FRAME SEQUENCES AND REPRESENTATIONS FOR SAMPLABLE RANDOM PROCESSES.

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Abstract. In this work we characterize random processes which can be linearly determined using a frame sequence of random variables. In particular, these could be the discrete samples of a continuous time process. We study the stable representation of continuous time processes by means of discrete samples or measurements of the original process. Finally, we study how these representations can be applied to reduce the effects of reconstructing a random signal from samples corrupted by additive noise.

1. Introduction

One of the most relevant mathematical problems in communication engineering, and other fields of applications, is the reconstruction of a signal from sampled data. Usually these samples are the values of a function, its derivatives at certain points or other linear operations applied to the function. The fundamental result in sampling theory is the Whittaker-Shannon-Kotelnikov (WSK) theorem [26], which says that if $f \in L^2(\mathbb{R})$ is such that $\hat{f}$ is concentrated in a finite interval $[-B, B]$, then

$$f(t) = \sum_{n\in\mathbb{Z}} \frac{\sin(Bt - \pi n)}{Bt - \pi n} f(\frac{\pi n}{B}) \,, \tag{1.1}$$

where the convergence is uniform and in the L^2-norm. One generalization of this result which, in part, motivates the present work is the Kramer sampling theorem [13] [26] for $L^2(I)$ functions:

Theorem 1.1. *Let $k(x,t)$ be a function, defined for all t in a suitable D of $\mathbb{R}$ such that, as a function of x, $k(\,.\,,t) \in L^2(I)$ for every $t \in D$, where I is an interval of the real line. Assume that there exists a sequence of distinct real numbers $\{t_n\}_{n\in\mathbb{Z}} \subset D$, such that $\{k(\,.\,,t_n)\}_{n\in\mathbb{Z}}$ is a complete orthogonal sequence of functions of $L^2(I)$. Then for any f of the form:*

$$f(t) = \int_I g(x)k(x,t)dx \,, \tag{1.2}$$

where $g \in L^2(I)$, we have:

$$f(t) = \lim_{N\to\infty} \sum_{|n|\le N} f(t_n)S_n(t) \,,$$

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with $S_n(t) = \frac{\langle k(.,t), k(.,t_n)\rangle_{L^2(I)}}{\|k(.,t_n)\|^2_{L^2(I)}}$.

On important fact of the generalization of the deterministic WSK theorem given by Kramer, is that it allows to treat the case of non uniform samples. It is also possible to prove a converse of this result [9]. We study similar conditions for second order random process or, when possible, in the general setting of an arbitrary (complex) Hilbert space H, considering a general indexed set, or "process": $\{x_t\}_{t\in\mathfrak{T}} \subset H$, indexed by an arbitrary set $\mathfrak{T}$. Kramer's result is strongly related to orthonormal bases, but as noted in [8] and [9], what is really needed is a stability condition, and frame sequences provide an appropriate framework for this. Hilbert space bases provide some natural representations of certain random process [17] [6]. Moreover these representations are very useful in the analysis of various problems in statistical communication theory, such as detection an estimation [21] [3]. Since frame sequences are a natural generalization of the concept of a basis, we shall study conditions for a countable set of samples to be a frame sequence of the Hilbert space spanned by the whole process. The results presented here are a natural generalization of [19], where the case of Riesz Bases was studied. As it will be shown, the redundancy present in frame sequences provides a good tool to reduce the effect of additive noise. We shall study different equivalences between several representations [20] for samplable processes. In this context, a samplable process, will mean a continuous time, or spatial process, which can be completely linearly determined by a series expansion, using a set of countable samples or measurements of the original process. To motivate this discussion, let us recall the Whittaker-Shannon-Kotelnikov (WSK) sampling theorem for *wide sense stationary processes* (*w.s.s.* processes):

Theorem 1.2. [22] *Let* $\mathscr{X} = \{x_t\}_{t\in\mathbb{R}}$ *be a w.s.s. random process defined over a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$*, such that its spectral measure is concentrated in a finite interval* $(-B, B)$*, then*

$$x_t = \sum_{n\in\mathbb{Z}} \frac{sin(Bt-\pi n)}{Bt-\pi n} x_{\frac{\pi n}{B}} \,. \tag{1.3}$$

Where the convergence is in the $L^2(\Omega, \mathcal{F}, \mathbb{P})$*-norm.*

In particular, note that, eq. 1.3 implies that the process is completely linearly determined by its samples, i.e. $\overline{\text{span}}\{x_k\}_{k\in\mathbb{Z}} = \overline{\text{span}}\{x_t\}_{t\in\mathbb{R}}$. Lloyd [15] gave necessary and sufficient conditions, in terms of the spectral measure, for a w.s.s. process to be completely linearly determined by its samples. This result can be extended for some non stationary processes [14]. However, this does not imply that one obtains a convergent series expansion for x_t. Convergent expansions as 1.3 can be obtained using bases or frames. The study of conditions for a w.s.s. process to have a basis or minimal system goes back to Kolmogorov [22] [24]. However, all these references, as in the case of the WSK theorem, deal with equidistant samples, and are mostly stated for w.s.s. processes. From the stochastic version of the WSK theorem, under additional conditions, we obtain a Riesz basis or a frame sequence of samples which span the Hilbert space spanned by the whole process [18]. The representation of signals using frames has many practical applications [7], in particular, dealing with additive noise. Related representations for random processes were studied in [11], where useful results were obtained for encoding.

Recalling Kramer's result [13] for $L^2(I)$ functions and its converse [9], it is interesting to find analogous conditions or results for random processes, since it would allow to treat the case of non uniform samples, among other additional properties. Noting that 1.3 can be written as

$$x_t = \sum_{n\in\mathbb{Z}} x_{t_n} S_n(t) \ ,$$

with $t_n = \frac{\pi n}{B}$, we would like to study similar conditions to those of theorem 1.1. In this work we obtain analogous results for finite variance processes. We shall see here that these results, as they generally rely only on second order properties of the processes, most of them can be stated for any indexed set $\{x_t\}_{t\in\mathfrak{T}} \subset H$, where H is an arbitrary Hilbert space. Here, as a convention, we shall refer unambiguously to such a set as a "process". By means of the reproducing kernel Hilbert space [25] associated to the process, we will give an analogous to Kramer's result, and its converse (theorems 3.1, 3.2, 3.3 respectively), for these processes, which naturally appear to be the integral transform of an appropriate kernel function with respect a certain type of vector measure, indeed a *countably additive orthogonally scattered-c.a.o.s.* measure. In section 2.1.3 we give a brief summary of the basic definitions and results for c.a.o.s. measures used in this work. Finally, in section 4 we discuss some applications to the reduction of the effects of additive noise.

2. Preliminary .

In order to make this work self contained, we shall fix some notation, review some definitions and results and prove, if necessary, some auxiliary results which are going to be used in the sequel.

2.1. **Some definitions.** Let H be a Hilbert space, with an inner product $\langle\, .\, ,\, .\, \rangle_H$. In the following, if U is any subset of H, we will denote lin U to the subspace of all the finite linear combinations of elements of U, and $H(U) = \overline{\text{span}}\, U$, the closed linear span of U in H. We shall study some congruences and related properties for the closed linear span $H(\mathscr{X})$ of indexed sets $\mathscr{X} = \{x_t\}_{t\in\mathfrak{T}} \subset H$, where the set of indexes $\mathfrak{T}$, can be considered uncountable as occurs in some sampling problems. As our results are, in particular, aimed at finite variance random processes, we shall consider, sometimes, the case when $H = L^2(\Omega, \mathcal{F}, \mathbb{P})$, where $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space. If x is an integrable random variable, we denote $\mathbb{E}(x) = \int_\Omega x(\omega) d\mathbb{P}(\omega)$.
We recall that in the case $H = L^2(\Omega, \mathcal{F}, \mathbb{P})$ the correlation function is defined by $R(t,t') = \mathbb{E}(x_t\, \overline{x_{t'}})$. This positive definite function R defines a *reproducing kernel Hilbert space (RKHS)* [20] [25], which we will denote $H(R)$. However, we note that the same occurs in an arbitrary Hilbert space, if we set $R(t,t') = \langle x_t, x_{t'} \rangle_H$.
We will study series representations and sequences, so to simplify we will sometimes write $\sum_n x_n$ or $(a_n)_n$ respectively , without mention of the set of indexes when this is clear form the context.

