Weighted norm estimates for the maximal operator of the Laguerre functions heat diffusion semigroup *

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Abstract

We obtain weighted L^p boundedness for the maximal operator with weights of the type y^δ, $\delta > -1$, for the heat semigroups associated to the Laguerre differential operator

$$\mathbf{L}^\alpha = -y\frac{d^2}{dy^2} - \frac{d}{dy} + \frac{y}{4} + \frac{\alpha^2}{4y},\ y > 0,$$

for $\alpha > -1$. It is proved that:
when $-1 < \alpha < 0$, the maximal operator is of strong type (p,p) if

$$p > 1 \quad \text{and} \quad \frac{2\,(1+\delta)}{(2+\alpha)} < p < \frac{2\,(1+\delta)}{(-\alpha)}.$$

If $\alpha \geq 0$ there is strong type for

$$1 < p \leq \infty \quad \text{and} \quad \frac{2\,(1+\delta)}{(2+\alpha)} < p.$$

The behavior at the end points of these intervals where there is strong type is studied in detail and results about the existence or not of strong, weak or restricted types are given.

1 Introduction

The Laguerre polynomials $L_k^\alpha(y)$ are given by

$$e^{-y}y^\alpha L_k^\alpha(y) = \frac{1}{k!}\frac{d}{dy^k}\left(e^{-y}y^{k+\alpha}\right),$$

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where y is positive. We assume that $\alpha > -1$. The Laguerre polynomials $\{L_k^\alpha(y)\}_{k=0}^\infty$ form a orthogonal system with respect to the measure $e^{-y}y^\alpha dy$. More precisely,

$$\int_0^\infty L_k^\alpha(y)\, L_j^\alpha(y)\, e^{-y} y^\alpha dy = \frac{\Gamma(k+\alpha+1)}{\Gamma(k+1)}\delta_{kj},$$

The Laguerre functions $\mathcal{L}_k^\alpha(y)$ are defined by

$$\mathcal{L}_k^\alpha(y) = \left(\frac{\Gamma(k+1)}{\Gamma(k+\alpha+1)}\right)^{1/2} e^{-y/2} y^{\alpha/2} L_k^\alpha(y).$$

Standard references for Laguerre functions and polynomials are [1], [9] and [10].

We define the heat diffusion kernel $W^\alpha(t,y,z)$ for $\alpha > -1$, $t > 0$, $y > 0$, and $z > 0$, as

$$W^\alpha(t,y,z) = \sum\nolimits_{n=0}^\infty e^{-t(n+(\alpha+1)/2)} \mathcal{L}_n^\alpha(y)\, \mathcal{L}_n^\alpha(z),$$

and the heat diffusion integral $W^\alpha f(t,y)$, as

$$W^\alpha f(t,y) = \int_0^\infty W^\alpha(t,y,z)\, f(z)\, dz.$$

The heat diffusion integral $W^\alpha f(t,y)$ satisfies the semigroup property

$$W^\alpha f(t_1+t_2,y) = \int_0^\infty W^\alpha(t_1,y,z)\, W^\alpha f(t_2,z)\, dz.$$

The maximal operator $W^{\alpha,*}f(t)$ associated to the heat diffusion integral $W^\alpha f(t,y)$ is given by

$$W^{\alpha,*}f(t) = sup_{t>0} |W^\alpha f(t,y)|.$$

We define the fractional maximal function $M_\theta f(y)$ for $0 \leq \theta < 1$ as

$$M_\theta f(y) = \sup\nolimits_{h>0} \frac{1}{(2h)^{1-\theta}} \int_{|z|\leq h} |f(y-z)|\, dz.$$

If $\theta = 0$, $M_0 f(y)$ is the Hardy-Littlewood maximal function. It is well known that if y^δ is a weight with $-1 < \delta < p-1$, then $M_0 f$ is of strong type (p,p) for $p > 1$ and of weak type $(1,1)$ if $p = 1$ with the measure $y^\delta dy$. We will need the right sided maximal function

$$M^+ f(y) = \sup\nolimits_{h>0} \frac{1}{h} \int_y^{y+h} |f(z)|\, dz.$$

We denote by A_p the class of all weights $\omega(y)$ such that $M_0 f$ is of strong type (p,p) for $p > 1$, and of weak type for $p = 1$, with the measure $\omega(y)\, dy$, and by A_p^+ the class of all weights $\omega(y)$ such that $M^+ f$ is of strong type (p,p) for $p > 1$, and of weak type for $p = 1$ with the measure $\omega(y)\, dy$. It is well known that $A_1 \subset A_p$ and $A_1^+ \subset A_p^+$ for every $p > 1$. For $M^+ f$ we need to know that it is of weak type $(1,1)$ with the measure $y^\delta dy$ for any $\delta > -1$.

This is true because for $\delta \geq 0$ the weight is a non-decreasing function, and for $-1 < \delta \leq 0$, because $M^{+}f(y) \leq M_0 f(y)$. As references see [4], [5], [6].

The purpose of this paper is to study the action of the maximal operator $W^{\alpha,*}f$ just defined on the spaces $L^p\left((0,\infty), y^\delta dy\right)$, $\delta > -1$. For $\alpha \geq 0$ and $\delta = 0$ the results we give here were obtained by K. Stempak in [8], and for $-1 < \alpha < 0$ and $\delta = 0$ by Macías, Segovia and Torrea in [3]. For the case when $\alpha \geq 0$ and $\delta > 0$ we can majorized the maximal operator $W^{\alpha,*}f(y)$ by a constant times $W^{0,*}f(y)$ and thus we obtain the strong type (p,p) whenever $p > 1+\delta > 0$ for the operator $W^{\alpha,*}f(y)$. However we can do better, in fact, in Theorem 1 we show that $W^{\alpha,*}f(y)$ is of strong type (p,p) for the possibly greater interval $p > 1$ and $p > \dfrac{2(1+\delta)}{(\alpha+2)}$.

2 Statement of the results

Let N_α denote the interval

$$N_\alpha = \begin{cases} \left(\dfrac{2(1+\delta)}{2+\alpha}, \dfrac{2(1+\delta)}{-\alpha}\right) \cap (1,\infty) & , \text{ if } -1 < \alpha < 0, \\ \text{and} & \\ \left(\dfrac{2(1+\delta)}{2+\alpha}, \infty\right] \cap (1,\infty] & , \text{ if } \alpha \geq 0. \end{cases}$$

We will assume that N_α is not empty. This implies that $1+\delta+\alpha/2 > 0$. With this notation, we have

Theorem 1 *Let $-1 < \alpha < \infty$ and $-1 < \delta < \infty$. If $p \in N_\alpha$, then the maximal operator $W^{\alpha,*}f(y)$ is of strong type (p,p) with respect to the measure $y^\delta dy$, that is to say,*

$$\int_0^\infty W^{\alpha,*}f(y)^p \, y^\delta dy \leq C_{\alpha,\delta,p} \int_0^\infty |f(y)|^p \, y^\delta dy$$

holds with a constant $C_{\alpha,\delta}$ depending on α and δ only, provided that

(a) *If* $-1 < \alpha < 0$, *then* $p > 1$, *and* $\dfrac{2(1+\delta)}{2+\alpha} < p < \dfrac{2(1+\delta)}{-\alpha}$.

(b) *If* $\alpha \geq 0$, *then* $p > 1$ *and* $\dfrac{2(1+\delta)}{2+\alpha} < p \leq \infty$.

The following theorem gives the behavior of $W^{\alpha,*}f$ at the end points of N_α.

Theorem 2 *Let $-1 < \delta$. At the end points of N_α, we have:*

(a) If $-1 < \alpha < 0$ and $\dfrac{2(1+\delta)}{(-\alpha)} > 1$, the upper end point of N_α is equal to $\dfrac{2(1+\delta)}{(-\alpha)}$, and the operator $W^{\alpha,*}f$ is of weak type and not of strong type $\left(\dfrac{2(1+\delta)}{(-\alpha)}, \dfrac{2(1+\delta)}{(-\alpha)}\right)$ with respect to the measure $y^\delta dy$.

