MINIMAL LENGTH CURVES IN UNITARY ORBITS OF A HERMITIAN COMPACT OPERATOR

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ABSTRACT. We study some examples of minimal length curves in homogeneous spaces under a left action of a unitary group. Recent results relate this curves with the existence of minimal (with respect to the quotient norm) anti-Hermitian operators Z in the tangent space of the starting point. By instance, we show minimal curves that are not of this type but nevertheless can be approximated uniformly by those.

1. Introduction

Let $\mathcal{H}$ be a separable Hilbert space and $\mathcal{K}(\mathcal{H})$ be the algebra of compact operators. In this paper we study the orbit manifold of a self-adjoint compact operator A by a particular unitary group, that is

$$\mathcal{O}_A = \{uAu^* : u \text{ unitary in } \mathcal{B}(\mathcal{H}) \text{ and } u - 1 \in \mathcal{K}(\mathcal{H})\}.$$

Given two points, $x, y \in \mathcal{O}_A$, the rectifiable distance between them is the infimum of the lengths of all the smooth curves in $\mathcal{O}_A$ that join x and y. Our purpose is to study the existence and properties of minimal length curves in $\mathcal{O}_A$.

The tangent space at any $b \in \mathcal{O}_A$ is

$$(T\mathcal{O}_A)_b = \{zb - bz : z \in \mathcal{K}(\mathcal{H}),\ z^* = -z\}$$

endowed with the Finsler metric given by the usual operator norm $\|\cdot\|$. If $x \in (T\mathcal{O}_A)_b$, the existence of a (not necessarily unique) minimal element z_0 such that

$$\|x\|_b = \|z_0\| = \inf\{\|z\| : z \in \mathcal{K}(\mathcal{H}),\ z^* = -z,\ zb - bz = x\}$$

allows in [1] the description of minimal length curves of the manifold by the parametrization

$$\gamma(t) = e^{tz_0}\ b\ e^{-tz_0},\ t \in \left[-\frac{\pi}{2\|z_0\|}, \frac{\pi}{2\|z_0\|}\right].$$

These z_0 can be described as $i(C + D)$, with $C \in \mathcal{K}(\mathcal{H})$, $C^* = C$ and D a real diagonal operator in the orthonormal basis of eigenvectors of A.

If we consider $\mathcal{B} \subset \mathcal{A}$ von Neumann algebras, it has been proved in [5] that for each $a \in \mathcal{A}$, $a^* = a$, there always exists a minimal element b_0 in $\mathcal{B}$. That is $\|a + b_0\| \le \|a + b\|$, for all $b \in \mathcal{B}$. However, in the case of $\mathcal{K}(\mathcal{H})$, a C^*-algebra which is not a von Neumann algebra, we proved in [4] that there is not always a minimal compact operator. In this case, the existence of a best approximant for $C \in \mathcal{K}(\mathcal{H})$, $C^* = C$ is guaranteed when C, for example, has finite rank (see Proposition 5.1 in[1]).

The above motivates us to study the following, among other issues in the unitary orbit of a Hermitian operator: let $b \in \mathcal{O}_A$ and $x \in (T\mathcal{O}_A)_b$ and suppose that there exists an uniparametric

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curve $\psi(t) = e^{tZ} b e^{-tZ}$ which is a minimal length curve among all the smooth curves joining b and $\psi(t)$ in $\mathcal{O}_A$ for $t \in \left[-\frac{\pi}{2\|Z\|}, \frac{\pi}{2\|Z\|}\right]$.

- Would Z be a compact minimal lifting of x (i.e $x = Zb - bZ$ and $\|Z\| = \|X\|_b$)?
- Can ψ be approximated in $\mathcal{O}_A$ by a sequence of minimal lenght curves of matrices?

The present work continues the analysis made in [1] of this homogeneous spaces and we use minimality properties that we developed in [4].

The results in this paper are divided in two parts. In the first we describe and study minimal length curves in the orbit of a particular compact Hermitian operator. In the second part we construct a sequence of minimal length curves of matrices which converges uniformly to the minimal length curves found in the first part.

2. Preliminaries and notation

Let $(\mathcal{H}, \langle , \rangle)$ be a separable Hilbert space. We denote by $\|h\| = \langle h, h\rangle^{1/2}$ the norm for each $h \in \mathcal{H}$. Let $\mathcal{B}(\mathcal{H})$ denote the set of bounded operators (with the identity operator I) and $\mathcal{K}(\mathcal{H})$, the two-sided closed ideal of compact operators on $\mathcal{H}$. Given $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$, we use the superscript ^{ah} (resp. ^{h}) to note the subset of anti-Hermitian (resp. Hermitian) elements of $\mathcal{A}$.

We consider the group of unitary operators in $\mathcal{B}(\mathcal{H})$

$$\mathcal{U}(\mathcal{H}) = \{u \in \mathcal{B}(\mathcal{H}) : \ uu^* = u^*u = I\}$$

and the unitary Fredholm group, defined as

$$\mathcal{U}_c(\mathcal{H}) = \{u \in \mathcal{U}(\mathcal{H}) : u - I \in \mathcal{K}(\mathcal{H})\}.$$

We denote with $\|\cdot\|$ the usual operator norm in $\mathcal{B}(\mathcal{H})$ and with $[\ ,\ ]$ the commutator operator, that is, for any $T, S \in \mathcal{B}(\mathcal{H})$

$$[T, S] = TS - ST.$$

It should be clear from the context the use of the same notation $\|\cdot\|$ to refer to the operator norm or a norm on $\mathcal{H}$.

We define the unitary orbit for any fixed $A \in \mathcal{K}(\mathcal{H})$, $A = A^*$, as

$$\mathcal{O}_A = \{uAu^* : u \in \mathcal{U}_c(\mathcal{H})\} \ \subset \ A + \mathcal{K}(\mathcal{H}).$$

$\mathcal{O}_A$ is an homogeneous space if we consider the action $\pi_b : \mathcal{U}_c(\mathcal{H}) \to \mathcal{O}_A$, $\pi_b(u) = ubu^*$. For each $b \in \mathcal{O}_A$, the isotropy group $\mathcal{I}_b$ is

$$\mathcal{I}_b = \{u \in \mathcal{U}_c(\mathcal{H}) : \ ubu^* = b\}.$$

Since for each $u \in \mathcal{U}_c(\mathcal{H})$ there always exists $X \in \mathcal{K}(\mathcal{H})^{ah}$ such that $u = e^X$ (see Proposition 2), the isotropy can be redefined by

$$\mathcal{I}_b = \{e^X \in \mathcal{U}_c(\mathcal{H}) : \ X \in \mathcal{K}(\mathcal{H})^{ah}, \ [X, b] = 0\}.$$

For each $b \in \mathcal{O}_A$, its tangent space is

$$(T\mathcal{O}_A)_b = \{Yb - bY : Y \in \mathcal{K}(\mathcal{H})^{ah}\} \subset \mathcal{K}(\mathcal{H})^{ah}.$$

Consider a smooth curve (i.e. C^1 and with derivative non equal to zero) $u : [0,1] \to \mathcal{U}_c(\mathcal{H})$ such that $u(0) = 1$ y $u'(0) = Y$, then the differential of the surjective map π_b at 1 is

$$\begin{aligned}(d\pi_b)_1(Y) &= \frac{d}{dt}\, \pi_b(u(t))|_{t=0} = u'(0)b\ u^*(0) + u(0)b\ u'(0)^* \\ &= Yb1^* + 1bY^* = Yb - bY = [Y, b].\end{aligned}$$

For every $b \in \mathcal{O}_A$ we consider each tangent space as

$$(T\mathcal{O}_A)_b \cong (T\mathcal{U}_c(\mathcal{H}))_1/(T\mathcal{I}_b)1 \cong \mathcal{K}(\mathcal{H})^{ah}/(\{b\}')^{ah},$$

being $\{b\}'$ the set of elements that commute with b in a C^*-algebra $\mathcal{A}$ (in this particular case $\mathcal{A} = \mathcal{K}(\mathcal{H})$). Let us consider the Finsler metric, defined for each $x \in (T\mathcal{O}_A)_b$ as

$$\|x\|_b = \inf\{\|Y\| : Y \in \mathcal{K}(\mathcal{H})^{ah} \text{ such that } [Y, b] = x\}$$

and it can be expressed in terms of the projection to the quotient $\mathcal{K}(\mathcal{H})^{ah}/(\{b\}')^{ah}$ as

$$\|Yb - bY\|_b = \|[Y]\| = \inf_{C \in (\{b\}')^{ah}} \|Y + C\|$$

for each class $[Y] = \left\{Y + C : \ C \in (\{b\}')^{ah}\right\}$. This Finsler norm is unitarily invariant under the action of the Fredholm group.

