Bu_0 = P_{A_2,R(B))}y$.

In a similar way, by proposition 4.8 and i), u_0 is an A_1A_2-LSS of $Bx = y$ and a solution of (6) if and only if there exists $Q \in P(A_1, N(B))$ such that $u_0 = (I - Q)B^\dagger P_{A_2,R(B)}y$ and ii) follows.

To prove iii) observe that the minimum of the set

$$\{\|(I-Q)B^{\dagger}P_{A_2,R(B)}\| : Q \in P(A_1, N(B))\}$$

is attained, by Theorem 2.3, in $(I - P_{A_1,N(B)})B^{\dagger}P_{A_2,R(B)}$. □

In [20] Morley solved the following problem: Given a (densely defined unbounded) linear operator $B : \mathcal{H} \to \mathcal{K}$, with $R(B^*)$ closed, $c \in R(B^*)$ and $V \in L(\mathcal{K})$ such that V^2 positive semidefinite, find

$$\min\{< V^2y, y >: B^*y = c\}. \tag{7}$$

If g is a solution of this minimizing problem, g is called a *best linear unbiased estimator* (BLUE).

This result is equivalent to solving the following least squares problem with linear equality constraints: given $C \in L(\mathcal{H}, \mathcal{K})$, a closed subspace $\mathcal{S}$ of $\mathcal{H}$, $x_0 \in \mathcal{H}$ and $y \in \mathcal{K}$, find

$$\inf\{\|Cx - y\| : x \in x_0 + \mathcal{S}\}.$$

In fact,

$$\min\{< V^2y, y >: B^*y = c\} = \min\{\|y\|_{V^2} : B^*y = c\}.$$

Observe that $B^*y = c$ if and only if there exists $w \in N(B^*)$ such that $y = B^{*\dagger}c + w$; so that (7) is equivalent to the problem of finding

$$\min\{\|B^{*\dagger}c + w\|_{V^2} : w \in N(B^*)\}.$$

The next proposition is a proof of Morley's result in terms of compatible pairs, for the case of bounded operators.

Proposition 4.10. *Consider $C \in CR(\mathcal{H}, \mathcal{K})$, $\mathcal{S}$ a closed subspace of $\mathcal{H}$, $x_0 \in \mathcal{H}$ and $y \in \mathcal{K}$ such that the pair $(C^*C, \mathcal{S})$ is compatible. Then $\|Cu - y\| \leq \|Cx - y\|$ for every $x \in x_0 + \mathcal{S}$ if and only if there exists $Q \in P(C^*C, \mathcal{S})$ such that $u = (I - Q)(x_0 - C^{\dagger}y)$.*

Proof. Observe that $\|Cx - y\|^2 = \|Cx - P_{R(C)}y\|^2 + \|P_{R(C)^\perp}y\|^2$ so that

$$\inf_{x\in x_0+\mathcal{S}} \|Cx - y\|^2 = \|P_{R(C)^\perp}y\|^2 + \inf_{x\in x_0+\mathcal{S}} \|Cx - P_{R(C)}y\|^2.$$

If $u_0 = C^{\dagger}P_{R(C)}y = C^{\dagger}y$ then $\|Cx - P_{R(C)}y\| = \|C(x - C^{\dagger}y)\| = \|x - C^{\dagger}y\|_{C^*C} = \|x - u_0\|_{C^*C}$
so that

$$\inf_{x\in x_0+\mathcal{S}} \|Cx - y\|^2 = \|P_{R(C)^\perp}y\|^2 + \inf_{x\in x_0+\mathcal{S}} \|x - x_0\|^2_{C^*C} = \|P_{R(C)^\perp}y\|^2 + \inf_{x\in x_0-u_0+\mathcal{S}} \|x\|^2_{C^*C}.$$

By Lemma 4.6 it follows that $\|u\|_{C^*C} \leq \|x\|_{C^*C}$ for every $x \in x_0 - u_0 + \mathcal{S}$ if and only if there exists $Q \in P(C^*C, \mathcal{S})$ such that $u = (I - Q)(x_0 - u_0) = (I - Q)(x_0 - C^{\dagger}y)$. □

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