Some not so new Factorization Theorems for Hilbert-Schmidt Operators *

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Abstract

A natural Riemannian structure is introduced on the set of positive invertible (unitized) Hilbert-Schmidt operators, in order to obtain several decomposition theorems by means of geodesically convex submanifolds. We also give an intrinsic (algebraic) characterization of such submanifolds, and we study the group of isometries.[1]

1 Introduction

This paper deals with a geometric construction based in functional analysis and operator theoretic tools, which can be situated in the context of the theory of infinite dimensional Riemannian manifolds (for a systematic approach to the subject, see J. McAlpin's PhD Thesis [McA65]). In this context (of Hilbert manifolds), there is a natural class of Lie algebras known as L^*-algebras: an L^*-algebra is a complex Hilbert space $\mathcal{L}$ provided with a Lie algebra structure such that for any $x \in \mathcal{L}$ there is an $x^* \in \mathcal{L}$ such that

$$\langle [x,y], z\rangle = \langle y, [x^*, z]\rangle \qquad \text{for any } \ y, z \in \mathcal{L}$$

The Hilbert-Schmidt operators (*i.e.* the Schatten class $\mathcal{S}_2$ of compact operators) with the usual involution of operators, the Lie product given by ordinary commutators, and the inner product given by the trace, form a simple (*i.e.* noncommutative and with no nontrivial closed ideals) L^*-algebra. The reader should note that this algebra of compact operators is not a C^*-algebra, and there is no (topologically) equivalent norm which makes it a C^*-algebra.

There is a classification theory of simple L^*-algebras [Sch60] [Sch61] due to J. Schue (see also [CGM90] by J.A. Cuenca Mira, A. García Martín and C. Martín González and the paper [Neh93] by E. Neher). There is also a classification theory of symmetric Hilbert manifolds [Kaup81], [Kaup83] due to W. Kaup. Both theories provide an

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abstract framework to do classical Riemannian geometry of manifolds modelled on infinite dimensional L^*-algebras, but we work with a concrete algebra (the Hilbert-Schmidt operators with the trace inner product) because we don't think this is too much of a restriction, since any simple L^*-algebra can be naturally embedded into this algebra of compact operators.

Through the years, several authors have studied the geometry of positive operators with different approaches that lead to a variety of results: in his 1955's paper [Mos55], G.D. Mostow introduced a natural Riemannian structure in the set M_n^+ of positive invertible matrices; the metric induced makes M_n^+ an hyperbolic (*i.e.* nonpositively curved) space. He obtained several results relating the geometrical and algebraic aspects of the set M_n^+. These results had the form of decomposition theorems for matrices by means of geodesically convex subsets, and a main (algebraic) concept behind the notion of convexity was that of a *Lie triple system*, which is basically the real part of a given complex Lie algebra $\mathfrak{g}$. The reader must note that the present paper's title is a deserved homage for Mostow's paper and results.

Stepping to the infinite dimensional context was not immediate (in time) neither in the techniques involved, and several years came by until a group of authors (G. Corach, H. Porta and L. Recht among others) started working with geometry of operators using functional analysis techniques [PR87a], [CPR90a]. This area of research is currently very active (see the papers [CM99], [CM00] for a list of references).

It is widely known that the main obstruction behind the differential geometry approach for sets of operators in $\mathcal{L}(H)$ with the usual norm is that when H is infinite dimensional, the norm $\| \|_\infty$ is not smooth neither equivalent to a smooth norm (this is also true for an arbitrary infinite dimensional C^*-algebra). This obstruction usually leads to ad-hoc techniques such as the action of a natural Lie group, an action which (under certain regularity hypothesis) provides bundles to do differential geometry (see for instance [Re92] by L. Recht or [AFHS90] by E. Andruchow, L.A Fialkow, D.A. Herrero, M. Pecuch de Herero and D. Stojanoff). A way around this obstruction is to consider the topology and the curves as the foundational objects of geometry, an approach sustained in [Kr-Mi] and nicely carried out in the papers [DMR04a], [DMR04b] by D. Durán, L. Mata-Lorenzo and L. Recht.

A result which is closer in spirit to the factorization theorem we prove in this paper is the infinite dimensional decomposition theorem for operators proved in the paper [PR94] by H. Porta and L. Recht, though we should stress that the treatment in that paper is in the context of C^*-algebras and conditional expectations, hence the geometry is not Riemannian neither differentiable in the (strict) classical sense. However, there is a common general principle in these factorizations that the reader will recognize:

- a) The tangent space of the submanifold where we project identifies naturally with a particular subalgebra. This subalgebra splits conveniently the total algebra (which is the tangent space of the full manifold).
- b) The kernel of the the conditional expectation is the vertical space (in this way the expectation is the local analogue of the orthogonal projection), and the mapping induced in the manifold by the normal geodesics is a contraction.

We summarize briefly the contents of this paper: we study the geometry of a Hilbert manifold Σ_∞ which is modelled on the classical Lie Group $GL_{\mathcal{H}_\mathbb{C}}$ of invertible (unitized) Hilbert-Schmidt operators. This manifold consists of positive invertible operators, and has a natural $GL_{\mathcal{H}_\mathbb{C}}$-invariant metric which makes it nonpositely curved. We show (Theorem 3.21) that for any geodesically convex submanifold M (which is also nonpositively curved) there exists a natural Lie subgroup $G_M \subset GL_{\mathcal{H}_\mathbb{C}}$ which acts isometrically and transitively on M, and we prove that G_M has a polar decomposition where $G_M^+ = M$. The manifold Σ_∞ can be decomposed by means of a foliation $\{\Sigma_\lambda\}_{\lambda>0}$ of totally geodesic submanifolds (see section 6), and can be also decomposed (Theorem 4.6) by means of any convex, closed submanifold M. We exhibit explicit decompositon theorems for the submanifold of positive invertible $n \times n$ matrices (Theorem 6.10), and also for the submanifold of diagonal operators. The latter decomposition theorem (Theorem 5.2) takes the form of a factorization theorem for operators $e^a = de^v d$, where v has null diagonal and d is an invertible diagonal operator.

This work is inspired mainly by two sources: the exposition on nonpositively curved symmetric spaces given by Patrick Eberlein in his Lecture Notes [Eb96] and the splitting theorems (for the set of positive invertible elements in a C^*-algebra) of the paper [CPR91] by Gustavo Corach, Horacio Porta and Lázaro Recht.

2 The main objects involved

The main framework of this paper is the von Neumann algebra $\mathcal{L}(H)$ of bounded operators acting on a complex, infinite dimensional and separable Hilbert space H, and throughout $\mathcal{HS}$ stands for the bilateral ideal of Hilbert-Schmidt operators of $\mathcal{L}(H)$. Recall that $\mathcal{HS}$ is a Banach algebra (without unit) when given the norm $\|a\|_2 = 2\, tr(a^*a)^{\frac{1}{2}}$ (see [Simon89] for a detailed exposition). Inside $\mathcal{L}(H)$ we consider a certain kind of Fredholm operators, namely

$$\mathcal{H}_\mathbb{C} = \{a + \lambda : \ a \in \mathcal{HS},\ \lambda \in \mathbb{C}\},$$

the complex linear subalgebra consisting of Hilbert-Schmidt perturbations of scalar multiples of the identity (the closure of this algebra in the operator norm is the set of compact perturbations of scalar multiples of the identity). There is a natural Hilbert space structure for this subspace, where scalar operators are orthogonal to Hilbert-Schmidt operators, which is given by the inner product (see [Har72])

$$\langle a + \lambda, b + \beta \rangle_2 = 4tr(ab^*) + \lambda\overline{\beta}$$

This product is well defined and positive definite; $\mathcal{H}_\mathbb{C}$ is complete with this norm, due to the completeness of the $\mathcal{HS}$ operators with the trace inner product.

The model space that we are interested in is the real part of $\mathcal{H}_\mathbb{C}$,

$$\mathcal{H}_\mathbb{R} = \{a + \lambda : \ a^* = a,\ a \in \mathcal{HS},\ \lambda \in \mathbb{R}\},$$

which inherits the structure of real Banach space, and with the same inner product, becomes a real Hilbert space. We will use $\mathcal{HS}^h$ to denote the closed subspace of selfadjoint Hilbert-Schmidt operators. Inside $\mathcal{H}_{\mathbb{R}}$, consider the subset of positive invertible operators, *i.e*

$$\Sigma_\infty := \{A > 0, A \in \mathcal{H}_{\mathbb{R}}\}$$

It is easy relatively easy to see that Σ_∞ is an open set of $\mathcal{H}_{\mathbb{R}}$ (for instance, use lower semicontinuity of the spectrum).

For $p \in \Sigma_\infty$, we identify $T_p\Sigma_\infty$ with $\mathcal{H}_{\mathbb{R}}$, and endow this manifold with a (real) Riemannian metric by means of the formula

$$\langle X, Y \rangle_p = \langle p^{-1}X, Yp^{-1} \rangle_2 = \langle Xp^{-1}, p^{-1}Y \rangle_2$$

Note that this metric arises naturally when we consider the action of the group of invertible operators, see Lemma 2.2.

We collect some facts that can be found in [Wilk94], [AV03] and [CPR94]:

- Covariant derivative is given by the expression

$$\nabla_X Y = X(Y) - \frac{1}{2}\left(Xp^{-1}Y + Yp^{-1}X\right) \tag{1}$$

  where $X(Y)$ denotes derivation of the vector field Y in the direction of X (performed in the ambient space $\mathcal{H}_{\mathbb{R}}$).

- With this metric Σ_∞ has nonpositive sectional curvature; moreover, the curvature tensor is given by the following commutant:

$$\mathcal{R}_p(X,Y)Z = -\frac{1}{4}\, p\left[\left[p^{-1}X, p^{-1}Y\right], p^{-1}Z\right] \tag{2}$$

- Euler's equation for geodesics ($\nabla_{\dot\gamma}\dot\gamma = 0$) reads

$$\ddot\gamma \; - \dot\gamma\gamma^{-1}\dot\gamma = 0, \tag{3}$$

  and the unique geodesic joining $\gamma_{pq}(0) = p$ with $\gamma_{pq}(1) = q$ is given by the expression

$$\gamma_{pq}(t) = p^{\frac{1}{2}}\left(p^{-\frac{1}{2}}qp^{-\frac{1}{2}}\right)^t p^{\frac{1}{2}} \tag{4}$$

  These curves look formally equal to the geodesics between positive definite matrices (regarded as a symmetric space, see [Mos55]).

