Jensen's inequality and majorization.

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Dedicated to the memory of Gert K. Pedersen

Abstract

Let $\mathcal{A}$ be a C^*-algebra and $\phi : \mathcal{A} \to L(H)$ be a positive unital map. Then, for a convex function $f : I \to \mathbb{R}$ defined on some open interval and a self-adjoint element $a \in \mathcal{A}$ whose spectrum lies in I, we obtain a Jensen's-type inequality $f(\phi(a)) \preceq \phi(f(a))$ where $\preceq$ denotes an operator preorder (usual order, spectral preorder, majorization) and depends on the class of convex functions considered i.e., operator convex, monotone convex and arbitrary convex functions. Some extensions of Jensen's-type inequalities to the multi-variable case are considered.

Keywords: Jensen's inequality, convex functions, positive maps, majorization, spectral preorder.

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1 Introduction

Jensen's inequality is the continuous version of the usual definition of convex function and it can be stated in the following way: let (X, P) be a probability space and let g be a bounded function such that for all $x \in X$, $g(x) \in I$. Then

$$f\left(\int_X g\, dP\right) \leq \int_X f \circ g\, dP.$$

In the context of C^*-algebras the simplest generalization of Jensen's inequality can be made by taking a state ϕ and a selfadjoint element a of a C^*-algebra $\mathcal{A}$ such that the spectrum of a is contained in I, then it holds that

$$f(\varphi(a)) \leq \varphi(f(a)). \tag{1}$$

The same inequality, replacing the state φ by a positive unital map $\phi : \mathcal{A} \to \mathcal{B}$ between two C^*-algebras ($\phi(a) \geq 0$ whenever $a \geq 0$ and $\phi(I) = I$) is false unless f be an operator convex function. Indeed T. Ando proved that, given a Hilbert space $\mathcal{H}$ and a function F defined on some open interval I, the following statements are equivalent:

1. F is operator convex.

2. $F(v^*av) \leq v^*F(a)v$ for every $a \in L_{sa}(\mathcal{K})$ such that $\sigma(a) \subseteq I$, where $\mathcal{K}$ is some Hilbert space and v is any isometry from $\mathcal{H}$ into $\mathcal{K}$.

3. $F(\phi(a)) \leq \phi(F(a))$ for every $a \in L_{sa}(\mathcal{H})$ such that $\sigma(a) \subseteq I$, and for every unital completely positive map $\phi : L(\mathcal{H}) \to L(\mathcal{H})$.

4. $F(\phi(a)) \leq \phi(F(a))$ for every $a \in L_{sa}(\mathcal{H})$ such that $\sigma(a) \subseteq I$, and for every unital positive map $\phi : L(\mathcal{H}) \to L(\mathcal{H})$.

Although in some particular cases (e.g. when $\phi(a)$ and $\phi(f(a))$ commute) we can obtain Jensen's type inequalities like in item 4 for arbitrary convex functions, Ando's result tells us that in general we can not consider the usual order for these kind of inequalities and convex funtions which are not operator convex.

However, previous works on the matter, such as Brown-Kosaki's and Hansen-Pedersen's ([6], [7]), suggest the idea of considering Jensen's type inequalities with respect to other preorders, such as the spectral and majorization preorders (see subsection 2.3 for their definitions).

In this paper we study different Jensen's-type inequalities in which, as in the case of Ando's Theorem, the roll of non-commutative integral is played by a positive (unital) map.

The paper is organized as follows. In section 2 we recall some definitions and we introduce notation used throughout this paper. For the sake of completness, we sketch the proof of Ando's Theorem and give a short proof of Jensen's inequality for states in a C^*-algebra.

Section 3 is divided in two subsections, depending on the preorder relation and the assumptions on the convex functions. In subsection 3.1 we consider monotone convex functions and, following Brown-Kosaki's ideas on the matters, we prove Jensen's type inequalities with respect to the spectral preorder. In subsection 3.2 we obtain Jensen's type inequalities for arbitrary convex functions either by taking restrictions on the algebra $\mathcal{B}$ or by using the majorization preorder.

We collect the results of this section in the following list: Let $\mathcal{A}$, $\mathcal{B}$ be unital C^*-algebras, $\phi : \mathcal{A} \to \mathcal{B}$ a positive unital map, f a convex function defined on an open interval I and $a \in \mathcal{A}$, such that $a = a^*$ and $\sigma(a) \subseteq I$. Then

1. If f is monotone and $\mathcal{B}$ is a von Neumann algebra, then
$$f(\phi(a)) \precsim \phi(f(a)) \text{ (spectral preorder).}$$

2. If $\mathcal{B}$ is abelian or, more generally, if $\phi(f(a))$ and $\phi(a)$ commute, then
$$f(\phi(a)) \leq \phi(f(a)).$$

3. If $E : \mathcal{B} \to \mathcal{C}$ is a conditional expectation and $\phi(a)$ belongs to the centralizer of E, then
$$\mathcal{E}(g[f(\phi(a))]) \leq \mathcal{E}(g[\phi(f(a))])$$
for every continuous increasing convex function g such that $\mathrm{Im}(f) \subseteq \mathrm{Dom}(g)$.

4. if $\mathcal{B}$ a finite factor, then

$$f(\phi(a))) \preccurlyeq \phi(f(a)) \ \ \text{(majorization).}$$

We remark that some of this inequalities still hold for *contractive* positive maps, under the assumption that $0 \in I$ and $f(0) \leq 0$.

We should also mention that in the item 3 of the previous list, a similar inequality without the convex function g can be obtain as a corollary of Hansen and Pedersen's results [7] and Stinespring's representation theorem. However, this convex function is meaningful because it let us connect Hansen and Pedersen's work with the theory of majorization in finite factors.

In section 4 we describe briefly the multi-variable functional calculus and obtain in this context similar results to those of section 3.2 by using essentially the same techniques.

In section 5 we apply the results obtained to the finite dimensional case, where there exist fairly simple expressions for the spectral preorder and for the majorization; we show that our results generalize those appeared in a recent work by J. S. Aujla and F. C. Silva [2].

Finally, section 6 is devoted to obtain some inequalities by choosing particular functions and (unital) positive maps. For instance, we prove non-commutative versions of the information inequality, Liapunov's inequality and Hölder's inequality.

2 Preliminaries

Let $\mathcal{A}$ be a C^*-algebra, throughout this paper $\mathcal{A}_{sa}$ will be the real vector space of self-adjoint elements of $\mathcal{A}$, $\mathcal{A}^+$ the cone of positive elements, $\mathrm{GL}(A)$ the group of invertible elements and $M_n(\mathcal{A})$ the $n \times n$ matrices whose entries are elements of $\mathcal{A}$. We also suppose that all the C^*-algebras in consideration are unital. In a similar way given a Hilbert space $\mathcal{H}$, $L(\mathcal{H})$ will be the algebra of all linear bounded operators on $\mathcal{H}$, $L_{sa}(\mathcal{H})$ the real vector space of self-adjoint operators, and $L(\mathcal{H})^+$ the cone of all positive operators. For every $C \in L(\mathcal{H})$ its range will be denoted by $R(C)$, its null space will be denoted by $\ker(C)$ and its spectrum by $\sigma(C)$.

On the other hand, if p and q are orthogonal projections, the orthogonal projection onto the intersection of their ranges will be denoted $p \wedge q$ and the orthogonal projection onto the closed subspace generated by their ranges will be denoted $p \vee q$

2.1 Positive and Completely Positive Maps.

To begin with, let us recall some definitions.