2.1.1. *Conjugate vectors and RKHS's.* Let H be an arbitrary Hilbert space, and let $\{v_l\}_{l\in\mathbb{J}}$ any orthonormal basis of H. Such a basis always exists, however it may be uncountable, and then the dimension of H, is defined as $\dim(H) = card(\mathbb{J})$. If $x \in H$, then we can write $x = \sum_{l\in\mathbb{J}} \langle x, v_l \rangle_H\, v_l$. From this we define the *conjugate of x* as:

$$\overline{x} = \sum_{l\in\mathbb{J}} \overline{\langle x, v_l\rangle}_H v_l \, .$$

Then, for $x, x' \in H$, from Parseval's identity :

$$\langle x, x'\rangle_H = \sum_{l\in\mathbb{J}} \langle x, v_l\rangle_H \overline{\langle x', v_l\rangle}_H = \langle \overline{x}', \overline{x}\rangle_H \, . \tag{2.1}$$

From this, it is immediate that $R(t,s) = \langle x_t, x_s\rangle_H = \langle \overline{x}_s, \overline{x}_t\rangle_H$. So the conjugate process $\overline{x}_t$ gives the same RKHS $H(R)$. On the other hand, recalling the theory of reproducing kernels, for every $f \in H(R)$, there exists $y \in H(\mathscr{X})$ such that $f(t) = \langle y, \overline{x}_t\rangle_H$. This can be written as $f(t) = \widetilde{J}(y)(t)$, where

$$\underset{Y \longmapsto \widetilde{J}(y)(t) = \langle y, \overline{x}_t\rangle_H}{\widetilde{J} : H(\mathscr{X}) \longrightarrow H(R)} \, , \tag{2.2}$$

Moreover the reproducing kernel Hilbert space $H(R)$ coincides with $Ran(\widetilde{J})$, equipped with the norm [25]: $\|v\|_{H(R)} = \inf\left\{ \|w\|_{H(\mathscr{X})} : \widetilde{J}(w) = v \right\}$. From these facts, given $z \in H(\mathscr{X})$, we have that:

$$\langle x_t, z\rangle_H = \langle \overline{z}, \overline{x}_t\rangle_H = \widetilde{J}(\overline{z})(t) \, . \tag{2.3}$$

Is clear, that $\widetilde{J}$ give up to an isometry between $H(R)$ and $H(\mathscr{X})$.

2.1.2. *Frames and Riesz bases.* We are interested in some frame sequences and Riesz bases [7] for $H(\mathscr{X}) = \overline{\text{span}}\mathscr{X}$ and some of their properties.

Definition 1. A sequence $\{f_n\}_{n\in\mathbb{N}} \subset H$ is a frame (or *frame sequence*) for H if there exists constants $A, B > 0$ such that

$$A \|f\|^2 \le \sum_{n=1}^{\infty} |\langle f, f_n\rangle|^2 \le B \|f\|^2 \, .$$

In particular, if a frame sequence is also a basis, or a minimal system, then it is also a Riesz basis:

Definition 2. *A Riesz basis* for H, a Hilbert space, is a family of vectors $\{v_n\}_{n\in\mathbb{N}} \subset H$, such that $v_n = Te_n$, where $\{e_n\}_{n\in\mathbb{N}}$ is an orthonormal basis of H, and $T : H \longrightarrow H$ is a bounded bijective operator.

2.1.3. *Countably additive orthogonally scattered measures (c.a.o.s. measures).* As we shall give conditions in terms of some particular integral transforms, we shall recall the concept of c.a.o.s. measures over a pre-ring $\mathcal{P}$, and the concept of an integral with respect to that measure, following [16]. In particular, σ-algebras, σ-rings and rings are pre-rings. Moreover, here we will only need the following case: Let $(U, \mathcal{R}, \mu)$ be a measure space, with μ a non-negative measure on a σ-algebra or σ-ring $\mathcal{R}$, then we have the sub-ring (and then a pre-ring) of $\mathcal{R}$: $\mathcal{R}_\mu = \{A \in \mathcal{R} : \mu(A) < \infty\}$.

Definition 3. Let $\mathcal{P}$ be a pre-ring and H a (complex) Hilbert space, we say that $M : \mathcal{P} \longrightarrow H$ is a countably additive orthogonally scattered (c.a.o.s.) measure if: (i) given a disjoint family $\{A_k\}_{k\in\mathbb{N}} \subset \mathcal{P}$ such that $\bigcup_{k=1}^{\infty} A_k \in \mathcal{P}$ then $\sum_{k=1}^{\infty} M(A_k)$

converges unconditionally to $M\left(\overset{\infty}{\underset{k=1}{\cup}} A_k\right)$. (ii) given disjoint $A, B \in \mathcal{P}$, then $M(A) \perp M(B)$.

2.1.4. *The integral with respect to a c.a.o.s. measure and isometry.* We have that $\mu(\,.\,) := \|M(\,.\,)\|^2$ defines a countably additive non negative measure on $\mathcal{P}$. Then, the measure μ is called the *control measure* of the c.a.o.s. measure M. Following [16] one can first define the integral $I(.) = \int\limits_U (.) dM \,:\, L^2(U, \mathcal{R}, \mu) \longrightarrow H$, for simple functions and then extend this notion to the whole $L^2(U, \mathcal{R}, \mu)$ space. Hence, $I(\,.\,)$ finally defines a bounded linear operator, and an isometry is established between $Ran(I) = \overline{\text{span}}\{M(A) : A \in \mathcal{R}_\mu\}$ and $L^2(U, \mathcal{R}, \mu)$, i.e. $\forall\, f \in L^2(U, \mathcal{R}, \mu)$:

$$\left\| \int\limits_U f\, dM \right\|_H^2 = \int\limits_U |f|^2 d\mu\,. \tag{2.4}$$

This is a generalization of the notion of stochastic integral with respect to an orthogonal random measure or with respect to a process with orthogonal increments, such as the Wiener process.

2.2. **A Representation Theorem.** We will prove later that every samplable process admits a representation as an stochastic integral of a certain kernel as in Kramer's result. First, we need the following theorem, which is a Hilbert space version of the generalization given in [4] of Karhunen's classic result for covariance functions [10]:

Theorem 2.1. *Let $\mathscr{X} = \{x_t\}_{t\in\mathfrak{T}} \subset H$, and let $(U, \mathcal{R}, \mu)$ be a measure space, with $\mathcal{G} = \{g(t,\,.\,)\}_{t\in\mathfrak{T}} \subset L^2(U, \mathcal{R}, \mu)$ such that,*

$$\langle x_t, x_{t'}\rangle_H = \int\limits_U g(t,\,.\,)\overline{g(t',\,.\,)} d\mu \;\forall\, t, t' \in \mathfrak{T}, \tag{2.5}$$

then there exists a c.a.o.s. measure $M : \mathcal{R}_\mu \longrightarrow H$ such that: i) $x_t = \int\limits_U g(t,\,.\,) dM$, ii) $\mu(A) = \|M(A)\|^2\ \forall\, A \in \mathcal{R}_\mu$.

Proof. Let $f_1 = \sum\limits_{k=1}^{n} \alpha_k g(t_k,\,.\,)$, $f_2 = \sum\limits_{k=1}^{m} \beta_k g(t'_k,\,.\,)$, $\psi_1 = \sum\limits_{k=1}^{n} \alpha_k x_{t_k}$ and $\psi_2 = \sum\limits_{k=1}^{m} \beta_k x_{t'_k}$. Then, since the inner product is bilinear:

$$\langle f_1, f_2\rangle_{L^2(\mu)} = \sum_{k=1}^{n}\sum_{j=1}^{m} \alpha_k \overline{\beta}_j \langle g(t_k,\,.\,) g(t'_j,\,.\,)\rangle_{L^2(\mu)}$$

$$= \sum_{k=1}^{n}\sum_{j=1}^{m} \alpha_k \overline{\beta}_j \langle x_{t_k}, x_{t'_j}\rangle_H = \langle \psi_1, \psi_2\rangle_H\,. \tag{2.6}$$

From eq. 2.6 and the polar identity, we obtain:

$$\|f_1 - f_2\|_{L^2(\mu)} = \|\psi_1 - \psi_2\|_H\,. \tag{2.7}$$

Then, if $H(\mathcal{G}) = \overline{\text{span}}\,\mathcal{G}$, by a density argument, one can define a linear operator $T : H(\mathcal{G}) \longrightarrow H(\mathscr{X})$, in a way such that for any $f = \sum\limits_{k=1}^{n} \alpha_k g(t_k,\,.\,) \in \text{lin}\,\mathcal{G} \subset H(\mathcal{G})$,

