THE COMPLETE SOLUTION TO THE WEAK-TYPE BOUNDEDNESS OF HARDY-LITTLEWOOD MAXIMAL OPERATOR ON WEIGHTED LORENTZ SPACES

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Abstract. The main goal of this paper is to provide a complete characterization of the weak-type boundedness of the Hardy-Littlewood maximal operator, M, on weighted Lorentz spaces $\Lambda^p_u(w)$, whenever $p > 1$. This solves a problem left open in [5]. Moreover, with this result, we complete the program of unifying the study of the boundedness of M on weighted Lebesgue spaces and classical Lorentz spaces, which was initiated in the aforementioned monograph.

1. Introduction

The classical Hardy-Littlewood maximal operator M, is defined by

$$Mf(x) = \sup_{x\in Q} \frac{1}{|Q|}\int_Q |f(y)|dy,$$

where the supremum is taken over all cubes Q containing $x \in \mathbb{R}^d$. This operator is related with several problems in analysis, and in some sense it controls the boundedness of many other operators. For these reasons, it has been widely studied in different settings.

In 1972, Muckenhoupt [9] gave the complete characterization of the boundedness of M on weighted Lebesgue spaces $L^p(u)$, defined by the set of all Lebesgue measurable functions f such that

$$||f||_{L^p(u)} := \left(\int_{\mathbb{R}^d} |f(x)|^p u(x)dx\right)^{1/p} < \infty,$$

where u is a positive and locally integrable function on $\mathbb{R}^d$ (we call it weight). For $p > 1$, the characterization was given in terms of the so called A_p class of weights [9]; that is

$$\sup_Q \left(\frac{1}{|Q|}\int_Q u(x)dx\right)\left(\frac{1}{|Q|}\int_Q u^{-1/(p-1)}(x)dx\right)^{p-1} < \infty,$$

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where the supremum is considered over all cubes Q of $\mathbb{R}^d$. It was also proved in [9] that, if $p > 1$,

$$M : L^p(u) \to L^p(u) \iff M : L^p(u) \to L^{p,\infty}(u) \iff u \in A_p,$$

where the weak-type space $L^{p,\infty}(u)$ is defined through the quasi norm

$$||f||_{L^{p,\infty}(u)} := \sup_{t>0} t\, u(\{x \in \mathbb{R}^d : |f(x)| > t\})^{\frac{1}{p}} < \infty,$$

and $u(E) = \displaystyle\int_E u(x)dx$, for any measurable set $E \subset \mathbb{R}^d$. If $p = 1$ the only case that makes sense is the weak-type boundedness

$$M : L^1(u) \to L^{1,\infty}(u),$$

characterized by the A_1 class of weights defined by

$$Mu(x) \le Cu(x), \quad \text{a.e. } x \in \mathbb{R}^d.$$

If $p < 1$ there are no weights so that $M : L^p(u) \to L^{p,\infty}(u)$ is bounded [5].

Later on, the development of the interpolation theory motivated the study of the boundedness of M on the so called Lorentz spaces. The (classical) Lorentz space $\Lambda^p(w)$ is defined as the class of all functions satisfying

$$||f||_{\Lambda^p(w)} := \left(\int_0^\infty p\, t^{p-1} W(|\{x \in \mathbb{R}^d : |f(x)| > t\}|)dt\right)^{1/p} < \infty,$$

where w is a weight in $\mathbb{R}^+$, $W(t) = \displaystyle\int_0^t w(s)ds$ and $|E|$ denotes the Lebesgue measure of E. The weak-type Lorentz space $\Lambda^{p,\infty}(w)$ is defined by the following quasi norm

$$||f||_{\Lambda^{p,\infty}(w)} := \sup_{t>0} t\, W(|\{x \in \mathbb{R}^d : |f(x)| > t\}|)^{\frac{1}{p}} < \infty.$$

Ariño and Muckenhoupt characterized in [2] the boundedness of M on $\Lambda^p(w)$. The key idea to study the boundedness of M on these spaces is the existence of $c, C > 0$ such that

$$cPf^*(t) \le (Mf)^*(t) \le CPf^*(t). \tag{1.1}$$

In these inequalities f^* is the decreasing rearrangement of f, which is defined in $[0, +\infty)$ by

$$f^*(t) = \inf\left\{s > 0 : |\{x \in \mathbb{R}^d : |f(x)| > s\}| \le t\right\},$$

and P is the Hardy operator defined by

$$Pf(t) = \frac{1}{t}\int_0^t f(s)ds, \quad t > 0, \tag{1.2}$$

(see [3] for more details). Consequently, the boundedness of M on $\Lambda^p(w)$ is equivalent to the boundedness of P on the cone of decreasing functions of $L^p(w)$. Given $p>0$, the class of weights satisfying

$$M : \Lambda^p(w) \to \Lambda^p(w)$$

is known as B_p, and it can be proved [2] that $w \in B_p$ if and only if

$$r^p \int_r^\infty \frac{w(t)}{t^p}\, dt \leq C \int_0^r w(s)ds, \quad \text{for every } \ r>0.$$

