On the distributions $\left[\delta^{(l)}(m^2+P)\right]^r$ and $\left[\delta^{(l)}(P\pm io)\right]^r$

Susana Elena Trione

Abstract: In this note we give a sense to certain classes of distributions. In fact, we evaluate the distributions $\left[\delta^{(l)}(m^2+P)\right]^r$ and $\left[\delta^{(l)}(P\pm io)\right]^r$.

The results are the following formulas:

i) $\left[\delta^{(k)}(m^2+P)\right]^l$ (form. (II, 2)),

ii) $\left[\delta^{(k)}(m^2+P+io)\right]^l=\left[\delta^{(k)}(m^2+P-io)\right]^l=\left[\delta^{(k)}(m^2+P)\right]^l$ (form. (II, 5)),

iii) $\left[\delta^{(k)}(P+io)\right]^l=\left[\delta^{(k)}(P-io)\right]^l=\left[\delta^{(k)}(P)\right]^l$ (form. (II, 6)),

iv) $P^k\delta^{(k)}(P)=(-1)^k k!\,\delta(P)$, (form. (III, 5)),

v) $\left[\delta^{(k)}(P\pm io)\right]^r$ (form. (IV, 5)),

and

vi) $\left[\delta^{(l)}(P_+^s)\right]^m+e^{\pm i\pi s}\left[\delta^{(l)}(P_-^s)\right]^m$ (form. (IV, 12)).

I. Definitions and others

Let $x=(x_1,x_2,\dots,x_n)$ be a point of the n-dimensional Euclidean space $\mathbb{R}^n$. Consider a non-degenerate quadratic form in n-variables of the form

$$P=P(x)=x_1^2+\cdots+x_p^2-x_{p+1}^2-\cdots-x_{p+q}^2\ , \qquad (\mathrm{I},1)$$

where $n=p+q$. The distributions $(P\pm io)^\lambda$ are defined by

$$(P\pm io)^\lambda=\lim_{\varepsilon\to 0}\left\{P\pm i\varepsilon|x|^2\right\}^\lambda\ , \qquad (\mathrm{I},2)$$

where $\varepsilon>0$, $|x|^2=x_1^2+\cdots x_n^2$, $\lambda\in\mathbb{C}$.

The distributions $(P\pm io)^\lambda$ are analytic in λ everywhere except at $\lambda=-\frac{n}{2}-k$, $k=0$, $1,\dots$, where they have simple poles (cf. [1], p. 275).

These generalized functions can be expressed in terms of P_+^λ and P_-^λ (cf. [1], formulae (2) and (2'), p. 276) by

$$(P+io)^\lambda=P_+^\lambda+e^{i\pi\lambda}P_-^\lambda\ , \qquad (\mathrm{I},3)$$

and

$$(P-io)^\lambda=P_+^\lambda+e^{-i\pi\lambda}P_-^\lambda\ , \qquad (\mathrm{I},4)$$

Here , for $Re\lambda > 0$, the functionals (P_+^λ, φ) and (P_-^λ, φ) corresponds to the functions

$$P_+^\lambda = \begin{cases} P^\lambda, & \text{where } P \geq 0, \\ 0, & \text{where } P < 0, \end{cases} \tag{I, 5}$$

and

$$P_-^\lambda = \begin{cases} 0, & \text{where } P > 0, \\ (-P)^\lambda, & \text{where } P \leq 0. \end{cases} \tag{I, 6}$$

By analytic continuation the formulas (I, 3) and (I, 4) must remain valid also for other λ. We observe that

$$(P + io)^\lambda = (P - io)^\lambda = P^\lambda \ , \tag{I, 7}$$

when λ is a non negative integer.