$Tf = \psi = \sum_{k=1}^{n} \alpha_k x_{t_k} \in \operatorname{lin} \mathscr{X}$. Indeed, T is bounded and unambiguously defined, from eq. 2.7, and moreover eq. 2.6 makes T an unitary operator.
Now we proceed to consider two cases:
Case I. If $H(\mathcal{G}) = L^2(U, \mathcal{R}, \mu)$, we define for $A \in R_\mu$: $M(A) = T(\mathbf{1}_A)$. For this M, one can define, as usual $\int_U f dM$ for $f \in L^2(U, \mathcal{R}, \mu)$, and in particular $\|M(A)\|_H^2 = \|T(\mathbf{1}_A)\|_H^2 = \|\mathbf{1}_A\|_{L^2(\mu)}^2 = \mu(A)$, and $\int_U f dM = T(f)$. Finally, from the definition of T:

$$x_t = T(g(t,\,.\,)) = \int_U g(t,\,.\,)\, dM$$

Case II. If $H(\mathcal{G})^\perp \neq \{0\}$ the idea is to consider $H(\mathscr{X})$ embedded in an appropriate larger Hilbert space H'. On one side, one has $L^2(U, \mathcal{R}, \mu) = H(\mathcal{G}) \bigoplus H(\mathcal{G})^\perp$ and one can take the direct sum of Hilbert spaces:
$H' = H(\mathscr{X}) \bigoplus H(\mathcal{G})^\perp$. That is take $H(\mathscr{X}) \times H(\mathcal{G})^\perp$ with the norm given for every $(u, v) \in H(\mathscr{X}) \times H(\mathcal{G})^\perp$ as $\|(u,v)\|^2_{H'} = \|u\|_H^2 + \|v\|^2_{L^2(\mu)}$. Now, take any set $\{k(t,\,.\,)\}_{t \in \mathfrak{T}'} \subset H(\mathcal{G})^\perp$ such that $\overline{\operatorname{span}}\{k(t,\,.\,)\}_{t\in\mathfrak{T}'} = H(\mathcal{G})^\perp$, with $\mathfrak{T}'$ any set of indexes verifying $\mathfrak{T}' \bigcap \mathfrak{T} = \oslash$, and then define for every $t \in \mathfrak{T}' \bigcup \mathfrak{T}$, two functions $\varphi(t)$ and x'_t taking values in $L^2(U, \mathcal{R}, \mu)$ and H' respectively, as:

$$\varphi(t,\,.\,) = g(t,\,.\,)\mathbf{1}_{\mathfrak{T}}(t) + k(t,\,.\,)\mathbf{1}_{\mathfrak{T}'}(t)\,,$$

and

$$x'_t = (\, x_t\,,\, 0\,)\mathbf{1}_{\mathfrak{T}}(t) + (\,0\,, k(t,\,.\,))\mathbf{1}_{\mathfrak{T}'}(t)\,.$$

From these definitions, we obtain:

$$\langle x'_t, x'_s\rangle_{H'} = \int_U \varphi(t,\,.\,)\overline{\varphi(s,\,.\,)} d\mu\,.$$

Then from the first case considered in the proof, there exists a c.a.o.s. measure: $M' : \mathcal{R}_\mu \longrightarrow H'$ such that $x'_t = \int_U \varphi(t,\,.\,) dM'$. Thus, if $\pi_{H(\mathscr{X})} : H(\mathscr{X}) \bigoplus H(\mathcal{G})^\perp \longrightarrow H(\mathscr{X})$ is the usual projection map, then we can take $M = \pi_{H(\mathscr{X})} \circ M' : \mathcal{R}_\mu \longrightarrow H$. It is easy to check that M defines also a c.a.o.s. measure which verifies, $\forall\, f \in L^2(U, \mathcal{R}, \mu)$: $\pi_{H(\mathscr{X})}\left(\int_U f\, dM'\right) = \int_U f\, dM$. So if $t \in \mathfrak{T}$:

$$x_t = \pi_{H(\mathscr{X})}(x'_t) = \pi_{H(\mathscr{X})}\left(\int_U \varphi(t,\,.\,) dM'\right)$$

$$= \pi_{H(\mathscr{X})}\left(\int_U g(t,\,.\,) dM'\right) = \int_U g(t,\,.\,) dM\,.$$

$\square$

3. A "stochastic" Kramer like theorem and its converse: Condition for the existence of a frame sequence of $H(\mathscr{X})$.

The Kramer sampling theorem [13] gives a method for obtaining orthogonal sampling formulas, for functions which are in the range of an appropriate integral operator. The WSK theorem for band limited functions is a particular case of this. In the random case, we can see that something similar happens if we consider processes which are the integral transform of a suitable random measure. Here, $\mathfrak{T}$ denotes a set of indexes, which we assume to be, in general, uncountable, as this is the case of interest in sampling problems. We shall see that every samplable process admits a representation as an stochastic integral of a certain kind of kernel.

Theorem 3.1. *Let $\{x_t\}_{t\in\mathfrak{T}} \subset H$ and a sequence $\{z_n\}_n \subset H(\mathscr{X})$, then the following are equivalent:*
i) $\{z_n\}_n$ is a frame of $H(\mathscr{X})$.
ii) Given $(U,\mathcal{R},\mu)$ a measure space, such that $\dim L^2(\mu) \geq \dim H(\mathscr{X})$. There exists a c.a.o.s. measure $M : \mathcal{R}_\mu \longrightarrow H$, with control measure μ, i.e, $\mu(\,.\,) = \|M(\,.\,)\|_H^2$, and a frame sequence $\{f_n\}_n$ (of $\overline{\operatorname{span}}\,\{f_n\}_n$) such that:

ii.a) $z_n = \int\limits_U f_n dM$.

ii.b)

$$x_t = \int\limits_U k(t,s)dM(s)\,,$$

with $k(t,s) = \sum\limits_{n\in\mathbb{Z}} S'_n(t) f_n(s)$ *and* $\{S'_n\}_n \subset H(R)$, $(S'_n(t))_n \in l^2(\mathbb{Z})$ $\forall\, t \in \mathfrak{T}$.

ii.c) $\overline{\operatorname{span}}\,\{k(t,.)\}_{t\in\,\mathfrak{T}} = \overline{\operatorname{span}}\,\{f_n\}_n$.

Proof. ((ii)$\Longrightarrow$(i)) If $\{f_n\}_{n\in\mathbb{Z}}$ is a frame of $\overline{\operatorname{span}}\,\{f_n\}_n$, then $z_n = \int\limits_U f_n dM$ is a frame of $\overline{\operatorname{span}}\,\{z_n\}_{n\in\mathbb{Z}}$ since the vector integral $f \longmapsto \int\limits_U f\,dM$ induces an isometry between $\overline{\operatorname{span}}\,\{z_n\}_{n\in\mathbb{Z}}$ and $\overline{\operatorname{span}}\,\{f_n\}_n = \overline{\operatorname{span}}\,\{k(t,.)\}_{t\in\,\mathfrak{T}}$. Thus we have to verify that $\overline{\operatorname{span}}\,\{z_n\}_{n\in\mathbb{Z}} = H(\mathscr{X})$, or equivalently that for all $t\in\mathfrak{T}$: $x_t \in \overline{\operatorname{span}}\,\{z_n\}_{n\in\mathbb{Z}}$. In fact,

$$(3.1)\qquad \left\| x_t - \sum_{|n|\leq N} S'_n(t) z_n \right\|_H^2 = \left\| k(t,\,.\,) - \sum_{|n|\leq N} S'_n(t) f_n \right\|_{L^2(\mu)}^2 \underset{N\to\infty}{\longrightarrow} 0\,,$$

with $\dim L^2(\mu) \geq \dim \overline{\operatorname{span}}\,\{f_n\}_n = \dim H(\mathscr{X})$.

((i)$\Longrightarrow$(ii)) If $\{z_n\}_{n\in\mathbb{Z}}$ is a frame of $H(\mathscr{X})$, then $H(\mathscr{X})$ is separable, and since $\dim L^2(\mu) \geq \dim H(\mathscr{X})$, then there exists a separable subspace $\mathcal{S} \subset L^2(\mu)$ and an isometric isomorphism $J : H(\mathscr{X}) \longrightarrow \mathcal{S}$, so that taking $f_n = J(z_n)$, then $\{f_n\}_{n\in\mathbb{Z}}$ is a frame of $\mathcal{S}$. If $x_t = \sum\limits_{n\in\mathbb{Z}} S'_n(t) z_n$ with $(S'_n(t))_n \in l^2(\mathbb{Z})$, then $J(x_t) =$

$\sum\limits_{n\in\mathbb{Z}} S'_n(t)J(z_n) = \sum\limits_{n\in\mathbb{Z}} S'_n(t)f_n$, so we can take $k(t,\,.\,) = J(x_t)$, and we have:

$$\langle x_t, x_{t'}\rangle_H = \int\limits_U J(x_t)\overline{J(x_{t'})}d\mu = \int\limits_U k(t,\,.\,)\overline{k(t',\,.\,)}d\mu\ ,$$

where $\{k(t,\,.\,)\}_{t\in\mathfrak{T}}$ is complete. Indeed, take $h \in \overline{\operatorname{span}}\,\{f_n\}_n = \mathcal{S}$ such that

$$\langle k(t,\,.\,), h\rangle_{L^2(\mu)} = 0\ \ \forall\, t\, \in\, \mathfrak{T}\,.$$

This is equivalent to $\langle x_t, J^{-1}(h)\rangle_H = 0$ for every $t \in \mathfrak{T}$. Then $J^{-1}(h) = 0$ a.s. and thus $h = 0$ a.e.-$[\mu]$. Finally, from the representation theorem 2.1, it follows that there exists a random measure $M : \mathcal{R}_\mu \longrightarrow H$, such that $x_t = \int\limits_U k(t,\,.\,)dM$, and $\|M(\,.\,)\|_H^2 = \mu(\,.\,)$.