(b) If $\alpha \geq 0$, then the upper end point of N_α is equal to ∞, and the operator $W^{\alpha,*}f$ is of strong type (∞,∞) with respect to the measure $y^\delta dy$.

(c) If $-1 < \alpha$ and $\dfrac{2(1+\delta)}{(2+\alpha)} > 1$, then the lower end point of N_α is equal to $\dfrac{2(1+\delta)}{(2+\alpha)}$. and the operator $W^{\alpha,*}f$ is of restricted weak type and not of weak type $\left(\dfrac{2(1+\delta)}{(2+\alpha)}, \dfrac{2(1+\delta)}{(2+\alpha)}\right)$ with respect to the measure $y^\delta dy$.

(d) If $-1 < \alpha$ and $\dfrac{2(1+\delta)}{(2+\alpha)} \leq 1$, then the lower end point of N_α is equal to 1. and the operator $W^{\alpha,*}f$ is of weak type and not of strong type $(1,1)$ with respect to the measure $y^\delta dy$.

Remark 1 *If $-1 < \alpha < 0$ and $\dfrac{2(1+\delta)}{(-\alpha)} = 1$, the interval N_α is empty. However since $\dfrac{2(1+\delta)}{(2+\alpha)} < \dfrac{2(1+\delta)}{(-\alpha)} = 1$, by part d) of Theorem 2, the operator $W^{\alpha,*}f$ is of weak type and not of strong type $(1,1)$ with respect to the measure $y^\delta dy$.*

Remark 2 *The results obtained in Theorem 2 do not depend on Theorem 1, and can be used to give a proof of Theorem 1 by interpolation, see [7] and [2].*

3 Lemmas

Throughout this paper we shall assume that $f(x)$ is a non-negative function. The constants will not have the same value in each occurrence.

Definition 1 *Let $f(y)$ be a locally integrable function on $(0,\infty)$. We define the maximal function $M^R f(y)$ as the function given for $0 < y < \infty$, by*

$$M^R f(y) = \sup_{J_y \subset (y/4,3y)} \frac{1}{|J_y|} \int_{J_y} f(z)\, dz, \tag{3.1}$$

where J_y is an interval containing y. Obviously, $M^R f(y) \leq M_0 f(y)$.

Lemma 1 *The maximal function $M^R f(y)$ is of weak type (p,p), $1 \leq p \leq \infty$, with respect to the measure $y^\delta dy$ for any real value of δ.*

Proof. The case $p = \infty$ is obvious. Let us represent $(0,\infty)$ as the union of the intervals $\left\{(8^k, 8^{k+1}]\right\}_{k=-\infty}^{\infty}$. If y belongs to the set $\left\{y : \lambda < M^R f(y)\right\} \cap (8^k, 8^{k+1}]$, then, there exists an interval J_y such that $y \in J_y \subset (y/4, 3y)$ and $M^R f(y) \leq 2\frac{1}{|J_y|}\int_{J_y} f(z)\, dz$. This interval J_y is contained in the interval $(8^{k-1}, 8^{k+2}]$. Then, by Hölder's inequality, we have,

$$\lambda^p < M^R f(y)^p \leq \left(2\frac{1}{|J_y|}\int_{J_y} f(z)\, dz\right)^p \leq 2^p \frac{1}{|J_y|}\int_{J_y} f(z)^p\, dz.$$

Given a compact subset K of $\left\{y : \lambda < M^R f(y)\right\} \cap (8^k, 8^{k+1}]$, we can find a finite sequence $\left\{J_{y_i}\right\}$ that covers K and such that no point of K belongs to more than three intervals of the sequence. Then

$$\begin{aligned}\int_K y^\delta dy &\leq \sum_i \int_{J_{y_i}} y^\delta dy \leq c_\delta 8^{k\delta} \sum_i |J_{y_i}| \leq c_\delta 2^p 8^{k\delta} \frac{1}{\lambda^p} \sum_i \int_{J_y} f(z)^p \, dz \\ &\leq 3c_\delta 2^p 8^{k\delta} \frac{1}{\lambda^p} \int_{\left[8^{k-1},8^{k+2}\right]} f(z)^p \, dz \leq c_{\delta,p} \frac{1}{\lambda^p} \int_{\left[8^{k-1},8^{k+2}\right]} f(z)^p \, z^\delta dz.\end{aligned}$$

Thus,

$$\int_{\{y:\lambda<M^R f(y)\}\cap\left[8^k,8^{k+1}\right]} y^\delta dy \leq c_{\delta,p} \frac{1}{\lambda^p} \int_{\left[8^{k-1},8^{k+2}\right]} f(z)^p \, z^\delta dz.$$

Hence,

$$\int_{\{y:\lambda<M^R f(y)\}} y^\delta dy \leq c_{\delta,p} \frac{1}{\lambda^p} \int_0^\infty f(z)^p \, z^\delta dz$$

holds, and Lemma 1 is proved. ∎

Lemma 2 *Given* β , $0 \leq \beta < 1$, *there exists a constant* C_β *such that for every* $y > 0$

$$\begin{aligned}(3.2)\quad y^{-\beta/2} M_\beta\left(f(z)\, z^{-\beta/2}\right)(y) &\leq C_\beta \left\{ y^{\beta/2} \frac{1}{y} \int_0^y f(z)\, z^{-\beta/2} dz \right. \\ &\left. + sup_{y\leq 2h} \left(\frac{y^{-\beta/2}}{(2h)^{1-\beta}} \int_y^{y+h} f(z)\, z^{-\beta/2} dz \right) + M_0 f(y) \right\},\end{aligned}$$

and

$$(3.3)\quad \sup_{h\geq y/2} \left(y^{-\beta/2} \frac{1}{(2h)^{1-\beta}} \int_y^{y+h} f(z)\, z^{-\beta/2} dz \right) \leq C_\beta y^{-\beta/2} M_0\left(f(z)\, z^{\beta/2}\right)(y).$$

For a proof of this lemma see [3], Lemma 1.

We shall introduce some notation. Let us consider the generating function for the Laguerre polynomials

$$(3.4)\ \sum\nolimits_0^\infty \frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)} L_n^\alpha(y)\, L_n^\alpha(z)\, r^n = \frac{1}{1-r} e^{-r(z+y)/(1-r)} (ryz)^{-\alpha/2} I_\alpha \left(2\frac{(ryz)^{1/2}}{1-r} \right),$$

where $0 \leq r < 1$, and $I_\alpha(y) = e^{-i\alpha\pi/2} J_\alpha(iy)$ is the modify Bessel function, see [1], p.189 (20). Let $Q_\alpha(y,z,r)$ be the function defined as

$$Q_\alpha(y,z,r) = \sum\nolimits_0^\infty \frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)} e^{-y/2} y^{\alpha/2} L_n^\alpha(y)\, e^{-z/2} z^{\alpha/2} L_n^\alpha(z)\, r^{n+(\alpha+1)/2},$$

then, by (3.4), $Q_\alpha(y,z,r)$ is equal to

$$\sum_0^\infty \mathcal{L}_n^\alpha(y)\,\mathcal{L}_n^\alpha(z)\,r^{n+(\alpha+1)/2} = \frac{r^{1/2}}{1-r}e^{-(z+y)/2}e^{-r(z+y)/(1-r)}I_\alpha\left(2\frac{(ryz)^{1/2}}{1-r}\right).$$

This shows that $Q_\alpha(y,z,e^{-t}) = W^\alpha(t,y,z)$. Let $e^{-t} = \left(\frac{1-s}{1+s}\right)^2$, then $0 < s \le 1$ holds if and only if $0 < t \le \infty$. If we denote $R_\alpha(y,z,s) = Q_\alpha\left(y,z,\left(\frac{1-s}{1+s}\right)^2\right)$ then, we get the expression

$$R_\alpha(y,z,s) = \frac{1}{2}\frac{1-s^2}{2s}e^{-\frac{1}{4}\left(s+\frac{1}{s}\right)\left(y^{1/2}-z^{1/2}\right)^2}e^{-\frac{1}{2}\left(s+\frac{1}{s}\right)(yz)^{1/2}}I_\alpha\left(\frac{1-s^2}{2s}(yz)^{1/2}\right). \tag{3.5}$$