Definition 2.1. Let $\mathcal{A}$ and $\mathcal{B}$ two C^*-algebras and $\phi : \mathcal{A} \to \mathcal{B}$ a linear map. We shall say that ϕ is **positive** if it maps positive elements of $\mathcal{A}$ to positive elements of $\mathcal{B}$.

Definition 2.2. Given a positive map, ϕ, between two C^*-algebra $\mathcal{A}$ and $\mathcal{B}$, let us call ϕ_n the map between $M_n(\mathcal{A})$ and $M_n(\mathcal{B})$ defined by $\phi_n((a_{ij})) = (\phi(a_{ij}))$. We say that ϕ is **n-positive** if ϕ_n is positive and **completely positive** if ϕ is n-positive for all $n \in \mathbb{N}$.

The following result due to Stinespring is one of the most general theorems which characterizes completely positive maps from a C^*-algebra into $L(\mathcal{H})$.

Theorem 2.3. *Let $\mathcal{A}$ be a C^*-algebra and let $\phi : \mathcal{A} \to L(\mathcal{H})$ be a completely positive map. Then, there exists a Hilbert space $\mathcal{K}$, a unital *-homomorphism $\pi : \mathcal{A} \to L(\mathcal{K})$, and a bounded operator $V : \mathcal{H} \to \mathcal{K}$ with $\|\phi(1)\| = \|V\|^2$ such that*

$$\phi(a) = V^*\pi(a)V.$$

The reader can find a proof of this theorem in [11] or in [13]. Note that in our case, since the map will be either unital or a contraction, the bounded operator V will be either an isometry or a contraction.

Another important result about completely positive maps is the next theorem, also due to Stinespring (See [11]).

Theorem 2.4. *Let X be a compact Hausdorff space and $C(X)$ the C^*-algebra of continuous function on X. Then, every positive map defined on $C(X)$ is completely positive.*

2.2 Convex and Operator Convex Maps.

Definition 2.5. A real function f defined on a segment (a, b), where $-\infty \leq a < b \leq \infty$, is called **convex** if the inequality

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)\,, \tag{2}$$

holds whenever $a < x < b$, $a < y < b$ and $0 \leq \lambda \leq 1$. A function g such that $-g$ is convex is called concave.

It is not difficult to prove that equation (2) is equivalent to the requirement that

$$\frac{f(t) - f(s)}{t - s} \leq \frac{f(u) - f(t)}{u - t}\,, \tag{3}$$

whenever $a < s < t < u < b$.

Proposition 2.6. *Let f be a convex function defined in an open interval I and let $K \subseteq I$ be compact. Then, there exist a (countable) family of linear functions $\{f_i\}_{i\geq 1}$ such that for each $x \in K$ it holds that*

$$f(x) = \sup_{i\geq 1} f_i(x).$$

Definition 2.7. A continuous function $F : I \to \mathbb{R}$ is said to be **operator convex** if

$$F(\lambda a + (1-\lambda)b) \leq \lambda F(a) + (1-\lambda)F(b)\,, \tag{4}$$

for each $\lambda \in [0, 1]$ and every pair of self-adjoint operators a, b on an infinite dimensional Hilbert space $\mathcal{H}$ with spectrum in I. As in the scalar case, F is called operator concave if $-F$ is operator convex.

Remarks 2.8.

1. The equation (4) is well defined because the set of self-adjoint operator with spectrum in a segment I is convex.

2. The function F is called **matrix convex of order n** if equation (4) is satisfied for operators belonging to an n dimensional Hilbert space. It is well known that a function is operator convex if and only if it is matrix convex of arbitrary orders. The interested reader can find a proof of this fact in [3].

3. Clearly, an operator convex function is convex but the converse is not true. Indeed it is known that the class of operator convex functions has a very rich structure. For example (see [4]), a function $f : [0, \infty) \to \mathbb{R}$ is an operator convex function if and only if there exist a positive finite measure μ on $[0, \infty)$ and real constants α, β, γ with $\gamma \geq 0$, such that

$$f(t) = \alpha + \beta t + \gamma t^2 + \int_0^\infty \frac{\lambda t^2}{1 + \lambda t}\, d\mu(\lambda)$$

In particular f is infinitely differentiable on $(0, \infty)$. ▲

Some well known functions that are operator convex and operator concave are listed below:

Examples 2.9.

1. $F(x) = x^r$ for $r \in [-1, 0] \cup [1, 2]$ is operator convex and for $r \in [0, 1]$ is operator concave.

2. $F(x) = \log(x)$ and $F(x) = -x \log(x)$ are operator concave in $(0, \infty)$. ▲

The notion of convexity is very close to Jensen's inequality. The following Jensen-type inequality for states defined on a C^*-algebra is the simplest generalization to a non-commutative context.

Theorem 2.10. *Let $\mathcal{A}$ be a C^*-algebra , $\varphi : \mathcal{A} \to \mathbb{C}$ a state and f a convex function defined on some interval (α, β) $(-\infty \leq \alpha < \beta \leq \infty)$. Then it holds that*

$$f(\varphi(a)) \leq \varphi(f(a)))\,,$$

for every $a \in \mathcal{A}_{sa}$ whose spectrum is contained in (α, β).

Proof. Put $\tau = \varphi(a)$. Then $\alpha < \tau < \beta$. If γ is the supremum of quotients on the left of (3), where $\alpha < s < \tau$, then γ is no larger than any of the quotients on the right of the same inequality, for $\mu \in (\tau, \beta)$. Then

$$f(\sigma) \geq f(\tau) + \gamma(\sigma - \tau) \qquad (\alpha < \sigma < \beta).$$

Hence

$$f(a) - f(\tau) + \gamma(a - \tau) \geq 0.$$

Finally, if we apply φ to this inequality, taking into account that φ is a state we get the desired inequality. ■

There also exist a closed relation between the notion of operator convexity and Jensen-type inequalities, as the following theorem due to Ando shows.

Theorem 2.11. *Let $\mathcal{H}$ be an infinite dimensional Hilbert space and F a function defined on some open interval I. Then, the following statements are equivalent:*

1. *F is operator convex.*

2. *$F(v^*av) \leq v^*F(a)v$ for every $a \in L_{sa}(\mathcal{K})$ such that $\sigma(a) \subseteq I$, where $\mathcal{K}$ is some Hilbert space and v is any isometry from $\mathcal{H}$ into $\mathcal{K}$.*

3. *$F(\phi(a)) \leq \phi(F(a))$ for every $a \in L_{sa}(\mathcal{H})$ such that $\sigma(a) \subseteq I$, and for every unital completely positive map $\phi : L(\mathcal{H}) \to L(\mathcal{H})$.*

4. *$F(\phi(a)) \leq \phi(F(a))$ for every $a \in L_{sa}(\mathcal{H})$ such that $\sigma(a) \subseteq I$, and for every unital positive map $\phi : L(\mathcal{H}) \to L(\mathcal{H})$.*

Sketch of proof.

1. $\Leftrightarrow$ 2. This is a well known fact. A proof can be found in [4].