- This geodesic is unique and realizes the distance: the manifold Σ_∞ turns out to be complete with this distance (in the sense that any two points can be joined by a geodesic).

- The distance function $f : \mathbb{R} \to \mathbb{R}_{\geq 0}$, $f(t) = \mathrm{dist}(\gamma_1(t), \gamma_2(t))$ (γ_i are geodesics) is a convex function.

- The exponential map increases distance, that is

$$\|X - Y\|_2 \leq \mathrm{dist}(\mathrm{e}^X, \mathrm{e}^Y) = \|\ln(\mathrm{e}^{-X/2}\mathrm{e}^X\mathrm{e}^{-X/2})\|_2$$

Remark 2.1. Throughout, $\|X\|_p^2 := \langle X, X \rangle_p$, namely

$$\|X\|_p^2 = \|p^{-1/2} X p^{-1/2}\|_2 = \left\langle Xp^{-1}, p^{-1}X \right\rangle_2 = \left\langle p^{-1}X, Xp^{-1} \right\rangle_2 ,$$

which is the norm of tangent vectors $X \in T_p\Sigma_\infty$. We will use $\exp_p$ to denote the exponential map of Σ_∞. Note that

$$\exp_p(V) = p^{\frac{1}{2}} \, \mathrm{e}^{\, p^{-\frac{1}{2}} \, V \, p^{-\frac{1}{2}}} p^{\frac{1}{2}},$$

but rearranging the exponential series we get the simpler expressions

$$\exp_p(V) = p \, \mathrm{e}^{p^{-1}V} = \mathrm{e}^{Vp^{-1}} p$$

Lemma 2.2. *The metric in Σ_∞ is invariant for the action of the group of invertible elements: if g is an invertible operator in $\mathcal{H}_{\mathbb{C}}$, then $I_g(p) = gpg^*$ is an isometry.*

Proof. Note that $d_r I_g(x) = gxg^*$ for any $x \in T_r\Sigma_\infty$, so

$$\|gxg^*\|_{grg^*}^2 = \left\langle gxg^*(g^*)^{-1}r^{-1}g^{-1}, (g^*)^{-1}r^{-1}g^{-1}gxg^* \right\rangle_2 =$$

$$= \left\langle gxr^{-1}g^{-1}, (g^*)^{-1}r^{-1}xg^* \right\rangle_2 = \left\langle xr^{-1}, r^{-1}x \right\rangle_2 = \|x\|_r^2$$

where the third equality follows from the ciclicity of the trace. □

3 Geodesically convex submanifolds

Convex sets are particulary useful in geometry, and play a major role in the theory of hyperbolic (*i.e.* nonpositevely curved) spaces. For convenience we recall the following definitions:

Definition 3.1. *A* geodesically convex *(also* totally convex*, or* convex*) set $M \subset \Sigma_\infty$ is a set such that, given any two points $p, q \in M$, the unique geodesic of Σ_∞ joining p to q lies entirely in M.*

A Riemannian submanifold $M \subset \Sigma_\infty$ (with the induced metric) is geodesic *at $p \in M$ if geodesics of the ambient space starting at p which have initial velocity in T_pM, are also geodesics of M. M is a* totally geodesic manifold *if it is geodesic at any $p \in M$. Equivalently, any geodesic of M is also a geodesic of the ambient space Σ_∞.*

A Riemannian submanifold $M \subset \Sigma_\infty$ i at $p \in M$ if $\exp_p^M$ is defined in the whole tangent space and maps onto M. M is a complete manifold *if it is complete at any pont.*

A Riemannian submanifold $M \subset \Sigma_\infty$ is flat at p *if sectional curvature vanishes for any 2-subspace of T_pM. M is a flat manifold if it is flat at any $p \in M$.*

Remark 3.2. Σ_∞ is complete; moreover, $\exp_p$ is a diffeomorphism onto Σ_∞ for any p. The reader should be careful with other notions of completeness, because, as C.J.

Atkin shows in [Atkin75] and [Atkin87], Hopf-Rinow theorem is not necessarily valid in this infinite dimensional context.

These previous notions are strongly related; it is easy to see that, for any Riemannian submanifold M of Σ_∞,

$$M \text{ geodesically convex } \iff M \text{ complete and totally geodesic}$$

3.1 An intrinsic characterization of convexity

Obviously, in this context curvature and commutativity are related; the following proposition makes this relation explicit:

Proposition 3.3. *Assume $M \subset \Sigma_\infty$ is a flat submanifold. Assume further that M is geodesic at p. Then the tangent space at each $p \in M$ is $p-$abelian, namely*

$$p^{-\frac{1}{2}} X p^{-\frac{1}{2}} \text{ commutes with } p^{-\frac{1}{2}} Y p^{-\frac{1}{2}} \text{ for any pair } X, Y \in T_pM$$

Proof. Since M is geodesic at p, the curvature tensor is the restriction of the curvature tensor of Σ_∞. Set $x = p^{-\frac{1}{2}} X p^{-\frac{1}{2}}$, $y = p^{-\frac{1}{2}} Y p^{-\frac{1}{2}}$. Then a straightforward computation shows that

$$\langle \mathcal{R}_p(X,Y)Y, X\rangle_p = -\frac{1}{4}\left\{ \left\langle xy^2, x\right\rangle_2 - 2\left\langle yxy, x\right\rangle_2 + \left\langle y^2x, x\right\rangle_2 \right\}$$

Now $x, y \in \mathcal{H}_\mathbb{R}$, so $x = \lambda + a$, $y = \beta + b$, and the equation reduces to

$$\langle \mathcal{R}_p(X,Y)Y, X\rangle_p = -\frac{1}{2}\left\{ tr(a^2b^2) - tr((ab)^2)\right\} \tag{5}$$

The Cauchy-Schwarz inequality for the trace tells us that curvature at $p \in \Sigma_\infty$ is always nonpositive, and it is zero if and only if a and b commute. Hence whenever M is flat, x and y commute for any pair of tangent vectors $X, Y \in T_pM$ as stated. □

To deal with convex sets the following definition will be useful:

Definition 3.4. *We say that a subspace $\mathfrak{m} \subset \mathcal{H}_\mathbb{R}$ is a* Lie triple system *if* $[[a,b],c] \in \mathfrak{m}$ *for any $a, b, c \in \mathfrak{m}$.*

Note that whenever a, b, c are selfadjoint operators, $d = [a,[b,c]]$ is also a selfadjoint operator. So, for any Lie subalgebra of operators $\mathfrak{a} \subset \mathcal{H}_\mathbb{C}$ (in particular: for any Banach subalgebra), $\mathfrak{m} = \mathfrak{Re}(\mathfrak{a})$ is a Lie triple system in $\mathcal{H}_\mathbb{R}$.

Assume $M \subset \Sigma_\infty$ is a submanifold such that $1 \in M$, and M is geodesic at $p = 1$. Then T_1M is a Lie triple system, because the curvature tensor at $p = 1$ is the restriction to T_1M of the curvature tensor of Σ_∞, and $\mathcal{R}_1(X,Y)Z = -\frac{1}{4}[[X,Y],Z]$. In particular, if M is geodesically convex, T_1M must be a Lie triple system. This particular condition on the tangent space turns out to be strong enough to ensure convexity (this result is well known, see [Har72]):

Theorem 3.5. *Assume* $\mathfrak{m} \subset \mathcal{H}_{\mathbb{R}}$ *is a closed subspace, set* $M = \exp(\mathfrak{m}) \subset \Sigma_\infty$ *with the induced topology and Riemannian metric.*

$$\textit{If } \mathfrak{m} \textit{ is a Lie triple system, then } p, q \in M \Rightarrow qpq \in M$$

Proof. As Pierre de la Harpe pointed out, the proof of G.D. Mostow for matrices in [Mos55] can be translated to Hilbert-Schmidt operators without any modification: we give a sketch of the proof here. Assume $p = \mathrm{e}^X$, $q = \mathrm{e}^Y$, and consider the curve $\mathrm{e}^{\alpha(t)} = \mathrm{e}^{tY} \cdot \mathrm{e}^X \cdot \mathrm{e}^{tY}$. Then it can be proved that $\dot{\alpha}(t) = G(\alpha(t))$ for a map G that maps $\mathfrak{m}$ into $\mathfrak{m}$ (this is nontrivial). Since $\alpha(0) = X \in \mathfrak{m}$ and G is a Lipschitz map by the uniqueness of the solutions of ordinary differential equations we have $\alpha \subset \mathfrak{m}$. Hence $\mathrm{e}^{\alpha(1)} = qpq \in M$ and the claim follows. □

Corollary 3.6. *Assume* $M = \exp(\mathfrak{m}) \subset \Sigma_\infty$ *as above, and* $\mathfrak{m}$ *is a Lie triple system. Then* M *is geodesically convex.*

Proof. Take $p, q \in M$. Then $p = \mathrm{e}^X$, $q = \mathrm{e}^Y$ with $X, Y \in \mathfrak{m}$. If we set $r = \mathrm{e}^{-X/2}\mathrm{e}^Y\mathrm{e}^{-X/2}$, then $r \in M$ because $\mathrm{e}^{-X/2}$ and e^Y are in M. Moreover, $Z = \ln(r) \in \mathfrak{m}$. But the unique geodesic of Σ_∞ joining p to q is

$$\gamma(t) = \mathrm{e}^{X/2}\mathrm{e}^{tZ}\mathrm{e}^{X/2}, \quad \text{so} \quad \gamma \subset M \quad \square$$

Corollary 3.7. *Assume* $\mathfrak{m} \subset \mathcal{H}_{\mathbb{R}}$ *is a closed abelian subalgebra of operators. Then the manifold* $M = \exp(\mathfrak{m}) \subset \Sigma_\infty$ *is a closed, convex and flat Riemannian submanifold. Moreover,* $M \subset \mathfrak{m}$ *is open and* M *is an abelian Banach-Lie group.*

Proof. The first assertion follows from the fact that $\mathfrak{m}$ is a Lie triple system; since curvature is given by commutators, M is flat. Since $\mathfrak{m}$ is a closed subalgebra, $e^X = \sum \frac{X^n}{n!} \in \mathfrak{m}$ for any $X \in \mathfrak{m}$, so $M \subset \mathfrak{m}$. Note that every $p = e^X \in M$ has a unique analytic logarithm (because it is a positive operator), so the exponential map is locally open. □

If M is geodesic at $p = 1$, $T_1M = \mathfrak{m}$ is abelian (by Proposition 3.3), so

Corollary 3.8. *Assume* $M = \exp(\mathfrak{m})$ *is closed and flat. If* M *is geodesic at* $p = 1$, *then* M *is a convex submanifold. Moreover,* M *is an abelian Banach-Lie group and* $M \subset \mathfrak{m}$ *is open.*

The definition of symmetric space we adopt is the usual definition for Riemannian manifolds (see [Hel62])

Definition 3.9. *A Hilbert manifold* M *is called a* globally symmetric space *if each point* $p \in M$ *is an isolated fixed point of an involutive isometry* $s_p : M \to M$. *The map* s_p *is called the* geodesic symmetry at p.