There always exists $Z \in \mathcal{B}(\mathcal{H})^{ah}$ such that $[Z, b] = x$ and $\|Z\| = \|x\|_b$. Such element Z is a called minimal lifting for x, and Z may not be compact and/or unique (see [4]). We consider piecewise smooth curves $\beta : [a, b] \to \mathcal{O}_A$. We define the rectifiable length of β as

$$\mathrm{L}(\beta) = \int_a^b \|\beta'(t)\|_{\beta(t)} \ dt,$$

and the rectifiable distance between two points of $\mathcal{O}_A$, named c_1 and c_2, is

$$\mathrm{dist}(c_1, c_2) = \inf\{\mathrm{L}(\beta) : \ \beta \text{ is smooth}, \ \beta(a) = c_1, \beta(b) = c_2\}.$$

If $\mathcal{A}$ is any C^*-algebra of $\mathcal{B}(\mathcal{H})$ and $\{e_k\}_{k=1}^{\infty}$ is a fixed orthonormal basis of $\mathcal{H}$, we denote with $\mathcal{D}(\mathcal{A})$ the set of diagonal operators, that is

$$\mathcal{D}(\mathcal{A}) = \{T \in \mathcal{A} : \ \langle Te_i, e_j\rangle = 0 \ , \quad \text{for all } i \neq j\}\,.$$

Given an operator $Z \in \mathcal{A}$, if there exists an operator $D_1 \in \mathcal{D}(\mathcal{A})$ such that

$$\|Z + D_1\| = \mathrm{dist}\left(Z, \mathcal{D}\left(\mathcal{A}\right)\right),$$

we say that D_1 is a best approximant of Z in $\mathcal{D}(\mathcal{A})$. In other terms, the operator $Z + D_1$ verifies the following inequality

$$\|Z + D_1\| \leq \|Z + D\|$$

for all $D \in \mathcal{D}(\mathcal{A})$. In this sense, we call $Z + D_1$ a minimal operator or similarly we say that D_1 is minimal for Z. If Z is anti-Hermitian it holds that

$$\mathrm{dist}\left(Z, \mathcal{D}\left(\mathcal{A}\right)\right) = \mathrm{dist}\left(Z, \mathcal{D}\left(\mathcal{A}^{ah}\right)\right),$$

since $\|Im(X)\| \leq \|X\|$ for every $X \in \mathcal{A}$.

Let $T \in \mathcal{B}(\mathcal{H})$ and consider the coefficients $T_{ij} = \langle Te_i, e_j\rangle$ for each $i, j \in \mathbb{N}$, that define an infinite matrix $(T_{ij})_{i,j\in\mathbb{N}}$. With this, the jth-column and ith-row of T are the vectors in ℓ^2 given by $c_j(T) = (T_{1j}, T_{2j}, ...)$ and $f_j(T) = (T_{i1}, T_{i2}, ...)$, respectively.

We use $\sigma(T)$ and $R(T)$ to denote the spectrum and range of $T \in \mathcal{B}(\mathcal{H})$, respectively.

We define $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{D}(\mathcal{B}(\mathcal{H}))$, $\Phi(X) = \mathrm{Diag}(X)$, which takes the main diagonal (i.e the elements of the form $\{\langle Xe_i, e_i\rangle\}_{i\in\mathbb{N}}$) of an operator X and builds a diagonal operator in the chosen fixed basis of $\mathcal{H}$. For a given bounded sequence $\{d_n\}_{n\in\mathbb{N}} \subset \mathbb{C}$ we denote with $\mathrm{Diag}\big(\{d_n\}_{n\in\mathbb{N}}\big)$ the diagonal (infinite) matrix with $\{d_n\}_{n\in\mathbb{N}}$ in its diagonal and 0 elsewhere.

3. The unitary Fredholm orbit of a Hermitian compact operator

We study the unitary orbit of a particular case of Hermitian operators, that is: $A \in \mathcal{K}(\mathcal{H})^h$, $A = u\mathrm{Diag}\left(\{\lambda_i\}_{i\in\mathbb{N}}\right) u^*$, with $u \in \mathcal{U}_c(\mathcal{H})$ and $\{\lambda_i\}_{i\in\mathbb{N}} \subset \mathbb{R}$ such that $\lambda_i \neq \lambda_j$ for each $i \neq j$. Consider $\mathcal{O}_A$ as defined in section 2 and $b = \mathrm{Diag}\left(\{\lambda_i\}_{i\in\mathbb{N}}\right) \in \mathcal{O}_A$. The isotropy $\mathcal{I}_b$ is the set $\{e^d : \ d \in \mathcal{D}(\mathcal{K}(\mathcal{H})^{ah})\}$ and $(T\mathcal{O}_A)_b$ can be identified with the quotient space $\mathcal{K}(\mathcal{H})^{ah}/\mathcal{D}(\mathcal{K}(\mathcal{H})^{ah})$. We state the first remark.

Proposition 1. *Let $b = \operatorname{Diag}(\{\lambda_i\}_{i\in\mathbb{N}}) \in \mathcal{O}_A$. For each $x \in (T\mathcal{O}_A)_b$, if $Z \in \mathcal{K}(\mathcal{H})^{ah}$ such that $[Z,b]=x$, then*

$$\|x\|_b = \inf_{D\in\mathcal{D}(\mathcal{K}(\mathcal{H})^{ah})} \|Z+D\| \tag{3.1}$$

Proof. If $Y_1, Y_2 \in \{Y \in \mathcal{K}(\mathcal{H})^{ah} : [Y,b]=x\}$ then

$$Y_1 - Y_2 \in \{D:\ [D,b] = Db - bD = 0\} = \{b\}'$$

and b is a diagonal operator, so every D is diagonal. Thus

$$Y_1 - Y_2 = D, \text{ with } D \text{ diagonal}$$

or equivalently

$$Y_1 = Y_2 + D, \text{ with } D \in \mathcal{D}(\mathcal{K}(\mathcal{H})^{ah}).$$

If $Y_2 \in \{Y \in \mathcal{K}(\mathcal{H})^{ah} : [Y,b]=x\}$, it follows that

$$\|x\|_b = \inf\{\|Y\| : Y \in \mathcal{K}(\mathcal{H})^{ah} \text{ such that } [Y,b]=x\}$$

$$= \inf\{\|Y\| : Y \in \mathcal{K}(\mathcal{H})^{ah} \text{ such that } Y = Y_2 + D, \text{ with } D \in \mathcal{D}(\mathcal{K}(\mathcal{H})^{ah})\}.$$

□

Fix $x = [Z_r, b] = Z_r b - bZ_r \in \mathcal{B}(\mathcal{H})^{ah}$, where Z_r is a anti-Hermitian operator defined as the infinite matrix given by

$$Z_r = i\begin{pmatrix} 0 & r\gamma & r\gamma^2 & r\gamma^3 & \cdots \\ r\gamma & d_2 & \gamma & \gamma^2 & \cdots \\ r\gamma^2 & \gamma & d_3 & \gamma^2 & \cdots \\ r\gamma^3 & \gamma^2 & \gamma^2 & d_4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{3.2}$$

$$= i\underbrace{\begin{pmatrix} 0 & r\gamma & r\gamma^2 & r\gamma^3 & \cdots \\ r\gamma & 0 & \gamma & \gamma^2 & \cdots \\ r\gamma^2 & \gamma & 0 & \gamma^2 & \cdots \\ r\gamma^3 & \gamma^2 & \gamma^2 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}}_{Y_r} + i\underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 0 & d_2 & 0 & 0 & \cdots \\ 0 & 0 & d_3 & 0 & \cdots \\ 0 & 0 & 0 & d_4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}}_{D_0} = Y_r + D_0. \tag{3.3}$$

This operator Z_r satisfies the conditions:

(1) $\gamma \in \mathbb{R}$ such that $|\gamma| < 1$.
(2) For each $j \in \mathbb{N}$, $j > 1$:

$$d_j = -\frac{1-\gamma^{j-2}}{1-\gamma} - \frac{\gamma^j}{1-\gamma^2}.$$

Notice that $\lim_{j\to\infty} d_j = \frac{1}{\gamma-1}$.

(3) $r \geq \frac{\left\|Y^{[1]}+D_0\right\|}{\left(\sum_{k=1}^{\infty}\gamma^{2k}\right)^{1/2}}$, where $Y^{[1]} = Y_r - \begin{pmatrix} 0 & r\gamma & r\gamma^2 & r\gamma^3 & \cdots \\ r\gamma & 0 & 0 & 0 & \cdots \\ r\gamma^2 & 0 & 0 & 0 & \cdots \\ r\gamma^3 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$.