2. $\Rightarrow$ 3. Given a unital completely positive map ϕ, by theorem 2.3 there exists a Hilbert space $\mathcal{K}$, a representation $\pi : \mathcal{A} \to L(\mathcal{K})$, and a isometry $K : \mathcal{H} \to \mathcal{K}$ such that:

$$\phi(a) = K^*\pi(a)K.$$

So, as $F(\pi(a)) = \pi(F(a))$, using (2.) we have that:

$$F(\phi(a)) = F(K^*\pi(a)K) \leq K^*F(\pi(a))K = K^*\pi(F(a))K = \phi(F(a)).$$

3. $\Rightarrow$ 4. The unital positive map restricted to the unital abelian C^*-algebra generated by a is completely positive according to the theorem 2.4. So (4.) is a consequence of (3.).

4. $\Rightarrow$ 2. It is clear. ■

2.3 Spectral Preorder and Majorization.

In what follows $E_a[I] = \chi_I(a)$ denotes the spectral projection of a self-adjoint operator a in a von Neumann algebra $\mathcal{A}$, corresponding to a (Borel) subset $I \subseteq \mathbb{R}$.

To begin with, let us recall the notion of spectral preorder.

Definition 2.12. Let $\mathcal{A}$ be a von Neumann algebra. Given $a, b \in \mathcal{A}_{sa}$, we shall say that $a \precsim b$ if $E_a[(\alpha, +\infty)]$ is equivalent, in the sense of Murray-von Neumann, to a subprojection of $E_b[(\alpha, +\infty)]$ for every real number α.

In finite factors the following result can be proved.

Proposition 2.13. *Let $\mathcal{A}$ be a finite factor, with normalized trace tr. Given $a, b \in \mathcal{A}_{sa}$, the following conditions are equivalent:*

1. $a \precsim b$.

2. $\operatorname{tr}(f(a)) \leq \operatorname{tr}(f(b))$ *for every continuous increasing function* f *defined on an interval containing both* $\sigma(a)$ *and* $\sigma(b)$.

Finally, Eizaburo Kamei defined the notion of majorization and submajorization in finite factors (see [9]):

Definition 2.14. Let $\mathcal{A}$ be a finite factor with normalized trace tr. Given $a, b \in \mathcal{A}_{sa}$, we shall say that a is submajorized by b, and denote $a \preccurlyeq_w b$, if the inequality:

$$\int_0^\alpha e_a(t)\, dt \leq \int_0^\alpha e_b(t)\, dt$$

holds for every real number α, where $e_c(t) = \inf\{\gamma : tr(E_c[(\gamma, \infty)]) \leq t\}$ for $c \in \mathcal{A}_{sa}$. We shall say that a is majorized by b if $a \preccurlyeq_w b$ and $\operatorname{tr}(a) = \operatorname{tr}(b)$.

The following characterizarion of submajorization also appears in [9].

Proposition 2.15. *Let* $\mathcal{A}$ *be a finite factor with normalized trace* tr*. Given* $a, b \in \mathcal{A}_{sa}$*, the following conditions are equivalent:*

1. $a \preccurlyeq_w b$.

2. $\operatorname{tr}(f(a)) \leq \operatorname{tr}(f(b))$ *for every increasing convex function* f *defined on an interval containing both* $\sigma(a)$ *and* $\sigma(b)$.

3 Jensen-type inequalities.

Throughout this section ϕ is a positive unital map from a C^*-algebra $\mathcal{A}$ to another C^*-algebra $\mathcal{B}$, $f : I \to \mathbb{R}$ is a convex function defined on the open interval I and $a \in \mathcal{A}_{sa}$ whose spectrum lies in I. Note that the spectrum of $\phi(a)$ is also contained in I.

As we mentioned in the introduction, by Ando's Theorem, we can not expect a Jensen-type inequality of the form

$$f(\phi(a)) \leq \phi(f(a)) \tag{5}$$

for an arbitrary convex function f without other assumptions. This is the reason why, in order to study inequalities similar to (5) for different subsets of convex function, we shall use the spectral and majorization (pre)orders, or we shall change the hypothesis made over the C^*-algebra where the positive map takes its values.

Although most of the inequalities considered in this section involve unital positive maps, similar results can be obtained for contractive positive maps by adding some extra hypothesis on f.

3.1 Monotone convex and concave functions. Spectral Preorder.

In this section we shall consider monotone convex and concave functions. The following result due to Brown and Kosaki ([6]) indicates that the appropriate order relation for this class of functions is the spectral preorder. Let $\mathcal{A}$ be a semi-finite von Neumann algebra, and let $v \in \mathcal{A}$ be a contraction; then, for every positive operator $a \in \mathcal{A}$ and every continuous monotone convex function f defined in $[0, +\infty)$ and such that $f(0) = 0$, they proved that

$$v^* f(a) v \precsim f(v^* a v).$$

The following Theorem is an analogue of Brown and Kosaki's result, in terms of positive unital maps and monotone convex functions. The proof we give below follows essentially the same lines.

Theorem 3.1. *Let $\mathcal{A}$ be an unital C^*-algebra, $\mathcal{B}$ a von Neumann algebra and $\phi : \mathcal{A} \to \mathcal{B}$ a positive unital map. Then, for every monotone convex function f, defined on some interval I, and for every self-adjoint element $a \in \mathcal{A}$ whose spectrum is contained in I it holds that*

$$f(\phi(a)) \precsim \phi(f(a)) \tag{6}$$

Proof. According to the definition, given $\alpha \in \mathbb{R}$ we have to prove that there exists a projection $q \in \mathcal{A}$ such that

$$E_{f(\phi(a))}\left[(\alpha, +\infty)\right] = E_{\phi(a)}\left[\{f > \alpha\}\right] \sim q \leq E_{\phi(f(a))}\left[(\alpha, +\infty)\right].$$

To begin with, we claim that $E_{\phi(a)}\left[\{f > \alpha\}\right] \wedge E_{\phi(f(a))}\left[(-\infty, \alpha]\right] = 0$. In fact, take $\overline{\eta} \in R\left(E_{\phi(a)}\left[\{f > \alpha\}\right]\right)$, $\|\overline{\eta}\| = 1$. Since f is monotone we have that

$$\alpha < f(\langle \phi(a)\overline{\eta},\, \overline{\eta}\rangle),$$

and using Theorem 2.10 we get

$$\alpha < \langle \phi(f(a))\overline{\eta},\, \overline{\eta}\rangle.$$

But, on the other hand, for every $\overline{\xi} \in R\left(E_{\phi(f(a))}\left[(-\infty, \alpha]\right]\right)$, $\|\overline{\xi}\| = 1$

$$\alpha \geq \left\langle \phi(f(a))\overline{\xi},\, \overline{\xi}\right\rangle.$$

Taking this into account, and using Kaplansky's formula we have that

$$\begin{aligned} E_{f(\phi(a))}\left[(\alpha, +\infty)\right] &= E_{f(\phi(a))}\left[(\alpha, +\infty)\right] - \left(E_{f(\phi(a))}\left[(\alpha, +\infty)\right] \wedge E_{\phi(f(a))}\left[(-\infty, \alpha]\right]\right) \\ &\sim \left(E_{f(\phi(a))}\left[(\alpha, +\infty)\right] \vee E_{\phi(f(a))}\left[(-\infty, \alpha]\right]\right) - E_{\phi(f(a))}\left[(-\infty, \alpha]\right] \\ &\leq I - E_{\phi(f(a))}\left[(-\infty, \alpha]\right] = E_{\phi(f(a))}\left[(\alpha, +\infty\right]. \end{aligned}$$

∎

Remarks 3.2.