Theorem 3.10. *Assume* $M = \exp(\mathfrak{m})$ *is closed and geodesically convex. Then* M *is a symmetric space; the geodesic symmetry at* $p \in M$ *is given by* $s_p(q) = pq^{-1}p$ *for any* $q \in M$. *In particular,* Σ_∞ *is a symmetric space.*

Proof. First observe that, for $p = \mathrm{e}^X$, $q = \mathrm{e}^Y$, $s_p(q) = \mathrm{e}^X \mathrm{e}^{-Y} \mathrm{e}^X$, which shows that s_p maps M into M. To prove that s_p is an isometry, take a geodesic α_V of M such that $\alpha(0) = q$ and $\dot{\alpha}(0) = V$. Then $\alpha(t) = q\mathrm{e}^{t\,q^{-1}V}$ and

$$d_q(s_p)(V) = (s_p \circ \alpha_V)^{\cdot}\big|_{t=0} = -pq^{-1}Vq^{-1}p$$

Since M has the induced metric, $\|pq^{-1}Vq^{-1}p\|^2_{pq^{-1}p} = \|V\|^2_q$ by Lemma 2.2 (take $g = pg^{-1}$). In particular, $d_p s_p = -id$, so p is an isolated fixed point of s_p for any $p \in M$. □

As we see from Theorem 3.5 and its corollaries, Σ_∞ (as any symmetric space) contains plenty of convex sets; in particular

Remark 3.11. We can embed isometrically any k-dimensional plane in Σ_∞ as a geodesically convex, closed submanifold: take an orthonormal set of k commuting operators (for instance, fix an orthonormal basis $\{e_i\}_{i\in\mathbb{M}}$ of H, and take $p_i = e_i \otimes e_i$, $i = 1, \cdots, k$), now take the exponential of this set. In the languaje of symmetric spaces, what we are saying is that $rank\,(\Sigma_\infty) = +\infty$.

Following the usual notation for symmetric spaces, we set $I_0(M)$=the path-connected component of the identity of the group of isometries of M.

Remark 3.12. Assume $1 \in M \subset \Sigma_\infty$ is closed and convex. Then, since any isometry φ is uniquely determined by its value at $1 \in M$ and its differential $d_1\varphi$, $I_0(M)$ carries a natural structure of Banach-Lie group: take $\varphi \in I_0(M)$, and consider

$$\overline{\varphi}\,(q) = \varphi(1)^{-1/2} \cdot \varphi(q) \cdot \varphi(1)^{-1/2}$$

note that $d_1\overline{\varphi}$ is a unitary operator of $T_1M = \mathfrak{m}$ (which carries a natural Hilbert-space structure), so there is an inclusion $J : I_0(M) \hookrightarrow M \times \mathcal{U}_{\mathcal{L}(\mathfrak{m})}$ given by the map $\varphi \mapsto (\varphi(1), d_1\overline{\varphi})$. We will see later that the unitaries of the form $x \mapsto gxg^*$ (inner automorphisms, see Lemma 2.2) are enough to act transitively on M (g must be in G_M, see Theorem 3.21).

Theorem 3.13. *Assume $M = \exp(\mathfrak{m})$ is closed and geodesically convex. Then $I_0(M)$ acts transitively on M.*

Proof. Take $p = \mathrm{e}^X$, $q = \mathrm{e}^Y$ two points in M. Take $\gamma(t) = p\mathrm{e}^{t\,p^{-1}V}$ the geodesic joining p to q. Note that

$$q = \gamma(1) = p\mathrm{e}^{p^{-1}V} = \mathrm{e}^{vp^{-1}}p$$

Consider the curve of isometries $\varphi_t = s_{\gamma(t/2)} \circ s_p$. Since $\varphi_0 = id$, $\varphi_t \subset I_0(M)$. Now

$$\varphi_1(p) = \mathrm{e}^{\frac{1}{2}V\mathrm{e}^{-X}}\mathrm{e}^X\mathrm{e}^{-X}\,\mathrm{e}^{\frac{1}{2}V\mathrm{e}^{-X}}\mathrm{e}^X = \mathrm{e}^{V\mathrm{e}^{-X}}\mathrm{e}^X = q$$

which proves that $I_0(M)$ acts transitively on M. □

Remark 3.14. If $M = \exp(\mathfrak{m})$ is closed and convex, in particular it is geodesic at p for any $p \in M$, so

$$T_pM = \exp_p^{-1}(M) = \{p^{\frac{1}{2}} \ln(p^{-\frac{1}{2}}\, q\, p^{-\frac{1}{2}}) p^{\frac{1}{2}} : q \in M\}$$

(see Remark 2.1). This observation together with Theorem 3.5 proves the identification

$$T_pM = p^{\frac{1}{2}}\,(T_1M)\,p^{\frac{1}{2}} = p^{\frac{1}{2}}\,\mathfrak{m}\,p^{\frac{1}{2}}$$

From the previous identifications of the tangent space follow easily that an operator $V \in \mathcal{H}_{\mathbb{R}}$ is orthogonal to M at p (that is, $V \in T_pM^{\perp}$) if and only if

$$\left\langle p^{-\frac{1}{2}}\, Z\, p^{-\frac{1}{2}}\ ,\ V \right\rangle_2 = \left\langle p^{-\frac{1}{2}}\, V\, p^{-\frac{1}{2}}, Z \right\rangle_2 = 0 \quad \text{for any} \quad Z \in \mathfrak{m}$$

In particular,

$$T_1M^{\perp} = \mathfrak{m}^{\perp} = \{V \in \mathcal{H}_{\mathbb{R}}\ :\ \langle V, Z\rangle_2 = 0 \ \text{ for any } \ Z \in \mathfrak{m}\}$$

Note that when $\mathfrak{m}$ is a closed abelian *subalgebra* of operators, $p^{\frac{1}{2}} = e^{X/2} \in \mathfrak{m}$ and also the map $Y \mapsto Y \cdot p^{\frac{1}{2}}$ is an isomorphism of $\mathfrak{m}$; so $T_pM = \mathfrak{m} = T_1M$ in this case (for any $p \in M$). This also follows easily from Corollary 3.7. In particular,

$$T_pM^{\perp} = T_1M^{\perp} = \mathfrak{m}^{\perp} \quad \text{for any } p \in M$$

Remark 3.15. Assume $M \subset \Sigma_\infty$ is geodesically convex. Then, if γ is the geodesic joining p to q, the isometry $\varphi_t = s_{\gamma(t/2)} \circ s_p$ translates along the curve γ, namely

$$\varphi_t(\gamma(s)) = p\,\mathrm{e}^{\frac{t}{2}p^{-1}V} \cdot p^{-1} \cdot p\,\mathrm{e}^{sp^{-1}V} \cdot p^{-1} \cdot p\,\mathrm{e}^{\frac{t}{2}p^{-1}V} =$$

$$= p\,\mathrm{e}^{\frac{t}{2}p^{-1}V} \cdot \mathrm{e}^{sp^{-1}V} \cdot\ \mathrm{e}^{\frac{t}{2}p^{-1}V} = p\,\mathrm{e}^{(s+t)p^{-1}V} = \gamma(s+t)$$

In particular, $\varphi_1(p) = q$. Now take any tangent vector $W \in T_{\gamma(s)}M$, and set

$$W(t) := (d\varphi_t)_{\gamma(s)}(W) = \mathrm{e}^{\frac{t}{2}Vp^{-1}} \cdot W \cdot \mathrm{e}^{\frac{t}{2}p^{-1}V}$$

Then $W(t)$ is the parallel translation of W from $\gamma(s)$ to $\gamma(s+t)$; namely $\nabla_{\dot\gamma}\, W \equiv 0$ (this follows from a straightforward computation using equation (1))

We conclude that the map $(d\varphi_t)_{\gamma(s)} : T_{\gamma(s)}M \to T_{\gamma(s+t)}M$ gives parallel translation along γ, namely $(d\varphi_t)_{\gamma(s)} = P_s^{t+s}(\gamma)$. In particular, since $q = \gamma(1) = p^{\frac{1}{2}}\ \mathrm{e}^{p^{-1/2}Vp^{1/2}} p^{\frac{1}{2}}$, the map

$$W \mapsto p^{\frac{1}{2}}(p^{-\frac{1}{2}}qp^{-\frac{1}{2}})^{\frac{1}{2}}p^{-\frac{1}{2}} \cdot W \cdot p^{-\frac{1}{2}}(p^{-\frac{1}{2}}qp^{-\frac{1}{2}})^{\frac{1}{2}}p^{\frac{1}{2}}$$

gives parallel translation from T_pM to T_qM.

3.1.1 A few examples of convex sets

We list several convex submanifolds of Σ_∞; for some of them we present on this paper an explicit factorization theorem. The general factorization theorems (Theorem 4.8

and Theorem 4.9) apply for any of these (to be precise, to their closures in the 2-norm):

1. For any subspace $\mathfrak{s} \subset \mathcal{H}_{\mathbb{R}}$, the subspace

$$\mathfrak{m}_{\mathfrak{s}} = \{X \in \mathcal{H}_{\mathbb{R}} \ : \ [X,Y] = 0 \ \forall\, Y \in \mathfrak{s}\}$$

   is a Lie triple system.

2. In particular, for any $Y \in \mathcal{H}_{\mathbb{R}}$,

$$\mathfrak{m}_Y = \{X \in \mathcal{H}_{\mathbb{R}} \ : \ [X,Y] = 0\}$$

   is a Lie triple system.

3. The family of operators in $\mathcal{H}_{\mathbb{R}}$ which act as endomorphisms of a closed subspace $S \subset H$ form a Lie triple system in $\mathcal{H}_{\mathbb{R}}$.