Observe that the definition of each d_j is independent from the parameter r.

The operator $-iZ_r$ it has been studied in [4] and fulfills the conditions of minimality stated in Theorem 1 of the same work. Therefore, it holds that

$$\|[Y_r]\| = \inf_{D\in\mathcal{D}(\mathcal{K}(\mathcal{H})^{ah})} \|Y_r + D\| = \|Y_r + D_0\| = \|Z_r\|.$$

Moreover, the diagonal operator D_0 is the unique minimal diagonal (bounded, but non compact) operator for Y_r. Since $D_0 b - bD_0 = 0$, then $x = Y_r b - bY_r \in (T\mathcal{O}_A)_b$ and it holds that

$$\|x\|_b = \|Z_r b - bZ_r\|_b = \inf_{D\in\mathcal{D}(\mathcal{K}(\mathcal{H})^{h})} \|Y_r + D\| = \|[Y_r]\| = \|Z_r\| < \|Y_r + D\|$$

for all $D \in \mathcal{D}(\mathcal{K}(\mathcal{H})^{ah})$. In other words, there is no compact minimal lifting for x.

The following Proposition is a characterization of the unitary Fredholm group in terms of operators in $\mathcal{K}(\mathcal{H})^{ah}$.

Proposition 2. *If $X \in \mathcal{K}(\mathcal{H})^{ah}$ then $e^X \in \mathcal{U}_c(\mathcal{H})$. On the other hand, given $w \in \mathcal{U}_c(\mathcal{H})$ there exists $X \in \mathcal{K}(\mathcal{H})^{ah}$ such that $w = e^X$.*

Proof. If we consider the series expansion of e^X with $X \in \mathcal{K}(\mathcal{H})^{ah}$, then

$$e^X = 1 + X + \frac{1}{2}X^2 + \frac{1}{3!}X^3 + ...$$

$$= 1 + \underbrace{X}_{\in\mathcal{K}(\mathcal{H})^{ah}} \left[1 + \frac{1}{2}X + \frac{1}{3!}X^2 + ...\right] = 1 + K \ , \ K \in \mathcal{K}(\mathcal{H}),$$

and therefore $e^X \in \mathcal{U}_c(\mathcal{H})$.

On the other hand, given $w \in \mathcal{U}_c(\mathcal{H}) \subset \mathcal{U}(\mathcal{H})$ by Lemma 2.1 in [1] there exists $X \in \mathcal{K}(\mathcal{H})$ such that $w = e^X$. Moreover, the operator X can be chosen anti-Hermitian. Indeed, let $u(t)$ be a smooth curve in $\mathcal{U}_c(\mathcal{H})$ such that $u(0) = 1$ and $u'(0) = v \in \mathcal{B}(\mathcal{H})$. We claim that $v = -v^*$. In particular $u(t) = e^{tX}$, with $X \in \mathcal{K}(\mathcal{H})$ and $t \in \mathbb{R}$, is a smooth curve in $\mathcal{U}_c(\mathcal{H})$ that satisfies the previous conditions, with $u(0) = e^{0X} = 1$ and $u'(0) = Xe^{0X} = X$. Therefore, X can be chosen in $\mathcal{K}(\mathcal{H})^{ah}$. $\square$

Remark 3. *Even if $Z \notin \mathcal{K}(\mathcal{H})^{ah}$, e^Z may belong to $\mathcal{U}_c(\mathcal{H})$. Indeed, let $X_0 \in \mathcal{K}(\mathcal{H})^{ah}$, then $X_0 + 2\pi i I \notin \mathcal{K}(\mathcal{H})^{ah}$ but*

$$e^{X_0 + 2\pi i I} = e^{X_0} \in \mathcal{U}_c(\mathcal{H}).$$

Define the uniparametric curve β given by

$$\beta(t) = e^{tZ_r} b e^{-tZ_r} \ , \ t \in \left[-\frac{\pi}{2\|Z_r\|}, \frac{\pi}{2\|Z_r\|}\right]. \tag{3.4}$$

To prove that $\beta \subset \mathcal{O}_A$, we introduce first the next result.

Lemma 4. *Let Z_r the operator defined in (3.2). Then for each $t \in \mathbb{R}$, there exist $z_t \in \mathbb{C}$, $|z_t| = 1$ and $U(t) \in \mathcal{U}_c(\mathcal{H})$ such that*

$$e^{tZ_r} = z_t U(t).$$

Proof. Let $\alpha = -i \lim\limits_{n\to\infty} d_n = \frac{i}{1-\gamma}$. Then

$$e^{tZ_r + \alpha I t} = e^{tZ_r} e^{t\alpha I}.$$

Observe that $e^{t\alpha I} = e^{t\alpha} I$. It follows that

$$e^{tZ_r} = e^{-t\alpha} e^{tZ_r + t\alpha I} = e^{-t\alpha} e^{tY_r + tD_0 + t\alpha I},$$

with $e^{-t\alpha}\in\mathbb{C}$, $|e^{-t\alpha}|=1$ for every $t\in\mathbb{R}$. Also, $D_0+\alpha I\in\mathcal{D}(\mathcal{K}(\mathcal{H})^{ah})$, since it is bounded, diagonal and

$$\left|(D_0+\alpha I)_{jj}\right|=\left|d_j+\frac{1}{1-\gamma}\right|=\left|-\frac{1-\gamma^{j-2}}{1-\gamma}-\frac{\gamma^j}{1-\gamma^2}+\frac{1}{1-\gamma}\right|=\left|\frac{\gamma^{j-2}}{1-\gamma}-\frac{\gamma^j}{1-\gamma^2}\right|\to 0$$

when $j\to\infty$. Therefore, since $tZ_r+t\alpha I\in\mathcal{K}(\mathcal{H})^{ah}$ for every $t\in\mathbb{R}$ then $U(t)=e^{tZ_r+t\alpha I}\in\mathcal{U}_c(\mathcal{H})$ and

$$e^{tZ_r}=z_tU(t),\ \text{with } z_t=e^{t\alpha}\in\mathbb{C}.$$

$\square$

Remark 5. *Observe that* $\|Z_r\|=\|[Y_r]\|_{\mathcal{B}(\mathcal{H})^{ah}/\mathcal{D}(\mathcal{B}(\mathcal{H}))^{ah}}$ *and* Z_r *is (the unique) minimal lifting of* $x=[Y_r,b]$ *in* $\mathcal{B}(\mathcal{H})$. *Consider* $\mathcal{P}=\{uAu^*:\ u\in\mathcal{U}(\mathcal{H})\}$, *then by Theorem II in* [5] *the curve* β *has minimal length over all the smooth curves in* $\mathcal{P}$ *that join* $\beta(0)=b$ *and* $\beta(t)$, *with* $|t|\le\frac{\pi}{2\|Z_r\|}$. *Since* $\mathcal{O}_A\subseteq\mathcal{P}$, *then for each* $t_0\in\left[-\frac{\pi}{2\|Z_r\|},\frac{\pi}{2\|Z_r\|}\right]$ *follows that*

$$\begin{aligned}\mathrm{L}(\beta)&=\inf\{\mathrm{L}(\chi):\ \chi\in\mathcal{P},\ \chi(0)=b \text{ and } \chi(t_0)=\beta(t_0)\}\le\\ &\le\inf\{\mathrm{L}(\chi):\ \chi\in\mathcal{O}_A,\ \chi \text{ is smooth},\ \chi(0)=b \text{ and } \chi(t_0)=\beta(t_0)\}\\ &=\mathrm{dist}(b,\beta(t_0)),\end{aligned}$$

where $\mathrm{dist}(b,\beta(t_0))$ *is the rectifiable distance between* b *and* $\beta(t_0)$ *defined in the preliminaries.*

Using the previous remark and Lemma 4 we can prove the following Theorem.