- Note that we can not infer from the above result a similar one for monotone concave function because it is not true that $a \precsim b \Rightarrow -b \precsim -a$. However, using the same arguments a similar result can be proved for monotone concave function.

- If $0 \in I$ and $f(0) \leq 0$ the same result holds for positive contractive maps. The proof follows essentially the same lines but we have to use Corollary 3.7, which will be proved in the next section, instead of Theorem 2.10. ▲

In the particular case when $\phi(a)$ and $\phi(f(a))$ are compact operators and $f(0) = 0$ we obtain the following Corollary:

Corollary 3.3. *Let $\mathcal{A}$ be an unital C^*-algebra, $\mathcal{H}$ a Hilbert space and $\phi : \mathcal{A} \to L(\mathcal{H})$ a positive unital map. Then, for each monotone convex function f defined on $[0, +\infty)$ such that $f(0) = 0$, and for each positive element a of $\mathcal{A}$ such that $\phi(a)$ and $\phi(f(a))$ are compact, there exists a partial isometry $u \in L(\mathcal{H})$ with initial space $\overline{R(f(\phi(a)))}$ such that:*

$$uf(\phi(a))u^* \leq \phi(f(a)) \qquad \text{and} \qquad \Big(uf(\phi(a))u^*\Big)\,\phi(f(a)) = \phi(f(a))\,\Big(uf(\phi(a))u^*\Big).$$

The proof of this Corollary is based in the next technical Lemma.

Lemma 3.4. *Let $\mathcal{H}$ be a Hilbert space. Given a, b positive operators of $L(\mathcal{H})$ such that a is compact, b is diagonalizable and $a \precsim b$. Then there exists a partial isometry $u \in L(\mathcal{H})$ with initial space $\overline{R(a)}$ such that*

$$uau^* \leq b \qquad\qquad \text{and} \qquad\qquad (uau^*)b = b(uau^*).$$

Moreover, if $\mathcal{H}$ has finite dimension, the same inequality holds for $a, b \in L_{sa}(\mathcal{H})$ and the isometry can be changed by a unitary operator.

Proof. Let $\{\xi_n\}_{n\in\mathbb{N}}$ be an orthonormal basis of $\overline{R(a)}$ which consists of eigenvectors associated to a decreasing sequence of positive eigenvalues $\{\lambda_n\}_{n\in\mathbb{N}}$ of a, counted with multiplicity.

On the other hand, let $\{\eta_m\}_{m\in M}$ be an orthonormal basis of $\overline{R(b)}$ which consists of eigenvectors associated to a sequence of positive eigenvalues $\{\mu_m\}_{m\in M}$ of b, counted with multiplicity.

Now, given $\alpha \geq 0$ let $\lambda_1, \ldots, \lambda_{n_\alpha}$ be the eigenvalues of a which are greater than α. In a similar way, let $M(\alpha) = \{m \in M,\ \mu_m > \alpha\}$. If m_α denotes be the cardinal of $M(\alpha)$ then notice that $0 \leq m_\alpha \leq \infty$ and if $\alpha \geq \beta \geq 0$ then $M(\alpha) \subseteq M(\beta)$.

By hypothesis $E_a[(\alpha, +\infty)]$ is equivalent to a subprojection of $E_b[(\alpha, +\infty)]$. Since $E_a[(\alpha, +\infty)]$ is the (orthogonal) projection onto the linear span of $\{\xi_n\}_{n=1}^{n_\alpha}$ and $E_b[(\alpha, +\infty)]$ is the projection onto the linear span of $\{\eta_m\}_{m\in M(\alpha)}$, then we have that $n_\alpha \leq m_\alpha$.

If we take this into account, we can define an injection $\psi : \mathbb{N} \to M$ such that for all $\alpha \geq 0$ it holds $\psi(k) \in M(\alpha)$, $k = 1, \ldots, n_\alpha$. But then, defining $u : \overline{R(a)} \to \overline{R(b)}$ by

$$u(\xi_n) = \eta_{\psi(n)}$$

and extending this definition to $\mathcal{H}$ as zero in $\ker(a)$, we obtain that $uau^* \leq b$ and $(uau^*)b = b(uau^*)$, since the set $(\eta_m)_{m\in M}$ is a system of eigenvectors for uau^* and b, which is complete for $\overline{R(uau^*)} = R(u)$, and $\ker(uau^*) = R(u)^\perp$ is an invariant subspace for b.

Finally, if $\dim \mathcal{H} = n < \infty$ and $a, b \in L_{sa}(\mathcal{H})$, let $\xi_1, \ldots, \xi_n$ be a basis of $\mathcal{H}$ which consists of eigenvectors of a associated to a decreasing sequence of eigenvalues $\lambda_1, \ldots, \lambda_n$ and let $\eta_1, \ldots, \eta_n$ be a basis of $\mathcal{H}$ which consists of eigenvectors of b associated to a decreasing sequence of eigenvalues $\mu_1, \ldots \mu_n$.

Then, if we define $u : \mathcal{H} \to \mathcal{H}$ by mean of $u(\xi_m) = \eta_m$, $1 \leq m \leq n$, the same argument used before shows that $uau^* \leq b$ and $(uau^*)b = b(uau^*)$. By construction, u is unitary. ■

3.2 Arbitrary convex functions.

As the following example due to J. S. Aujla and F. C. Silva shows, Theorem 3.1 may be false if the function f is not monotone.

Example 3.5. Consider the positive map $\phi : \mathcal{M}_4 \to \mathcal{M}_2$ given by

$$\phi\left(\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\right) = \frac{A_{11} + A_{22}}{2}$$

Take $f(t) = |t|$ and let A be the following matrix

$$A = \left(\begin{array}{cc|cc} -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

Then

$$\phi(f(A)) = \begin{pmatrix} 3/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \qquad \text{and} \qquad f(\phi(A)) = \begin{pmatrix} 1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} \end{pmatrix}.$$

Easy calculations shows that

$$\operatorname{rank}(E_{\phi(f(A))}\,[(0.5, +\infty)]) = 1 < 2 = \operatorname{rank}(E_{f(\phi(A))}\,[(0.5, +\infty)]).$$

▲

Nevertheless, a Jensen's type inequality holds with respect to the usual order for every convex function, if the map ϕ takes values in a commutative algebra $\mathcal{B}$. More generaly, it suffices to assume that the elements $\phi(f(a))$ and $f(\phi(a))$ of $\mathcal{B}$ commute.

Theorem 3.6. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras , $\phi : \mathcal{A} \to \mathcal{B}$ a positive unital map, and f a convex function defined on some open interval I. Given a self-adjoint element $a \in \mathcal{A}$ whose spectrum is contained in I, if $\phi(a)$ and $\phi(f(a))$ commute then it holds that:*

$$f(\phi(a)) \leq \phi(f(a)). \tag{7}$$

Moreover, if $0 \in I$ and $f(0) \leq 0$ then equation (7) holds if ϕ is contractive.