□

Theorem 1.1, indirectly shows that the samplable functions belong to an appropriate RKHS, that is, to the range of the integral operator given by eq. 1.2. On the other hand, eq. 2.5, shows that, if we take $k(t,\,.\,)$ as $g(t,\,.\,)$, is equivalent to say that $H(R) = Ran(T)$, with $Tf(t) = \int\limits_U f\,k(t,\,.\,)d\mu$.

Taking in account these facts, the following result shows that there is a relation between the ordinary Kramer sampling theorem and its "stochastic" counterpart.

Lemma 3.1. *Let $\{x_t\}_{t\in\mathfrak{T}} \subset H$ and a sequence $\{z_n\}_n \subset H(\mathscr{X})$, then the following are equivalent:*
i) $\{z_n\}_{n\in\mathbb{Z}}$ is a frame for $H(\mathscr{X})$.
ii) If $(U, \mathcal{R}, \mu)$ is a measure space such that there exists a c.a.o.s. vector measure $M : \mathcal{R}_\mu \longrightarrow H$, for which, for some $\{k(t,\,.\,)\}_{t\in\mathfrak{t}} \subset L^2(\mu)$:

$$x_t = \int\limits_U k(t,s)dM(s)\,,$$

and $\mu(\,.\,) = \|M(\,.\,)\|_H{}^2$. Then: $H(R) = Ran(T)$, where $T : \overline{\operatorname{span}}\,\{k(t,.)\}_{t\in\mathfrak{T}} \longrightarrow H(R)$ is defined by:

$$(Tf)(t) = \int\limits_U k(t,s)f(s)d\mu(s)\,.$$

The kernel $k(t,\,.\,) \in L^2(\mu)$ admits the following expansion: $k(t,.) = \sum\limits_{n\in\mathbb{Z}} S_n(t)f_n$, with $\{f_n\}_{n\in\mathbb{Z}}$ a frame of $\overline{\operatorname{span}}\,\{k(t,.)\}_{t\in\mathfrak{T}}$, $\{S_n\}_n$ a frame of $H(R)$ and $z_n = \int\limits_U f_n dM$.

Proof. i) $\Longrightarrow$ ii) If $\{z_n\}_{n\in\mathbb{Z}}$ is a frame of $H(\mathscr{X})$ then $x_t = \sum\limits_{n\in\mathbb{Z}} \langle x_t, z'_n\rangle_H z_n$, with $\{z'_n\}_{n\in\mathbb{Z}}$ a dual frame of $\{z_n\}_{n\in\mathbb{Z}}$. As $\int\limits_U (\,.\,)\,dM\ :\ \overline{\operatorname{span}}\,\{k(t,.)\}_{t\in\mathfrak{T}} \longrightarrow H(\mathscr{X})$ is an isometry, then there exists a frame $\{f_n\}_{n\in\mathbb{Z}} \subset \overline{\operatorname{span}}\,\{k(t,.)\}_{t\in\mathfrak{T}}$ such that $z_n = \int\limits_U f_n dM$. Thus in a similar way to eq. 3.1, one obtains:

$$x_t = \sum_{n\in\mathbb{Z}} S_n(t)\int\limits_U f_n dM = \int\limits_U \left(\sum_{n\in\mathbb{Z}} S_n(t)f_n\right) dM\,.$$

From this $k(t,\,.\,) = \sum_{n\in\mathbb{Z}} S_n(t) f_n$. Finally, the fact that $H(R) = Ran(T)$, follows from $\langle x_t, x_s\rangle_H = \langle k(t,\,.\,), k(s,\,.\,)\rangle_{L^2(\mu)}$.
ii) $\Longrightarrow$ i) is immediate. □

Note that in theorem 3.1, $\{k(t,\,.\,)\}_{t\in\mathfrak{T}}$ and $L^2(\mu)$ respectively, are both *representations* of $\mathscr{X}$, in the sense given by Parzen in [20]:

Definition 4. A Hilbert space H' is a representation of a random process $\mathscr{X} = \{x_t\}_{t\in\mathfrak{T}}$ if H' is congruent to $H(\mathscr{X})$, i.e. there exists an isomorphism which preserves inner products.

A related notion is the following:

Definition 5. A family of vectors $\{v_t\}_{t\in\mathfrak{T}}$ in a Hilbert space H', is a representation of a random process $\mathscr{X} = \{x_t\}_{t\in\mathfrak{T}}$ if for every $s,t \in \mathfrak{T}$: $\langle v_t, v_s\rangle_{H'} = \langle x_t, x_s\rangle_H$.

As in [9] it is possible to give a converse of theorem 3.1 (theorem 3.2 below). Giving appropriate conditions on the sampling functions, it is possible to obtain a frame sequence of the whole space $H(\mathscr{X})$. In particular, the random process is linearly determined by its samples. In contrast to Garcia's result [9], the hypothesis on the signal of being the image of an integral transform can be dropped in the random case. So one may conjecture that there exists a larger class of processes with this property. However, we have seen in theorem 3.1, that if there exists a frame sequence of $H(\mathscr{X})$ then the process is the integral transform of an appropriate vector measure, by the representation theorem 2.1 .

Next, similarly to [19], with appropriate modifications, we can prove the converse converse of theorem 3.1:

Theorem 3.2. *Let $\mathfrak{T}$ be a set of indexes, generally non countable, and let $\mathscr{X} = \{x_t\}_{t\in\mathfrak{T}} \subset H$. Let $H(\mathscr{X})$ be the closed subspace spanned by $\mathscr{X}$. Given $\{z_n\}_{n\in\mathbb{Z}} \subset H(\mathscr{X})$, the following assertions are equivalent:*

i) $(\langle y, z_n\rangle_H)_n \in l^2(\mathbb{Z})$ for all $y \in H(\mathscr{X})$ and there exists a subset of functions $\{S_n\}_{n\in\mathbb{Z}} \in H(R)$ such that:

$$x_t = \sum_{n\in\mathbb{Z}} S_n(t) z_n\,,$$

and $(\langle f, S_n\rangle_{H(R)})_n \in l^2(\mathbb{Z})$ for all $f \in H(R)$.

ii) $\{z_n\}_{n\in\mathbb{Z}}$ is a frame of $H(\mathscr{X})$.

Proof. ($(ii) \Longrightarrow (i)$) There exists $\{z'_n\}_{n\in\mathbb{Z}} \subset H(\mathscr{X})$, a dual frame for $\{z_n\}_{n\in\mathbb{Z}}$, such that $S_n(t) = \langle x_t, z'_n\rangle_H$ and $x_t = \sum_{n\in\mathbb{Z}} S_n(t) z_n$. It is immediate that $(\langle y, z_n\rangle_H)_{n\in\mathbb{Z}} \in l^2(\mathbb{Z})$, for all $y \in H(\mathscr{X})$ and that $\widetilde{J}(\overline{z'}_n) = S_n \in H(R)$, from the definition of a frame sequence and the previously discussed properties of $H(R)$. On the other hand, from the congruence between $H(R)$ and $H(\mathscr{X})$:

$$\langle f, S_n\rangle_{H(R)} = \langle f, \widetilde{J}(\overline{z'}_n)\rangle_{H(R)} = \langle y, \overline{z'}_n\rangle_H = \overline{\langle \overline{y}, z'_n\rangle}_H,$$

but $(\langle \overline{y}, z'_n\rangle_H)_n \in l^2(\mathbb{Z})$ because $\{z'_n\}_{n\in\mathbb{Z}}$ is also a frame sequence, and then $(\langle f, S_n\rangle_{H(R)})_{n\in\mathbb{Z}}$.