Theorem 6. *Let* $A=u\mathrm{Diag}\left(\{\lambda_i\}_{i\in\mathbb{N}}\right)u^*$, *with* $u\in\mathcal{U}_c(\mathcal{H})$ *and* $\{\lambda_i\}_{i\in\mathbb{N}}\subset\mathbb{R}$ *such that* $\lambda_i\ne\lambda_j$ *for each* $i\ne j$. *Let* $b=\mathrm{Diag}\left(\{\lambda_i\}_{i\in\mathbb{N}}\right)\in\mathcal{O}_A$ *and the parametric curve* β *defined in (3.4). Then* β *satisfies:*

(1) $\beta(t)=e^{t(Z_r+\frac{i}{1-\gamma}I)}be^{-t(Z_r+\frac{i}{1-\gamma}I)}$, *which means that* $\beta\subset\mathcal{O}_A$.
(2) $\beta'(0)=x=Y_rb-bY_r=Z_rb-bZ_r\in(T\mathcal{O}_A)_b$.
(3) β *has minimal length between all smooth curves in* $\mathcal{O}_A$ *joining* b *with* $\beta(t_0)$, *for every* $t_0\in\left[-\frac{\pi}{2\|Z_r\|},\frac{\pi}{2\|Z_r\|}\right]$. *That is*

$$\mathrm{L}\left(\beta|_{[0,t_0]}\right)=\inf\{\mathrm{L}(\chi):\ \chi \text{ is smooth},\ \chi(0)=b \text{ and } \chi(t_0)=\beta(t_0)\}=\mathrm{dist}(b,\beta(t_0)).$$

(4) $\mathrm{L}\left(\beta|_{[0,t_0]}\right)=|t_0|\,\|x\|_b$, *for each* $t_0\in\left[-\frac{\pi}{2\|Z_r\|},\frac{\pi}{2\|Z_r\|}\right]$.

Proof. (1) By Lemma 4, if $U(t)=e^{tZ_r+t\frac{i}{1-\gamma}I}$, then β can be rewritten as

$$\begin{aligned}\beta(t)&=z_tU(t)b(z_tU(t))^*=z_t\overline{z_t}U(t)bU^{-1}(t)\\ &=U(t)bU^{-1}(t)=e^{t(Z_r+\frac{i}{1-\gamma}I)}be^{-t(Z_r+\frac{i}{1-\gamma}I)}\end{aligned}$$

and $U(t)\in\mathcal{U}_c(\mathcal{H})$ for each $t\in\mathbb{R}$. It follows that $\beta(t)\in\mathcal{O}_A$ for every $t\in\mathbb{R}$.

(2) $\beta'(0)=e^{tZ_r}\left[Z_r,b\right]e^{-tZ_r}\big|_{t=0}$.

(3) It is a direct consequence of $\mathcal{O}_A\subseteq\mathcal{P}$ and the minimality of Z_r in $\mathcal{B}(\mathcal{H})^{ah}/\mathcal{D}(\mathcal{B}(\mathcal{H}))^{ah}$, as was stated in Remark 5.

(4) Observe that $\mathrm{L}(\beta)=\int_0^{t_0}\|\beta'(t)\|_{\beta(t)}\ dt=t_0\|Y_rb-bY_r\|_b$. Indeed,

$$\begin{aligned}\|\beta'(t)\|_{\beta(t)}&=\left\|Z_re^{tZ_r}be^{-tZ_r}-e^{tZ_r}bZ_re^{-tZ_r}\right\|_{\beta(t)}=\left\|e^{tZ_r}\left[Z_r,b\right]e^{-tZ_r}\right\|_{\beta(t)}\\ &=\left\|z\overline{z}U(t)\left[Z_r,b\right]U^{-1}(t)\right\|_{\beta(t)}=|z|^2\left\|U(t)\left[Z_r,b\right]U^{-1}(t)\right\|_{\beta(t)}\\ &=\left\|U(t)\left[Z_r,b\right]U^{-1}(t)\right\|_{U(t)bU^{-1}(t)}=\|Z_rb-bZ_r\|_b\\ &=\|Y_rb-bY_r\|_b=\|x\|_b\,.\end{aligned}$$

where the equality $\|U(t)\,[Z_r,b]\,U^{-1}(t)\|_{U(t)bU^{-1}(t)} = \|Z_rb - bZ_r\|_b$ holds due to the unitary invariance of the Finsler norm.

$\square$

Summarizing, we obtained that the parametric curve given by

$$\pi_b \circ (e^{tZ_\alpha}) = e^{tZ_\alpha} b e^{-tZ_\alpha},\ Z_\alpha = Z_r + \frac{i}{1-\gamma} I \in \mathcal{K}(\mathcal{H})^{ah},$$

has minimal length between elements of $\mathcal{O}_A$. Nevertheless, the operator Z_α is not a minimal element in its class (recall that $[Z_r] = \{Z_r + D : D \in \mathcal{D}(\mathcal{K}(\mathcal{H})^{ah}) = [Y_r]\}$). On the other hand,

$$e^{tZ_\alpha} b e^{-tZ_\alpha} = e^{tZ_r} b e^{-tZ_r}$$

and Z_r is minimal, but it does not belong to $\mathcal{K}(\mathcal{H})^{ah}$. We conclude with the following result.

Corollary 7. *Let $b \in \mathcal{O}_A$, $b = \operatorname{Diag}(\{\lambda_i\}_{i\in\mathbb{N}})$ such that $\lambda_i \neq \lambda_j$ for each $i \neq j$. Then, there exist minimal length curves of the form $\rho(t) = e^{tZ} b e^{-tZ}$ in $\mathcal{O}_A$ such that they join b with other points of the orbit, but with $Z \in \mathcal{K}(\mathcal{H})^{ah}$ and $\|Z\| > \|[Z]\|_{\mathcal{K}(\mathcal{H})^{ah}/\mathcal{D}(\mathcal{K}(\mathcal{H})^{ah})}$.*

3.1. **Bounded minimal operators $Z + D$ with $Z \in \mathcal{K}(\mathcal{H})$ and non compact diagonal D.** Let Y_r, D_0 be the operators defined in (3.3). To stablish the equality $\beta(t) = e^{Y_r + D_0 + \frac{i}{1-\gamma} I} b e^{-(Y_r + D_0 + \frac{i}{1-\gamma} I)}$ in Theorem 6 the following properties were essential:

(1) $D_0 + \frac{i}{1-\gamma} I \in \mathcal{D}(\mathcal{K}(\mathcal{H})^{ah})$ and
(2) $\frac{i}{1-\gamma} I$ commutes with Z_r and b.

Since $\frac{i}{1-\gamma} I \notin \mathcal{K}(\mathcal{H})$, then $\frac{i}{1-\gamma} I \notin \{Z_r\}' \cap \{b\}'$. This motivated us to study more cases in $\mathcal{K}(\mathcal{H})$.

Let $\{\lambda_j(A)\}_{j=1}^\infty$ be the eigenvalues of a Hermitian compact operator A (suppose all have multiplicity one) and

$$b = \operatorname{Diag}\left(\{\lambda_j(A)\}_{j=1}^\infty\right) \in \mathcal{O}_A.$$

For x in the tangent space $(T\mathcal{O}_A)_b \subseteq \mathcal{K}(\mathcal{H})^{ah}$ holds that

$$\|x\|_b = \inf\{\|Z + D\| : Z \in \mathcal{K}(\mathcal{H})^{ah},\ [Z,b] = x \text{ and } [D,b] = 0\} = \|[Z]\|_{\mathcal{K}(\mathcal{H})^{ah}/\mathcal{D}(\mathcal{K}(\mathcal{H})^{ah})}.$$

In this context, there always exists $D_1 \in \mathcal{D}(\mathcal{B}(\mathcal{H})^{ah})$ such that

$$\|[Z]\| = \|Z + D_1\|,$$

nevertheless, in [4] we proved that the best diagonal approximant might not be compact. Suppose that the minimal $D_1 \notin \mathcal{D}(\mathcal{K}(\mathcal{H})^{ah})$. Denote with $\{(D_1)_{jj}\}_{j\in\mathbb{N}}$ the sequence of the elements of the main diagonal of D_1. The following proposition shows that if this sequence has a limit then $e^{t(Z+D_1)} b e^{-t(Z+D_1)} \in \mathcal{O}_A$.

Proposition 8. *Let $Z \in \mathcal{K}(\mathcal{H})^{ah}$ and suppose that there exists a unique $D_1 \in \mathcal{D}(\mathcal{B}(\mathcal{H})^{ah})$ such that*

$$\|[Z]\|_{\mathcal{K}(\mathcal{H})^{ah}/\mathcal{D}(\mathcal{K}(\mathcal{H})^{ah})} = \|Z + D_1\|$$

and D_1 is not compact. If there exists $\lambda \in i\mathbb{R}$ such that $\lim_{j\to\infty} (D_1)_{jj} = \lambda$, then the curve

$$\chi(t) = e^{t(Z + D_1 - \lambda I)} b e^{-t(Z + D_1 - \lambda I)}$$

has minimal length between all the smooth curves in $\mathcal{O}_A$ joining b with $\chi(t_0)$, for $t_0 \in [\frac{\pi}{2\|Z\|}, \frac{\pi}{2\|[Z]\|}]$.