Now, we shall introduce the distributions $(m^2+P\pm io)^\lambda$, which are defined in an analogous manner as the distributions $(P \pm io)^\lambda$. Let us put (cf. [1], p. 289)

$$(m^2 + P \pm io)^\lambda \stackrel{def}{=} \lim_{\varepsilon \to 0} \left(m^2 + P \pm i\varepsilon|x|^2\right) \lambda \ , \tag{I, 8}$$

where ε is an arbitrary positive number.

It is useful to state an equivalent definition of the distributions $(m^2 + P \pm io)^\lambda$. In this definition appear the distributions

$$(m^2 + P)_+^\lambda = \begin{cases} (m^2 + P)^\lambda & \text{if } \ m^2 + P \geq 0, \\ 0 & \text{if } \ m^2 + P < 0, \end{cases} \tag{I, 9}$$

and

$$(m^2 + P)_-^\lambda = \begin{cases} 0 & \text{if } \ m^2 + P > 0, \\ (-m^2 - P)^\lambda & \text{if } \ m^2 + P \leq 0, \end{cases} \tag{I, 10}$$

We can prove, without difficulty, that the following formula is valid (cf. [3], p.566)

$$(m^2 + P \pm io)^\lambda = (m^2 + P)_+^\lambda + e^{\pm i\pi\lambda}(m^2 + P)_-^\lambda \ . \tag{I, 11}$$

From this formula we conclude immediately that

$$(m^2 + P + io)^\lambda = (m^2 + P - io)^\lambda = (m^2 + P)^\lambda \tag{I, 12}$$

when $\lambda = k =$ positive integer.

We observe that $(m^2+P\pm io)^\lambda$ are entire distributions of λ. This is the principal difference between the distributions, formally analogous, $(P \pm io)^\lambda$ which have poles at the points $\lambda = \frac{n}{2} - k$, $h = 0, 1, \ldots$.

It can prove (cf. [3], formula (2.14), p. 573 and formula (3.5), p. 575) that

$$(m^2 + P \pm io)^{-k} = Pf\ (m^2 + P)^{-k} \mp \frac{i\pi(-1)^{k-1}}{(k-1)!}\,\delta^{(k-1)}(m^2 + P)\ , \qquad (\mathrm{I}, 13)$$

$k = 0, 1, \ldots$.

Otherwise, the following theorems are valid:

Theorem 1. (cf. [4], formula (I, 3; 16), p. 23)
The following formula is true for every λ, $\mu \in \mathbb{C}$ and $m^2 \neq 0$:

$$(m^2 + P \pm io)^{\lambda} \cdot (m^2 + P \pm io)^{\mu} = (m^2 + P \pm io)^{\lambda+\mu}\ . \qquad (\mathrm{I}, 14)$$

From the formula (I, 14), we conclude immediately that, when $m^2 = 0$, the following theorem is valid

Theorem 2. *(cf. [4], formula (I, 3; 17)) The following formula is valid*

$$(P \pm io)^{\lambda} \cdot (P \pm io)^{\mu} = (P \pm io)^{\lambda+\mu}\ , \qquad (\mathrm{I}, 15)$$

when λ, $\mu \in \mathbb{C}$, λ, μ and $\lambda + \mu$ are different from $-\frac{n}{2} - k$, $k = 0, 1, \ldots$.

II. The distribution $\left[\delta^{(k)}(m^2 + P)\right]^l$

From [2], formula (I, 3; 2), p. 6, we have

$$\delta^{(k)}(m^2 + P) = \frac{k!(-1)^k}{(2\pi i)}\left\{(m^2 + P - io)^{-k-1} - (m^2 + P + io)^{-k-1}\right\}\ . \qquad (\mathrm{II}, 1)$$

From (II, 1), we obtain

$$\left[\delta^{(k)}(m^2 + P)\right]^l = \frac{(k!)^l(-1)^{kl}}{(2\pi i)^l}\sum_{p=0}^{l}\binom{l}{p}\left[(m^2 + P - io)^{-k-1}\right]^{l-p}\left[(m^2 + P + io)^{-k-1}\right]^{p}\ , \qquad (\mathrm{II}, 2)$$

here l and k are positive integer.

