Unitarization of uniformly bounded subgroups in finite von Neumann algebras

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Abstract

This note presents a new proof of the fact that every uniformly bounded group of invertible elements in a finite von Neumann algebra is similar to a unitary group. In 1974 Vasilescu and Zsido proved this result using the Ryll-Nardzewsky fixed point theorem. This new proof involves metric geometric arguments in the non-positively curved space of positive invertible operators of the algebra, which yield a more explicit unitarizer. [1]

1 Geometry of the cone of positive invertible operators in a finite algebra

The metric geometry of the cone of positive invertible operators in a finite von Neumann algebra was studied in [1, 4]. In this section we recall some facts from these papers.

Let $\mathcal{A}$ be a von Neumann algebra with a finite (normal, faithful) trace τ. Denote by $\mathcal{A}_{sa}$ the set of self-adjoint operators, by G the group of invertible operators, by U the group of unitary operators, and by P the set of positive invertible operators in $\mathcal{A}$.
Since P is an open subset of $\mathcal{A}_{sa}$ in the norm topology we can regard P as a submanifold of $\mathcal{A}_{sa}$. Therefore the tangent spaces of P identify with $\mathcal{A}_{sa}$ endowed with the uniform norm, which we denote by $\|\cdot\|$. The pre-Hilbert norm $\|x\|_2 = \tau(x^2)^{\frac{1}{2}}$ on $\mathcal{A}_{sa}$ is used to give P a Finslerian length structure (see [3] Section 2). The admissible paths of the length structure are the piecewise smooth curves in P and for each $a \in P$ the tangent space $T_aP \simeq \mathcal{A}_{sa}$ is endowed with the norm given by $\|x\|_{a,2} = \|a^{-\frac{1}{2}}xa^{-\frac{1}{2}}\|_2$ for $x \in \mathcal{A}_{sa}$. For $a, b \in P$ the unique geodesic $\gamma_{a,b} : [0,1] \to P$ between a and b is given by $\gamma_{a,b}(t) = a^{\frac{1}{2}}(a^{-\frac{1}{2}}ba^{-\frac{1}{2}})^t a^{\frac{1}{2}}$ and has length equal to $d(a,b) = Length(\gamma_{a,b}) = \|ln(a^{-\frac{1}{2}}ba^{-\frac{1}{2}})\|_2$, see [1, Th. 3.1 and Th. 3.2]. The interior metric space associated with this length structure will be denoted by (P, d).
If $\mathcal{A}$ is finite dimensional, and therefore a sum of matrix spaces, the metric d is the well-known non-positively curved Riemannian metric on the set of positive definite matrices [6].

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If $\mathcal{A}$ is of type II_1, the trace inner product is not complete, so P is not a Hilbert-Riemann manifold and (P,d) is not a complete metric space, see [4, Rem. 3.21].
The semi-parallelogram law, which states that if $a,b,c \in P$ then

$$d(a,b)^2 + 4d(c,\gamma_{a,b}(\frac{1}{2}))^2 \leq 2(d(c,a)^2 + d(c,b)^2),$$

holds in the metric space (P,d), see [4, Th. 4.4]. Hence the midpoint between a and b is $\gamma_{a,b}(\frac{1}{2})$. See the beginning of [5, Ch. XI, Section 3] for further discussion on the semi-parallelogram law and midpoint maps.
The action of G on P given by $I_g(a) = gag^*$ sends geodesic segments to geodesic segments and is isometric, i.e. $I_g \circ \gamma_{a,b} = \gamma_{I_g(a),I_g(b)}$ and $d(I_g(a), I_g(b)) = d(a,b)$ for all $a,b \in P$ and $g \in G$, see the introduction of [1].

2 Uniformly bounded subgroups

A subset $C \subseteq P$ is *geodesically convex* if $\gamma_{a,b}(t) \in C$ for every $a,b \in C$ and $t \in [0,1]$, and is *midpoint convex* if $\gamma_{a,b}(\frac{1}{2}) \in C$ for every $a,b \in C$. Note that a geodesically convex set is midpoint convex.

Lemma 2.1. *If $C \subseteq P$ is geodesically convex then its closure $\overline{C}$ in (P,d) is geodesically convex.*

Proof. By [1, Cor. 3.4] the distance between two geodesics is convex, i.e.

$$t \mapsto d(\gamma_{a_1,b_1}(t), \gamma_{a_2,b_2}(t)), \qquad [0,1] \to [0,+\infty)$$

is convex for all $a_1,b_1,a_2,b_2 \in P$. Hence, for a fixed $t \in [0,1]$, the function $(a,b) \mapsto \gamma_{a,b}(t)$ is d-continuous.
If $a,b \in \overline{C}$ and $t \in [0,1]$ let $(a_n)_n, (b_n)_n$ be sequences in C such that $a_n \to a$ and $b_n \to b$. Since C is geodesically convex $\gamma_{a_n,b_n}(t) \in C$ for all $n \in \mathbb{N}$. The d-continuity of $(a,b) \mapsto \gamma_{a,b}(t)$ implies that $\gamma_{a_n,b_n}(t) \to \gamma_{a,b}(t)$, so that $\gamma_{a,b}(t) \in \overline{C}$. $\square$

Lemma 2.2. *For $0 < c_1 < c_2$ the interval $P_{c_1,c_2} = \{a \in P : c_1 1 \leq a \leq c_2 1\}$ endowed with the metric d is a complete and bounded metric space.*