Proof. First observe that $Re\,((D_1)_{jj}) = 0$ for each $j \in \mathbb{N}$, since $D_1 \in \mathcal{D}(\mathcal{B}(\mathcal{H})^{ah})$. Then,

$$\lim_{j\to\infty} (D_1)_j = \lambda$$

and $\lambda \neq 0$ since D_1 is not compact. Therefore, using functional calculus

$$\|Z + D_1 - \lambda I\| = \max\{|-\|[Z]\| - |\lambda||\,;\, \|[Z]\| - |\lambda|\} > \|[Z]\|.$$

Also $D_1 - \lambda I \in \mathcal{D}(\mathcal{K}(\mathcal{H})^{ah})$, since

$$|(D_1 - \lambda I)_{jj}| = |(D_1)_{jj} - \lambda| \to 0$$

when $j \to \infty$. Then, $Z + D_1 - \lambda I$ is not minimal in $\mathcal{K}(\mathcal{H})^{ah}/\mathcal{D}(\mathcal{K}(\mathcal{H})^{ah})$ but the curve parameterized by

$$\chi(t) = e^{t(Z+D_1-\lambda I)} b e^{-t(Z+D_1-\lambda I)} \in \mathcal{O}_A$$

has minimal length, as χ is equal to the curve

$$\delta(t) = e^{t(Z+D_1)} b e^{-t(Z+D_1)}.$$

Then, by Theorem I in [5], δ has minimal length in the homogeneous space given by

$$\{uAu^* : \ u \in \mathcal{U}(\mathcal{H})\}$$

and clearly, this homogeneous space contains to $\mathcal{O}_A$. Therefore χ has minimal length in $\mathcal{O}_A$. □

Given $Z \in \mathcal{K}(\mathcal{H})^{ah}$, it is not true that every diagonal operator D_1 such that $Z + D_1$ is minimal fulfills the condition

$$\exists \lambda \in i\mathbb{R} \ / \ \lim_{j\to\infty} (D_1)_{jj} = \lambda.$$

Indeed, consider the following operator

$$T = \begin{pmatrix} 0 & -\delta & \gamma & -\delta^2 & \gamma^2 & -\delta^3 & \gamma^3 & \cdots \\ -\delta & 0 & \gamma & -\delta^2 & \gamma^2 & -\delta^3 & \gamma^3 & \cdots \\ \gamma & \gamma & 0 & -\delta^2 & \gamma^2 & -\delta^3 & \gamma^3 & \cdots \\ -\delta^2 & -\delta^2 & -\delta^2 & 0 & \gamma^2 & -\delta^3 & \gamma^3 & \cdots \\ \gamma^2 & \gamma^2 & \gamma^2 & \gamma^2 & 0 & -\delta^3 & \gamma^3 & \cdots \\ -\delta^3 & -\delta^3 & -\delta^3 & -\delta^3 & -\delta^3 & 0 & \gamma^3 & \cdots \\ \gamma^3 & \gamma^3 & \gamma^3 & \gamma^3 & \gamma^3 & \gamma^3 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \text{ with } \gamma, \delta \in (0,1).$$

Since

$$(T^*T)_{11} = (T^*T)_{22} = \frac{\gamma^4}{1-\gamma^2} + \frac{\delta^2}{1-\delta^2}$$

and for each $n \geq 3$

$$(T^*T)_{nn} = \begin{cases} (n-1)\delta^n + \frac{\gamma^n}{1-\gamma^2} + \frac{\delta^{n+2}}{1-\delta^2} & \text{if } \ n = 2k \\ (n-1)\gamma^n + \frac{\gamma^{n+1}}{1-\gamma^2} + \frac{\delta^{n+1}}{1-\delta^2} & \text{if } \ n = 2k-1, \end{cases}$$

follows that

$$\begin{aligned} tr(T^*T) &= \sum_{n=1}^{\infty} (T^*T)_{nn} = (T^*T)_{11} + (T^*T)_{22} + \sum_{k=2}^{\infty} (T^*T)_{(2k)(2k)} + \sum_{k=2}^{\infty} (T^*T)_{(2k-1)(2k-1)} \\ &= 2\frac{\gamma^4}{1-\gamma^2} + 2\frac{\delta^2}{1-\delta^2} + \frac{2\left(\gamma^3 - 2\gamma^3\delta^2 + (2+\gamma(-2+\gamma(-2+3\gamma)))\delta^4\right)}{(-1+\gamma)^2(1+\gamma)\left(-1+\delta^2\right)^2} < \infty. \end{aligned}$$

Thus, $Z_0 = iT \in \mathcal{K}(\mathcal{H})^{ah}$ (moreover, Z_0 is a Hilbert Schmidt operator).

Let $D' = i\mathrm{Diag}\left(\{d'_n\}_{n\in\mathbb{N}}\right)$ the unique bounded diagonal operator such that

$$\langle c_1(Z_0), c_n(Z_0 + D')\rangle = 0 \ \forall \ n \in \mathbb{N}. \tag{3.5}$$

Simple operations show that the condition (3.5) implies that $\{d'_n\}_{n\in\mathbb{N}} = \{d'_{2k}\}_{k\in\mathbb{N}} \cup \{d'_{2k-1}\}_{k\in\mathbb{N}}$ with

$$d'_{2k} = -\gamma\left(\sum_{j=0}^{k-2}\gamma^j\right) + \left(\frac{\gamma^2}{\delta}\right)^k \frac{\gamma^2}{1-\gamma^2} + \sum_{j=1}^{k-1}\delta^j + \frac{\delta^{k+2}}{1-\delta^2}$$

and

$$d'_{2k-1} = \delta\left(\sum_{j=0}^{k-2}\delta^j\right) - \left(\frac{\delta^2}{\gamma}\right)^k \frac{\gamma}{1-\delta^2} - \sum_{j=1}^{k-2}\gamma^j + \frac{\gamma^{k+1}}{1-\gamma^2}$$

for each $k \in \mathbb{N}$. If $\gamma^2 \le \delta$ and $\delta^2 \le \gamma$ both sequences, $\{d'_{2k}\}_{k\in\mathbb{N}}$ and $\{d'_{2k-1}\}_{k\in\mathbb{N}}$, are convergent. For example, if we fix $\delta = 1/4$ and $\gamma = 1/2$ then

$$k \to \infty : \begin{cases} d'_{2k} \to 2/3 \\ d'_{2k-1} \to -2/3 \end{cases}$$

If $Z_0^{[1]}$ is the operator Z_0 with zeros in its first column and row and $r = \frac{\left\|Z_0^{[1]}+D'\right\|}{c_1(Z_0)}$ then $r(Z_0 - Z_0^{[1]}) + Z_0^{[1]} + D'$ is a minimal operator by the same arguments as well as Z_r was minimal. Nevertheless, $D' = i\mathrm{Diag}\left(\{d'_{2k}\}_{k\in\mathbb{N}} \cup \{d'_{2k-1}\}_{k\in\mathbb{N}}\right)$ satisfies

$$\lim_{n\in\mathbb{N}}(D')_{nn} = \begin{cases} \frac{2}{3}i & \text{if } \; n = 2k \\ -\frac{2}{3}i & \text{if } \; n = 2k-1. \end{cases}$$

We conclude that the sequence $\{(D')_{nn}\}_{n\in N}$ has no limit. This allow us to state the following corollary.

Corollary 9 (Minimal operators with oscillating diagonal)**.** *There exists $Z \in \mathcal{K}(\mathcal{H})^{ah}$ such that its best approximant $D_1 \in \mathcal{D}(\mathcal{B}(\mathcal{H})^{ah})$ does not fulfills the condition*

$$\exists\lambda \in i\mathbb{R} \;/\; \lim_{j\to\infty}(D_1)_{jj} = \lambda.$$

As Z_0, there exist other operators such that its best bounded diagonal approximant oscilates. Moreover, there exist examples of minimal operators in which the elements on the main diagonal have subsequences convergent to m different limits (for every $m \in \mathbb{N}$).

4. Approximation with minimal length curves of matrices

There are two main objectives in this section: the first is to build two sequences of minimal matrices which approximate Z_r and $Z_r + \frac{i}{1-\gamma}I$ in the strong operator topology and in the operator norm, respectively. The second objetive is to find a family of minimal length curves of matrices which approximates the curve β defined in (3.4).