4. Any norm closed abelian subalgebra of $\mathcal{H}_{\mathbb{R}}$ is a Lie triple system, in particular
   (a) The diagonal operators (see section 5). This is a maximal abelian closed subspace of $\mathcal{H}_{\mathbb{R}}$, hence the manifold Δ (which is the exponential of this set) is a maximal flat submanifold of Σ_∞.
   (b) The scalar manifold $\Lambda = \{\lambda \cdot 1 \ \lambda > 0\}$
   (c) For fixed $a \in \mathcal{HS}^h$, the real part of the closed algebra generated by a, which is the closure in the 2-norm of the set of polynomials in a.

5. The real part of any Lie subalgebra of $\mathcal{H}_{\mathbb{C}}$ is a Lie triple system (in particular: the real part of any Banach subalgebra).

6. Any real Banach-Lie algebra $\mathfrak{g}$ with a compatible Riemannian product invariant by inner automorphisms has a complexification which leads to the structure of an L^*-algebra, and any L^*-algebra can be embedded as a closed Lie subalgebra of $\mathcal{HS}$ (see [CGM90] and [Neh93]).

7. If G is a simply connected semisimple locally compact Lie group, then any irreducible unitary representation of $C^*(G)$ into $\mathcal{L}(H)$ maps $C_K(G)$ (the continous functions with compact support) into $\mathcal{HS}$ (see [Bag69]). This inclusion is also true for any irreducible subrepresentation of the left regular representation of a unimodular group G.

3.2 Convex manifolds as homogeneous spaces

We recall (without proof, see for instance [Lang95]) a result for quotients of Banach-Lie groups:

Theorem 3.16. *Let G be an analytic Banach-Lie group (i.e. an algebraic group G together with a compatible Banach manifold structure), and K a Banach-Lie subgroup (i.e. an algebraic subgroup of G which is also a submanifold). Then on the left cosets space G/K there exists a unique analytic manifold strutcture such that the projection is a submersion. The canonical action $G \times G/K \to G/K$ is analytic.*

For any Banach algebra $\mathcal{B}$, we will denote $GL(\mathcal{B})$ the group of invertible elements. Note that this group has a natural structure of manifold as an open set of the algebra, so $GL(\mathcal{B})$ is always a Banach-Lie group with Lie algebra $\mathcal{B}$.

Remark 3.17. The group $GL_{\mathcal{H}_{\mathbb{C}}}$, having the homotopy type of the inductive limit of the groups $GL(n,\mathbb{C})$ (see [Har72], section II.6) is connected; moreover, there is an homotopy equivalence

$$GL_{\mathcal{H}_{\mathbb{C}}} \simeq S^1 \times S^1 \times SU(\infty)$$

Here $SU(\infty)$ stands for the inductive limit of the groups $SU(n,\mathbb{C})$

The following result is standard in finite dimensions (see for instance, [Hel62]); we say that G is a selfadjoint subgroup (shortly, $G^* = G$) if $g^* \in G$ whenever $g \in G$. Note that G is selfadjoint iff $\mathfrak{g}^* = \mathfrak{g}$, where the latter denotes the Lie algebra of G.

Theorem 3.18. *Fix a connected Lie subgroup $G \subset GL_{\mathcal{H}_{\mathbb{C}}}$ such that $G^* = G$. Let P be the analytic map*

$$P : GL_{\mathcal{H}_{\mathbb{C}}} \to GL_{\mathcal{H}_{\mathbb{C}}} \quad \text{where } P(g) = gg^* = \mid g \mid^2$$

If K denotes the isotropy group of P (namely $K = P^{-1}(1) \cap G$ with the induced analytic structure), then $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$, where $\mathfrak{k}$ is the Lie algebra of K and $\mathfrak{p}$ are the selfadjoint elements of $\mathfrak{g}$. In particular, K is a Lie subgroup of G.

Proof. Note that $\sigma(g) = g^*$ is involutive so its differential at $g = 1$ gives an involution Θ of $\mathfrak{g}$ that induces the desired splitting of the Lie algebra of G. Now K is a Lie subgroup because the Lie algebra splits conveniently. □

Remark 3.19. For $M = \exp(\mathfrak{m})$ a geodesically convex closed manifold in Σ_∞, consider

$$[\mathfrak{m},\mathfrak{m}] = span\{[a,b] : a, b \in \mathfrak{m}\} = \left\{ \sum_{i\in F} [a_i, b_i] \ : \ a_i, b_i \in \mathfrak{m}\,;\ F \text{ a finite set} \right\}$$

Note that all the operators in $[\mathfrak{m},\mathfrak{m}]$ are skewadjoint. Set $\mathfrak{g}_M = \mathfrak{m} \oplus \overline{[\mathfrak{m},\mathfrak{m}]}$. Then $\mathfrak{g}_M$ is a closed Lie subalgebra of $\mathcal{H}_{\mathbb{C}}$ because $\mathfrak{m}$ is a Lie triple system (see [Hel62]). Since $\mathcal{H}_{\mathbb{C}}$ is a Hilbert space and $\mathfrak{g}_M$ is closed, the Lie algebra splits: it follows that $\mathfrak{g}_M$ is integrable (see [Lang95]). Let $G_M \subset GL_{\mathcal{H}_{\mathbb{C}}}$ be the Lie subgroup which is the component of the identity of the Lie group which has $\mathfrak{g}_M$ as Lie algebra.

Note that $M \subset G_M$ and $G_M^* = G_M$ since $(a+[b,c])^* = a+[c,b]$ for any $a,b,c \in \mathfrak{m}$. It is also clear that $\mathfrak{k} = \overline{[\mathfrak{m},\mathfrak{m}]}$ (in the notation of Theorem 3.18). G_M is the smallest Lie group containing M.

The elements of M are indeed the positive elements of G_M, and the elements of K the unitary operators of G_M; we prove it below. Note that when $\mathfrak{m}$ is abelian, $\mathfrak{g}_M = \mathfrak{m}$ and also $G_M = M \subset \mathfrak{m}$ is an open set.

Lemma 3.20. *With the notation of the above Remark and the hypothesis of Theorem 3.18, $P(G_M) \subset M$.*

Proof. Since $\mathfrak{g}_M$ splits, there are neighbourhoods of zero $U_\mathfrak{m} \subset \mathfrak{m}$ and $U_\mathfrak{k} \subset \mathfrak{k} = \overline{[\mathfrak{m},\mathfrak{m}]}$ such that the map $X_\mathfrak{m} + Y_\mathfrak{k} \mapsto \mathrm{e}^{X_\mathfrak{m}}\mathrm{e}^{Y_\mathfrak{k}}$ is an isomorphism from $U_\mathfrak{m} \oplus U_\mathfrak{k}$ onto an open neighbourhood V_M of $1 \in G_M$. Clearly, the group generated by V_M is open (and closed) in G_M, so $\langle V_M \rangle = G_M$. So, for any $g \in G_M$, $g = (\mathrm{e}^{x_1}\mathrm{e}^{y_1})^{\alpha_1} \cdots (\mathrm{e}^{x_n}\mathrm{e}^{y_n})^{\alpha_n}$ for some selfadjoint $x_i \in U_\mathfrak{m}$, some skewadjoint $y_i \in U_\mathfrak{k}$, and some $\alpha_i = \pm 1$.

Now $\mathrm{e}^x\mathrm{e}^y\mathrm{e}^x \in M$ whenever $x, y \in \mathfrak{m}$ (see Theorem 3.5), so an inspection of the expression for $P(g) = gg^*$ shows that $P(g)$ will be in M if we can prove that $\mathrm{e}^y\mathrm{e}^x\mathrm{e}^{-y} \in M$ whenever $x \in \mathfrak{m}$ and $y \in \mathfrak{k}$ (namely, if we can prove that $kMk^* \subset M$ for any $k \in K$). It will be enough to show this is valid for $x \in \mathfrak{m}$ and $y = \sum_i [a_i, b_i] \in [\mathfrak{m},\mathfrak{m}]$ because M is closed. We assert that this is true, but to avoid cumbersome notations we write the proof for $y = [a,b]$. The proof of the general case is identical and therefore we omit it.

Consider the map $F : \mathcal{H}_\mathbb{R} \to \mathcal{H}_\mathbb{R}$ given by $F(z) = [[a,b], z]$. Since F maps $\mathfrak{m}$ into $\mathfrak{m}$, the flow of F in $\mathfrak{m}$ stays in $\mathfrak{m}$, so the ordinary differential equation $\dot{x}(t) = F(x(t))$ has unique solution in $\mathfrak{m}$ if $x(0) \in \mathfrak{m}$ is given (see [Lang95]). Take $\alpha(t) = \mathrm{e}^{\,t[a,b]}x\,\mathrm{e}^{-t[a,b]}$. Then $\alpha(0) = x \in \mathfrak{m}$; moreover

$$\dot{\alpha}(t) = \mathrm{e}^{\,t[a,b]}[[a,b], x]\,\mathrm{e}^{-t[a,b]} = [[a,b], \mathrm{e}^{\,t[a,b]}x\,\mathrm{e}^{-t[a,b]}] = F(\alpha(t))$$

which proves that $\alpha(t) \subset \mathfrak{m}$ for any $t \geq 0$. In particular, $\alpha(1) = \mathrm{e}^{\,[a,b]}x\,\mathrm{e}^{-[a,b]} \in \mathfrak{m}$. Taking the exponential and using that $\mathrm{e}^{\,[a,b]}$ is a unitary operator, we get $\mathrm{e}^{\,[a,b]}\mathrm{e}^x\,\mathrm{e}^{-[a,b]} \in M$, which finishes the proof. $\square$

Theorem 3.21. *If $M = \exp(\mathfrak{m})$ is convex and closed, and $G_M \subset GL_{\mathcal{H}_\mathbb{C}}$ is the Lie subgroup with Lie algebra $\mathfrak{g}_M = \mathfrak{m} \oplus \overline{[\mathfrak{m},\mathfrak{m}]}$, then*

(a) $P(G_M) = M$, so M is an homogeneous space for G_M.