Proof. Let $\widehat{\mathcal{B}}$ denote the abelian C*-subalgebra of $\mathcal{B}$ generated by $\phi(a)$ and $\phi(f(a))$. On the other hand, let $\{f_i\}_{i\geq 1}$ be the linear functions given by proposition 2.6, such that for each $x \in \sigma(a)$

$$f(x) = \sup_{i\geq 1} f_i(x).$$

Since $f \geq f_i$ $(i \geq 1)$ we have that $f(a) \geq f_i(a)$ and therefore we obtain that

$$\phi(f(a)) \geq \phi(f_i(a)) = f_i(\phi(a))\,, \tag{8}$$

where the last equality holds because f_i is linear.

As $f_i(\phi(a))$ belongs to the abelian C^*-algebra $\widehat{\mathcal{B}}$ for every $i \geq 1$, and $\phi(f(a))$ also belongs to the abelian C^*-algebra $\widehat{\mathcal{B}}$ we have that

$$\phi(f(a)) \geq \max_{1\leq i\leq n} [f_i(\phi(a))] = \max_{1\leq i\leq n} [f_i] (\phi(a)).$$

Now, since $\max\limits_{1\leq i\leq n} [f_i] \to f$ uniformly on compact sets by Dini's theorem, we obtain

$$\phi(f(a)) \geq f(\phi(a)).$$

If ϕ is contractive and $f(0) \leq 0$, the functions f_i also satisfy that $f_i(0) \leq 0$ and we can replace (8) by:

$$\phi(f(a)) \geq \phi(f_i(a)) \geq f_i(\phi(a))\,,$$

and then we can repeat the same argument to get the desired inequality. ■

Corollary 3.7. *Let $\mathcal{A}$, $\mathcal{B}$ be C^*-algebras and suppose that $\mathcal{B}$ is commutative. Let f be a convex function defined on some open interval I and $\phi : \mathcal{A} \to \mathcal{B}$ positive unital. Then*

$$f(\phi(a)) \leq \phi(f(a)))\,, \tag{9}$$

for every $a \in \mathcal{A}_{sa}$ whose spectrum is contained in I. Moreover, if $0 \in I$ and $f(0) \leq 0$ then equation (9) holds even if ϕ is contractive.

Let $\mathcal{C} \subseteq \mathcal{B}$ be C^*-algebras. A **conditional expectation** $\mathcal{E} : \mathcal{B} \to \mathcal{C}$ is a positive $\mathcal{C}$-linear projection from $\mathcal{B}$ onto $\mathcal{C}$ of norm 1. For instance, states are conditional expectations. It is well known that a conditional expectation is a completely positive map. The **centralizer** of $\mathcal{E}$ is the C^*-subalgebra of $\mathcal{B}$ defined by:

$$\mathcal{B}^{\mathcal{E}} = \{b \in \mathcal{B} : \; \mathcal{E}(ba) = \mathcal{E}(ab), \; \forall a \in \mathcal{B}\}$$

Notice that, for every $b \in \mathcal{B}^{\mathcal{E}}$, $\mathcal{E}(b) \in \mathcal{Z}(\mathcal{C})$; where $\mathcal{Z}(\mathcal{C}) = \{c \in \mathcal{C} : \; cb = bc, \; \forall b \in \mathcal{C}\}$ is the **center** of $\mathcal{C}$.

In [7] Hansen and Pedersen considered the vector space $C_b(X, \mathcal{A})$ of all continuous and norm bounded function from a locally compact Hausdorff space X, endowed with a borel measure μ, to a unital C^*-algebra $\mathcal{A}$. As it is well known, this space is a C^*-algebra with the pointwise sum, multiplication, involution and the norm

$$\|g\|_\infty = \sup_X \|g(x)\|.$$

If the function $x \to \|g(x)\|$ is integrable, the function g is called integrable and we can consider the Bochner's integral

$$\int_X g(x) d\mu(x).$$

A function $d \in C_b(X, \mathcal{A})$ is called **density** whenever d^*d is integrable and it holds that

$$\int_X d^*(x)\, d(x)\, d\mu(x) = I.$$

To each density d there is associated a positive unital map, $\phi : C_b(X, \mathcal{A}) \to \mathcal{A}$, defined in the following way

$$\phi(g) = \int_X d^*(x)\, g(x)\, d(x)\, d\mu(x).$$

In this context, given a state φ of $\mathcal{A}$ and a convex function f defined on an open interval I, Hansen and Pedersen proved ([7]) the following Jensen's type inequality

$$\varphi\left(f\left(\int_X d^*(X)\, g(x)\, d(x)\, d\mu(x)\right)\right) \leq \varphi\left(\int_X d^*(x)\, f(g(x))\, d(x)\, d\mu(x)\right)$$

for each $g \in C_b(X, \mathcal{A})_{sa}$ such that $h = \int_X d^*(x)\, g(x)\, d(x)\, d\mu(x) \in \mathcal{A}^\varphi$ and the spectrum of $g(x)$ is contained in I for every $x \in X$.

Using Stinespring's Theorems, we can rewrite the above result in the following way:

$$\varphi(f(\phi(a))) \leq \varphi(\phi(f(a))) \tag{10}$$

where $\phi : \mathcal{A} \to \mathcal{B}$ is a positive unital map between C^*-algebras, and we suppose that $\phi(a) \in \mathcal{B}^\varphi$.

The following Theorem is a slight improvement of (10), which let us connect Hansen-Pedersen's result with the theory of majorization in finite factors.

Theorem 3.8. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras and $\phi : \mathcal{A} \to \mathcal{B}$ be a positive unital map. Let $\mathcal{E} : \mathcal{B} \to \mathcal{C}$ be a conditional expectation from $\mathcal{B}$ onto a C^*-subalgebra $\mathcal{C}$. Then, for every convex function f defined on some open interval I,*

$$\mathcal{E}(g[f(\phi(a))]) \leq \mathcal{E}(g[\phi(f(a))]) \tag{11}$$

where $a \in \mathcal{A}_{sa}$ such that $\sigma(a) \subseteq I$ and $\phi(a)$ belongs to $\mathcal{B}^{\mathcal{E}}$, and $g : J \to \mathbb{R}$ is any increasing convex function from some open interval J such that $Im(f) \subseteq J$.

Before starting the proof of this Theorem, we need to prove the following Lemma.

Lemma 3.9. *Let $\mathcal{B}$ be a C^*-algebra, φ a state defined on $\mathcal{B}$, and $b \in \mathcal{B}^\varphi$. Then, there exist a Borel measure μ defined on $\sigma(b)$ and a positive unital linear map $\Psi : \mathcal{B} \to L^\infty(\sigma(b), \mu)$ such that:*

i. $\Psi(f(b)) = f$ *for every* $f \in C(\sigma(b))$.

ii. $\varphi(x) = \displaystyle\int_{\sigma(b)} \Psi(x)(t)\, d\mu(t)$ *for every* $x \in \mathcal{B}$.