Let Y_r be the anti-Hermitian compact operator defined in (3.3) and consider the following decomposition

$$Y_r = rL + Y^{[1]}, \text{ with } L = i\begin{pmatrix} 0 & \gamma & \gamma^2 & \gamma^3 & \cdots \\ \gamma & 0 & 0 & 0 & \cdots \\ \gamma^2 & 0 & 0 & 0 & \cdots \\ \gamma^3 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{4.1}$$

Let D_0 the diagonal bounded operator defined also in the previous section. If $r \ge \frac{\left\|Y^{[1]}+D_0\right\|}{\|c_1(L)\|}$, then $Z_r = rL + Y^{[1]} + D_0$ is minimal.

Let us consider for each $n \in \mathbb{N}$ the orthogonal projection P_n over the space generated by $\{e_1, ..., e_n\}$, that is $P_n = P_n I P_n$. We define the following finite range operators

$$Y_n = r_n P_n L P_n + P_n Y^{[1]} P_n, \tag{4.2}$$

with $r_n \in \mathbb{R}_{>0}$ for each $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ we define a diagonal operator $D_n = i\mathrm{Diag}(d_k^{(n)})$ such that

(1) $d_1^{(n)} = 0$;
(2) $\langle c_1(Y_n + D_n), c_j(Y_n + D_n)\rangle = 0$, for each $j \in \mathbb{N}$, $j \neq 1$;
(3) $d_k^{(n)} = 0$, for every $k > n$.

Thus, if we take for convention that $\sum_{j=0}^{2-3} \gamma^j = 0 = \sum_{j=n}^{n-1} \gamma^{2j-n+1}$, each $d_k^{(n)}$ is univocally determinated as

$$D_n = i\mathrm{Diag}(d_k^{(n)})/ \left\{ \begin{array}{l} d_k^{(n)} = -\sum_{j=0}^{k-3} \gamma^j - \sum_{j=i}^{n-1} \gamma^{2j-i} < 0 \\ d_k^{(n)} = 0 \text{ for all } k \in \mathbb{N} - \{2; 3; ...; n\}. \end{array} \right. \tag{4.3}$$

The proof is by induction over the indices k for every $n \in \mathbb{N}$. Observe that the choice of each $d_k^{(n)}$ is independent from the parameter r_n.

Consider the following lemma which will be used to prove the minimality of each $Y_n + D_n$ as matrices for a fixed r_n.

Lemma 10. *Let $Y_n = r_n P_n L P_n + P_n Y^{[1]} P_n$ and D_n as defined in (4.2) and (4.3) for each $n \in \mathbb{N}$, respectively. Then*

$$\sup_{n\in\mathbb{N}} \left\| P_n Y^{[1]} P_n + D_n \right\| < \infty.$$

Proof. Fix $n \in \mathbb{N}$, then

$$\begin{aligned} \left\| P_n Y^{[1]} P_n + D_n \right\| &\leq \left\| P_n Y^{[1]} P_n \right\| + \|D_n\| \leq \|P_n\|^2 \left\| Y^{[1]} \right\| + \sup_{1\leq k\leq n} \left| d_k^{(n)} \right| \\ &\leq \left\| Y^{[1]} \right\| + \left| d_n^{(n)} \right| \leq \left\| Y^{[1]} \right\| + \sup_{n\in\mathbb{N}} \left| d_n^{(n)} \right| \leq \left\| Y^{[1]} \right\| + \|D_0\| < \infty. \end{aligned}$$

□

As a consequence of this lemma, there exists a constant $M_0 \in \mathbb{R}_{>0}$ such that:

$$M_0 = \max\left\{ \sup_{n\in\mathbb{N}} \left\| P_n Y^{[1]} P_n + D_n \right\|, \left\| Y^{[1]} + D_0 \right\| \right\}. \tag{4.4}$$

Now we can prove the minimality of each $Y_n + D_n$.

Proposition 11. *Let $Y_n = r_n P_n L P_n + P_n Y^{[1]} P_n$ and D_n as defined in (4.2) and (4.3) for each $n \in \mathbb{N}$, respectively. Consider the real constant M_0 as in (4.4) and fix $r_n = \frac{M_0}{\|c_1(P_n L P_n)\|}$. Then for each $n \in \mathbb{N}$ the operator $Z_n = Y_n + D_n$ is minimal in $\mathcal{K}(\mathcal{H})^{ah}/\mathcal{D}(\mathcal{K}(\mathcal{H})^{ah})$, that is*

$$\|[Y_n]\| = \inf_{\tilde{D_n}\in\mathcal{D}(\mathcal{K}(\mathcal{H})^{ah})} \left\| Y_n + \tilde{D_n} \right\| = \|Y_n + D_n\| = M_0.$$

Proof. Fix $n \in \mathbb{N}$. Without loss of generality, we can consider $Y_n + D_n$ as a $n \times n$ matrix. Then

- $d_1^{(n)} = 0$;
- $\langle c_1(Y_n + D_n), c_j(Y_n + D_n)\rangle = 0$, for each $j \in \mathbb{N}$, $2 \leq j \leq n$;
- $\|c_1(Y_n + D_n)\| = \|c_1(Y_n)\| = r_n \|c_1(P_n L P_n)\| = M_0 \geq \left\| P_n Y^{[1]} P_n + D_n \right\|$.

Therefore, using Theorem 8 in [7] D_n is the unique minimal diagonal for $n \times n$ for Y_n. Since

$$\inf_{D\in\mathcal{D}(\mathcal{K}(\mathcal{H})^{ah})} \|Y_n + D\| = \min_{\tilde{D_n}\in\mathcal{D}(M_n(\mathbb{C})^{ah})} \|Y_n + D_n\|,$$

it follows that

$$\|[Y_n]\| = \|Y_n + D_n\|.$$

□

Observe that the norm of the minimal operator $Y_n + D_n$ is M_0 for every $n \in \mathbb{N}$.

Remark 12. *For every $n \in \mathbb{N}$ holds that*

$$\inf_{D \in \mathcal{D}(\mathcal{K}(\mathcal{H})^{ah})} \|Y_n + D\| = \min_{D' \in \mathcal{D}(M_n(\mathbb{C})^{ah})} \|Y_n + D'\| = \|Y_n + D_n\|,$$

but there is no uniqueness in $\mathcal{D}(\mathcal{K}(\mathcal{H})^{ah})$ even when it occurs in $\mathcal{D}(M_n(\mathbb{C})^{ah})$. Moreover, every block operator of the form
$C_n = \begin{pmatrix} D_n & 0 \\ 0 & D_c \end{pmatrix}$, *with D_c diagonal and such that $\|D_c\| \leq \|c_1(Y_n)\|$ satisfies*

$$\|Y_n + C_n\| = \max\{\|Y_n + D_n\|; \|D_c\|\} = \|Y_n + D_n\| = \|[Y_n]\|.$$

Reconsider the operator $Y_r = rL + Y^{[1]}$ fixing $r = \frac{M_0}{\|c_1(L)\|}$. Note that

$$\frac{\left\|Y^{[1]} + D_0\right\|}{\|c_1(L)\|} \leq r < \infty$$

where the inequality on the right holds due to Lemma 10. Then, $Z_r = Y_r + D_0$ satisfies the hypothesis of Theorem 1 in [4] and is a minimal operator with D_0, the unique (non compact) bounded diagonal operator such that

$$\|[Y_r]\| = \inf_{D \in \mathcal{D}(\mathcal{K}(\mathcal{H})^{ah})} \|Y_r + D\| = \|Z_r\|.$$

Moreover,

$$\|[Y_r]\| = \|c_1(Z_r)\| = \|c_1(Y_r)\| = M_0.$$

Therefore,

$$\|[Y_r]\| = \|[Y_n]\|. \tag{4.5}$$

The following result vinculate Y_r with Y_n.

Proposition 13. *Let Y_r the operator defined in (4.1) and $\{Y_n\}_{n=1}^{\infty}$ the family of finite range operators defined in (4.2). If M_0 is the real constant defined in (4.4) such that $r = \frac{M_0}{\|c_1(Y_r)\|}$ and $r_n = \frac{M_0}{\|c_1(P_nLP_n)\|}$ for each $n \in \mathbb{N}$, are fixed. Then*

1. $\lim_{n\to\infty} r_n = r$.
2. $Y_n \to Y_r$ *when $n \to \infty$ in the operator norm.*

Proof. (1) Since $\|c_1(Y_n)\| = \left(\sum_{i=1}^{n-1} \gamma^{2i}\right)^{\frac{1}{2}}$ and $\|c_1(Y_r)\| = \left(\sum_{i=1}^{\infty} \gamma^{2i}\right)^{\frac{1}{2}}$, follows that $\lim_{n\to\infty} r_n = r$.

(2) Fix $\epsilon > 0$. The condition $\lim_{n\to\infty} r_n = r$ implies that there exists $n_1 \in \mathbb{N}$ such that if $n \geq n_1$ then $|r_n - r| < \epsilon$. Since the sequence is bounded, there exists $M_1 > 0$ such that $|r_n| \leq M_1$ for each $n \in \mathbb{N}$.