Observe also that

$$W^\alpha(f,t,y) = \int_0^\infty R_\alpha(y,z,s)\,f(z)\,dz, \text{ for } s = \left(1-e^{-t/2}\right)/\left(1+e^{-t/2}\right). \tag{3.6}$$

Moreover,

$$1-s^2 = 4e^{-t/2}/\left(1+e^{-t/2}\right)^2 \le 4e^{-t/2}. \tag{3.7}$$

We shall need in the sequel the following estimations for $I_\alpha(y)$: Let $\alpha > -1$, there exist two constants c_α and C_α such that

$$\begin{array}{lll} (1) \text{ If } 0 \le y \le 1, & \text{then} & c_\alpha y^\alpha \le I_\alpha(y) \le C_\alpha y^\alpha. \\ (2) \text{ If } y \ge 1, & \text{then} & c_\alpha \frac{1}{y^{1/2}}e^y \le I_\alpha(y) \le C_\alpha \frac{1}{y^{1/2}}e^y. \end{array} \tag{3.8}$$

See, [1], p.5, (12) and [1], p.86, (5).

Let $D_s = \left\{y : \left(\frac{1-s^2}{2s}\right)^2 y \ge 1\right\}$. Then $\chi_{D_s}(yz) = 1$ if and only if $\left(\frac{1-s^2}{2s}\right)^2 yz \ge 1$. By (3.8) and (3.5) we have

$$\begin{aligned} \chi_{D_s}(yz)\,R_\alpha(y,z,s) &\le C\frac{1}{2}\frac{1-s^2}{2s}e^{-\frac{1}{4}\left(s+\frac{1}{s}\right)\left(z^{1/2}-y^{1/2}\right)^2-\frac{1}{2}\left(s+\frac{1}{s}\right)(zy)^{1/2}}\chi_{D_s}(yz)\,\frac{e^{\left(\frac{1-s^2}{2s}\right)(zy)^{1/2}}}{\left(\frac{1-s^2}{2s}\right)^{1/2}(zy)^{1/4}} \\ &\le \frac{1}{2}\frac{1-s^2}{2s}e^{-\frac{1}{4s}\left(z^{1/2}-y^{1/2}\right)^2}\chi_{D_s}(yz)\,\frac{1}{\left(\frac{1-s^2}{2s}\right)^{1/2}(zy)^{1/4}}. \end{aligned} \tag{3.9}$$

Where we have used that $-\frac{1}{2}\left(s+\frac{1}{s}\right)+\left(\frac{1-s^2}{2s}\right) = -s$.

Analogously, by (3.8), $\chi_{D_s^c}(yz)\,R_\alpha(y,z,s)$ is bounded by constant times

$$\frac{1}{2}\frac{1-s^2}{2s}e^{-\frac{1}{4s}\left(z^{1/2}-y^{1/2}\right)^2}\chi_{D_s^c}(yz)\left(\frac{1-s^2}{2s}(yz)^{1/2}\right)^\alpha. \tag{3.10}$$

We denote

$$H_{\alpha,1}(s,y)=\int_0^\infty \chi_{D_s}(yz)\,R_\alpha(y,z,s)\,f(z)\,dz,$$

and

$$H_{\alpha,2}(s,y)=\int_0^\infty \chi_{D_s^c}(yz)\,R_\alpha(y,z,s)\,f(z)\,dz.$$

Given $y>0$, and $s>0$. For every integer k we define

$$B_k(y)=\left\{z: 2^k s^{1/2}<\left|z^{1/2}-y^{1/2}\right|\le 2^{k+1}s^{1/2}\right\}.$$

Let k_0 be an integer to be fixed later, then

$$\begin{aligned}
H_{\alpha,1}(s,y) &\le C_\alpha\sum_{-\infty}^{k_0}\frac{1-s^2}{2s}e^{-2^{2k}/4}\int_{B_k(y)}\chi_{D_s}(yz)\frac{f(z)\,dz}{\left(\frac{1-s^2}{2s}\right)^{1/2}(zy)^{1/4}}\\
&\quad +C_\alpha\sum_{k_0+1}^{+\infty}\frac{1-s^2}{2s}e^{-2^{2k}/4}\int_{B_k(y)}\chi_{D_s}(yz)\frac{f(z)\,dz}{\left(\frac{1-s^2}{2s}\right)^{1/2}(zy)^{1/4}}\\
&= H_{\alpha,11}(s,y)+H_{\alpha,12}(s,y).
\end{aligned}\tag{3.11}$$

For k_0 and $B_k(y)$ having the same meaning as before, we have that $H_{\alpha,2}(s,y)$ is bounded by a constant times

$$\begin{aligned}
&\sum_{-\infty}^{k_0}\frac{1-s^2}{2s}e^{-2^{2k}/4}\int_{B_k(y)}\chi_{D_s^c}(yz)\left(\frac{1-s^2}{2s}(yz)^{1/2}\right)^\alpha f(z)\,dz\\
&\quad +\sum_{k_0+1}^{+\infty}\frac{1-s^2}{2s}e^{-2^{2k}/4}\int_{B_k(y)}\chi_{D_s^c}(yz)\left(\frac{1-s^2}{2s}(yz)^{1/2}\right)^\alpha f(z)\,dz\\
&= H_{\alpha,21}(s,y)+H_{\alpha,22}(s,y).
\end{aligned}\tag{3.12}$$

Given $y>0$ and $s>0$, let us fix k_0 as the unique integer satisfying

$$2^{k_0+2}s^{1/2}<y^{1/2}\le 2^{k_0+3}s^{1/2}.$$

If $k\le k_0$ and $z\in B_k(y)$ then, since $\left|z^{1/2}-y^{1/2}\right|\le 2^{k+1}s^{1/2}$, we get that

$$y/4\le\left(y^{1/2}-2^{k+1}s^{1/2}\right)^2\le z\le\left(y^{1/2}+2^{k+1}s^{1/2}\right)^2\le 3y.\tag{3.13}$$

In particular,

$$y/4\le z\le 3y.\tag{3.14}$$

If $k\ge k_0$ and $z\in B_k(y)$, since $\left|z^{1/2}-y^{1/2}\right|\le 2^{k+1}s^{1/2}$, we get that

$$0<z\le 100.\,2^{2k}s\quad\text{and}\quad 0<y\le 100.\,2^{2k}s.\tag{3.15}$$

Lemma 3 *Let $-1 < \alpha$. We have the following estimates for the heat diffusion integral $W^{\alpha}f(t,y)$:*

a) If $-1 < \alpha \leq 0$, we denote $\beta = -\alpha$. Then, we get

$$W^{\alpha}f(t,y) \leq C_{\alpha}\left\{e^{-t/4}M_0 f(y) + e^{-t(1-\beta)/2}y^{-\beta/2}M_{\beta}\left(z^{-\beta/2}f(z)\right)(y)\right\}. \tag{3.16}$$

b) If $\alpha \geq 0$, then, we have

$$W^{\alpha}f(t,y) \leq C_{\alpha}e^{-t/4}\left\{M^{R}f(y) + M^{+}f(y) + y^{-\alpha/2}\frac{1}{y}\int_0^y z^{\alpha/2}f(z)\,dz\right\}. \tag{3.17}$$

Proof. We will estimate $H_{\alpha,11}(s,y)$, $H_{\alpha,12}(s,y)$, $H_{\alpha,21}(s,y)$, and $H_{\alpha,22}(s,y)$ for $\alpha > -1$. We observe that

$$\sum\nolimits_{-\infty}^{\infty} e^{-2^{2k}/4}2^{\rho k} < \infty,$$

if $\rho > 0$.