Moreover, for every $p>0$, the condition $B_{p,\infty}$ characterizes the boundedness

$$M : \Lambda^p(w) \longrightarrow \Lambda^{p,\infty}(w),$$

where for $p>1$, $B_{p,\infty}=B_p$, and for $p\leq 1$ a weight $w\in B_{p,\infty}$ if and only if

$$\frac{W(t)}{t^p} \leq C\frac{W(r)}{r^p}, \quad \text{for every } 0<r<t<\infty.$$

These classes of weights have been well studied in [2, 5, 10].

Some analogies between the boundedness properties of M in $L^p(u)$ and in $\Lambda^p(w)$ suggested that there might be a unifying theory behind. A natural framework for this unification is provided by the weighted Lorentz spaces defined by Lorentz in [7, 8]. Given u, a weight in $\mathbb{R}^d$ and given a weight w in $\mathbb{R}^+$,

$$\Lambda^p_u(w) = \left\{ f\in\mathcal{M} : ||f||^p_{\Lambda^p_u(w)} := \int_0^\infty pt^{p-1}W(u(\{x\in\mathbb{R}^d : |f(x)|>t\}))dt<\infty \right\},$$

where $\mathcal{M}=\mathcal{M}(\mathbb{R}^d)$ is the set of Lebesgue measurable functions on $\mathbb{R}^d$, and the weak-type Lorentz space is defined as follows

$$\Lambda^{p,\infty}_u(w) = \left\{ f\in\mathcal{M} : ||f||^p_{\Lambda^{p,\infty}_u(w)} := \sup_{t>0} tW^{1/p}(u(\{x\in\mathbb{R}^d : |f(x)|>t\}))<\infty \right\}.$$

Note that these spaces include, as particular examples, the weighted Lebesgue spaces $L^p(u)$, $L^{p,\infty}(u)$ (when $w=1$) and the Lorentz spaces $\Lambda^p(w)$, $\Lambda^{p,\infty}(w)$ (when $u=1$).

In [5] the strong-type boundedness

$$M : \Lambda^p_u(w) \to \Lambda^p_u(w) \tag{1.3}$$

was completely characterized as follows.

Theorem 1.1 ([5], Theorem 3.3.5)**.** *For every* $0<p<\infty$*,*

$$M : \Lambda^p_u(w) \longrightarrow \Lambda^p_u(w)$$

is bounded if and only if there exists $q \in (0,p)$ such that, for every finite family of cubes $(Q_j)_{j=1}^J$, and every family of measurable sets $(S_j)_{j=1}^J$, with $S_j \subset Q_j$, for every j, we have that

$$\frac{W\left(u\left(\bigcup_{j=1}^J Q_j\right)\right)}{W\left(u\left(\bigcup_{j=1}^J S_j\right)\right)} \leq C \max_{1\leq j\leq J} \left(\frac{|Q_j|}{|S_j|}\right)^q. \tag{1.4}$$

for some universal positive constant C depending only on p and the dimension.

It is easy to see that condition (1.4) recovers $u \in A_p$ if $w = 1$, and $w \in B_p$ if $u = 1$. Later on, Lerner and Pérez found in [6] other equivalent conditions to the strong boundedness of M in $\Lambda_u^p(w)$ in terms of the so called local maximal operator.

In [5], the weak-type boundedness of M was also characterized for $p \leq 1$. In this case, the solution is given by condition (1.4), but with the exponent p instead of q. However, the weak-type boundedness

$$M : \Lambda_u^p(w) \to \Lambda_u^{p,\infty}(w) \tag{1.5}$$

remained open for $p > 1$. The main result in this paper is the following theorem that completely solves this problem.

Theorem 1.2. *If $p > 1$, then*

$$M : \Lambda_u^p(w) \longrightarrow \Lambda_u^{p,\infty}(w)$$

is bounded if and only if (1.4) holds. In particular,

$$M : \Lambda_u^p(w) \to \Lambda_u^{p,\infty}(w) \text{ is bounded} \iff M : \Lambda_u^p(w) \to \Lambda_u^p(w) \text{ is bounded.}$$

Finally, we have to mention that, if $d = 1$, Theorem 1.2 was proved in [1], and the proof uses the explicit construction of a function, which together with the weak-type boundedness lead to the geometric condition (1.4). Even though this paper is inspired on [1], we have to use a different approach, since the same method cannot be extended to the multi-dimensional case.