(b) For any $g = \mid g \mid u_g$ (Cauchy polar decomposition) in G_M, we have

$$\mid g \mid = \sqrt{gg^*} \in M \subset G_M,$$

and also $u_g \in K \subset G_M$ where K is the isotropy Lie subgroup

$$K = \{g \in G_M \ : \ gg^* = 1\} \text{ with Lie algebra } \mathfrak{k} = \overline{[\mathfrak{m},\mathfrak{m}]}$$

In particular, G_M has a polar decomposition

$$G_M \simeq M \times K = P(G_M) \times U(G_M)$$

(c) $M = P(G_M) \simeq G_M/K$.

(d) M has nonpositive sectional curvature.

(e) For $g \in G_M$, consider $I_g(r) = grg^*$. Then $I : G_M \to I_0(M)$.

(f) Take $p, q \in M$, and set $g = p^{\frac{1}{2}}(p^{-\frac{1}{2}}qp^{-\frac{1}{2}})^{\frac{1}{2}}p^{-\frac{1}{2}} \in G_M$. Then I_g is an isometry in $I_0(M)$ which sends p to q, namely G_M acts transitively and isometrically on M.

Proof. Since any $p \in M$ is the exponential of some $x \in \mathfrak{m}$, we get $p = P(\mathrm{e}^{x/2})$, which proves that $M \subset P(G_M)$; the other inclusion is given by Lemma 3.20. To prove (b), note that $P(G_M) = M = \exp(\mathfrak{m})$; namely for any $g \in G_M$, $gg^* = P(g) \in M$; hence $gg^* = e^{x_0}$ for some $x_0 \in \mathfrak{m}$ and then also $| \, g \, | = \mathrm{e}^{x_0/2} \in M \subset G_M$. Now $u_g = | \, g \, |^{-1} \cdot g \in G_M$ also (and clearly $u_g \in K$). Statement (c) follows from Theorem 3.18, Remark 3.19 and statement (b). The assertion in (d) follows from (a) and the fact that M is totally geodesic, together with equation (5) in the proof of Proposition 3.3. To prove (e) , note that I_g is an isometry of M because M has the induced metric so Lema 2.2 applies; from Lemma 3.20 we deduce that $kMk^* \subset M$ for any $k \in K$; from Theorem 3.5 and statement (b) follows easily that I_g maps M into M; since G_M is connected, we have the assertion. Statement (*f*) follows from statement (e) and the proof of Theorem 3.13 (see also Remark 3.15). $\square$

From general arguments (or from the classification of L^*-algebras, see [Neh93]) follows that

$$\overline{[\mathcal{HS}, \mathcal{HS}]} = \mathcal{HS} \qquad \text{and} \qquad \overline{[\mathcal{HS}^h, \mathcal{HS}^h]} = i\mathcal{HS}^h,$$

so taking $\mathfrak{m} = \mathcal{H}_{\mathbb{R}} = \mathbb{R} \oplus \mathcal{HS}^h$ we get $\mathfrak{k} = i\mathcal{HS}^h$, henceforth $\mathfrak{g} = \mathbb{R} \oplus \mathcal{HS} = \mathcal{H}_{\mathbb{C}}/i\mathbb{R}$, which leads to

$$G_{\Sigma_\infty} = GL_{\mathcal{H}_{\mathbb{C}}}/S^1 = \{\alpha + a \ ; \ \alpha \in \mathbb{R}_{>0}, \ a \in \mathcal{HS} \text{ and } -\alpha \notin \sigma(a)\}$$

Clearly $P(G_{\Sigma_\infty}) = P(GL_{\mathcal{H}_{\mathbb{C}}}) = \Sigma_\infty$ since any positive invertible operator has an invertible square root. On the other hand it is also obvious that the isotropy group K equals $\mathcal{U}_{\mathcal{H}_{\mathbb{C}}}$ (the unitary group of $\mathcal{H}_{\mathbb{C}}$), so it is clear that there is an analytic isomorphism given by polar decomposition

$$\Sigma_\infty \simeq GL_{\mathcal{H}_{\mathbb{C}}}/\mathcal{U}_{\mathcal{H}_{\mathbb{C}}}$$

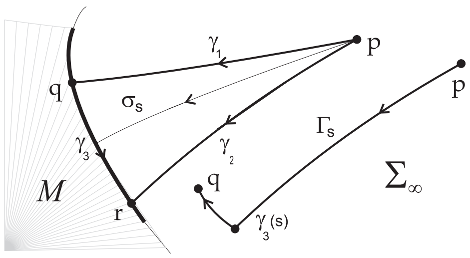
The manifold of positive invertible operators Σ_∞ is an homogeneous space for the group of invertible operators $GL_{\mathcal{H}_{\mathbb{C}}}$, which acts isometrically and transitively on Σ_∞. This last statement is well known, and Theorem 3.21 can be seen as a natural generalization.

4 Projecting to closed convex submanifolds

We refer the reader to [Lang95] for the first and second variation formulas.

Proposition 4.1. *Let M be a geodesically convex subset of Σ_∞. Then there is at most one geodesic γ joining p and M such that $L(\gamma) = \operatorname{dist}(p, M)$. In other words, there is at most one point $q \in M$ such that* $\operatorname{dist}(p,q) = \operatorname{dist}(p, M)$.

Proof. Suppose there are two such points, q and $r \in M$, joined by a geodesic $\gamma_3 \in M$, such that $L(\gamma_1) = \operatorname{dist}(p,q) = L(\gamma_2) = \operatorname{dist}(p,r) = d\,(p, M)$. We construct a proper variation of $\gamma \equiv \gamma_1$, which we call Γ_s; the construction follows the figure below, $\sigma_s(t) = \sigma(s,t)$ is the minimal geodesic joining p with $\gamma_3(s)$.



$$\Gamma(s,t) = \begin{cases} \sigma(s,t) & \text{if} \quad 0 \le t \le 1 \\ \gamma_3\left(s(2-t)\right) & \text{if} \quad 1 \le t \le 2 \end{cases}$$

Note that

$$\gamma(t) = \Gamma(0,t) = \begin{cases} \gamma_1(t) & \text{if} \quad 0 \le t \le 1 \\ q & \text{if} \quad 1 \le t \le 2 \end{cases}, \text{ so } \dot\gamma(t) = \begin{cases} \dot\gamma_1(t) & \text{if} \quad 0 \le t \le 1 \\ 0 & \text{if} \quad 1 \le t \le 2 \end{cases}$$

Also note that the variation vector field (which is a piecewise Jacobi field for the curve γ) is given by equations

$$V(t) = \frac{\partial \Gamma}{\partial s}(t,0) = \begin{cases} \frac{\partial \sigma}{\partial s}(t,0) & \text{if} \quad 0 \le t \le 1 \\ \\ (2-t)\dot\gamma_3(0) & \text{if} \quad 1 \le t \le 2 \end{cases}$$

If $\Delta_i \dot\gamma$ denotes the jump of the tangent vector field to γ at t_i, namely $\dot\gamma(t_i^+) - \dot\gamma(t_i^-)$, and Γ is a proper variation of γ, then the first variation formula for the curve $\gamma : [0,2] \to \Sigma_\infty$ reads

$$\|\dot\gamma\| \, \frac{d}{ds}|_{s=0} L\left(\Gamma_s\right) = -\int_a^b \langle V(t), D_t\dot\gamma(t)\rangle \, dt \; - \sum_{i=1}^{k-1} \langle V(t_i), \Delta_i \dot\gamma\rangle$$

where $D_t = \nabla_{\dot\gamma}$ stands for the covariant derivative along γ. In this case, $D_t\dot\gamma$ is null in the whole $[0,2]$, because γ consists (piecewise) of geodesics. The jump points are $t_0 = 0$, $t_1 = 1$ and $t_2 = 2$, so the formula reduces to

$$\langle \dot\gamma_3(0), \dot\gamma_1(1)\rangle = \frac{d}{ds}|_{s=0} L\left(\Gamma_s\right) \|\dot\gamma\|$$

Recall that $\gamma_3 \subset M$, and that γ_1 is minimizing. Then the right hand term is zero, which proves that γ_1 and γ_3 are orthogonal at q. Similarly, γ_2 and γ_3 are orthogonal at r. Henceforth, the sum of the three inner angles of this geodesic triangle is *at least* π. If we can prove that the sum cannot exceed π (see the Lemma below), it will follow that the angle at p must be null, which proves that γ_1 and γ_2 are the same geodesic, and uniqueness follows. $\square$

To prove the upper bound we need a hyperbolic triangle comparison lemma, which is known to be valid in finite dimensional manifolds; however in our context it is a consequence of the fact that the exponential map increases distance, which can be viewed as a particular case of the theory developed by McAlpin in his thesis [McA65], or as a delicate computation for compact operators (see the papers [AV03], [CPR94] and [CPR92]):

Lemma 4.2. *Let p,q and r be the vertices of a geodesic triangle in Σ_∞. Then the sum of the inner angles of this triangle cannot exceed π.*

Proof. Naming the sides of the triangle as in the geodesic triangle figure of Proposition 4.1, if we introduce the notation $L(\gamma_i) = l_i$, α_i =angle opposite to γ_i , then the length of the Euclidean geodesic (*i.e.* segment) joining $V = \exp_p^{-1}(q)$ and $W = \exp_p^{-1}(r)$ cannot exceed l_3 (see [CPR94], [CPR92]). Thus we get the following known formula for manifols of nonpositive curvature (expanding $\|W - V\|_p = \langle W - V, W - V\rangle_p$, and using that $\|W\|_p = l_2$, $\|V\|_p = l_1$)

$$l_3^2 \geq l_2^2 + l_1^2 - 2l_1 l_2 \cos(\alpha_3) \tag{6}$$

If we construct an Euclidean triangle of side lengths l_i, the Cosine Law tells us that

$$l_3^2 = l_2^2 + l_1^2 - 2l_1 l_2 \cos(\Delta_3)$$

being Δ_i the angle opposite to l_i. The two formulas put together prove that $\cos(\alpha_i) \geq \cos(\Delta_i)$, which implies (noting that all angles are less than π) that $\alpha_i \leq \Delta_i$. Therefore,

$$\alpha_1 + \alpha_2 + \alpha_3 \leq \Delta_1 + \Delta_2 + \Delta_3 = \pi$$

This inequality also finishes the proof of the uniqueness of the minimizing geodesic. □

Now, we consider the problem of the existence of the minimizing geodesic. We can rephrase the problem in the following way:

Proposition 4.3. *Let M be a geodesically convex submanifold of Σ_∞, and p a point not in M. Then existence of a minimizing geodesic joining p with M such that $L(\gamma) = \operatorname{dist}(p, M)$ is equivalent to the existence of a geodesic joining p with M, with the property that γ is orthogonal to M.*

Proof. In fact, the existence of such a geodesic is equivalent to the existence of a point $q_p \in M$ such that $\operatorname{dist}(p, M) = \operatorname{dist}(p, q_p)$, and we will show that if $q \in M$ is a point such that γ_{qp} is orthogonal to M at q, then $\operatorname{dist}(q, p) = \operatorname{dist}(D, p)$. The other implication follows from the uniqueness theorem. For this, consider the geodesic triangle generated by p, q and d, where d is any point in M different form q. As γ_{qp} is orthogonal to T_qM, it is, in particular, orthogonal to γ_{qd}. Then, by virtue of the hyperbolic Cosine Law (equation (6)), we have

$$L(\gamma_{dp})^2 \geq L(\gamma_{qp})^2 + L(\gamma_{qd})^2 > L(\gamma_{qp})^2$$

which implies $\operatorname{dist}(q, p) < \operatorname{dist}(d, p)$. □

We conclude that the existence problem is equivalent to the question: Is NM, the normal bundle of M, diffeomorphic to the whole Σ_∞, *via* the exponential map? The answer to the local version of this question is yes, by virtue of the inverse function theorem (see [Lang95]).