Estimate of $H_{\alpha,11}(s,y)$ for $\alpha > -1$. By (3.11), (3.14) and (3.13), $H_{\alpha,11}(s,y)$ is less than or equal to a constant times the sum for $k \leq k_0$ of the terms

$$\left(\frac{1-s^2}{2s}\right)^{1/2} e^{-2^{2k}/4}y^{-1/2}\int_{\left(y^{1/2}-2^{k+1}s^{1/2}\right)^2}^{\left(y^{1/2}+2^{k+1}s^{1/2}\right)^2} f(z)\,dz. \tag{3.18}$$

Since $\left(y^{1/2}+2^{k+1}s^{1/2}\right)^2 - \left(y^{1/2}-2^{k+1}s^{1/2}\right)^2 = 4y^{1/2}2^{k+1}s^{1/2}$, we have that (3.18) is bounded by a constant times

$$\left(1-s^2\right)^{1/2} e^{-2^{2k}/4}2^{k}\frac{1}{4y^{1/2}2^{k+1}s^{1/2}}\int_{\left(y^{1/2}-2^{k+1}s^{1/2}\right)^2}^{\left(y^{1/2}+2^{k+1}s^{1/2}\right)^2} f(z)\,dz. \tag{3.19}$$

Then, considering (3.13) and (3.19), by (3.1), we get

$$H_{\alpha,11}(s,y) \leq C_{\alpha}\left(1-s^2\right)^{1/2}M^{R}f(y). \tag{3.20}$$

Estimate of $H_{\alpha,12}(s,y)$, for $\alpha > -1$. By (3.11), we have that $H_{\alpha,12}(s,y)$ is bounded by a constant times the sum for $k > k_0$ of the terms

$$\frac{1-s^2}{2s}e^{-2^{2k}/4}\int\limits_{B_k(y)}\chi_{D_s}(yz)\frac{f(z)\,dz}{\left(\frac{1-s^2}{2s}\right)^{1/2}(zy)^{1/4}}. \tag{3.21}$$

The condition $\chi_{D_s}(yz) = 1$ is equivalent to $z \geq \frac{1}{y}\left(\frac{2s}{1-s^2}\right)^2$, and by (3.15), $y \leq 100\ 2^{2k}s$. Let $\gamma \geq 0$, hence (3.21) is bounded by a constant times

$$\frac{1-s^2}{2s}e^{-2^{2k}/4}\int_{\frac{1}{y}\left(\frac{2s}{1-s^2}\right)^2}^{100\,2^{2k}s}\chi_{D_s}(yz)\frac{f(z)\,dz}{\left(\frac{1-s^2}{2s}\right)^{1/2}(zy)^{1/4}}$$

$$
\begin{aligned}
&= \left(\frac{1-s^2}{2s}\right) e^{-2^{2k}/4} \int_{\frac{1}{y}\left(\frac{2s}{1-s^2}\right)^2}^{y} \chi_{D_s}(yz) \frac{z^\gamma f(z)\,dz}{\left(\frac{1-s^2}{2s}\right)^{1/2} (zy)^{1/4} z^\gamma} \\
&\quad + \left(\frac{1-s^2}{2s}\right) e^{-2^{2k}/4} \int_{y}^{100\,2^{2k}s} \chi_{D_s}(yz) \frac{f(z)\,dz}{\left(\frac{1-s^2}{2s}\right)^{1/2} (zy)^{1/4}} \qquad (3.22)\\
&\le C\left(1-s^2\right)^{1+2\gamma} e^{-2^{2k}/4} 2^{2(1+2\gamma)k} y^{-\gamma} \frac{1}{y} \int_0^y z^\gamma f(z)\,dz \\
&\quad + C\left(1-s^2\right) e^{-2^{2k}/4} 2^{2k} \frac{1}{100\,2^{2k}s} \int_y^{100\,2^{2k}s} f(z)\,dz.
\end{aligned}
$$

Then, for any $\gamma \ge 0$, we get that $H_{\alpha,12}(s,y)$ is bounded by a constant times

$$
\left(1-s^2\right)^{1+2\gamma} y^{-\gamma} \frac{1}{y} \int_0^y z^\gamma f(z)\,dz + \left(1-s^2\right) M^+ f(y).
$$

This implies that for $\gamma = 0$, we get

$$
H_{\alpha,12}(s,y) \le C_\alpha \left(1-s^2\right) M_0 f(y), \tag{3.23}
$$

and that for $\alpha \ge 0$, taking $\gamma = \alpha/2$, we get

$$
H_{\alpha,12}(s,y) \le C_\alpha \left(1-s^2\right)^{1+\alpha} y^{-\alpha/2} \frac{1}{y} \int_0^y z^{\alpha/2} f(z)\,dz + \left(1-s^2\right) M^+ f(y). \tag{3.24}
$$

Estimate of $H_{\alpha,21}(s,y)$ for $-1 < \alpha < 0$. Let $\beta = -\alpha$. By (3.12), (3.13) and (3.14), we have that $H_{\alpha,21}(s,y)$ is bounded by a constant times the sum for $k \le k_0$ of the terms

$$
\begin{aligned}
&\frac{1-s^2}{2s} e^{-2^{2k}/4} \int_{B_k(y)} \chi_{D_s^c}(yz) \left(\frac{1-s^2}{2s}(yz)^{1/2}\right)^\alpha f(z)\,dz \\
&\le C\left(\frac{1-s^2}{2s}\right)^{1+\alpha} e^{-2^{2k}/4} y^{\alpha/2} \int_{\left(y^{1/2}-2^{k+1}s^{1/2}\right)^2}^{\left(y^{1/2}+2^{k+1}s^{1/2}\right)^2} \chi_{D_s^c}(yz) z^{\alpha/2} f(z).
\end{aligned}
$$

If $\frac{1-s^2}{2s}(yy/4)^{1/2} > 1$, then for $z \ge y/4$ it turns out that $\frac{1-s^2}{2s}(yz)^{1/2} > 1$, thus $\chi_{D_s^c}(yz) = 0$ and the integral above is equal to zero. Therefore, we can assume that $\frac{1-s^2}{2s}(yy/4)^{1/2} \le 1$, which implies that

$$
\left(1-s^2\right) y \le 4s. \tag{3.25}
$$

Therefore $H_{\alpha,21}(s,y)$ is bounded by a constant times the sum for $k \le k_0$ of the terms

$$
\begin{aligned}
&\left(\frac{1-s^2}{2s}\right)^{1+\alpha} e^{-2^{2k}/4} y^{\alpha/2} \frac{\left(4y^{1/2}2^{k+1}s^{1/2}\right)^{1+\alpha}}{\left(4y^{1/2}2^{k+1}s^{1/2}\right)^{1+\alpha}} \int_{\left(y^{1/2}-2^{k+1}s^{1/2}\right)^2}^{\left(y^{1/2}+2^{k+1}s^{1/2}\right)^2} f(z) z^{\alpha/2} dz \\
&\le C_\alpha \left(1-s^2\right)^{\frac{1-\beta}{2}} e^{-2^{2k}/4} 2^{2k(1+\alpha)} y^{-\beta/2} M_\beta\left(f(z) z^{-\beta/2}\right)(y).
\end{aligned}
$$

Thus, summing up for $k \leq k_0$, we get that

$$H_{\alpha,21}(s,y) \leq C_\alpha \left(1-s^2\right)^{\frac{1-\beta}{2}} y^{-\beta/2} M_\beta\left(f(z)\,z^{-\beta/2}\right)(y) \tag{3.26}$$

holds for $-1 < \alpha < 0$.