Otherwise, taking into account the formula (I, 3; 1), p. 6 of [2], we know that

$$\delta^{(k)}(m^2 + P) = \frac{1}{2^{2k+\frac{n}{2}}\Gamma\left(\frac{n}{2} + k\right)}\mathcal{H}\left\{(m^2 + Q)_+^{-\frac{n}{2}-k}\right\} \qquad (\mathrm{II}, 3)$$

Here

$$Q = Q(y) = y_1^2 + \cdots + y_\mu^2 - y_{\mu+1}^2 - \cdots y_{\mu+\nu}^2 \ , \tag{II, 4}$$

$\mu + \nu = n$ (n dimension of the space) and $\mathcal{H}[U(y)]$ is the Hankel transform of $U(y)$ (cf. formula (I, 1; 1), p. 6 of [2])

Putting $\lambda = 1$ in (I, 12), we obtain,

$$\left[\delta^{(k)}(m^2 + P + io)\right]^l = \left[\delta^{(k)}(m^2 + P - io)\right]^l = \left[\delta^{(k)}(m^2 + P)\right]^l \ . \tag{II, 5}$$

So, the formula (II, 2) is valid for $\left[\delta^{(k)}(m^2 + P + io)\right]^l$ and $\left[\delta^{(k)}(m^2 + P - io)\right]^l$.

Otherwise, putting $m^2 = 0$ in the formulae (II, 2), we have

$$\left[\delta^{(k)}(P)\right]^l = \frac{(k!)^l(-1)^{kl}}{(2\pi i)^l} \sum_{p=0}^{l} \binom{l}{p} \left[(P - io)^{-k-1}\right]^{l-p} \left[(P + io)^{-k-1}\right]^p \ . \tag{II, 6}$$

Formula (II, 6) is valid for every $(-k-1)(l-p)$ and $(-k-1)p$ different from $-\frac{n}{2} - s$, n dimension of the space and $s = 0, 1, \ldots$.

Note.

We know that $(P \pm io)^\lambda = (|x|^2)^\lambda$ when $q = 0$, therefore all the above formulae are valid for $[\delta^{(k)}(|x|^2)]^l$.

III. The distribution $\delta^{(k)}(m^2 + P)$.

We know (cf. [5] formula (8), p. 3)

$$(m^2 + P) \cdot \delta^{(k)}(m^2 + P) + k\delta^{(k-1)}(m^2 + P) = 0 \ . \tag{III, 1}$$

Putting $k = 1$ in (III, 1), we have

$$(m^2 + P) \cdot \delta^{(1)}(m^2 + P) = -\delta(m^2 + P). \tag{III, 2}$$

Also, we know (cf. [5] and [6] formula (12), p. 4) that

$$(m^2 + P)^k \cdot \delta^{(k)}(m^2 + P) = (-1)^k k! \ \delta(m^2 + P) \tag{III, 3}$$

$k = 0, 1, \ldots$.

Taking into account the formula (I, 2; 10), p. 5, [2], we have

$$\delta^{(k)}(m^2+P)_{m^2\to 0}\delta^{(k)}(P) \qquad \text{(III, 4)}$$

Therefore, from (III, 3) and (III, 4), we obtain

$$P^k\cdot\delta^{(k)}(P)=(-1)^k k!\;\delta(P)\ , \qquad \text{(III, 5)}$$

k positive integer.