Proof. First of all, note that for every continuous function g defined on the spectrum of b, the map

$$g \to \varphi(g(b)),$$

is a bounded linear functional on $C(\sigma(b))$. Therefore, by the Riesz's representation theorem, there exists a Borel measure μ defined on the Borel subsets of $\sigma(b)$, such that for every continuous function g on $\sigma(b)$,

$$\varphi(g(b)) = \int_{\sigma(b)} g(t)\, d\mu(t).$$

Now, given $x \in \mathcal{B}^+$, define the following functional on $C(\sigma(B))$

$$g \to \varphi(xg(b)).$$

Since for every positive element y of $C(b)$, $\varphi(xy) = \varphi(y^{1/2}xy^{1/2}) \leq \|x\|\varphi(y)$, this functional is not only bounded but also dominated by the functional defined before. So, there exists an element h_x of $L^\infty(\sigma(b), \mu)$ such that, for every $g \in C(\sigma(b))$,

$$\varphi(xg(b)) = \int_{\sigma(b)} g(t)\, h_x(t)\, d\mu(t).$$

The map $x \mapsto h_x$, extended by linearization, defines a positive unital linear map $\Psi : \mathcal{B} \to L^\infty(\sigma(b), \mu)$ which satisfies condition (i), because

$$\varphi(f(b)g(b)) = \varphi(fg(b)) = \int_{\sigma(b)} g(t)f(t)\, d\mu(t).$$

In order to prove (ii), note that

$$\varphi(x) = \varphi(x\ 1(b)) = \int_{\sigma(b)} 1\ \Psi(x)(t)\, d\mu(t) = \int_{\sigma(b)} \Psi(x)(t)\, d\mu(t).$$

■

Proof of Theorem 3.8. Suppose firstly that $\mathcal{E}$ is a state. Define $b = \phi(a)$. Since $b \in \mathcal{B}^\varphi$, by the previous lemma there exist a Borel measure μ defined on $\sigma(b)$ and a positive unital linear map $\Psi : \mathcal{B} \to L^\infty(\sigma(b), \mu)$ such that:

i. $\Psi(f(b)) = f$ for every $f \in C(\sigma(b))$.

ii. $\varphi(x) = \displaystyle\int_{\sigma(b)} \Psi(x)(t)\, d\mu(t)$ for every $x \in \mathcal{B}$.

Now, consider the map $\Phi : C(\sigma(a)) \to L^\infty(\sigma(b), \mu)$ defined by $\Phi(h) = \Psi(\phi(h(a)))$. Then, Φ is bounded, unital and positive. Moreover,

$$\varphi(\phi(h(a))) = \int_{\sigma(b)} \Psi(\phi(h(a)))(t)\, d\mu(t) = \int_{\sigma(b)} \Phi(h)(t)\, d\mu(t).$$

Let g be a increasing convex function. Then, using Theorem 3.6,

$$\begin{aligned}
\varphi(g[f(\phi(a))]) = \varphi(g \circ f(b)) &= \int_{\sigma(b)} g \circ f(t)\, d\mu(t) = \int_{\sigma(b)} g[f(\Phi(Id))]\, d\mu(t) \\
&\leq \int_{\sigma(b)} g[\Phi(f)(t)]\, d\mu(t) = \int_{\sigma(b)} g[\Psi(\phi(f(a)))(t)]\, d\mu(t) \\
&\leq \int_{\sigma(b)} \Psi(g[\phi(f(a))])(t)\, d\mu(t) = \varphi(g[\phi(f(a))]).
\end{aligned}$$

The general case can be reduced to the case already proved by composing the conditional expectation with the states of the C^*-algebra $\mathcal{C}$. ■

Using Theorem 3.8, we can show a Jensen's type inequality for arbitrary convex functions with respect to the majorization preorder:

Theorem 3.10. *Let $\mathcal{A}$ be a unital C^*-algebra, $\mathcal{B}$ a finite factor and $\phi : \mathcal{A} \to \mathcal{B}$ a positive unital map. Then, for every convex function f, defined on some open interval I, and for every self-adjoint element a of $\mathcal{A}$ whose spectrum is contained in I,*

$$f(\phi(a))) \preccurlyeq \phi(f(a)). \tag{12}$$

Proof. By Theorem 3.8, applied to the tracial state of the finite factor $\mathcal{B}$,

$$\operatorname{tr}[g(f(\phi(a)))] \leq \operatorname{tr}[g(\phi(f(a)))]$$

for every increasing convex function g. By Proposition 2.15, we get $f(\phi(a))) \preccurlyeq \phi(f(a))$. ■

Remark 3.11. As it happens with Theorems 3.6 and 3.1, in all other Theorems of this section we can ask ϕ to be contractive instead of unital. In that case the function f has to satisfy some condition at zero, for example $f(0) \leq 0$. ▲

4 The multi-variable case.

In this section we shall be concerned with the restatement, in the multi-variable context, of several results obtained in section 3.2. For related results, see [8] and [12].

4.1 Multivariated functional calculus

In what follows we consider the functional calculus for a function of several variables, so we state a few facts about it in order to keep the text selfcontained. Let $\mathcal{A}$ be a unital C*-algebra and let $a_1, \ldots, a_n$ be mutually commuting elements of $\mathcal{A}_{sa}$ i.e., the self-adjoint part of $\mathcal{A}$. If $\mathcal{B} = C^*(a_1, \ldots, a_n)$ denotes the unital C*-subalgebra of $\mathcal{A}$ generated by these elements, then $\mathcal{B}$ is an abelian C^*-algebra. So there exists a compact Hausdorff space X such that $\mathcal{B}$ is *-isomorphic to $C(X)$. Actually X is (up to homeomorphism) the space of characters of $\mathcal{B}$ i.e, the set of homomorphisms $\gamma : \mathcal{B} \to \mathbb{C}$, endowed with the weak*- topology.

Recall that in the case of one operator $a \in \mathcal{A}_{sa}$, X is homeomorphic to $\sigma(a)$. In general, characters of the algebra $\mathcal{B}$ are associated in a continuous and injective way to n-tuples $(\lambda_1, \ldots, \lambda_n) \in \prod_{i=1}^n \sigma(a_i)$ by restriction to the abelian C*-subalgebras of $\mathcal{B}$ generated by the $a_i' s$. Thus X is homeomorphic to its image under this map, which we call joint spectrum and denote $\sigma(a_1, \ldots, a_n)$.

Example 4.1. Let $\mathcal{A}$ be a unital C^*-algebra and a, $b \in \mathcal{A}_{sa}$ such that $ab = ba$. Although the joint spectrum $\sigma(a, b)$ is a closed subset of the product $\sigma(a) \times \sigma(b)$ it may be, in general, quite *thin* , for example $\sigma(a, b)^0 = \emptyset$. Indeed if $b = f(a)$ for a continuous function $f : \sigma(a) \to \mathbb{R}$ then is easy to see that $\sigma(a, b) = \{(x, f(x)),\ x \in \sigma(a)\}$ =Graph(f). Notice that in this case $C^*(a) = C^*(a, b)$ and $\sigma(a)$ is homeomorphic to $\sigma(a, b)$ in the obvious way. ▲

We can now describe the functional calculus in several variables. Let $f : \sigma(a_1, \ldots, a_n) \to \mathbb{R}$ be a continuous function defined on the joint spectrum of the a_i's. Then there exists an element, denoted by $f(a_1, \ldots, a_n)$, that corresponds to the continuous function f by the above *-isomorphism. Note that by Tietze's extension theorem we can consider functions defined on $\prod_{i=1}^n \sigma(a_i) \subseteq \mathbb{R}^n$ without loss of generality. Therefore the association $f \mapsto f(a_1, \ldots, a_n)$ is a *-homomorphism from $C(\prod_{i=1}^n \sigma(a_i))$ onto $\mathcal{B}$, which extends the usual functional calculus of one variable.