Estimate of $H_{\alpha,21}(s,y)$ for $\alpha \geq 0$. By (3.12), (3.13) and (3.14), we have that $H_{\alpha,21}(s,y)$ is bounded by a constant times the sum for $k \leq k_0$ of the terms

$$\begin{aligned}
&\frac{1-s^2}{2s} e^{-2^{2k}/4} \int_{B_k(y)} \chi_{D_s^c}(yz) \left(\frac{1-s^2}{2s}(yz)^{1/2}\right)^\alpha f(z)\,dz \\
&\leq C_\alpha \left(\frac{1-s^2}{2s}\right)^{1+\alpha} e^{-2^{2k}/4} y^\alpha \int_{\left(y^{1/2}-2^{k+1}s^{1/2}\right)^2}^{\left(y^{1/2}+2^{k+1}s^{1/2}\right)^2} \chi_{D_s^c}(yz) f(z) \\
&= C_\alpha \left(\frac{1-s^2}{2s}\right)^{1+\alpha} e^{-2^{2k}/4} y^\alpha \frac{4y^{1/2}2^{k+1}s^{1/2}}{4y^{1/2}2^{k+1}s^{1/2}} \int_{\left(y^{1/2}-2^{k+1}s^{1/2}\right)^2}^{\left(y^{1/2}+2^{k+1}s^{1/2}\right)^2} \chi_{D_s^c}(yz) f(z)\,dz.
\end{aligned} \tag{3.27}$$

Then, by using (3.25) and the fact that $\alpha \geq 0$, we get that (3.27) is bounded by a constant times

$$\left(1-s^2\right)^{1/2} \frac{e^{-2^{2k}/4}2^k}{4y^{1/2}2^{k+1}s^{1/2}} \int_{\left(y^{1/2}-2^{k+1}s^{1/2}\right)^2}^{\left(y^{1/2}+2^{k+1}s^{1/2}\right)^2} f(z)\,dz \leq C\left(1-s^2\right)^{1/2} e^{-2^{2k}/4}2^k M^R f(y).$$

Thus,

$$H_{\alpha,21}(s,y) \leq C_\alpha \left(1-s^2\right)^{1/2} M^R f(y). \tag{3.28}$$

Estimate of $H_{\alpha,22}(s,y)$ for the case $-1 < \alpha < 0$. Let $\beta = -\alpha$. By (3.12) and (3.15) we have that $H_{\alpha,22}(s,y)$ is bounded by a constant times the sum for $k > k_0$ of the terms

$$\begin{aligned}
&\left(\frac{1-s^2}{2s}\right) e^{-2^{2k}/4} \int_{B_k(y)} \chi_{D_s^c}(zy) \left(\left(\frac{1-s^2}{2s}\right)(zy)^{1/2}\right)^{-\beta} f(z)\,dz \\
&\leq \left(\frac{1-s^2}{2s}\right)^{1-\beta} e^{-2^{2k}/4} y^{-\beta/2} \left(\frac{100\,2^{2k}s}{100\,2^{2k}s}\right)^{1-\beta} \int_0^{100\,2^{2k}s} \chi_{D_s^c}(zy)\, z^{-\beta/2} f(z)\,dz.
\end{aligned}$$

The former expression is smaller than or equal to a constant times

$$\begin{aligned}
&\left(1-s^2\right)^{1-\beta} e^{-2^{2k}/4} 2^{2k(1-\beta)} y^{-\beta/2} \frac{1}{\left(100\,2^{2k}s\right)^{1-\beta}} \int_0^{100\,2^{2k}s} z^{-\beta/2} f(z)\,dz \\
&\leq \left(1-s^2\right)^{1-\beta} e^{-2^{2k}/4} 2^{2k(1-\beta)} y^{-\beta/2} M_\beta\left(z^{-\beta/2} f(z)\right)(y).
\end{aligned}$$

Hence, for $-1 < \alpha < 0$, $H_{\alpha,22}(s,y)$ is bounded by a constant times

$$\left(1-s^2\right)^{1-\beta} y^{-\beta/2} M_\beta\left(z^{-\beta/2} f(z)\right)(y). \tag{3.29}$$

Estimate of $H_{\alpha,22}(s,y)$ for $\alpha \geq 0$. By (3.12) and (3.15), $H_{\alpha,22}(s,y)$ is bounded by a constant times the sum for $k > k_0$ of the terms

$$\begin{aligned}
&\frac{1-s^2}{2s} e^{-2^{2k}/4} \int_{B_k(y)} \chi_{D_s^c}(yz) \left(\frac{1-s^2}{2s}(yz)^{1/2}\right)^{\alpha} f(z)\,dz \\
&\leq \left(\frac{1-s^2}{2s}\right)^{1+\alpha} e^{-2^{2k}/4} y^{\alpha/2} \int_0^y \chi_{D_s^c}(yz)\, z^{\alpha/2} f(z)\,dz \\
&\quad + \left(\frac{1-s^2}{2s}\right)^{1+\alpha} e^{-2^{2k}/4} y^{\alpha/2} \int_y^{100\,2^{2k}s} \chi_{D_s^c}(yz)\, z^{\alpha/2} f(z)\,dz \\
&\leq \left(\frac{1-s^2}{2s}\right)^{1+\alpha} e^{-2^{2k}/4} y^{\alpha/2} \int_0^y f(z)\, z^{\alpha/2} dz \\
&\quad + \left(\frac{1-s^2}{2s}\right)^{1+\alpha} e^{-2^{2k}/4} y^{\alpha/2} \left(100\,2^{2k}s\right)^{\alpha/2} \int_y^{100\,2^{2k}s} f(z)\,dz.
\end{aligned} \tag{3.30}$$

Since $y \leq 100\,2^{2k}s$, we obtain that (3.30) is bounded by a constant times

$$\begin{aligned}
&\left(1-s^2\right)^{1+\alpha} e^{-2^{2k}/4} 2^{2(1+\alpha)k} y^{-\alpha/2} \frac{1}{y} \int_0^y z^{\alpha/2} f(z)\,dz \\
&\quad + \left(1-s^2\right)^{1+\alpha} e^{-2^{2k}/4} 2^{2(1+\alpha)k} \frac{1}{100\,2^{2k}s} \int_y^{100\,2^{2k}s} f(z)\,dz,
\end{aligned}$$

thus, we have shown that, for $\alpha \geq 0$, $H_{\alpha,22}(s,y)$ is bounded by a constant times

$$\left(1-s^2\right)^{1+\alpha} \left(y^{-\alpha/2} \frac{1}{y} \int_0^y z^{\alpha/2} f(z)\,dz + M^+ f(y)\right). \tag{3.31}$$

Thus, taking into account (3.6) and (3.7), part *a)*, (3.16) follows from (3.20), (3.23), (3.26), (3.29), and, Part *b)*, (3.17) follows from (3.20), (3.24), (3.28),and (3.31) .Therefore, Lemma 3 is proved. ∎

4 Proof of the results

Proof of Theorem 1. By Lemma 3, where as usual we denote $\beta = -\alpha$, we have

$$W^{\alpha,*} f(y) \leq C_\alpha \left\{ M_0 f(y) + y^{-\beta/2} M_\beta \left(z^{-\beta/2} f(z)\right)(y) \right\},$$

Thus, applying Lemma 2, since

$$y^{\beta/2} \frac{1}{y} \int_0^y f(z)\, z^{-\beta/2} dz \leq y^{\beta/2} M_0 \left(f(z)\, z^{-\beta/2}\right)(y),$$

we get

$$W^{\alpha,*} f(y) \leq C_\beta \left\{ y^{\beta/2} M_0 \left(f(z)\, z^{-\beta/2}\right)(y) + y^{-\beta/2} M_0 \left(f(z)\, z^{\beta/2}\right)(y) + M_0 f(y) \right\}.$$

The hypothesis " if $-1<\alpha<0$, then $p\in\left(\dfrac{2(1+\delta)}{2+\alpha},\dfrac{2(1+\delta)}{(-\alpha)}\right)\cap(1,\infty)$, " is equivalent to $-1<\delta-p\,\beta/2\le\delta+p\,\beta/2<p-1$, and $p>1$. Under these conditions, the weights $y^{(\delta+p\beta/2)}$, $y^{(\delta-p\beta/2)}$ and y^{δ} belong to the class A_p of Muckenhoupt, thus

$$\begin{aligned}
&\int_0^\infty\left(y^{\beta/2}M\left(f(z)\,z^{-\beta/2}\right)(y)\right)^p y^\delta dy\le C_{\alpha,p}\int_0^\infty f(y)^p\,y^\delta dy,\\
&\int_0^\infty\left(y^{-\beta/2}M\left(f(z)\,z^{\beta/2}\right)(y)\right)^p y^\delta dy\le C_{\alpha,p}\int_0^\infty f(y)^p\,y^\delta dy,\qquad\text{and}\\
&\int_0^\infty M_0f(y)^p\,y^\delta dy\le C_{\alpha,p}\int_0^\infty f(y)^p\,y^\delta dy,
\end{aligned}$$

proving that $W^{\alpha,*}f$ is of strong type $(p.p)$ for $p\in N_\alpha$ with respect to the measure $y^\delta dy$ if $-1<\alpha<0$, this proves part (a) of the Theorem.