$((i) \Longrightarrow (ii))$We shall see that $\{z_n\}_{n\in\mathbb{Z}}$ of (i) is, indeed, a frame.
(Step I) Let T_k and T be defined as:

$$\begin{array}{c} T_k : H(\mathscr{X}) \longrightarrow l^2(\mathbb{Z}) \\ {\scriptstyle y \longmapsto \left(\langle y, z'_n\rangle_H \mathbf{1}_{[-k,k]}(n)\right)_n} \end{array} \quad \text{and} \quad \begin{array}{c} T : H(\mathscr{X}) \longrightarrow l^2(\mathbb{Z}) \\ {\scriptstyle y \longmapsto (\langle y, z'_n\rangle_H)_n} \end{array},$$

where the z'_n's are such that $\langle x_t, z'_n\rangle_H = S_n(t)$, thus,

$$\|T_k(y) - T(y)\|^2_{l^2(\mathbb{Z})} = \sum_{|n|>k} |\langle y, z'_n\rangle_H|^2 \underset{k\longrightarrow\infty}{\longrightarrow} 0\,,$$

and then pointwise convergence follows from this. On the other hand

$$\|T_k(y)\|^2_{l^2(\mathbb{Z})} \le \left(\sum_{|n|\le k} \|z'_n\|^2_H\right) \|y\|^2_H\ ,$$

then, by the Banach-Steinhaus theorem, T is a bounded operator, so there exists $B_1 > 0$ such that :

$$\sum_{n\in\mathbb{Z}} |\langle y, z'_n\rangle_H|^2 \le B_1\, \|y\|^2_H\ . \tag{3.2}$$

By the same argument, one proves that there exists $B_2 > 0$ such that:

$$\sum_{n\in\mathbb{Z}} |\langle y, z_n\rangle_H|^2 \le B_2\, \|y\|^2_H\ . \tag{3.3}$$

Moreover, by Lemma 3.1.6. of [7], and the definition of conjugate vectors , we have that $\{z'_n\}_n$ and $\{\overline{z'}_n\}_n$ are Besselian:

$$\left\|\sum_{n\in\mathbb{Z}} \overline{a}_n \overline{z'}_n\right\|^2_{H(\mathscr{X})} = \left\|\sum_{n\in\mathbb{Z}} a_n z'_n\right\|^2_{H(\mathscr{X})} \le B_1 \sum_{n\in\mathbb{Z}} |a_n|^2\,, \tag{3.4}$$

for all $(a_n)_n \in l^2(\mathbb{Z})$.
(Step II) Let $H(S)$ be the reproducing kernel Hilbert space induced by the linear operator

$$\begin{array}{c} J : l^2(\mathbb{Z}) \longrightarrow H(S) \\ {\scriptstyle (\alpha_k)_k \longmapsto J(\alpha) = \sum_k \alpha_k S_k(t)} \end{array}.$$

Note that J is well defined, and $H(S)$ is the range of J: $Ran(J) = H(S)$ equipped with the norm $\|v\|_{H(S)} = \inf\left\{ \|w\|_{l^2(\mathbb{Z})}\ :\ J(w) = v\ \right\}$.
Let us see that $H(S) = H(R)$, in the sense of set inclusions. Let $v \in H(S)$, then $v(t) = \sum_{n\in\mathbb{Z}} \alpha_n S_n(t)$, taking in account that $S_n(t) = \langle x_t, z'_n\rangle_H = \widetilde{J}(\overline{z'}_n)(t)$ and that $\sum_{n\in\mathbb{Z}} \overline{\alpha}_n \overline{z'}_n$ converges by eq. 3.4 (Step I), so that by eq. 2.2 $v(t) = \widetilde{J}\left(\sum_{n\in\mathbb{Z}} \overline{\alpha}_n \overline{z'}_n\right)(t)$, and thus $v \in Ran(\widetilde{J}) = H(R)$. On the other hand if $v \in H(R)$, then $v = \widetilde{J}(y)$, for some $y \in H(\mathscr{X})$, but if $x_t = \sum_{n\in\mathbb{Z}} z_n S_n(t)$ and recalling again eq. 2.2: $v(t) = \sum_{n\in\mathbb{Z}} \langle z_n, \overline{y}\rangle_H S_n(t) = J(\beta)(t)$, with $\beta_n = \langle z_n, \overline{y}\rangle_H$, $\beta \in l^2(\mathbb{Z})$, then $v \in Ran(J) = H(S)$.
(Step III) Now, let us see that the norms are equivalent. In fact, consider the

inclusion map $H(R) \overset{i}{\hookrightarrow} H(S)$ and $v_n = i(v_n) \underset{n\to\infty}{\longrightarrow} w$ in the $H(S)$-norm and such that $v_n \underset{n\to\infty}{\longrightarrow} 0$ in the $H(R)$-norm. But as $H(R)$ and $H(S)$ are both RKHS, then convergence in norm implies pointwise convergence for each $t \in \mathfrak{T}$, so we have that $v_n(t) = i(v_n)(t) \underset{n\to\infty}{\longrightarrow} w(t) = 0$ for all $t \in \mathfrak{T}$. Thus $w = 0$, and then by the closed graph theorem i is continuous, but i is also a bijective map, then there exist constants $0 < A \le B < \infty$ such that:

$$A \|v\|_{H(R)} \le \|i(v)\|_{H(S)} \le B \|v\|_{H(R)} .$$

Then

$$A \|v\|_{H(R)} \le \|v\|_{H(S)} = \inf\left\{ \|w\|_{l^2(\mathbb{Z})} \,:\, J(w) = v \right\} \le \sum_{n\in\mathbb{Z}} |\langle z_n, \overline{y}\rangle_H|^2$$

since we have seen that $v = J(\beta)$, with $\beta_n = \langle z_n, \overline{y}\rangle_H$. On the other hand, $v = \widetilde{J}(y)$, and taking in account eq. 3.3 we get:

$$A \|y\|_{H(\mathscr{X})} \le \left(\sum_{n\in\mathbb{Z}} |\langle z_n, \overline{y}\rangle_H|^2 \right)^{\frac{1}{2}} \le B_2 \|y\|_{H(\mathscr{X})} ,$$

and then $\{z_n\}_n$ is a frame.

□

3.0.1. *Remark:* If $\{z_n\}_n$ and $\{S_n\}_n$ are the same of the previous theorem, and if the sequence $\{z'_n\}_n$ is such that $S_n = \widetilde{J}(\overline{z'}_n)$ then $\{z'_n\}_n$ is a dual frame for $\{z_n\}_n$. Indeed, from eqs. 3.2 and 3.3, $\{z_n\}_n$ and $\{z'_n\}_n$ are Besselian. Thus, by Lemma 5.7.1 of [7], it suffices to show that $\sum_{n\in\mathbb{Z}} \langle y, z'_n\rangle_H z_n = y$ for all $y \in H(\mathscr{X})$. In a similar way as before, we can see that $T' : \underset{y \longmapsto (\langle y, z_n\rangle_H)_n}{H(\mathscr{X}) \longrightarrow l^2(\mathbb{Z})}$ is a bounded operator as well as its adjoint $(T')^*$. Then, equivalently, we shall see that $(T')^* \circ T = I$ over a dense subset. Take $\psi = \sum_{k=1}^{n} \alpha_k x_{t_k} \in \operatorname{lin} \mathscr{X}$ then $(T(\psi))_n = \langle z'_n, \sum_{k=1}^{n} \alpha_k x_{t_k}\rangle_H = \sum_{k=1}^{n} \alpha_k S_n(t_k)$, but on the other hand $(T')^*((\langle z'_n, x_{t_k}\rangle_H)_n) = \sum_n \langle z'_n, x_{t_k}\rangle_H z_n = x_{t_k}$, then $(T')^* \circ T(\psi) = (T')^*(T(\psi)) = \sum_{k=1}^{n} \alpha_k x_{t_k} = \psi$.

3.0.2. *Remark:* Despite theorem 2.1 is a more general result, one may construct the c.a.o.s. measure M, and the frame sequence $\{f_n\}_{n\in\mathbb{Z}}$ of theorem 3.1, in the following way: as $H(\mathscr{X})$ and $\mathcal{S}$ are both separable, take any pair of orthonormal basis $\{y_l\}_{l\in\mathbb{J}}$ and $\{g_l\}_{l\in\mathbb{J}}$ of $H(\mathscr{X}) \bigoplus \mathcal{S}^\perp$ and $L^2(\mu)$ respectively, and define over $\mathcal{R}_\mu$, the vector measure M and a frame sequence $\{f_n\}_{n\in\mathbb{Z}}$ as :

$$M(A) = \sum_{l\in\mathbb{J}} \langle \mathbf{1}_A, g_l\rangle_{L^2(\mu)} y_l \,, \text{and } f_n = \sum_{l\in\mathbb{J}} \langle z_n, y_l\rangle_H g_l \,.$$

Then one can verify that if $A \bigcap B = \varnothing$ then $M(A \bigcup B) = M(A) + M(B)$, and

$$\|M(A)\|_H^2 = \sum_{l\in\mathbb{J}} |\langle g_l, \mathbf{1}_A\rangle_{L^2(\mu)}|^2 = \int_U |\mathbf{1}_A|^2 d\mu = \mu(A).$$

From this, the measure M extends as usual. Now, given M, the stochastic integral $\int_U f dM$, for $f \in L^2(\mu)$ is constructed in the standard way, defining an isometry. Finally, take $f \in \mathcal{S}$, then

$$\langle f, f_n \rangle_{L^2(\mu)} = \sum_{l \in \mathbb{J}} \langle f, g_l \rangle_{L^2(\mu)} \overline{\langle z_n, y_l \rangle}_H = \langle \sum_{l \in \mathbb{J}} \langle f, g_l \rangle_{L^2(\mu)} y_l, z_n \rangle_H \, .$$

So, as $\{z_n\}_n$ is indeed a frame, we have:

$$A \left\| \sum_{l \in \mathbb{J}} \langle f, g_l \rangle_{L^2(\mu)} y_l \right\|_H^2 \leq \sum_{n \in \mathbb{Z}} |\langle f, f_n \rangle_{L^2(\mu)}|^2 \leq B \left\| \sum_{l \in \mathbb{J}} \langle f, g_l \rangle_{L^2(\mu)} y_l \right\|_H^2 \, . \tag{3.5}$$

But on the other hand, by Parseval's identity:

$$\left\| \sum_{l \in \mathbb{J}} \langle f, g_l \rangle_{L^2(\mu)} y_l \right\|_H^2 = \sum_{l \in \mathbb{J}} |\langle f, g_l \rangle_{L^2(\mu)}|^2 = \|f\|^2_{L^2(\mu)} \, ,$$

and combining this with eq. 3.5, we obtain that $\{f_n\}_n$ is a frame.