IV. The distribution $[\left[\delta^{(k)}(P\pm io)\right]^r$

We know (cf. [7]) under several conditions, that

$$\delta^{(k)}(P\pm io)=\left\{\delta^{(k)}(P_+)+e^{\pm i\pi(k+1)}\delta^{(k)}P_-\right\}, \qquad \text{(IV, 1)}$$

Here (cf. [1], p. 278),

$$\delta^{(k)}(P_+)=(-1)^k k!\underset{\lambda=-k-1}{\operatorname{res}}P_+^\lambda, \qquad \text{(IV, 2)}$$

and

$$\delta^{(k)}(P_-)=(-1)^k k!\underset{\lambda=-k-1}{\operatorname{res}}P_-^\lambda\ . \qquad \text{(IV, 3)}$$

The residues of P_+^λ and P_-^λ are given in chapter III, paragraph (2.2) of [1].
We also know (cf. [1], p. 279) that

$$\delta^{(k)}(P_-)=(-1)^k\delta^{(k)}(P_+). \qquad \text{(IV, 4)}$$

We have, from (IV, 1), the following formula

$$\left[\delta^k(P\pm io)\right]^r=\left[\delta^{(k)}(P_+)+e^{\pm i\pi(k+1)}\delta^{(k)}(P_-)\right]^r=$$

$$=\sum_{p=0}^{r}\binom{r}{p}\left[\delta^{(k)}(P_+)\right]^{r-p}\left[e^{\pm i\pi(k+1)}\delta^{(k)}(P_-)\right]^p. \qquad \text{(IV, 5)}$$

Taking into account the formula (4. 13), p. 11 of [8], we know that

$$\left[\delta^{(l)}(P_+^s)\right]^m=\left(\frac{1}{(-1)^{s-1}}\right)^m(-1)^{m-1}\cdot A_{s,p,q,n,l,m}\delta^{(m(l+s-1))}(P_+), \qquad \text{(IV, 6)}$$

under the following conditions,

i) p and q are both odd, (IV, 7)

and

ii) $0 \leq m(l+s) - \frac{n}{2} + 2 < \frac{n}{2}$, (IV, 8)

where

$$A_{s,p,q,n,l,m} = \left(\frac{1}{2}\frac{s}{(s-1)!}\right)^m \cdot \frac{((l+s-1)!)^m}{(m(l+s)-1)!}\left(\frac{1}{2\left[\psi\left(\frac{p}{2}\right) - \psi\left(\frac{n}{2}\right)\right]}\right)^m \qquad \text{(IV, 9)}$$

for $m = 1, 2, 3, \ldots$.

Here

$$\psi(k) = -C + \frac{1}{2} + \cdots + \frac{1}{k-1}\ , \qquad (IV, 10)$$

where C is the Euler's constant.

As we know, (cf. [1], p. 269), all that we have said above about $(P_+)^\lambda$ remains true also for P_-^λ except that p and q must interchanged, so, we obtain

$$\left[\delta^{(l)}(P_-^s)\right]^m = \left(\frac{1}{(-1)^{s-1}}\right)^{m-1} (-1)^{m-1} \cdot A_{s,p,q,n,l,m}\delta^{(m(l+s-1))}(P_-), \qquad \text{(IV, 11)}$$

where $A_{s,p,q,n,l,m}$ is given by the formula (IV, 9).

Finally, taking into account the formula (IV, 6) and (IV, 11), we obtain the explicit final result, this is,

$$\left[\delta^{(l)}(P_+^s)\right]^m + e^{\pm i\pi s}\left[\delta^{(l)}(P_-^s)\right]^m = (-1)^{m-1} A_{s,p,q,n,l,m}\left\{\left(\frac{1}{(-1)^{s-1}}\right)^m \delta^{(m(l+s-1))}(P_+^s) + \right.$$
$$\left. + e^{\pm i\pi s}\left(\frac{1}{(-1)^{s-1}}\right)^{m-1} \cdot \delta^{(m(l+s-1))}(P_-^s)\right\}\ . \qquad \text{(IV, 12)}$$

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Susana Elena Trione

Saavedra 15 – 3^{er} Piso

(1083) Buenos Aires, Argentina

Tel: 495-46781, Fax: 4954-6782

E-mail: trione@iamba.edu.ar