Lemma 4.4. *For every point $q \in M$, there exists an open neighbourhood $V_q \subset \Sigma_\infty$, and an open neighbourhood NM_q in the normal bundle of M, such that V_q is diffeomorphic to NM_q* via *the exponential map $E : NM \to \Sigma_\infty$, which assigns $(q, V) \mapsto \exp_q(V)$.*

Proof. Observe that $T(NM)$ is isomorphic to $T\Sigma_\infty$, because for every $(q, V) \in NM$, the tangent space $T_{(q,V)}NM$ can be decomposed in the tangent space T_qM and the normal space $T_qM^\perp$, and $T_qM \oplus T_qM^\perp = T_q\Sigma_\infty$. We will show that the differential of E at any point $(q, 0)$ is the identity (in particular, an isomorphism) and the inverse map theorem will provide us with the desired neighbourhoods. Let's consider the curve $\alpha(t) = (\delta_q(t), tv)$, where $\delta_q(0) = q$ and $\dot{\delta}(0) = W \in T_qM$. Then $(DE)_{(p,0)}(W, V) = (E \circ \alpha)^\cdot_{t=0}$; but

$$(E \circ \alpha)^\cdot(t) = \left(\delta(t)\mathrm{e}^{t\delta^{-1}(t)V}\right)^\cdot$$

so, recalling that $d_0\exp = id$, we get

$$(DE)_{(p,0)}(W, V) = (E \circ \alpha)^\cdot_{t=0} = W + V \quad \square$$

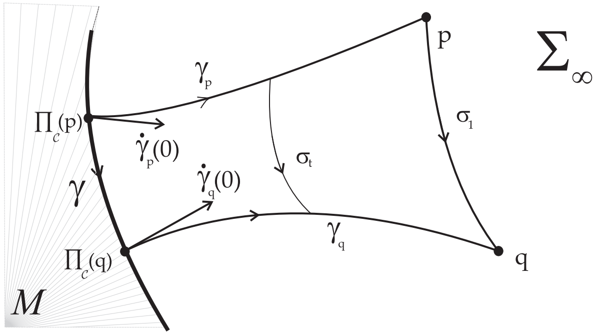
The preceeding result says that the map E is, in fact, an open mapping, so its image (which we will call $B_M = E(NM)$) is a tubular neighbouhood of M. In particular, B_M is an open neighbourhood of M in Σ_∞, and putting together the preceeding results, we see that for every point $p \in B_M$ there exists a unique minimizing geodesic which realizes distance between the operator p and the manifold M. It must also be noted (this is a consequence of the previous results of this section) that B_M can be described by the following property: a point $p \in B_M$ *if and only if* there exists a point $q \in M$ such that $\mathrm{dist}(q, p) = \mathrm{dist}\,(M, p)$.

With the above construction, we have proved the existence of a map projecting $\Pi_M : B_M \to M$, which assigns to any point $p \in B_M$ the unique point $q \in M$ such that $\mathrm{dist}(q, p) = \mathrm{dist}\,(M, p)$. Observe that this map is obtained *via* a geodesic which joins p and M, and this geodesic is orthogonal to M. The point $\Pi_M(p)$ is called the *foot of the perpendicular* from M to p.

Theorem 4.5. *The map Π_M is a contraction, namely*

$$\mathrm{dist}\,(\Pi_M(p), \Pi_M(q)) \leq \mathrm{dist}(p, q)$$

Proof. We may assume that $p, q \notin M$, and that $\Pi_M(p) \neq \Pi_M(q)$. If γ_p is a geodesic that joins $\Pi_M(p)$ to p and γ_q joins $\Pi_M(q)$ to q, set $f(t) = d\,(\gamma_p(t), \gamma_q(t))$. Note that $f(0) = d\,(\Pi_M(p), \Pi_M(q))$ and $f(1) = \mathrm{dist}(p, q)$. We also know that $f\,''(t) \geq 0$ (this is because distance is a convex function; this can be deduced from the variation formulas, or it can be found in [CPR94]). If we can prove that $f\,'(0) \geq 0$, it will follow that f is monotone increasing, and we will have proved the assertion. Take a variation $\sigma(t, s)$, being $\sigma_t(s)$ the unique geodesic joining $\gamma_p(t)$ to $\gamma_q(t)$. Then $\sigma(t, 0) = \gamma_p(t)$, $\sigma(t, 1) = \gamma_q(t)$, $\sigma(0, s) = \gamma(s) =$ the geodesic joining $\Pi_M(p)$ to $\Pi_M(q)$ (which is contained in M by virtue of the convexity), and finally $\sigma(1, s) =$ the geodesic joining p to q. This construction is better shown in the following figure:



Note that $f(t) = L(\sigma_t)$. We apply the first variation formula to this particular σ, to get

$$\|\dot{\gamma}\| \frac{d}{dt}|_{t=0} L(\sigma_t) = -\int_0^1 \langle V(s), D_s\dot{\gamma}(s)\rangle \, ds \; + \langle V(1), \dot{\gamma}(1)\rangle - \langle V(0), \dot{\gamma}(0)\rangle$$

The fact that γ is a geodesic and observation of the figure reduces the formula to $\|\dot{\gamma}\| \, f\,'(0) = -\langle V(1), -\dot{\gamma}(1)\rangle + \langle -V(0), \dot{\gamma}(0)\rangle$. Looking at the figure, it is also obvious that $V(0) = \dot{\gamma}_p(0)$, $V(1) = \dot{\gamma}_q(0)$. Recalling that the angles at M are right angles, we get $f\,'(0) = 0$. $\square$

Now we are ready to prove the main result of this section:

Theorem 4.6. *Let M be a geodesically convex, closed submanifold of Σ_∞. Then for every point $p \in \Sigma_\infty$, there is a unique minimizing geodesic γ joining p to M such that $L(\gamma) = \operatorname{dist}(p, M)$. Moreover, this geodesic is orthogonal to M, and if we call $\Pi_M : \Sigma_\infty \to M$ to the map that assigns the endpoint of the minimizing geodesic starting at p, then Π_M is a contraction.*

Proof. The theorem will hold true when we prove that $B_M = \Sigma_\infty$. But since Σ_∞ is connected and B_M is open, equality will immediately follow if we can prove that B_M is also closed. For this, take a point $p \in \overline{B_M}$. Then there exist points $q_n \in M$, $V_n \in T_{q_n} D^\perp$ such that $p = \lim_n p_n = \lim_n \exp_{q_n}(V_n)$. Now observe that $q_n = \Pi_M(p_n)$, so $\operatorname{dist}(q_n, q_m) \leq \operatorname{dist}(p_n, p_m)$. But $\{p_n\}$ converges to p, so it is a Cauchy sequence. It follows that $\{q_n\}$ is Cauchy and, since M is closed (and Σ_∞ is a complete metric space), there must exist a point $q \in M$ such that $q = \lim_n q_n$. We assert that $\operatorname{dist}(p, q) = \operatorname{dist}(p, M)$. For this, observe that

$$\operatorname{dist}(p, q_n) \leq \operatorname{dist}(p, p_n) + \operatorname{dist}(p_n, q_n)$$

and $\operatorname{dist}(p_n, q_n) = \operatorname{dist}(p_n, M)$, so

$$\operatorname{dist}(p, q_n) \leq \operatorname{dist}(p, p_n) + \operatorname{dist}(p_n, M)$$

Taking limits gets us to the inequality $\operatorname{dist}(p, q) \leq \operatorname{dist}(p, M)$, which tells us that the distance between p and M is given by $\operatorname{dist}(p, q)$. This concludes the proof. $\square$

Note that Σ_∞ decomposes as a direct product: with the contraction Π_M, we can decompose Σ_∞ by picking, for fixed p,

1. the unique point $q = \Pi_M(p)$ such that $\operatorname{dist}(p,q) = \operatorname{dist}(p,M)$
2. a vector V_p normal to T_qM such that the ambient geodesic with this initial velocity starting at q passes through p.

Note that $V_p = \exp^{-1}_{\Pi_M(p)}(p)$, and also $\|V_p\|_p = \operatorname{dist}(p,M)$.

Since the exponential map is an analytic function of both of its variables (recall that, for any $q \in \Sigma_\infty$, and any $V \in \mathcal{H}_\mathbb{R}$, $\exp_q(V) = q\mathrm{e}^{q^{-1}V}$), we get

Theorem 4.7. *The map $p \mapsto (\Pi_M(p), V_p)$ is the inverse of the exponential map $(q, V_q) \mapsto \exp_q(V_q)$, and it is, in fact, a real-analytic isomorphism between the manifolds NM and Σ_∞.*

This is a remarkable global analogue of the (linear) orthogonal decomposition of tangent spaces; we can restate the theorem in a different way if we recall that all points and tangent vectors are operators, see the theorem below. We should stress that this theorem is inpired mainly by the results of C^*-algebra decompositions that can be found in the paper [CPR91] by Corach, Porta and Recht.

Theorem 4.8. *Fix a closed, geodesically convex submanifold M of Σ_∞. Take any operator $A \in \Sigma_\infty$. Then there exist unique operators $C, V \in \Sigma_\infty$ such that $C \in M$, $V \in T_CM^\perp$, and the following formula holds:*

$$A = C\,\mathrm{e}^{C^{-1}V} \tag{7}$$

Naming $B = C^{\frac{1}{2}}$, $W = C^{-\frac{1}{2}}VC^{-\frac{1}{2}}$, equation (7) reads $A = Be^WB$ for unique B, W.