3.0.3. *An Equivalence Theorem.* Finally we can join these results together:

Theorem 3.3. *Let $\mathfrak{T}$ be a set of indexes, generally non countable, and let $\mathscr{X} = \{x_t\}_{t \in \mathfrak{T}} \subset H$. Let $H(\mathscr{X})$ be the closed subspace spanned by $\mathscr{X}$. Given $\{z_n\}_{n \in \mathbb{Z}} \subset H(\mathscr{X})$. The following assertions are equivalent:*

i) $(\langle y, z_n \rangle_H)_n \in l^2(\mathbb{Z})$ for all $y \in H(\mathscr{X})$ and there exists a subset of functions $\{S_n\}_{n \in \mathbb{Z}} \in H(R)$ such that:

$$x_t = \sum_{n \in \mathbb{Z}} S_n(t) z_n \, ,$$

and $(\langle f, S_n \rangle_{H(R)})_n \in l^2(\mathbb{Z})$ for all $f \in H(R)$.

ii) $\{z_n\}_{n \in \mathbb{Z}}$ is a frame of $H(\mathscr{X})$.

iii) Given $(U, \mathcal{R}, \mu)$ a measure space, such that $\dim L^2(\mu) \geq \dim H(\mathscr{X})$. There exists a c.a.o.s. measure $M : \mathcal{R}_\mu \longrightarrow H$, with control measure μ, i.e, $\mu(\,.\,) = \|M(\,.\,)\|_H^2$, and a frame sequence $\{f_n\}_n$ (of $\overline{\text{span}}\,\{f_n\}_n$) such that:
iii.a) $z_n = \int_U f_n dM$. iii.b)

$$x_t = \int_U k(t, s) dM(s) \, , \tag{3.6}$$

with $k(t, s) = \sum_{n \in \mathbb{Z}} S'_n(t) f_n(s)$ and $\{S'_n\}_n \subset H(R)$, $(S'_n(t))_n \in l^2(\mathbb{Z})$ $\forall t \in \mathfrak{T}$, iii.c) $\{k(t, .)\}_{t \in \mathfrak{T}} = \overline{\text{span}}\,\{f_n\}_n$.

iv) Let $(U, \mathcal{R}, \mu)$ be a measure space such that there exists a c.a.o.s. vector measure $M : \mathcal{R}_\mu \longrightarrow H$, for which eq. 3.6 holds for some $\{k(t, .)\}_{t \in \mathfrak{T}} \subset L^2(\mu)$, with

$\mu(\,.\,) = \|M(\,.\,)\|_H{}^2$. Then: $H(R) = Ran(T)$, where $T : \overline{\text{span}}\,\{k(t,.)\}_{t\in\mathfrak{T}} \longrightarrow H(R)$ is defined by:

$$(Tf)(t) = \int_U k(t,s)f(s)d\mu(s)\,.$$

The kernel $k(t,\,.\,)$ admits the expansion: $k(t,.) = \sum_{n\in\mathbb{Z}} S_n(t)f_n$, with $\{f_n\}_{n\in\mathbb{Z}}$ a frame of $\overline{\text{span}}\,\{k(t,.)\}_{t\in\mathfrak{T}}$, $\{S_n\}_n$ a frame of $H(R)$ and $z_n = \int_U f_n dM$.

Proof. Immediate from theorems 3.1 and 3.2. □

3.0.4. *The case of Riesz Bases.* Let us see the form of this last result when the frame sequence is also a basis:

Corollary 3.1. *Let $\mathfrak{T}$ be a set of indexes, generally non countable, and let $\mathscr{X} = \{x_t\}_{t\in\mathfrak{T}} \subset H$. Let $H(\mathscr{X})$ be the closed subspace spanned by $\mathscr{X}$. Given $\{z_n\}_{n\in\mathbb{Z}} \subset H(\mathscr{X})$. The following assertions are equivalent:*
i) $(\langle y, z_n\rangle_H)_n \in l^2(\mathbb{Z})$ for all $y \in H(\mathscr{X})$ and there exists a subset of functions $\{S_n\}_{n\in\mathbb{Z}} \in H(R)$ such that:

$$x_t = \sum_{n\in\mathbb{Z}} S_n(t)z_n\,,$$

and $(\langle f, S_n\rangle_{H(R)})_n \in l^2(\mathbb{Z})$ for all $f \in H(R)$. Additionally, the $\{S_n\}_{n\in\mathbb{Z}} \in H(R)$ verify that if $(c_n)_n \in l^2(\mathbb{Z})$ is such that $\sum_n c_n S_n(t) = 0\ \forall\, t$ then $c_n = 0\ \forall\, n$.
ii) $\{z_n\}_{n\in\mathbb{Z}}$ is a Riesz basis of $H(\mathscr{X})$.
iii) Given $(U, \mathcal{R}, \mu)$ a measure space, such that $\dim L^2(\mu) \geq \dim H(\mathscr{X})$. There exists a c.a.o.s. measure $M : \mathcal{R}_\mu \longrightarrow H$, with control measure μ, i.e, $\mu(\,.\,) = \|M(\,.\,)\|^2_H$, and a Riesz basis $\{f_n\}_n$ (of $\overline{\text{span}}\,\{f_n\}_n$) such that:
iii.a) $z_n = \int_U f_n dM$; iii.b)

$$x_t = \int_U k(t,s)dM(s)\,,$$

with $k(t,s) = \sum_{n\in\mathbb{Z}} S'_n(t)f_n(s)$ and $\{S'_n\}_n \subset H(R)$, $(S'_n(t))_n \in l^2(\mathbb{Z})\ \forall\, t \in \mathfrak{T}$, iii.c) $\{k(t,.)\}_{t\in\mathfrak{T}} = \overline{\text{span}}\,\{f_n\}_n$.
iv) If $(U, \mathcal{R}, \mu)$ is a measure space such that there exists a c.a.o.s. vector measure $M : \mathcal{R}_\mu \longrightarrow H$, for which eq. 3.6 holds for some $\{k(t,.)\}_{t\in\mathfrak{T}} \subset L^2(\mu)$, with $\mu(\,.\,) = \|M(\,.\,)\|_H{}^2$, then: $H(R) = Ran(T)$, where $T : \overline{\text{span}}\,\{k(t,.)\}_{t\in\mathfrak{T}} \longrightarrow H(R)$ is defined by:

$$(Tf)(t) = \int_U k(t,s)f(s)d\mu(s)\,.$$

The kernel $k(t,\,.\,)$ admits the following expansion: $k(t,.) = \sum_{n\in\mathbb{Z}} S_n(t)f_n$, with $\{f_n\}_{n\in\mathbb{Z}}$ a Riesz basis of $\overline{\text{span}}\,\{k(t,.)\}_{t\in\mathfrak{T}}$, $\{S_n\}_n$ a Riesz basis of $H(R)$ and $z_n = \int_U f_n dM$.

Proof. ii)$\Longrightarrow$ i) It follows from theorem 3.3 since if $\{z_n\}_{n\in\mathbb{Z}}$ is a Riesz basis, then it is also a frame [7]. So, we have only to verify the last condition of i). In fact, take $(c_n)_n \in l^2(\mathbb{Z})$ such that $0 = \sum_n c_n S_n(t) = \sum_n c_n \langle x_t, z'_n\rangle_H\ \forall t$, but from the remark 3.0.1 we have that these z_n's constitute the dual basis of $\{z_n\}_n$, and as $(c_n)_n \in l^2(\mathbb{Z})$, we have: $\langle x_t, \sum_n c_n z_n' \rangle_H = 0$ for all t. Since $\sum_n c_n z_n' \in H(\mathscr{X})$ then $\sum_n c_n z_n' = 0$, so that $c_n = 0$, since $\{z_n'\}_n$ is a basis.

i) $\Longrightarrow$ ii) By theorem 3.3, under these conditions we have seen that $\{z_n\}_n$ is a frame, and that there exists a dual frame $\{z_n'\}_n$ such that $S_n(t) = \langle x_t, z_n' \rangle_H$. So it is sufficient to show that both sequences are bi orthogonal. In fact, if $x_t = \sum_n S_n(t) z_n$, for $n \in \mathbb{Z}$: $S_m(t) = \langle x_t, z_n' \rangle_H = \sum_n S_n(t) \langle z_n, z_m' \rangle_H$, thus from the hypothesis we obtain that: $\langle z_n, z_m' \rangle_H = \delta_{n\,m}$. □