On the other hand, it is not difficult to see that L and $Y^{[1]}$ are Hilbert-Schmidt operators, then there exists $n_2 \in \mathbb{N}$ such that

$$n \geq n_2 \Rightarrow \|L - P_nLP_n\| < \epsilon$$

and

$$n \geq n_2 \Rightarrow \left\|Y^{[1]} - P_nY^{[1]}P_n\right\| < \epsilon.$$

Therefore, for every $n \geq n_0 = \max\{n_1; n_2\}$

$$\begin{aligned}\|Y_r - Y_n\| &= \left\|rL + Y^{[1]} - r_nP_nLP_n - P_nY^{[1]}P_n\right\| \leq \|rL - r_nP_nLP_n\| + \left\|Y^{[1]} - P_nY^{[1]}P_n\right\| \\ &\leq |r - r_n|\,\|L\| + \|r_n\|\,\|L - P_nLP_n\| + \left\|Y^{[1]} - P_nY^{[1]}P_n\right\| \\ &< \epsilon\,\|L\| + M_1\epsilon + \epsilon = \epsilon\,(\|L\| + M_1 + 1).\end{aligned}$$

We conclude that $\|Y_n - Y_r\| \to 0$ when $n \to \infty$.

□

Observe that the numerical sequence $\{d_k^{(n)}\}_{n\in\mathbb{N}}$ converges to d_k when $n\to\infty$, for each $k\in\mathbb{N}$

$$d_k^{(n)} \searrow -\sum_{j=0}^{k-3}\gamma^j - \sum_{j=k}^{\infty}\gamma^{2j-k} = -\sum_{j=0}^{i-3}\gamma^j - \frac{\gamma^k}{1-\gamma^2} = d_k.$$

As a consequence, the sequence of diagonal operators $\{D_n\}_{n\in\mathbb{N}}$ converges strongly to the unique best approximant (non compact) diagonal $D_0 \in \mathcal{D}(\mathcal{B}(\mathcal{H}))$ for Y_r.

Proposition 14. *Let Y_r be the operator defined in (3.3) and D_0 the unique bounded diagonal operator such that $\|Y_r + D_0\| < \|Y_r + D\|$ for all $D \in \mathcal{D}(\mathcal{K}(\mathcal{H})^{ah})$. Consider the sequence of finite range diagonal operators $\{D_n\}_{n\in\mathbb{N}}$ defined in (4.3). Then*

$$D_n \to D_0$$

in the strong operator topology.

Proof. Let $x\in\mathcal{H}$ and $\epsilon>0$. For each $n\in\mathbb{N}$,

$$\left|d_k^{(n)} - d_k\right| = \begin{cases} \left|\sum_{j=n}^{\infty}\gamma^{2j-k}\right| \ if \ k\le n \\ |d_k| \ if \ k>n. \end{cases}$$

If $k\le n$ y $\gamma\le 1$, the series $\sum_{j=n}^{\infty}\gamma^{2j-i}$ is convergent and then, there exists $n_0\in\mathbb{N}$ such that

$$n\ge n_0 \Rightarrow \left|\sum_{j=n}^{\infty}\gamma^{2j-k}\right| < \epsilon.$$

If $i>n$, then $|d_k|\le\|D_0\|$. Thus,

$$\|(D_n-D_0)\,x\|^2 = \langle (D_n-D_0)\,x,(D_n-D_0)\,x\rangle \le \epsilon^2\sum_{i=1}^{n}|\langle x,e_i\rangle|^2 + \|D_0\|^2\sum_{i=n}^{\infty}|\langle x,e_i\rangle|^2.$$

$\langle x,e_i\rangle$ are the coordinates of x in the basis $\{e_i\}_{i\in\mathbb{N}}$ so they form a sequence $\{\langle x,e_i\rangle\}_{i\in\mathbb{N}}\subset\ell^2$. Then, there exists $n_1\in\mathbb{N}$ such that

$$n\ge n_1 \Rightarrow \sum_{i=n}^{\infty}|\langle x,e_i\rangle|^2 < \epsilon^2.$$

Finally, for every $n\ge\max\{n_0;n_1\}$

$$\|(D_n-D_0)\,x\|^2 < \epsilon^2\|x\|^2 + \|D_0\|^2\epsilon^2 = \epsilon^2\left(\|x\|^2+\|D_0\|\right),$$

and it holds for each $x\in\mathcal{H}$ fixed. □

Observe that Propositions 13 and 14 imply that $Z_n = Y_n + D_n$ tends to Z_r in the strong operator topology. Since $D_n\in\mathcal{K}(\mathcal{H})^{ah}$ for all n and $D_0\in\mathcal{D}(\mathcal{B}(\mathcal{H})^{ah})\setminus\mathcal{D}(\mathcal{K}(\mathcal{H})^{ah})$, the convergence can not be in the operator norm. To establish the second main result of this section we prove first the convergence in the operator norm of $Z_n+\alpha I$ to $Z_r+\alpha I$, for a particular $\alpha\in\mathbb{R}$.

Proposition 15. *Let $Y_r, D_0, \{Y_n\}_{n\in\mathbb{N}}, \{D_n\}_{n\in\mathbb{N}}, \{P_n\}_{n\in\mathbb{N}}$ be the operators and sequence of operators defined previously. Then*

$$Y_n + D_n + \frac{i}{1-\gamma}P_n \to Y_r + D_0 + \frac{i}{1-\gamma}I,$$

in the operator norm when $n\to\infty$.

Proof. Let $\epsilon > 0$, then

$$\left\|Y_r + D_0 + \frac{i}{1-\gamma}I - Y_n - D_n - \frac{i}{1-\gamma}P_n\right\| \leq \|Y_r - Y_n\| + \left\|D + \frac{i}{1-\gamma}I - D_n - \frac{i}{1-\gamma}P_n\right\|.$$

By Proposition 13, there exists $n_1 \in \mathbb{N}$ such that

$$n \geq n_1 \Rightarrow \|Y_r - Y_n\| < \epsilon.$$

We focus on the second term. For each $n \in \mathbb{N}$

$$\left\|D_0 + \frac{i}{1-\gamma}I - D_n - \frac{i}{1-\gamma}P_n\right\| = \sup_{k\in\mathbb{N}} \left|d_k + \frac{1}{1-\gamma} - d_k^{(n)} - \left(\frac{1}{1-\gamma}P_n\right)_{kk}\right|,$$

but

$$\left|d_k + \frac{1}{1-\gamma} - d_k^{(n)} - \left(\frac{1}{1-\gamma}P_n\right)_{kk}\right| = \begin{cases} \left|\sum_{j=n}^{\infty} \gamma^{2j-k}\right| & if\ k \leq n \\ \left|d_k + \frac{1}{1-\gamma}\right| & if\ k > n \end{cases}.$$

Which means that there are two possibilities:

(1) If $k \leq n$, then $\left|d_k + \frac{1}{1-\gamma} - d_k^{(n)} - \left(\frac{1}{1-\gamma}P_n\right)_{kk}\right| = \left|\sum_{j=n}^{\infty} \gamma^{2j-k}\right|$. It is a convergent series, so there exists $n_2 \in \mathbb{N}$ such that

$$n \geq n_2 \Rightarrow \left|d_k + \frac{1}{1-\gamma} - d_k^{(n)} - \left(\frac{1}{1-\gamma}P_n\right)_{kk}\right| = \left|\sum_{j=n}^{\infty} \gamma^{2j-k}\right| < \frac{\epsilon}{2}.$$

(2) If $k > n$, then $\left|d_k + \frac{1}{1-\gamma} - d_k^{(n)} - \left(\frac{1}{1-\gamma}P_n\right)_{kk}\right| = \left|d_k + \frac{1}{1-\gamma}\right|$. Since $(id_k) \to -\frac{i}{1-\gamma}$ when $k \to \infty$, there exists $k_1 \in \mathbb{N}$ such that

$$k \geq k_1 \Rightarrow \left|d_k + \frac{1}{1-\gamma}\right| < \frac{\epsilon}{2}.$$

If $n \geq k_1$, then $k > n \geq k_1$ and

$$\left|d_k + \frac{1}{1-\gamma} - d_k^{(n)} - \left(\frac{1}{1-\gamma}P_n\right)_{kk}\right| = \left|d_k + \frac{1}{1-\gamma}\right| < \frac{\epsilon}{2}.$$