Proof. In P_{c_1,c_2} the linear metric and the rectifiable distance are equivalent [4, Prop. 3.2], i.e. there are $C, C' > 0$ such that $\|a-b\|_2 \leq Cd(a,b)$ and $d(a,b) \leq C'\|a-b\|_2$ for all $a,b \in P_{c_1,c_2}$.
Since $\|\cdot\|_2$ induces a complete metric on subsets of $\mathcal{A}$ which are closed and bounded in the uniform norm, and P_{c_1,c_2} is closed and bounded in the uniform norm, we conclude that (P_{c_1,c_2}, d) is a complete metric space.
Also, (P_{c_1,c_2}, d) is a bounded metric space because $d(a,b) \leq C'\|a-b\|_2 \leq C'\|a-b\| \leq 2C'c_2$ for all $a,b \in P_{c_1,c_2}$. $\square$

Theorem 2.3. *If $H \subseteq G$ is a subgroup such that $\sup_{h\in H} \|h\| = M < \infty$ then there exists $s \in P_{M^{-1},M}$ such that $s^{-1}Hs \subseteq U$.*

Proof. Consider the isometric action $I : H \to Isom(P)$ given by $I_h(a) = hah^*$ for $h \in H$ and $a \in P$. We denote the action by $h \cdot a = I_h(a)$. Take $X_1 = H \cdot 1$ and define inductively $X_{n+1} = \{\gamma_{a,b}(t) : a,b \in X_n, t \in [0,1]\}$ for $n \geq 1$. Let

$$conv(H\cdot 1)=\bigcup_{n\in\mathbb{N}} X_n.$$

Since P_{M^{-2},M^2} is geodesically convex [2] and the action sends geodesic segments to geodesic segments, if $X_n\subseteq P_{M^{-2},M^2}$ then $X_{n+1}\subseteq P_{M^{-2},M^2}$ for all $n\in\mathbb{N}$. Therefore $conv(H\cdot 1)\subseteq P_{M^{-2},M^2}$ follows from $X_1=H\cdot 1=\{hh^*\}_{h\in H}\subseteq P_{M^{-2},M^2}$. Using the fact that P_{M^{-2},M^2} is closed in (P,d) we conclude that $\overline{conv}(H\cdot 1)\subseteq P_{M^{-2},M^2}$.
Since the action sends geodesic segments to geodesic segments, if X_n is invariant under the action I then X_{n+1} is invariant for all $n\in\mathbb{N}$. Since $X_1=H\cdot 1$ is invariant, we conclude that $conv(H\cdot 1)$ is invariant. The action is also isometric, hence $\overline{conv}(H\cdot 1)$ is invariant and we can restrict the action I to this subset.
The space $(\overline{conv}(H\cdot 1),d)$ is midpoint convex and the semi-parallelogram law holds in P, hence this law also holds in $(\overline{conv}(H\cdot 1),d)$. Since $\overline{conv}(H\cdot 1)$ is a closed subset of the complete metric space (P_{M^{-2},M^2},d) the space $(\overline{conv}(H\cdot 1),d)$ is complete. We conclude that $(\overline{conv}(H\cdot 1),d)$ is a complete metric space in which the semi-parallelogram holds.
Since (P_{M^{-2},M^2},d) is a bounded metric space $\overline{conv}(H\cdot 1)$ is a bounded set. Therefore the restricted action has bounded orbits, and the Bruhat-Tits fixed point theorem [5, Ch. XI, Th 3.2] states that there exists $a\in\overline{conv}(H\cdot 1)$ such that $I_h(a)=hah^*=a$ for all $h\in H$. Then

$$\begin{aligned}1 &= a^{-\frac{1}{2}}aa^{-\frac{1}{2}}=a^{-\frac{1}{2}}hah^*a^{-\frac{1}{2}}=(a^{-\frac{1}{2}}ha^{\frac{1}{2}})(a^{\frac{1}{2}}h^*a^{-\frac{1}{2}})\\ &= (a^{-\frac{1}{2}}ha^{\frac{1}{2}})(a^{-\frac{1}{2}}ha^{\frac{1}{2}})^* \qquad \text{for all } h\in H.\end{aligned}$$

Therefore $a^{-\frac{1}{2}}Ha^{\frac{1}{2}}\subseteq U$, i.e. $s=a^{\frac{1}{2}}$ is a unitarizer of H.
Since the square root is an operator monotone function and $a\in P_{M^{-2},M^2}$, it follows that $s=a^{\frac{1}{2}}\in P_{M^{-1},M}$. □

Remark 2.4. *A fixed point of the action I in the above theorem is the circumcenter of the set $H\cdot 1=\{hh^*\}_{h\in H}$ in $(\overline{conv}(H\cdot 1),d)$, see [5, Ch. XI, Th 3.1 and Th 3.2]. Therefore, a unitarizer s is the square root of the circumcenter of $\{hh^*\}_{h\in H}$. The unitarizability of a uniformly bounded subgroup H of the group of bounded linear invertible operators acting on a Hilbert space was obtained independently in the 50s by Day, Dixmier, Nakamura and Takeda assuming that H is amenable (see [7] and the references therein). In that context the unitarizer s was obtained as the square root of the center of mass of $\{hh^*\}_{h\in H}$.*

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