Using the tools of section 3, we can restate the theorem in terms of intrinsic operator equations (see [Mos55] for the finite dimensional analogue):

Theorem 4.9. *Assume $\mathfrak{m} \subset \mathcal{H}_\mathbb{R}$ is a closed Lie triple system. Then for any operator $A \in \mathcal{H}_\mathbb{R}$, there exist unique operators $X \in \mathfrak{m}$ and $V \in \mathfrak{m}^\perp$ such that the following decomposition holds:*

$$\mathrm{e}^A = \mathrm{e}^X\,\mathrm{e}^V\,\mathrm{e}^X$$

5 Some applications of the decomposition

We will use the factorization theorem in several ways; for convenience we first state the following lemma, which we will be useful later on several ocasions:

Lemma 5.1. *For the exponential map in Σ_∞, we have*

$$\exp_{\alpha+a}(\beta+b) = (\alpha+a)e^{(\alpha+a)^{-1}(\beta+b)} = (\alpha+a)[1 + (\alpha+a)^{-1}(\beta+b) + \cdots]$$

$$= (\alpha+a)\left[1 + \frac{\beta}{\alpha} + \frac{1}{2}\left(\frac{\beta}{\alpha}\right)^2 + \cdots + k\right] = \alpha\,\mathrm{e}^{\beta/\alpha} + k$$

where k is a Hilbert-Schmidt operator.

We need some remarks before we proceed with the main applications. Fix an orthonormal basis of H.

1. The diagonal manifold $\Delta \subset \Sigma_\infty$ (defined below) is closed and geodesically convex.
$$\Delta = \{d + \alpha : d \text{ is a diagonal, Hilbert-Schmidt operator}\}$$
This is due to the fact that the diagonal operators form a closed abelian subalgebra.
2. If $p_0 \in \Delta$, then $T_{p_0}\Delta = \{\alpha + d;\ \alpha \in \mathbb{R},\ d \text{ is diagonal}\} = T_1\Delta$ by Remark 3.14.
3. Consider the map $A \mapsto A^D =$ the diagonal part of A. Then
   (a) For Hilbert-Schmidt operators we have $A^D = \sum_i p_i A p_i$ where convergence is in the 2-norm (and hence in the operator norm); here $p_i = e_i \otimes e_i = \langle e_i, \cdot \rangle\, e_i$ is the orthogonal projection in the real line generated by e_i
   (b) $(A^D)^D = A^D$ and $tr\left(A^D A\right) = tr((A^D)^2)$
4. The scalar manifold $\Lambda = \{\lambda.1 : \lambda > 0\}$, is geodesically convex and closed in Σ_∞, with tangent space $\mathbb{R} \cdot 1$
5. A vector $V = \mu + u \in T_{p_0}\Delta^\perp$ if and only if $\mu = 0$ and $u^D = 0$. This follows from: Remark 3.14, the fact that $\mu + u^D \in T_{p_0}\Delta$, and Remark (3) of this list.

Theorem 5.2. *Take any selfadjoint Hilbert-Schmidt operator a. Then there exist real $\lambda > 0$, a positive invertible Hilbert-Schmidt diagonal operator d and a Hilbert-Schmidt selfadjoint operator with null diagonal V such that the following formula holds:*

$$a + \lambda = (d+\lambda) e^{(d+\lambda)^{-1} V} = (d+\lambda)^{\frac{1}{2}} e^{(d+\lambda)^{-\frac{1}{2}} V (d+\lambda)^{-\frac{1}{2}}} (d+\lambda)^{\frac{1}{2}}$$

Moreover (for fixed λ) d and V are unique and the map $a + \lambda \mapsto (d, V)$ (wich maps $\Sigma_\infty \to N\Delta$) is a real analytic isomorphism between manifolds.

Proof. Take $\lambda = \|a\|_\infty + \epsilon$, for any $\epsilon > 0$. Then $p = a + \lambda \in \Sigma_\infty$, and $\Pi_\Delta(p) = d + \alpha$. This is the d we need. Now pick the unique $V \in T_{d+\alpha}\Delta^\perp$ such that $\exp_{d+\alpha}(V) = p$. This V has the desired form because of Remark (5) above. As a consequence of the 'exponential formula' (Lemma 5.1), $\alpha = \lambda$, for in this case, $\beta = 0$. $\square$

This theorem can be rephrased saying that, given a selfadjoint Hilbert-Schmidt operator a, for any λ that makes $a + \lambda > 0$, one has a unique factorization

$$a + \lambda = D\, \mathrm{e}^W D$$

where $D = (\lambda + d)^{\frac{1}{2}} > 0$ is diagonal and $W = D^{-1} V D^{-1}$ is symmetric and has null diagonal.

6 A foliation of Σ_∞ of codimension one

In this section we describe a foliation of the total manifold, and show how to translate the results from previous sections to a particular leaf (the submanifold Σ_1). We begin with a description of the leaves.

6.1 The leaves Σ_λ

Recall that we use $\mathcal{HS}^h$ (hermitian Hilbert-Schmidt operators) to denote the closed vector space of operators in $\mathcal{H}_{\mathbb{R}}$ with no scalar part. We define the following family of submanifolds:

$$\Sigma_\lambda = \{a + \lambda \in \Sigma_\infty,\ a \in \mathcal{HS}^h \text{ and } \lambda > 0 \text{ fixed } \}$$

We first observe that, by virtue of the 'exponential formula' (Lemma 5.1), for any real $\lambda > 0$ and any $p \in \Sigma_\lambda$, there is an identification *via* the inverse exponential map at p, $T_p\Sigma_\lambda = \mathcal{HS}^h$.

Observe that $\Sigma_\lambda \cap \Sigma_\beta = \emptyset$ when $\lambda \neq \beta$, since $a + \lambda = b + \beta$ implies $a - b = \beta - \lambda$.

In this way, we can decompose the total space by means of these leaves,

$$\Sigma_\infty = \dot{\bigcup_{\lambda>0}} \Sigma_\lambda$$

Proposition 6.1. *The leaves Σ_λ are closed and geodesically convex submanifolds.*

Proof. That Σ_λ is closed is a consequence of the fact that the projection to Λ is a contraction, and as consequence, continuous map. One must only observe that $\Sigma_\lambda = \Pi_\Lambda^{-1}(\lambda)$. That Σ_λ is geodesically convex follows from inspection of the formula for a geodesic joining $a + \lambda$, $b + \lambda$. $\square$

Remark 6.2. Take $\delta + c \in T_{a+\lambda}\Sigma_\lambda{}^\perp$. Since $T_{a+\lambda}\Sigma_\lambda$ can be identified with $\mathcal{HS}^h$, condition

$$\langle \delta + c, d\rangle_{a+\lambda} = 0 \qquad \forall \quad d \in \mathcal{HS}^h$$

immediately translates into

$$tr\left[(a+\lambda)^{-1}\left[(\delta + c)(a+\lambda)^{-1} - \frac{\delta}{\lambda}\right]d\right] = 0 \qquad \forall \quad d \in \mathcal{HS}^h$$

This says that $T_{a+\lambda}\Sigma_\lambda^\perp = span(a+\lambda)$; shortly $T_p\Sigma_\lambda^\perp = span(p)$ for any $p \in \Sigma_\lambda$.

Proposition 6.3. *Fix real $\alpha, \lambda > 0$. Set $\Pi_{\alpha,\lambda} = \Pi_{\Sigma_\lambda}|_{\Sigma_\alpha} : \Sigma_\alpha \to \Sigma_\lambda$. Then*

1. *$\Pi_{\alpha,\lambda}(p) = \frac{\lambda}{\alpha}p$, so $\Pi_{\alpha,\lambda}(p)$ commutes with p*
2. *$\Pi_{\alpha,\lambda}$ is an isometric bijection between Σ_α and Σ_λ, with inverse $\Pi_{\lambda,\alpha}$.*
3. *$\Pi_{\alpha,\lambda}$ gives parallel translation along 'vertical' geodesics joining both leaves.*

Proof. Notice that for a point $b+\alpha \in \Sigma_\alpha$ to be the endpoint of the geodesic γ starting at $a + \lambda \in \Sigma_\lambda$ such that $L(\gamma) = \operatorname{dist}(b + \alpha, \Sigma_\lambda)$, we must have

$$b + \alpha = \exp_{a+\lambda}(x + c) = \exp_{a+\lambda}(k.(a + \lambda)) = e^k(a + \lambda)$$

because $x + c \in T_{a+\lambda}\Sigma_\lambda{}^\perp$. From Lemma 5.1, we deduce that $k = \ln\left(\frac{\alpha}{\lambda}\right)$, and $a = \frac{\lambda}{\alpha}b$. So, $b + \alpha = \frac{\alpha}{\lambda}(a + \lambda)$ and also

$$\gamma(t) = (a + \lambda)\left(\frac{\alpha}{\lambda}\right)^t$$

Now it is obvious that $\Pi_\lambda(b+\alpha) = \frac{\lambda}{\alpha}(b+\alpha)$ and commutes with $b+\alpha$. To prove that Π is isometric, observe that $\mathrm{dist}(\Pi_{\alpha,\lambda}(p), \Pi_{\alpha,\lambda}(q)) = \mathrm{dist}(\frac{\lambda}{\alpha}p, \frac{\lambda}{\alpha}q) = \mathrm{dist}(p,q)$ by inspection of the geodesic equation (3) of section 2. That Π gives parallel translation along γ follows from $q = \frac{\lambda}{\alpha}p$ and Remark 3.15. $\square$

The normal bundle in the case of $M = \Sigma_1$ can be thought of as a direct product:

Proposition 6.4. *The map $T : \Sigma_\infty \to \Sigma_1 \times \Lambda$, which assigns $a+\alpha \mapsto \left(\frac{1}{\alpha}(a+\alpha), \alpha\right)$ is bijective and isometric (Σ_1 and Λ have the induced submanifold metric). In other words, there is a Riemannian isomorphism*

$$\Sigma_\infty \simeq \Sigma_1 \times \Lambda$$

Proposition 6.5. *The leaves Σ_α, Σ_λ are also parallel in the following sense: any minimizing geodesic joining a point in one of them with its projection in the other is orthogonal to both of them.*