3.1. **Application: A sampling theorem.** The following result, shows how the previous results may be applied to the problem of characterizing processes which are linearly determined by their samples, and which also form a Riesz basis

Theorem 3.4. *Let $\{x_t\}_{t\in\mathfrak{T}} \subset H$. For sequences $\{t_n\}_{n\in\mathbb{Z}} \subset \mathfrak{T}$, $\{x_{t_n}\}_{n\in\mathbb{Z}} \subset H(\mathscr{X})$, the following are equivalent:*

i) $\{x_{t_n}\}_{n\in\mathbb{Z}}$ is a Riesz basis of $H(\mathscr{X})$.

ii) $S_n(t_r) = \delta_{n\,r}$ and $(\langle f, S_n \rangle_{H(R)})_n \in l^2(\mathbb{Z})$ for all $f \in H(R)$, $(\langle y, x_{t_n} \rangle_H)_n \in l^2(\mathbb{Z})$ $\forall\, y \in H(\mathscr{X})$ and

$$x_t = \sum_{n\in\mathbb{Z}} x_{t_n} S_n(t)\,.$$

iii) Given $(U, \mathcal{A}, \mu)$ a measure space, such that $\dim\, L^2(\mu) \geq \dim\, H(\mathscr{X})$, there exists a c.a.o.s. measure M over $(U, \mathcal{A})$ such that $\|M(\,.\,)\|_H^2 = \mu(\,.\,)$, and $k(t,\,.\,) \in L^2(\mu)$ such that $\{k(t_n,\,.\,)\}_{n\in\mathbb{Z}}$ is a Riesz basis of its span, $\{S_n\}_n$ is a Riesz basis of $H(R)$ such that $k(t,\,.\,) = \sum_{n\in\mathbb{Z}} k(t_n,\,.\,) S_n(t)$, and

$$x_t = \int_U k(t,\, s\,) dM(s)\,.$$

iv) If $(U, \mathcal{R}, \mu)$ is a measure space such that there exists a c.a.o.s. vector measure $M : \mathcal{R}_\mu \longrightarrow H$, for which eq. 3.6 holds for some $\{k(t,.)\}_{t\in\mathfrak{T}} \subset L^2(\mu)$, with $\mu(\,.\,) = \|M(\,.\,)\|_H{}^2$. Then: $H(R) = Ran(T)$, where $T : \overline{\operatorname{span}}\,\{k(t,.)\}_{t\in\mathfrak{T}} \longrightarrow H(R)$ is defined by:

$$(Tf)(t) = \int_U k(t,s) f(s) d\mu(s)\,.$$

The kernel $k(t,\,.\,)$ admits the following expansion: $k(t,.) = \sum_{n\in\mathbb{Z}} S_n(t) k(t_n,\,.\,)$, with $\{k(t_n,\,.\,)\}_{n\in\mathbb{Z}}$ a Riesz basis of $\overline{\operatorname{span}}\,\{k(t,.)\}_{t\in\mathfrak{T}}$ and $\{S_n\}_n$ is a Riesz basis of $H(R)$.

3.2. **Examples:**

3.2.1. *The SKW theorem with a Riesz basis.* Let $\mathscr{X} = \{x_t\}_{t\in\mathbb{R}}$, be defined by:

$$x_t = \int_{\mathbb{R}} e^{it\lambda} dM(\lambda)\,,$$

where $M(\,.\,)$ is an orthogonal random measure, such that $\mu(A) = \mathbb{E}|M(A)|^2 = \int_A \phi(\lambda) d\lambda$, for some non negative $\phi \in L^1(\mathbb{R})$. Is an standard result that such M, exists [10]. Moreover we can take ϕ, such that $A \leq \phi \leq B$, a.e. on $[-\pi, \pi]$, for some

$A, B > 0$, and $\phi = 0$ a.e. on $[-\pi, \pi]^c$. The resulting process is w.s.s. and it is easy to verify that $\{\frac{e^{in\lambda}}{\sqrt{2\pi}}\}_{n\in\mathbb{Z}}$ is a Riesz basis of $L^2(\mathbb{R}, d\mu)$ and that, $\{x_n\}_{n\in\mathbb{Z}}$ is also a Riesz basis of $H(\mathscr{X})$. Moreover, the dual basis is given by $\{\frac{e^{in\lambda}(\phi(\lambda))^{-1}}{\sqrt{2\pi}}\}_{n\in\mathbb{Z}}$, and then $S_n(t) = \mathbb{E}(x_t z_n') = \frac{sin(\pi(t-n))}{\pi(t-n)}$, and as μ is equivalent to the restriction of the ordinary Lebesgue measure on $[-\pi, \pi]$, one may take the operator T defined by the kernel: $k(t, \lambda) = e^{it\lambda}\mathbf{1}_{[-\pi,\pi]}(\lambda)$. Finally, we recall that in this case, the condition on μ of being absolutely continuous with respect to the Lebesgue measure, is necessary, as it was proved in [18].

3.2.2. *A case from Generalized Harmonic Analysis.* We shall see with this example, that it is worth studying conditions for which we have stable sampling formulas. In fact, we will give here an example where we can completely determine a "process" from a set of samples, but which it does not constitute a frame sequence. Let $H = B^2 = \overline{\text{span}}\{e^{i\lambda x}\}_{\lambda\in\mathbb{R}}$ [1], with respect to the metric defined by the inner product

$$\langle f, g\rangle_{B^2} = \lim_{T\longrightarrow\infty} \frac{1}{2T}\int_{-T}^{T} f(x)\overline{g(x)}dx\,.$$

Moreover [1], the system $\{e^{i\lambda x}\}_{\lambda\in\mathbb{R}}$ is a non countable orthonormal system with respect to this inner product. Let us consider a *fixed* almost periodic function [23] $f \neq 0$, such that

$$\hat{f}(\lambda) = \lim_{T\longrightarrow\infty} \frac{1}{2T}\int_{-T}^{T} f(x)e^{-i\lambda x}dx\,, \tag{3.7}$$

verifies $\hat{f}(\lambda) = 0$ for all $|\lambda| > a$, for some constant $0 < a < \pi$. Define $H(f) = \overline{\text{span}}\{f(x-t)\}_{t\in\mathbb{R}}$. First, let us verify that $H(f) = \overline{\text{span}}\{f(x-n)\}_{n\in\mathbb{Z}}$. In fact, as $\hat{f}(\lambda) = 0$, except for a countable set of indexes λ's, we can write:

$$f(x) = \sum_{k\in\mathbb{Z}} \hat{f}(\lambda_k)e^{i\lambda_k x}\,. \tag{3.8}$$

On the other hand we have the ordinary Fourier series expansion:

$$\lim_{N\longrightarrow\infty} \sum_{|n|\leq N} \frac{sin(\pi(t-n))}{\pi(t-n)}e^{inx} = e^{ixt}\,, \tag{3.9}$$

which converges uniformly on $[-a, a]$. From eq. 3.8, we have that $f(x-t) = \sum_{k\in\mathbb{Z}} \hat{f}(\lambda_k)e^{-i\lambda_k t}e^{i\lambda_k x}$. From this:

$$\sum_{|n|\leq N} \frac{sin(\pi(t-n))}{\pi(t-n)}f(x-n) = \sum_{k\in\mathbb{Z}} \hat{f}(\lambda_k)\left(\sum_{|n|\leq N} \frac{sin(\pi(t-n))}{\pi(t-n)}e^{-in\lambda_k}\right)e^{i\lambda_k x}\,,$$

and then by Bohr's theorem [1], [23]:

$$\lim_{T\longrightarrow\infty} \frac{1}{2T}\int_{-T}^{T} \left| f(x-t) - \sum_{|n|\leq N} \frac{sin(\pi(t-n))}{\pi(t-n)}f(x-n)\right|^2 dx$$

$$= \sum_{k\in\mathbb{Z}} \left| \sum_{|n|\leq N} \frac{sin(\pi(t-n))}{\pi(t-n)} e^{-in\lambda_k} - e^{-i\lambda_k t} \right|^2 |\hat{f}(\lambda_k)|^2 \underset{N\longrightarrow\infty}{\longrightarrow} 0 \,,$$

form, eq. 3.9. And then $f(\,.\,-t) \in \overline{\text{span}}\{f(x-n)\}_{n\in\mathbb{Z}}$ for all $t\in\mathbb{R}$.
Again, by Bohr's theorem, we can set the Fourier transform of μ:

$$\hat{\mu}(t) = \langle f, f(\,.\,-t)\rangle_{B^2} = \sum_{k\in\mathbb{Z}} |\hat{f}(\lambda_k)|^2 e^{-i\lambda_k t} \,,$$

where μ is the singular measure given by

$$\mu\left((-\infty,\lambda]\right) = \sum_{k\in\mathbb{Z}} |\hat{f}(\lambda_k)|^2 \mathbf{1}_{(-\infty,\lambda_k)}(\lambda) \,.$$

Let us see that if $\overline{\text{span}}\{f(x-k)\}_{k\in\mathbb{Z}}$ is a frame sequence then we must have that μ is absolutely continuous with respect to the ordinary Lebesgue measure, i.e. $\mu \ll \mathcal{L}$, and then it would be a contradiction. Indeed, by the frame condition one should have:

$$\sum_{k\in\mathbb{Z}} |\hat{\mu}(k)|^2 = \sum_{k\in\mathbb{Z}} |\langle f, f(\,.\,-k)\rangle_{B^2}|^2 \leq B\,\|f\|^2_{B^2} < \infty \,.$$

Thus, since the support of μ is a subset of $[-\pi,\pi]$, the $\hat{\mu}(k)$'s should be the Fourier coefficients of a function in $L^2[-\pi,\pi] \subset L^1[-\pi,\pi]$, so that contradiction follows form this result.