Now, let us consider part (b), that is to say, $\alpha\ge0$. By (3.17) of Lemma 3, we have

$$W^\alpha f(t,y)\le C_\alpha\left\{M^Rf(y)+M^+f(y)+y^{-\alpha/2}\frac{1}{y}\int_0^y z^{\alpha/2}f(z)\,dz\right\}.$$

For $M^Rf(y)$, by Lemma 1, we have that for any $p>1$, and any $\delta>-1$,

$$\int_0^\infty M^Rf(y)^p\,y^\delta dy\le C_{p,\delta}\int_0^\infty f(y)^p\,y^\delta dy$$

holds. For $M^+f(y)$, since $y^\delta\in A_1^+\subset A_p^+$ for any $\delta>-1$, as we mention in the introduction, we have that

$$\int_0^\infty M^+f(y)^p\,y^\delta dy\le C_{p,\delta}\int_0^\infty f(y)^p\,y^\delta dy$$

holds. For $y^{-\alpha/2}\frac{1}{y}\int_0^y z^{\alpha/2}f(z)\,dz$, we have that

$$y^{-\alpha/2}\frac{1}{y}\int_0^y z^{\alpha/2}f(z)\,dz\le y^{-\alpha/2}M_0\left(f(z)\,z^{\alpha/2}\right)(y).$$

Thus, if $-1<\delta-p\,\alpha/2<p-1$, and $p>1$ we have

$$\begin{aligned}
\int_0^\infty\left(y^{-\alpha/2}\frac{1}{y}\int_0^y z^{\alpha/2}f(z)\,dz\right)^p y^\delta dy &\le \int_0^\infty\left(y^{-\alpha/2}M_0\left(f(z)\,z^{\alpha/2}\right)(y)\right)^p y^\delta dy\\
&\le C_{p,\alpha,\delta}\int_0^\infty f(y)^p\,y^\delta dy.
\end{aligned}$$

The conditions $-1<\delta-p\,\alpha/2<p-1$, $p>1$ are equivalent to $p>2(1+\delta)/(\alpha+2)$, $p>1$, and $p<2(1+\delta)/\alpha$. In order to finish the proof of (b) we need to show that condition $p<2(1+\delta)/\alpha$ can be removed.

Observe that $W^{0,*}f(y)$ is bounded by a constant times $M_0f(y)$, then $W^{0,*}f(y)$ is of strong type (p,p) for $-1<\delta<p-1$ and $p>1$, with the measure $y^\delta dy$.

Assume that $0 \leq \alpha < 2$ and $p \geq 2(1+\delta)/\alpha$, then $p \geq 2(1+\delta)/\alpha > 2(1+\delta)/(0+2) = 1+\delta$. Since by (3.8) we have that $W^{\alpha,*}f(y) \leq C_\alpha W^{0,*}f(y)$ then $W^{\alpha,*}f(y)$ is of strong type (p,p) for p in the range $p \geq 2(1+\delta)/\alpha$. We have showed the result for α in the range $0 \leq \alpha < 0+2$. Now the result follows by induction on $[j, j+2)$. ■

Proof of Theorem 2.

Proof of part (a). If $-1 < \alpha < 0$ and $2(1+\delta)/(-\alpha) > 1$, the upper end point of N_α is equal to $2(1+\delta)/(-\alpha)$. For s fixed, $0 < s < 1$, let us consider the points y and z satisfying $\left(\frac{1-s^2}{2s}\right) y \leq 1$ and $\left(\frac{1-s^2}{2s}\right) z \leq 1$. By (3.5), using (3.8), we have

$$R_\alpha(y,z,s) \geq C_{\alpha,s} y^{\alpha/2} z^{\alpha/2}.$$

Thus, denoting $a = \frac{2s}{1-s^2}$, we get

$$W^\alpha\left(\chi_{(0,a)}\right)(s,y) \geq C_{\alpha,s} y^{\alpha/2} \int_0^a z^{\alpha/2} dz = C_{\alpha,s} y^{\alpha/2},$$

for every $0 \leq y \leq a$. Since

$$\int_0^a \left(y^{\alpha/2}\right)^{2(1+\delta)/(-\alpha)} y^\delta dy = \int_0^a y^{-1} dy = \infty,$$

it follows that the operator $W^{\alpha,*}f$ is not of strong type $(2(1+\delta)/(-\alpha), 2(1+\delta)/(-\alpha))$ with respect to the measure $y^\delta dy$. However, the operator $W^{\alpha,*}f$ is of weak type. In fact, let $\alpha = -\beta$, it will be enough to show that the three terms on the right hand side of (3.2) satisfy the weak type condition. Since $-1 < \alpha < 0$ implies $-1 < \delta < 2(1+\delta)/(-\alpha) - 1$, the third term of (3.2) is of strong type $(2(1+\delta)/(-\alpha), 2(1+\delta)/(-\alpha))$ with respect to the measure $y^\delta dy$. The first term is bounded by $y^{(-\alpha)/2} M_0(f(z) z^{\alpha/2})(y)$ and since we have

$$-1 < ((-\alpha)/2)\, 2(1+\delta)/(-\alpha) + \delta < 2(1+\delta)/(-\alpha) - 1,$$

the weight $y^{((-\alpha)/2)2(1+\delta)/(-\alpha)+\delta} \in A_{2(1+\delta)/(-\alpha)}$. This shows that

$$\int_0^\infty \left(y^{(-\alpha)/2} M_0\left(f(z) z^{\alpha/2}\right)(y)\right)^{2(1+\delta)/(-\alpha)} y^\delta dy \leq C_{\alpha,\delta} \int f(y)^{2(1+\delta)/(-\alpha)} y^\delta dy,$$

which implies the strong type $(2(1+\delta)/(-\alpha), 2(1+\delta)/(-\alpha))$ with respect to the measure $y^\delta dy$ of the first term of (3.2)). Let us consider now the second term of (3.2). If we denote $2(1+\delta)/(-\alpha)$ by p, then $p' = 2(1+\delta)/(2(1+\delta)+\alpha)$. By Hölder's inequality, we obtain that

$$\frac{1}{(2h)^{1+\alpha}} \int_y^{y+h} f(z) z^{\alpha/2} dz \leq \frac{1}{(2h)^{1+\alpha}} \|f\|_{L^p\left((y,y+h), z^\delta dz\right)} \left\|z^{\alpha/2-\delta}\right\|_{L^{p'}\left((y,y+h), z^\delta dz\right)}.$$

In order to estimate $\left\|z^{\alpha/2-\delta}\right\|_{L^{p'}\left((y,y+h), z^\delta dz\right)}$ we observe that

$$\delta + (\alpha/2 - \delta) p' > -1 \quad \text{and} \quad (\delta + (\alpha/2-\delta) p' + 1)/p' = 1+\alpha$$

hold. Then $\left\|z^{\alpha/2-\delta}\right\|_{L^{p'}\left((y,y+h),z^{\delta}dz\right)} \leq c_{\delta,\beta}\,(y+h)^{1+\alpha}$. Thus, since $y \leq 2h$, we have

$$\frac{1}{(2h)^{1+\alpha}}\int_{y}^{y+h} f(z)\, z^{\alpha/2}dz \;\leq\; C_{\delta,\alpha}\left(\frac{y+h}{h}\right)^{1+\alpha}\|f\|_{L^{p}\left((y,y+h),z^{\delta}dz\right)} \leq C_{\delta,\alpha}\,\|f\|_{L^{p}\left((0,\infty),z^{\delta}dz\right)}.$$

Multiplying by $y^{-\beta/2}$ and taking the supremum in $h \geq y/2$, we obtain

$$\sup_{h\geq y/2}\left(y^{\alpha/2}\frac{1}{(2h)^{1+\alpha}}\int_{y}^{y+h} f(z)\, z^{-\beta/2}dz\right) \leq C_{\delta,\beta}y^{\alpha/2}\,\|f\|_{L^{p}((0,\infty),z^{\delta}dz)}.$$

From this inequality the weak type (p,p) for $p = 2(1+\delta)/(-\alpha)$ with respect to the measure $y^{\delta}dy$ is readily obtained.