Notation. As usual, we shall use the symbol $A \lesssim B$ to indicate that there exists a universal constant C, independent of all important parameters, such that $A \leq CB$. Also $A \approx B$ will indicate that $A \lesssim B$ and $B \lesssim A$. It is known that the space $\Lambda_u^p(w)$ is a quasi-normed space if and only if $w \in \Delta_2$ (see [4]); that is,

$$W(2r) \lesssim W(r).$$

This condition will be assumed all over the paper.

2. Proof of the main result

This section is devoted to the proof of Theorem 1.2. In some sense, the strategy of the proof combines ideas of [10] and [1]. We begin with the following two lemmas.

Lemma 2.1. *Let us assume that*

$$M : \Lambda_u^p(w) \to \Lambda_u^{p,\infty}(w) \tag{2.1}$$

is bounded. Then, for every $0 < \lambda < 1$ *and every Borel set* $E \subset \mathbb{R}^d$*,*

$$\| \chi_{\{M\chi_E > \lambda\}} \; M\chi_E \|_{\Lambda_u^p(w)}^p \lesssim \Big(1 + \log \frac{1}{\lambda}\Big) \; \|\chi_E\|_{\Lambda_u^p(w)}^p. \tag{2.2}$$

Proof. Fix $0 < \lambda < 1$. Then

$$\begin{aligned} \| \chi_{\{M\chi_E > \lambda\}} \; M\chi_E \|_{\Lambda_u^p(w)}^p &= \int_0^\lambda pt^{p-1} W(u(\{x : \chi_{\{M\chi_E > \lambda\}}(x) \; M\chi_E(x) > t\}))dt \\ &\quad + \int_\lambda^1 pt^{p-1} W(u(\{x : \chi_{\{M\chi_E > \lambda\}}(x) \; M\chi_E(x) > t\}))dt \\ &= I + II. \end{aligned}$$

On the one hand, note that for $t \leq \lambda$ we have that

$$\{x : \chi_{\{M\chi_E > \lambda\}}(x) \; M\chi_E(x) > t\} = \{x : M\chi_E(x) > \lambda\}.$$

Hence, by (2.1),

$$\begin{aligned} I &= \int_0^\lambda pt^{p-1} W\big(u(\{M\chi_E > \lambda\})\big)dt = \lambda^p W(u(\{M\chi_E > \lambda\})) \\ &\leq ||M\chi_E||_{\Lambda_u^{p,\infty}(w)}^p \lesssim ||\chi_E||_{\Lambda_u^p(w)}^p. \end{aligned}$$

On the other hand,

$$\begin{aligned} II &\leq \int_\lambda^1 pt^{p-1} W\big(u(\{M\chi_E > t\})\big)dt = p\int_\lambda^1 t^p W\big(u(\{M\chi_E > t\})\big)\frac{dt}{t} \\ &\lesssim \int_\lambda^1 ||\chi_E||_{\Lambda_u^p(w)}^p \frac{dt}{t} = \log\frac{1}{\lambda} ||\chi_E||_{\Lambda_u^p(w)}^p, \end{aligned}$$

and the result follows. ∎

The proof of the following lemma is motived by a result in [6]. It provides the extra decay that we shall need to go from the weak-type to the strong-type boundedness.

Lemma 2.2. *For any* $0 < \lambda < 1$ *and any Borel subset* $E \subset \mathbb{R}^d$*, it holds that*

$$\chi_{\{M\chi_E > \lambda\}}(x) \lesssim \frac{1}{\lambda(1 - \log \lambda)} M\big(\chi_{\{M\chi_E > \lambda\}} \; M\chi_E\big)(x) \quad (x \in \mathbb{R}^d). \tag{2.3}$$