4.2 Jensen's type inequality in several variables. Majorization

Definition 4.2. Let U be a convex subset of $\mathbb{R}^n$. A function $f : U \to \mathbb{R}$ is called convex if for all x, $y \in U$ and for all $0 \le \lambda \le 1$ it holds that

$$f(\lambda x + (1 - \lambda)y) \le \lambda f(x) + (1 - \lambda)f(y).$$

Proposition 4.3. *Let f be a convex function defined on an open convex set $U \subseteq \mathbb{R}^n$ and let $K \subseteq U$ be compact. Then, there exists a countable family of linear functions $\{f_i\}_{i\ge1}$ such that for each $x \in K$ it holds that*

$$f(x) = \sup_{i\ge1} f_i(x).$$

The following results are the multi-variable versions of Theorems 3.6, 3.8 and 3.10 and Lemma 3.9. The proofs of those results were chosen in such a way that they still hold in the multivariable case without substantial differences.

Theorem 4.4. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras, $\phi : \mathcal{A} \to \mathcal{B}$ be a positive unital map and $f : U \to \mathbb{R}$ a convex function. Let $a_1, \ldots, a_n \in \mathcal{A}_{sa}$ be mutually commuting, with $\prod_{i=1}^n \sigma(a_i) \subseteq U$ and such that $\phi(a_1), \ldots, \phi(a_n), \phi(f(a_1, \ldots, a_n))$ are also mutually commuting then*

$$f(\phi(a_1), \ldots, \phi(a_n)) \le \phi(f(a_1, \ldots, a_n)). \tag{13}$$

Moreover, if $\tilde{0} = (0, \ldots, 0) \in U$ and $f(\tilde{0}) \le 0$ then equation (13) holds even if ϕ is positive contractive.

Lemma 4.5. *Let $\mathcal{B}$ be a C^*-algebra, φ a state defined on $\mathcal{B}$, and $b_1, \dots, b_n \in \mathcal{B}^\varphi$ mutually commuting. Then, there exist a Borel measure μ defined on $K := \sigma(b_1, \dots, b_n)$ and a positive unital linear map $\Psi : \mathcal{B} \to L^\infty(K, \mu)$ such that:*

i. $\Psi(f(b_1, \dots, b_n)) = f$ *for every* $f \in C(K)$.

ii. $\varphi(x) = \displaystyle\int_K \Psi(x)(t)\, d\mu(t)$ *for every* $x \in \mathcal{B}$.

Theorem 4.6. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras, $\phi : \mathcal{A} \to \mathcal{B}$ be a positive unital map. Suppose there exists a conditional expectation $\mathcal{E} : \mathcal{B} \to \mathcal{C}$, from $\mathcal{B}$ onto the C^*-subalgebra $\mathcal{C}$. Then for every convex function $f : U \to \mathbb{R}$ defined on some open convex set $U \subseteq \mathbb{R}^n$ it holds that*

$$\mathcal{E}(g[f(\phi(a_1), \dots, \phi(a_n))]) \leq \mathcal{E}(g[\phi(f(a_1, \dots, a_n))]). \tag{14}$$

where $a_1, \dots, a_n \in \mathcal{A}_{sa}$ are mutually commuting such that $\prod_{i=1}^n \sigma(a_i) \subseteq U$, $\phi(a_1), \dots, \phi(a_n) \in \mathcal{B}^{\mathcal{E}}$ are mutually commuting, and $g : I \to \mathbb{R}$ is a convex increasing function defined on some open interval, for which $Im(f) \subseteq I$.

Theorem 4.7. *In the conditions of the above theorem let suppose further that $\mathcal{B}$ is a finite factor, then we have*

$$f(\phi(a_1), \dots, \phi(a_n))) \preccurlyeq \phi(f(a_1, \dots, a_n)). \tag{15}$$

Note that in these Theorems, the requirement that the elements $a_1, \dots, a_n$ (respectively $\phi(a_1), \dots, \phi(a_n)$) commute is needed in order to compute $f(a_1, \dots, a_n)$ (respectively $f(\phi(a_1), \dots, \phi(a_n))$). On the other hand the hypothesis that $\phi(f(a_1, \dots, a_n))$ commutes with the elements $\phi(a_1), \dots, \phi(a_n)$ corresponds to a technical reason just as in theorem 3.6.

5 The finite dimensional case.

Both the spectral preorder and the majorization have well known characterizations in the finite dimensional case. The main goal of this section is to rewrite the already obtained Jensen's inequalities in terms of this characterizations. Throughout this section we identify the space $L(\mathbb{C}^n)$ with the space of complex matrices $\mathcal{M}_n$, the real vector space $L_{sa}(\mathcal{H})$ with the real vector space $\mathcal{M}_n^{sa}$ of selfadjoint matrices and the positive cone $L(\mathcal{H})^+$ with the positive cone of positive matrices $\mathcal{M}_n^+$. Given a selfadjoint matrix A, by means of $\lambda_1(A), \dots, \lambda_n(A)$ we denote the eigenvalues of A counted with multiplicity and arranged in non-increasing order.

Let us start by recalling the aspect of the spectral preorder and majorization in $\mathcal{M}_n^{sa}$.

5.1. Let $A, B \in \mathcal{M}_n^{sa}$.

1. Using Corollary 3.4 it is easy to see that the following conditions are equivalent

(a) $A \precsim B$.

(b) There is an unitary matrix U such that $(UAU^*)B = B(UAU^*)$ and $UAU^* \leq B$.

(c) $\lambda_i(A) \leq \lambda_i(B)$ $(1 \leq i \leq n)$.

2. Straightforward calculations show that, given a selfadjoint matrix C, the functions $e_C(t)$ considered in the definition of majorization have the following form

$$e_C(t) = \begin{cases} \lambda_1(C) & \text{if } 0 \leq t < \frac{1}{n} \\ \vdots & \\ \lambda_n(C) & \text{if } \frac{n-1}{n} \leq t < 1 \end{cases}$$

Therefore, it holds

$$\int_0^\alpha e_A(t)\, dt \leq \int_0^\alpha e_B(t)\, dt \ \ (\forall\, \alpha \in \mathbb{R}^+) \iff \sum_{i=1}^k \lambda_i(A) \leq \sum_{i=1}^k \lambda_i(B) \ \ (k = 1, \dots, n).$$

In the next Proposition, we summarize the different versions of Jensen's inequality obtained in section 3 using the above characterizations of the spectral preorder and majorization.

Proposition 5.2. *Let $\mathcal{A}$ be an unital C^*-algebra and $\phi : \mathcal{A} \to \mathcal{M}_n$ a positive unital map. Suppose that $a \in \mathcal{A}_{sa}$ and that $f : I \to \mathbb{R}$ is a function whose domain is an interval which contains the spectrum of a. Then*

1. *If f is an operator convex function*

$$\phi(f(a)) \leq f(\phi(a)).$$

2. *If f is a monotone convex function, for every $i \in \{1, \dots, n\}$*

$$\lambda_i(f(\phi(a))) \leq \lambda_i(\phi(f(a))).$$

3. *If f is a convex function*

$$\sum_{i=1}^k \lambda_i(f(\phi(a))) \leq \sum_{i=1}^k \lambda_i(\phi(f(a))).$$

Moreover, if $0 \in I$ and $f(0) \leq 0$ the above inequalities also hold for contractive positive maps.