Proof of part (b). If $\alpha \geq 0$, the upper end point of N_{α} is equal to ∞ and, by (3.8) and (3.17), we have $W^{\alpha,*}f(y) \leq C_{\alpha}W^{0,*}f(y) \leq C_{\alpha}M_{0}f(y)$. Therefore since for $\delta > -1$, $L^{\infty}\left((0,\infty), y^{\delta}dy\right) = L^{\infty}((0,\infty), dy)$, the operator $W^{\alpha,*}f$ is of strong type (∞,∞) with respect to the measure $y^{\delta}dy$.

Proof of part (c). If the lower end point of N_{α} is greater than 1, then it coincides with $2(1+\delta)/(2+\alpha)$. This implies that $2\delta-\alpha > 0$. If for a given $a > 0$ the integral $\int_{0}^{a} f(z)\, z^{\alpha/2}dz = \int_{0}^{a} f(z)\, z^{\alpha/2-\delta}z^{\delta}dz$ is finite for every $f(z)$ in $L^{2(1+\delta)/(2-\beta)}\left((0,a), z^{\delta}dz\right)$, then since

$$\left(\frac{2(1+\delta)}{2+\alpha}\right)' = \frac{2(1+\delta)}{2\delta-\alpha},$$

by uniform boundedness, it follows that $z^{\alpha/2-\delta} \in L^{2(1+\delta)/(2\delta-\alpha)}\left((0,a), z^{\delta}dz\right)$. This is a contradiction since $z^{(\alpha/2-\delta)2(1+\delta)/(2\delta-\alpha)+\delta} = z^{-1}$. Therefore, there exists a function f belonging to $L^{2(1+\delta)/(2+\alpha)}\left((0,a), z^{\delta}dz\right)$ such that $\int_{0}^{a} f(z)\, z^{\alpha/2}dz = \infty$. Thus for this f, if $a = \dfrac{2s}{1-s^{2}}$, then

$$\int_{0}^{a} R(s,y,z)\, f(z)\, dz \geq C_{\alpha,s}y^{\alpha/2}\int_{0}^{a} z^{\alpha/2}f(z)\, dz = \infty,$$

showing that $W^{\alpha,*}f(y) = \infty$ for every $y \leq a$. This is telling us that the operator $W^{\alpha,*}$ cannot be of weak type at the lower end point $2(1+\delta)/(2-\beta) > 1$ with respect to the measure $y^{\delta}dy$.

Now we shall prove the restricted type. Let $-1 < \alpha < 0$ and $\beta = -\alpha$. By (3.16) and Lemma maxfrac, we have

$$W^{\alpha,*}f(y) \leq C_{\beta}\left\{M_{0}f(y) + y^{\beta/2}\frac{1}{y}\int_{0}^{y} f(z)\, z^{-\beta/2}dz + y^{-\beta/2}M_{0}\left(f(z)\, z^{\beta/2}\right)(y)\right\}. \tag{4.1}$$

It is easy to see that

$$\begin{aligned} -1 &< \delta < 2(1+\delta)/(2-\beta) - 1 \qquad \text{and} \\ -1 &< \delta - (\beta/2)\,2(1+\delta)/(2-\beta) < 2(1+\delta)/(2-\beta) - 1, \end{aligned}$$

hold. These inequalities imply that the weights y^{δ} and $y^{\delta-(\beta/2)2(1+\delta)/(2-\beta)}$ belong to $A_{2(1+\delta)/(2-\beta)}$. Therefore, the operators

$$M_0 f(y) \quad \text{and} \quad y^{-\beta/2} M_0\left(f(z)\, z^{\beta/2}\right)(y),$$

are of strong type $(2(1+\delta)/(2-\beta), 2(1+\delta)/(2-\beta))$, with respect to the measure $y^{\delta}dy$. We have not considered yet the second term of (4.1). If $\alpha \geq 0$, by (3.17), we have

$$W^{\alpha,*} f(y) \leq C_{\alpha}\left\{M^R f(y) + y^{-\alpha/2}\frac{1}{y}\int_0^y z^{\alpha/2} f(z)\, dz + M^+ f(y)\right\}. \tag{4.2}$$

By Lemma 1, the first term on the right hand side of (4.2) is of weak type for any $p \geq 1$ and any $\delta > -1$. As we mention in the introduction the weight y^{δ} belongs to the class $A_1^+ \subset A_p^+$ of Sawyer for $-1 < \delta$, thus we get that the operator $M^+ f(y)$ is of weak type (p, p) for any $p \geq 1$ with respect to the measure $y^{\delta}dy$ for any $\delta > -1$. Now we are going to consider the second terms on the right hand side of both (4.1) and (4.2). They are of the form $y^{-\alpha/2}\frac{1}{y}\int_0^y z^{\alpha/2} f(z)\, dz$ allowing $\alpha > -1$. Let E be a measurable set contained in $(0, \infty)$ and F the set defined by $\chi_E\left(u^{1/(1+\delta)}\right) = \chi_F(u)$ By the change of variables $z = u^{1/(1+\delta)}$, we have

$$\int_0^{\infty} \chi_E(z)\, z^{\delta} dz = \frac{1}{(1+\delta)}\int_0^{\infty} \chi_E\left(u^{1/(1+\delta)}\right) du = \frac{1}{(1+\delta)}|F|, \tag{4.3}$$

and

$$\begin{aligned}
\int_0^y \chi_E(z)\, z^{\alpha/2} dz &= \frac{1}{(1+\delta)}\int_0^{y^{1+\delta}} \chi_E\left(u^{1/(1+\delta)}\right) u^{(\alpha/2-\delta)/(1+\delta)} du \\
&= \frac{1}{(1+\delta)}\int_0^{y^{1+\delta}} \chi_F(u)\, u^{(\alpha/2-\delta)/(1+\delta)} du.
\end{aligned}$$

Since $2(1+\delta)/(2+\alpha) > 1$ implies $\alpha/2 - \delta < 0$, thus, it follows that

$$\int_0^{y^{1+\delta}} \chi_F(u)\, u^{(\alpha/2-\delta)/(1+\delta)} du \leq \int_0^{\infty} \chi_F(u)\, u^{(\alpha/2-\delta)/(1+\delta)} du \leq \int_0^{|F|} u^{(\alpha/2-\delta)/(1+\delta)} du.$$

Taking into account that $\alpha > -1$ implies $(\alpha/2-\delta)/(1+\delta) > -1$, we can compute the last integral above obtaining

$$\int_0^{|F|} u^{(\alpha/2-\delta)/(1+\delta)} du = \frac{2(1+\delta)}{(\alpha+2)}|F|^{(\alpha+2)/2(1+\delta)}.$$

Then, by (4.3), we get

$$\begin{aligned}
\int_0^{|F|} u^{(\alpha/2-\delta)/(1+\delta)} du &= \frac{2(1+\delta)}{(2-\beta)}\left((1+\delta)\int_0^{\infty} \chi_E(z)\, z^{\delta} dz\right)^{(\alpha+2)/2(1+\delta)} \\
&= c_{\alpha,\delta}\left(\int_0^{\infty} \chi_E(z)\, z^{\delta} dz\right)^{(\alpha+2)/2(1+\delta)}.
\end{aligned}$$

In consequence,

$$y^{-\alpha/2}\frac{1}{y}\int_0^y \chi_E(z)\, z^{\alpha/2}dz \le c_{\alpha,\delta}y^{-\alpha/2}\frac{1}{y}\left(\int_0^\infty \chi_E(u)\, u^\delta du\right)^{(\alpha+2)/2(1+\delta)}.$$

From this inequality the restricted weak type $(2(1+\delta)/(2+\alpha), 2(1+\delta)/(2+\alpha))$ for the operator $W^{\alpha,*}f$ with respect to the measure $y^\delta dy$ is readily obtained.