Proof. Fix a Borel set $E \subset \mathbb{R}^d$, $\lambda \in (0,1)$ and $x \in \mathbb{R}^d$ such that $M\chi_E(x) > \lambda$. Then there exists a cube Q so that $x \in Q$ and

$$\lambda < \frac{|E \cap Q|}{|Q|}.$$

Since the function $\phi(x) = x\left(1 + \log \frac{1}{x}\right)$ is increasing in $(0,1)$, we have that

$$\begin{aligned}\lambda\Big(1 + \log \frac{1}{\lambda}\Big) = \phi(\lambda) \le \phi\left(\frac{|E \cap Q|}{|Q|}\right) &= \frac{1}{|Q|}\int_0^{|Q|} \frac{\min(t, |E \cap Q|)}{t} dt \\ &= \frac{1}{|Q|}\int_0^{|Q|} P(\chi_{E\cap Q})^*(t)dt,\end{aligned}$$

where P denotes the Hardy operator defined by (1.2). Hence, by (1.1), we obtain

$$\begin{aligned}\lambda\Big(1 + \log \frac{1}{\lambda}\Big) &\approx \frac{1}{|Q|}\int_0^{|Q|} (M\chi_{E\cap Q})^*(t)dt \\ &\le \frac{1}{|Q|}\int_0^{|Q|} \big(\chi_{3Q} M\chi_{E\cap Q}\big)^*(t)dt + \frac{1}{|Q|}\int_0^{|Q|} \big(\chi_{(3Q)^c} M\chi_{E\cap Q}\big)^*(t)dt \\ &\le \frac{1}{|Q|}\int_{3Q} M\chi_{E\cap Q}(y)dy + \frac{1}{|Q|}\int_0^{|Q|} \big(\chi_{(3Q)^c} M\chi_{E\cap Q}\big)^*(t)dt.\end{aligned}$$

Now, the standard estimate

$$\chi_{(3Q)^c}(z) M\chi_{E\cap Q}(z) \lesssim \inf_{y\in Q} M\chi_{E\cap Q}(y) \le \inf_{y \in Q} M\chi_E(y), \quad z \in \mathbb{R}^d$$

implies that

$$\begin{aligned}\lambda\Big(1 + \log \frac{1}{\lambda}\Big) &\lesssim \frac{1}{|Q|}\int_{3Q} M\chi_E(y)dy + \frac{1}{|Q|}\int_Q M\chi_E(y)dy \\ &\lesssim M(M\chi_E)(x) \le M(\chi_{\{M\chi_E > \lambda\}} M\chi_E)(x) + M(\chi_{\{M\chi_E \le \lambda\}} M\chi_E)(x) \\ &\le M(\chi_{\{M\chi_E > \lambda\}} M\chi_E)(x) + \lambda.\end{aligned}$$

Finally, since $\{M\chi_E > \lambda\}$ is an open set, we obviously have that

$$\lambda \le M(\chi_{\{M\chi_E > \lambda\}} M\chi_E)(x)$$

and hence the result follows. ■

Equivalently, we can write the inequality (2.3) as an inclusion of level sets in the following way.

Corollary 2.3. *There exists $c > 0$ such that, for every Borel subset $E \subset \mathbb{R}^d$ and every $0 < \lambda < 1$,*

$$\{M\chi_E > \lambda\} \subseteq \{M(\,\chi_{\{M\chi_E > \lambda\}}\; M\chi_E\,) > c\,\lambda(1 - \log \lambda)\}.$$

Now, in order to proceed to the proof of our main theorem, we need to recall the following result proved in [5] (see Theorems 3.3.3 and 3.3.5).

Proposition 2.4. *If there exists $0 < r < \infty$ such that*

$$\int_0^1 \lambda^{r-1} W^{r/p}\big(u(\{M\chi_E > \lambda\})\big) d\lambda \lesssim ||\chi_E||^p_{\Lambda^p_u(w)},$$

then (1.4) *holds.*

Proof of Theorem 1.2. Let $0 < \lambda < 1$ and $f = \chi_{\{M\chi_E > \lambda\}}\, M\chi_E$. By Corollary 2.3, we have that

$$W\big(u(\{\, M\chi_E > \lambda\})\big) \le W\big(u(\{\, Mf > c\lambda(1 - \log\lambda)\})\big),$$

and using the weak-type boundedness of M, it holds that

$$W\big(u(\{Mf > c\lambda(1 - \log\lambda)\})\big) \lesssim \frac{1}{\lambda^p(1 - \log\lambda)^p} \|f\|^p_{\Lambda^p_u(w)}.$$

By (2.2) we obtain that

$$W\big(u(\{\, M\chi_E > \lambda\})\big) \lesssim \frac{1}{\lambda^p(1 - \log\lambda)^{p-1}} \|\chi_E\|^p_{\Lambda^p_u(w)},$$

and hence, if we take $r > 0$ such that $p/(p-1) < r < \infty$, we have that

$$\int_0^1 \lambda^{r-1} W^{r/p}\big(u(\{M\chi_E > \lambda\})\big) d\lambda \lesssim ||\chi_E||^p_{\Lambda^p_u(w)}$$

and the result follows by Proposition 2.4. ■

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