Example 5.3. Given two $n \times n$ matrices $A = (a_{ij})$ and $B = (b_{ij})$, we denote $A \circ B = (a_{ij}b_{ij})$ their Schur's product. It is a well known fact that the map $A \mapsto A \circ B$ is completely positive for each positive matrix B, and if we further assume that $I \circ B = I$ then it is also unital. So the above inequalities can be rewritten taking $\phi : \mathcal{M}_n \to \mathcal{M}_n$ given by $\phi(A) = A \circ B$ where B satisfies the mentioned properties. ▲

Example 5.4. Another example can be obtained by taking $\phi : \mathcal{M}_m \to \mathcal{M}_n$ given by

$$\phi(A) = \sum_{i=1}^{r} W_i^* A\, W_i$$

where $W_1, \ldots, W_r$ are $m \times n$ matrices such that $\sum_{i=1}^{r} W_i^* W_i = I$. These maps are completely positive by Choi's Theorem and they are unital as a result of the condition over the rectangular matrices W_i. ▲

Example 5.5. As a final application let us consider for each $\alpha \in (0,1)$ the positive unital map $\phi_\alpha : \mathcal{M}_{2n} \to \mathcal{M}_n$ defined by

$$\phi\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}\right) = \alpha A + (1-\alpha) D.$$

Using these maps in Proposition 5.2 and taking diagonal block matrices we get that for every monotone convex functions f

$$\lambda_i\left(f(\alpha A + (1-\alpha)D)\right) \leq \lambda_i\left(\alpha f(A) + (1-\alpha) f(D)\right) \qquad (i = 1, \ldots, n)$$

and for general convex functions f

$$\sum_{i=1}^{r} \lambda_i\left(f(\alpha A + (1-\alpha)D)\right) \leq \sum_{i=1}^{r} \lambda_i\left(\alpha f(A) + (1-\alpha) f(D)\right) \qquad (r = 1, \ldots, n)$$

where we are assuming that A and D are selfadjoint matrices whose spectra are contained in the domain of f. This inequalities were proved by J. S. Aujla and F. C. Silva in [2]. ▲

6 Some inequalities.

In this section we deduce some inequalities from the results developed in previous sections by choosing particular (operator) convex functions and positive maps. See [5] and [10] for related inequalities.

First of all, let us recall that given a locally compact Hausdorff space X and a unital C^*-algebra $\mathcal{A}$, $C_b(X, \mathcal{A})$ is the space of functions $g : X \to \mathcal{A}$ which are norm continuous and norm bounded.

When a Borel measure μ defined on the Borel subsets of X, for those functions $g \in C_b(X, \mathcal{A})$ such that $x \to \|g(x)\|$ is integrable, we can consider the Bochner integral

$$\int_X g(x) d\mu(x)$$

In particular, we shall say that $d \in C_b(X, \mathcal{A})$ is a **density** whenever it holds that

$$\int_X d^*(x)\, d(x)\, d\mu(x) = I.$$

Also recall that to each density d there is associated a positive unital map, $\phi : C_b(X, \mathcal{A}) \to \mathcal{A}$, defined in the following way

$$\phi(g) = \int_X d^*(x)\, g(x)\, d(x)\; d\mu(x).$$

We can now state the following non-commutative version of the Information inequality.

Proposition 6.1 (Information inequality). *Let X be a locally compact Hausdorff space, μ a Borel measure on X, $\mathcal{A}$ a unital C^*-algebra, and $a, b \in C_b(X, \mathcal{A})$ densities such that $a(x), b(x) \in \mathrm{GL}(\mathcal{A})$ for every $x \in X$. Then, it holds that*

$$\int_X a^* \,\log(a^{*-1}\, b^*\, b\, a^{-1})\, a\; d\mu \leq 0.$$

Proof. By Theorem 2.11, and using the fact that the function log is operator concave we obtain

$$\begin{aligned}\int_X a^* \,\log(a^{*-1}\, b^*\, b\, a^{-1})\, a\; d\mu &\leq \log\left[\int_X a^*\, a^{*-1}\, b^*\, b\, a^{-1}\, a\; d\mu\right] \\ &= \log\left[\int_X b^*\, b\; d\mu\right] \leq \log(I) = 0.\end{aligned}$$

■

Returning to the general case, the following is a non-commutative version of Liapounov's inequality.

Proposition 6.2 (Liapunov's inequality). *Let $\mathcal{A}$ a unital C^*-algebra and $\mathcal{H}$ a Hilbert space. Given a positive contractive map $\phi : \mathcal{A} \to L(\mathcal{H})$ and $1 \leq r \leq s$ it holds that*

$$\phi(a^r)^{1/r} \leq \phi(a^s)^{1/s} \qquad \forall\, a \in \mathcal{A}^+.$$

Proof. Let $b = a^r$ and $t = s/r$. Then, as the function $f(x) = x^{1/t}$ is operator concave

$$\phi(b^t)^{1/t} \geq \phi(b).$$

So we obtain

$$\phi(a^s)^{r/s} \geq \phi(a^r)\,,$$

and using that the function $f(x) = x^{1/r}$ is operator monotone it holds that

$$\phi(a^s)^{1/s} \geq \phi(a^r)^{1/r}.$$

■

Finally let us prove the following Hōlder's type inequality whose matrix version were proved by T. Ando and F. Hiai in [1]

Proposition 6.3. *Let $\mathcal{A}$ be a finite factor and $1 < p, q < \infty$ such that $1/p + 1/q = 1$. If $c, d \in \mathcal{A}^+$ and $c^q + d^q \leq I$, then*

$$\mathrm{tr}(ca + db) \leq \Big(\,\mathrm{tr}(a^p + b^p)\Big)^{1/p}$$

for every $a, b \in \mathcal{A}^+$

Proof. When $c, d \geq 0$ and $c^q + d^q \leq I$, if $c_1 = (I - d^q)^{1/q}$, then $c_1^q + d^q = I$ and $\operatorname{tr}(ca) \leq \operatorname{tr}(c_1 a)$ because $c \leq c_1$. Hence we may assume that $c^q + d^q = I$ so that c, d commute.

On the other hand, by Lemma 3.9, there exist a measure of probability μ defined in $\sigma(c)$ and a positive unital map $\Psi : \mathcal{A} \to L^\infty(\sigma(c), \mu)$ such that

i. $\Psi(f(c)) = f$ for every $f \in C(\sigma(c))$.

ii. $\operatorname{tr}(x) = \displaystyle\int_{\sigma(c)} \Psi(x)(t)\, d\mu(t)$ for every $x \in \mathcal{B}$.

Then

$$\operatorname{tr}(ca + db) = \int_{\sigma(c)} t\Psi(a)(t) + (1 - t^q)^{1/q}\Psi(b)(t)\, d\mu(t)$$

using Hōlder inequality for each $t \in \sigma(c)$

$$\begin{aligned}
&= \int_{\sigma(c)} \Big(t^q + (1 - t^q)\Big)^{1/q} \Big(\Psi(a)^p(t) + \Psi(b)^p(t)\Big)^{1/p} d\mu(t) \\
&= \int_{\sigma(c)} \Big(\Psi(a)^p(t) + \Psi(b)^p(t)\Big)^{1/p} d\mu(t) \\
&\leq \left(\int_{\sigma(c)} \Psi(a^p)(t) + \Psi(b^p)(t)\, d\mu(t)\right)^{1/p} \\
&= \Big(\operatorname{tr}(a^p + b^p)\Big)^{1/p}.
\end{aligned}$$

where the inequality follows from the standard version of Jensen's inequality for integrals and Theorem 3.6. ■

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