4. An application: Noise reduction.

Frames are a good tool when dealing with noise [7] in signal detection [2], or when reconstructing a signal form corrupted samples as we will see briefly now. Suppose that $(\Omega,\mathcal{F},\mathbb{P})$ is a probability space, $x_t \in H = L^2(\Omega,\mathcal{F},\mathbb{P})$, and that, as in the previous section, we have the representation: $x_t = \sum\limits_{n\in\mathbb{Z}} z_n S_n(t)$. We can consider the following problem. Suppose that instead of z_n we get the "corrupted" sequence: $\{z_n + w_n\}_{n\in\mathbb{Z}}$, where the w_n's are independent identically distributed random variables, with $\mathbb{E}(w_n) = 0$ and $\mathbb{E}|w_n|^2 = \sigma^2$. The sequence $\{w_n\}_{n\in\mathbb{Z}}$ is supposed to be independent of $\{z_n\}_{n\in\mathbb{Z}}$. Now, if we make a synthesis with this new sequence we get a process $\hat{x}_t = x_t + n_t$, where $n_t = \sum\limits_{n\in\mathbb{Z}} w_n S_n(t)$, where this series converges a.s. by Kolmogorov's theorem for sums of finite variance independent random variables . Then $\mathbb{E}|n_t|^2 = \sum\limits_{n\in\mathbb{Z}} \sigma^2|S_n(t)|^2$, and if $S_n(t) = \mathbb{E}(z'_n x_t)$, with $\{z'_n\}_{n\in\mathbb{Z}}$ the canonical dual frame, if we consider the signal to noise ratio for each t: $(SNR)_t = \frac{\mathbb{E}|x_t|^2}{\mathbb{E}|n_t|^2}$ and taking in account that $\frac{\mathbb{E}|x_t|^2}{A} \geq \sum\limits_{n\in\mathbb{Z}} |\mathbb{E}(z'_n x_t)|^2 \geq \frac{\mathbb{E}|x_t|^2}{B}$ then we have

$$(SNR)_t = \frac{\mathbb{E}|x_t|^2}{\sum\limits_{n\in\mathbb{Z}} \sigma^2 |S_n(t)|^2} \geq \frac{A}{\sigma^2} \,.$$

Hence we can see directly how the frame lower constant affects the signal to noise ratio.

4.0.3. *Example: The effect of oversampling.* We return to example 3.2.1. Let us consider the same w.s.s. process, and some $L \in \mathbb{N}$. Now, if we take the samples $\{x_{\frac{n}{L}}\}_n$, by a direct application of Parseval's theorem, if $y = \int_{\mathbb{R}} f(\lambda) dM(\lambda)$:

$$B'\mathbb{E}|y|^2 = B2\pi L\mathbb{E}|y|^2 \geq \sum_{n\in\mathbb{Z}} \left| \int_{[-\pi,\pi]} e^{i\frac{n}{L}\lambda} f(\lambda)\phi(\lambda)d\lambda \right|^2$$

$$= 2\pi L \int_{[-\pi,\pi]} |f(\lambda)\phi(\lambda)|^2 d\lambda \geq A2\pi L\mathbb{E}|y|^2 = A'\mathbb{E}|y|^2 .$$

Then the new frame constant is $A' > A$, and thus we improve the SNR as one should expect after introducing some redundancy.

Moreover, we can prove that under suitable conditions, the difference $n_t = \hat{x}_t - x_t$ can be uniformly controlled in probability by the frame bound A. Previously we recall Kolmogorov's condition:

Lemma 4.1. [10]*(p. 191) Let $\{x_t\}_{t\in J}$ be a separable random process satisfying the following condition: there exists a non-negative monotonically non-decreasing function $g(t)$ and a function $q(x,t)$, $t \geq 0$ such that $\mathbb{P}(|x_{t_0+t} - x_{t_0}| > xg(t)) \leq q(x,t)$ and $G = \sum_{n\geq 0} g(2^{-n}T) < \infty$ and $Q(x) = \sum_{n\geq 1} 2^n q(x, 2^{-n}T) < \infty$. Then:*

$$\mathbb{P}\left(\underset{0\leq t' < t'' \leq T}{sup} |x_{t'} - x_{t''}| > \lambda \right) \leq Q\left(\frac{\lambda}{2G}\right) .$$

By a direct application of this, again taking σ as in the beginning of this section, one can prove:

Proposition 4.1. *If $x_t \in L^2(\Omega, \mathcal{F}, \mathbb{P})$, verifies the Lipschitz condition $\mathbb{E}|x_t - x_s|^2 \leq C|t-s|^\alpha$ for some constants $C > 0$, $\alpha \in (1,2]$, independent of $t, s \in \mathcal{T} = [0,T]$. Then:*

$$\mathbb{P}\left(\underset{t\in\mathcal{T}}{sup} |n_t| > \sigma \right) \leq \frac{K(T,\alpha,C)}{A} .$$

Proof. By Tchevichev's inequality:

$$\mathbb{P}\left(|n_0| > \sigma\right) \leq \frac{\mathbb{E}|n_0|^2}{\sigma^2} = \frac{\sigma^2 \sum_{n\in\mathbb{Z}} |S_n(0)|^2}{\sigma^2} \leq \frac{\mathbb{E}|x_0|^2}{A} \tag{4.1}$$

Let $x > 0$ and $\beta > 0$:

$$\mathbb{P}\left(|n_t - n_0| > x\, t^\beta\right) \leq \frac{\mathbb{E}|n_t - n_0|^2}{x^2\, t^{2\beta}} = \frac{\sigma^2}{x^2\, t^{2\beta}} \sum_{k\in\mathbb{Z}} |S_k(t) - S_k(0)|^2$$

$$\leq \frac{\sigma^2 \mathbb{E}|x_t - x_0|^2}{A\, x^2\, t^{2\beta}} \leq \frac{C\, \sigma^2\, t^{\alpha-2\beta}}{A\, x^2} .$$

Taking $G := \sum_{n=0}^{\infty} (2^{-n}T)^\beta = \frac{T^\beta}{1-2^{-\beta}}$, $Q(x) = \sum_{n=1}^{\infty} \frac{C\sigma^2}{x^2\, A} 2^{n(2\beta-\alpha+1)} T^{\alpha-2\beta}$. Thus, if $2\beta - \alpha + 1 < 0$, by lemma 4.1 then:

$$\mathbb{P}\left(\underset{t\in\mathcal{T}\setminus\{0\}}{sup} |n_t - n_0| > \sigma \right) \leq Q\left(\frac{\sigma}{2G}\right) = \frac{4G^2 C}{A} \frac{2^{(2\beta-\alpha+1)}}{1 - 2^{(2\beta-\alpha+1)}} T^{\alpha-2\beta}$$

Taking $\beta = \frac{\alpha-1}{4}$, this equals

$$\frac{4CT^{\alpha}}{A(1-2^{-\beta})^2}\frac{2^{\left(\frac{1}{2}-\frac{\alpha}{2}\right)}}{(1-2^{\frac{1-\alpha}{2}})} = \frac{K'(T,\alpha,C)}{A}.$$

So, recalling eq. 4.1 we get the desired result, since

$$\mathbb{P}\left(\sup_{t\in\mathcal{T}}|n_t| > \sigma\right) \leq \mathbb{P}\left(\sup_{t\in\mathcal{T}\setminus\{0\}}|n_t - n_0| > \frac{\sigma}{2}\right) + \mathbb{P}\left(|n_0| > \frac{\sigma}{2}\right).$$

□

5. Conclusion

We gave conditions for a finite variance random process, or a more general Hilbert space process, to be reconstructed from its discrete samples or a countable set of measurements if they form a frame sequence of the closed linear span of $\mathscr{X}$. The conditions are analogous to the conditions for $L^2(I)$ functions using Riesz and orthogonal bases, studied by Kramer in [13] and Garcia [9]. Finally, we studied the application of frames in noise reduction for the case of random signals.

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