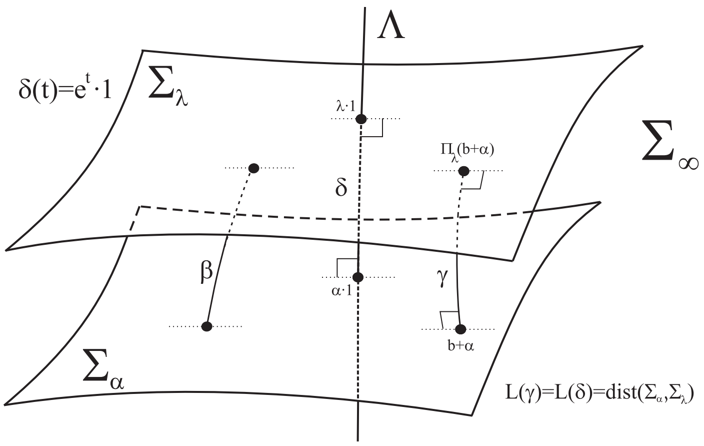


Figure 1: The geodesics γ and δ are minimizing, the geodesic β is not

For any $b+\alpha \in \Sigma_\alpha$ we have $\mathrm{dist}(b+\alpha, \Sigma_\lambda) = \mathrm{dist}(\Sigma_\alpha, \Sigma_\lambda) = \mid \ln\left(\frac{\alpha}{\lambda}\right) \mid$. In particular, the distance between α, λ in the scalar manifold Λ is given by the Haar measure of the open interval (α, β) on $\mathbb{R}_{>0}$.

Proof. It is a straightforward computation that follows from the previous results; the last statement was observed by E. Vesentini in [Ves76]. $\square$

Since Σ_∞ is a symmetric space, curvature is preserved when we parallel-translate bidimensional planes. Note also that vertical planes (*i.e.* planes generated by a vector $v \in \mathcal{HS}^h = T_\lambda\Sigma_\lambda$ and λ) are abelian, so

Proposition 6.6. *For any point $p \in \Sigma_\lambda$, sectional curvature of vertical 2-planes is trivial.*

The previous theorems show that the geometry of Σ_∞ is essentially the geometry of Σ_1; in particular, the factorization theorem inside Σ_1 has a simpler form; we state it below

Theorem 6.7. *Fix a closed, geodesically convex submanifold M of Σ_1. For any $a+1 \in \Sigma_1$, there is a selfadjoint Hilbert-Schmidt operator d such that $d+1 \in M$, and a selfadjoint Hilbert-Schmidt operator V, such that $V \in T_{d+1}M^\perp$, which make the following formula hold:*

$$1+a = [1+d]\, e^{(1+d)^{-1} V}$$

Moreover d and V are unique, and the map $1+a \mapsto (1+d, V)$ (which maps Σ_1 to NM) is a real analytic isomorphism between manifolds. Equivalently,

$$1+a = [1+d]^{\frac{1}{2}}\, e^{(1+d)^{-\frac{1}{2}} V\, (1+d)^{-\frac{1}{2}}} [1+d]^{\frac{1}{2}}$$

The intrinsic version of the theorem reads (see Theorem 3.6):

Theorem 6.8. *Assume $\mathfrak{m} \subset \mathcal{HS}^h$ is a closed subspace such that*

$$[X,[X,Y]] \in \mathfrak{m} \quad \textit{for any} \quad X, Y \in \mathfrak{m}$$

Then for any $A \in \mathcal{HS}^h$ there is a unique decomposition of the form

$$\mathrm{e}^A = \mathrm{e}^X\, \mathrm{e}^V\, \mathrm{e}^X$$

where $X \in \mathfrak{m}$ and $V \in \mathcal{HS}^h$ is such that $tr\,(VZ) = 0$ for any $Z \in \mathfrak{m}$.

6.2 The embedding of M_n^+ in Σ_1

We are going to state and prove a projection theorem for $M = M_n^+$. First note that we can embed $M_n^+ \hookrightarrow \Sigma_1$ for any $n \in \mathbb{N}$: fix an orthonormal basis $\{e_n\}_{n \in \mathbb{B}}$ of H, set $p_{ij} = e_i \otimes e_j$, and identify M_n with the set

$$\mathcal{T} = \left\{ \sum_{i,j=1}^{n} a_{ij}\, p_{ij} \;:\; a_{ij} = a_{ji} \in \mathbb{R} \right\} \subset \mathcal{HS}^h$$

In this way, we can identify isometrically (see Proposition 4.1 of [AV03]) the manifolds M_n^+ with the set

$$\mathcal{P} = \left\{ \mathrm{e}^T : T \in \mathcal{T} \right\} \subset \Sigma_1$$

and the tangent space at each $\mathrm{e}^T \in \mathcal{P}$ is $\mathcal{T}$. $\mathcal{P}$ is closed and geodesically convex in Σ_1 by Corollary 3.6

Let's call $S = \mathrm{span}(e_1, \cdots, e_n)$, $S^\perp = \mathrm{span}(e_{n+1}, e_{n+2} \cdots)$. The operator P_S is the orthogonal projection to S and $Q_S = 1 - P_S$ is the orthogonal projection to $S^\perp$.

Using matrix blocks, for any operator $A \in \mathcal{L}(S)$, we identify

$$\mathcal{T} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathcal{P} = \begin{pmatrix} \mathrm{e}^A & 0 \\ 0 & 1 \end{pmatrix}$$

Remark 6.9. There is a direct sum decomposition of $\mathcal{HS}^h = \mathcal{T} \oplus \mathcal{J}$ where operators in $J \in \mathcal{J}$ are such that $P_S\, J\, P_S = 0$. A straightforward computation using the matrix-block representation shows that $tr(ab) = 0$ for any $a \in \mathcal{T}, b \in \mathcal{J}$, which says $\mathcal{T}^\perp = \mathcal{J}$. So the manifolds $\exp(\mathcal{J})$ and $\mathcal{P} = \exp(\mathcal{T})$ are orthogonal at 1, the unique intersection point.

Theorem 6.10. *(projection to positive invertible $n \times n$ matrices) : Set $\mathcal{P} \simeq M_n^+ \subset \Sigma_1$ with the above identification. Then for any positive invertible operator $e^b \in \Sigma_1$, ($b \in \mathcal{HS}^h$) there is a unique factorization of the form*

$$e^b = \begin{pmatrix} e^A & 0 \\ 0 & 1 \end{pmatrix} \exp\left\{ \begin{pmatrix} e^{-A} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbb{O}_{n\times n} & Y^* \\ Y & X \end{pmatrix} \right\}$$

where $e^a = e^A P_S + Q_S \in \mathcal{P} \simeq M_n^+$, ($a \in \mathcal{T}$), $X^* = X$ acts on the subspace $S^\perp$ and $Y \in \mathcal{L}(S, S^\perp)$.

An equivalent expression for the factorization is

$$\mathrm{e}^b = \begin{pmatrix} \mathrm{e}^{A/2} & 0 \\ 0 & 1 \end{pmatrix} \exp\left\{ \begin{pmatrix} \mathbb{O}_{n\times n} & \mathrm{e}^{-A/2}Y^* \\ Y\mathrm{e}^{-A/2} & X \end{pmatrix} \right\} \begin{pmatrix} \mathrm{e}^{A/2} & 0 \\ 0 & 1 \end{pmatrix}$$

Yet another form is the following: for any $p \in \Sigma_1$ exist unique $V \in \mathcal{HS}^h$ such that $P_S V P_S = 0$, and unique $q \in \Sigma_1$ such that $P_S\, q\, Q_S = Q_S\, q\, P_S = 0$ and $Q_S\, q\, Q_S = Q_S$ which make the following equation valid

$$p = q\, \mathrm{e}^V q$$

Proof. From previous theorems and the observations we made, we know that

$$\mathrm{e}^b = \begin{pmatrix} \mathrm{e}^{A/2} & 0 \\ 0 & 1 \end{pmatrix} \cdot C \cdot \begin{pmatrix} \mathrm{e}^{A/2} & 0 \\ 0 & 1 \end{pmatrix}$$

where

$$C = \exp\left\{ \begin{pmatrix} \mathrm{e}^{-A/2} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} V_{11} & V_{21}^* \\ V_{21} & V_{22} \end{pmatrix} \begin{pmatrix} \mathrm{e}^{-A/2} & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

for some $A \in \mathcal{L}(S)$ and some $V \in \mathcal{HS}^h$. That $V_{11} = 0$ follows from the fact (see Remark 6.9) that $\mathcal{T}^\perp = \mathcal{J}$, and $V \in T_{\mathrm{e}^a}\mathcal{P}^\perp$ iff $tr(\mathrm{e}^{-A}B\mathrm{e}^{-A}V_{11}) = 0$ for any $B \in \mathcal{T}$. This says that V has the desired form. $\square$

Remark 6.11. Since V is orthogonal to $\mathcal{P}$ at any point, in particular it is orthogonal to $\mathcal{P}$ at 1; so 1 is the foot of the perpendicular from e^V to $\mathcal{P}$. In other words, 1 is the point in $\mathcal{P}$ closer to e^V; the distance between 1 and e^V is exactly $\|V\|_2 = tr(V^2)$.

In the notation of Theorem 6.10, $e^a = 1$ if and only if $A = 0$, if and only if $V = b$, and we conclude that for any $b \in \mathcal{HS}^h$ such that $P_S \, b \, P_S = 0$, the point in $\mathcal{P}$ closer to e^b is 1. This is nothing but Remark 6.9 in disguise.

For any $b \in \mathcal{HS}^h$, it holds true that the operator

$$e^a = e^A P_{S_n} + P_{S_n^\perp} = \begin{pmatrix} e^A & 0 \\ 0 & 1 \end{pmatrix} = \exp \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$$

is the 'first block' $n \times n$ matrix which is closer to e^b in Σ_∞, and with a slight abuse of notation for the traces of $\mathcal{L}(S_n)$ and $\mathcal{L}(S_n^\perp)$, we have

$$\operatorname{dist}(\mathcal{P}, e^b) = \operatorname{dist}(e^a, e^b) = \left\| \begin{pmatrix} \mathbb{O}_{n\times n} & Y^* \\ Y & X \end{pmatrix} \right\|_{e^a} = \sqrt{\|Y \, e^{-A/2}\|_2^2 + \|X\|_2^2}$$

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