Proof of part (d). Let us show that if the lower end point of N_α is equal to 1, then the operator $W^{\alpha,*}f$ cannot be of strong type $(1,1)$ with respect to the measure $y^\alpha dy$. In fact, by (3.8), we have

$$\chi_{D_s}(yz)\, R_\alpha(y,z,s) \ge C_\alpha\left(\frac{1-s^2}{2s}\right)^{1/2} e^{-\frac{1}{4s}\left(y^{1/2}-z^{1/2}\right)^2} e^{-\frac{s}{4}\left(y^{1/2}-z^{1/2}\right)^2} e^{-s(yz)^{1/2}}\chi_{D_s}(yz)\,\frac{1}{(yz)^{1/4}}.$$

Take $0<\varepsilon\le 1$. Let us assume that $1<z\le 1+\varepsilon$, $1+2\varepsilon\le y\le 2$, and $s=(y-1)^2/4$. Then it follows that $s\le 1/4$, $\left(\frac{1-s^2}{2s}\right)\ge 1$, and $\left(\frac{1-s^2}{2s}\right)^{1/2}(yz)^{1/4}\ge 1$. Thus $\chi_{D_s}(yz)=1$ and since

$$\begin{aligned}\frac{1}{4s}\left(y^{1/2}-z^{1/2}\right)^2 &= \frac{\left(y^{1/2}-z^{1/2}\right)^2}{(y-1)^2} \le \left(\frac{(y-z)}{2(y-1)}\right)^2\\ &\le \left(\frac{1}{2}+\frac{|1-z|}{2(y-1)}\right)^2 \le \left(\frac{1}{2}+\frac{\varepsilon}{4\varepsilon}\right)^2 \le 1,\end{aligned}$$

we get $R_\alpha(y,z,s)\ge \frac{C_\alpha}{(y-1)}$, and therefore

$$W^{\alpha,*}\left(\chi_{(1,1+\varepsilon)}\right)(y) \ge \frac{C_\alpha}{(y-1)}\int_0^\infty \chi_{(1,1+\varepsilon)}(z)\, dz = C_\alpha\frac{\varepsilon}{(y-1)},$$

for $1+2\varepsilon\le y\le 2$. Then, if the operator $W^{\alpha,*}$ were of strong type $(1,1)$ with respect to the measure $y^\delta dy$, and recalling that $\delta>-1$, we would have that

$$\int_0^\infty W^{\alpha,*}\left(\chi_{(1,1+\varepsilon)}\right)(y)\, y^\delta dy \le A_\alpha\int_0^\infty \chi_{(1,1+\varepsilon)}(y)\, y^\delta dy = A_\alpha\frac{(1+\varepsilon)^{1+\delta}-1}{1+\delta} \le A_{\alpha,\delta}\,\varepsilon \tag{4.4}$$

holds for a finite constant $A_{\alpha,\delta}$ depending on α and δ only. On the other hand, we get

$$\begin{aligned}&\int_{1+2\varepsilon}^2 W^{\alpha,*}\left(\chi_{(1,1+\varepsilon)}\right)(y)\, y^\delta dy\\ &\ge C_\alpha\int_{1+2\varepsilon}^2 \frac{\varepsilon}{y-1}y^\delta dy \ge C_{\alpha,\delta}\int_{1+2\varepsilon}^2 \frac{\varepsilon}{y-1}dy = C_{\alpha,\delta}\,\varepsilon\log\left(\frac{1}{2\varepsilon}\right).\end{aligned} \tag{4.5}$$

In consequence, from (4.4) and (4.5), it follows that $C_{\alpha,\delta}\,\varepsilon\log\left(\frac{1}{2\varepsilon}\right)\le A_\alpha\,\varepsilon$, or also, that $C_{\alpha,\delta}\log\left(\frac{1}{2\varepsilon}\right)\le A_{\alpha,\delta}$. This is a contradiction since the left hand side of the inequality above

tends to ∞ when ε tends to 0, proving that there is no strong type $(1,1)$ for the operator $W^{\alpha,*}f$ with the measure $y^{\delta}dy$.
However, as we are going to show, the operator $W^{\alpha,*}f$ is of weak type $(1,1)$ with respect to the measure $y^{\delta}dy$. Since $2(1+\delta)/(2+\alpha)\leq 1$, it follows that $2\delta-\alpha\leq 0$. Notice that since N_α is not empty we always have that $2(1+\delta)+\alpha\geq 0$, which is equivalent to $1+\alpha+\delta-\alpha/2\geq 0$. Let us assume $-1<\alpha<0$, and let $\beta=-\alpha$. By (3.2) and (3.16) (Lemma 2), we have that $W^{\alpha,*}f$ is bounded by a constant times

$$M_0 f(y)+\sup_{y\leq 2h}\left(\frac{y^{-\beta/2}}{(2h)^{1-\beta}}\int_y^{y+h} f(z)\,z^{-\beta/2}dz\right)+y^{\beta/2}\frac{1}{y}\int_0^y f(z)\,z^{-\beta/2}dz. \tag{4.6}$$

Since $2\delta+\beta=2\delta-\alpha\leq 0$ it turns out that $-1<\delta<-\beta/2<0$. Then, $M_0 f(y)$ is of weak type $(1,1)$ with respect to the measure $y^{\delta}dy$. For the second term of (4.6), since $y\leq 2h$ and $2\delta+\beta\leq 0$, we have

$$\begin{aligned}
\frac{y^{-\beta/2}}{(2h)^{1-\beta}}\int_y^{y+h} f(z)\,z^{-\beta/2}dz &\leq \frac{y^{-\beta/2}}{(2h)^{1-\beta}}\int_0^{3h} f(t)\,z^{\delta}z^{-(\delta+\beta/2)}dz\\
&\leq y^{-\beta/2}\frac{(3h)^{-(\delta+\beta/2)}}{(2h)^{1-\beta}}\int_0^{3h} f(z)\,z^{\delta}dz\\
&= c_{\alpha,\delta}y^{-\beta/2}\frac{1}{h^{1-\beta+(\delta+\beta/2)}}\int_0^{3h} f(z)\,z^{\delta}dz \qquad (4.7)\\
&\leq c_{\beta,\delta}\frac{1}{y^{1+\delta}}\int_0^{\infty} f(z)\,z^{\delta}dz.
\end{aligned}$$

Which clearly implies the weak type $(1,1)$ of the second term. Still we have to estimate the third term of (4.6).
For the case $\alpha\geq 0$. From (3.17) we see that the operator $W^{\alpha,*}f(y)$ is bounded by

$$C_\alpha\left\{M^R f(y)+M^{+}f(y)+y^{-\alpha/2}\frac{1}{y}\int_0^y z^{\alpha/2}f(z)\,dz\right\}. \tag{4.8}$$

By Lemma 1, the first term of (4.8) is of weak type for any $1\leq p\leq\infty$ with the measure $y^{\delta}dy$ for any δ. As we mention before, the weight $y^{\delta}\in A_1^{+}$ for any $\delta>-1$, therefore $M^{+}f(y)$ is of weak type $(1,1)$ with respect to the same measure $y^{\delta}dy$. For the third terms of (4.6) and (4.8), we have that for $\alpha>-1$,

$$y^{-\alpha/2}\frac{1}{y}\int_0^y z^{\alpha/2}f(z)\,dz=y^{-\alpha/2}\frac{1}{y}\int_0^y z^{\alpha/2-\delta}f(z)\,z^{\delta}dz.$$

Since $\alpha/2-\delta\geq 0$, this expression is bounded by

$$y^{-\alpha/2}\frac{y^{\alpha/2-\delta}}{y}\int_0^y f(z)\,z^{\delta}dz\leq\frac{1}{y^{1+\delta}}\int_0^{\infty} f(z)\,z^{\delta}dz.$$

This inequality and (4.8) imply the $(1,1)$ weak type of the operator $W^{\alpha,*}f$ with respect to the measure $y^{\delta}dy$. ∎

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