Then, there exists $n_3 \in \mathbb{N}$ (by example, $n_3 = \max\{k_1; n_2\}$) such that

$$n \geq n_3 \Rightarrow \left\|D_0 + \frac{i}{1-\gamma}I - D_n - \frac{i}{1-\gamma}P_n\right\| \leq \frac{\epsilon}{2} < \epsilon,$$

since for each n holds that

$$\sup_{k\in\mathbb{N}} \left|d_k + \frac{1}{1-\gamma} - d_k^{(n)} - \left(\frac{1}{1-\gamma}P_n\right)_{kk}\right| = \max\left\{\max_{1\leq k\leq n} \left|\sum_{j=n}^{\infty} \gamma^{2j-k}\right|; \sup_{k>n} \left|d_k + \frac{1}{1-\gamma}\right|\right\}.$$

Finally, if $n_0 = \max\{n_1; n_2; n_3\}$ it follows that

$$n \geq n_0 \Rightarrow \left\|Y_r + D_0 + \frac{i}{1-\gamma}I - Y_n - D_n - \frac{i}{1-\gamma}P_n\right\| < 2\epsilon,$$

which means that $Y_n + D_n + \frac{i}{1-\gamma}P_n$ tends to $Y_r + D_0 + \frac{i}{1-\gamma}I$ when $n \to \infty$ in the operator norm. $\square$

In the above proof we also obtained that $\{D_n + \frac{i}{1-\gamma}P_n\}_{n\in\mathbb{N}}$, which is a finite range operator sequence tends in the operator norm to $D_0 + \frac{i}{1-\gamma}I \in \mathcal{D}(\mathcal{K}(\mathcal{H})^{ah})$. Even though $Y_n + D_n + \frac{i}{1-\gamma}P_n$ and $Y_r + D_0 + \frac{i}{1-\gamma}I$ are not minimal operators, they are useful to construct minimal length curves in the unitary orbit of A. We will also use the operators $Y_n + D_n + \frac{i}{1-\gamma}P_n$ to construct a sequence of minimal length curves that converge to β defined in (3.4).

The first result in this direction is the convergence of the sequence of exponential curves in $\mathcal{O}_A$.

Proposition 16. *Let $b \in \mathcal{O}_A$ and $\beta_n(t) = e^{tZ_n}be^{-tZ_n}$ a sequence of curves in $\mathcal{O}_A$ with $\{Z_n\}_{n\in\mathbb{N}} \subset \mathcal{K}(\mathcal{H})^{ah}$ such that $\|Z_n - Z\| \to 0$ when $n \to \infty$. If we define $\beta(t) = e^{tZ}be^{-tZ}$, then*

$$\beta_n \to \beta$$

uniformly in the operator norm when $n \to \infty$ for any closed real interval $[t_1, t_2]$.

Proof. Let $\epsilon > 0$.

$$\begin{aligned}\|\beta_n(t) - \beta(t)\| &= \left\|e^{tZ_n}be^{-tZ_n} - e^{tZ}be^{-tZ}\right\| \leq \left\|e^{tZ_n}be^{-tZ_n} - e^{tZ}be^{-tZ_n}\right\| + \left\|e^{tZ}be^{-tZ_n} - e^{tZ}be^{-tZ}\right\| \\ &\leq \left\|\left(e^{tZ_n} - e^{tZ}\right)be^{-tZ_n}\right\| + \left\|e^{tZ}b\left(e^{-tZ_n} - e^{-tZ}\right)\right\| \leq \left(\left\|e^{tZ_n} - e^{tZ}\right\| + \left\|e^{-tZ_n} - e^{-tZ}\right\|\right)\|b\|.\end{aligned}$$

It is known that the exponential map $exp : \mathcal{K}(\mathcal{H})^{ah} \to \mathcal{U}_c(\mathcal{H})$ is Lipschitz continuous in compact sets of $\mathcal{K}(\mathcal{H})$, then there exists $n_0 \in \mathbb{N}$ such that

$$n \geq n_0 \Rightarrow \begin{cases} \left\|e^{tZ_n} - e^{tZ}\right\| < \frac{\epsilon}{\|b\|} \\ \left\|e^{-tZ_n} - e^{-tZ}\right\| < \frac{\epsilon}{\|b\|}, \end{cases}$$

for each t in a closed interval $[t_1, t_2] \subset \mathbb{R}$. Therefore

$$\|\beta_n(t) - \beta(t)\| < \epsilon$$

for each $n \geq n_0$ and $t \in [t_1, t_2]$, which implies that $\beta_n \to \beta$ uniformly in the operator norm. $\square$

If we consider the sequence $\{Y_n + D_n + \frac{i}{1-\gamma}P_n\}_{n\in N}$, by Proposition 15 follows that

$$Y_n + D_n + \frac{i}{1-\gamma}P_n \to Y_r + D_0 + \frac{i}{1-\gamma}I$$

in the operator norm when $n \to \infty$. We define for each $n \in \mathbb{N}$ the curves parametrized by

$$\beta_n(t) = e^{t(Y_n + D_n + \frac{i}{1-\gamma}P_n)}be^{-t(Y_n + D_n + \frac{i}{1-\gamma}P_n)},\ t \in [0, t_0]. \tag{4.6}$$

Observe that these curves can be considered as matricial type, since each operator Y_n, D_n and P_n is a compresion.

Below, we state the following result.

Theorem 17. *Let A and $b \in \mathcal{O}_A$ as in Theorem 6. Let $\{\beta_n\}_{n\in\mathbb{N}}$, the sequence of curves defined as in (4.6), and β the curve defined in (3.4). Then, for each $n \in \mathbb{N}$*

(1) $\begin{cases} \beta_n(0) = b \\ \beta_n'(0) = Y_nb - bY_n \in (T\mathcal{O}_A)_b. \end{cases}$.

(2) $\beta_n(t) = e^{t(Y_n+D_n)}be^{-t(Y_n+D_n)}$ *for all t, since $\frac{i}{1-\gamma}P_n$ commutes with $Y_n + D_n$.*

(3) *For each $t_0 \in \left[-\frac{\pi}{2\|[Y_n]\|}, \frac{\pi}{2\|[Y_n]\|}\right] = \left[-\frac{\pi}{2M_0}, \frac{\pi}{2M_0}\right]$ holds that*

$$\mathrm{L}\left(\beta_n|_{[0,t_0]}\right) = |t_0|\,\|[Y_n]\| = |t_0|\,M_0 = \mathrm{L}(\beta).$$

(4) $\beta_n : [0, t_0] \to \mathcal{O}_A$ *with $t_0 \in \left[-\frac{\pi}{2M_0}, \frac{\pi}{2M_0}\right]$ is a minimal length curve in $\mathcal{O}_A$.*

(5) $\beta_n'(0) \to \beta'(0)$ *in the norm $\|.\|_b$ of $(T\mathcal{O}_A)_b$.*

Moreover, by Proposition 16 $\beta_n \to \beta$ uniformly in the operator norm in the interval $\left[-\frac{\pi}{2M_0}, \frac{\pi}{2M_0}\right]$.

Proof. The proof of items (1), (2), (3) is analogue to proof in Theorem 6. The equality $\|[Y_n]\| = M_0 = \|[Y_r]\|$ is due to Proposition 11.

Since for each $n \in \mathbb{N}$ fixed $Y_n + D_n$ is a minimal compact operator, Theorem I in [5] states that β_n is a minimal length curve between all curves in $\mathcal{O}_A$ joining $\beta_n(0) = b$ to $\beta_n(t)$ with $|t| \leq \frac{\pi}{2\|Y_n+D_n\|}$. This means that β_n is also a minimal curve in $\mathcal{O}_A$. Then, (4) is proved.

We proceed to prove (5): let $\epsilon > 0$. Then there exists $n_0 \in \mathbb{N}$ such that $n \geq n_0 \Rightarrow \|Y_n - Y_r\| < \epsilon$. Therefore,

$$\begin{aligned}\|\beta_n'(0) - \beta'(0)\|_b &= \inf\left\{\|Z\| : Z \in \mathcal{K}(\mathcal{H})^{ah},\ [Z,b] = (Y_n - Y_r)\,b - b\,(Y_n - Y_r)\right\} \\ &= \inf_{D \in \mathcal{D}\left(\mathcal{K}(\mathcal{H})^{ah}\right)} \|Y_n - Y_r + D\| \leq \|Y_n - Y_r\| < \epsilon\end{aligned}$$

for each $n \geq n_0$. It follows that $\|\beta_n'(0) - \beta'(0)\|_b \to 0$ when $n \to \infty$.

□

As a conclusion of this section, we obtain a minimal lenght curve $\beta \subset \mathcal{O}_A$ that can be uniformly approximated by minimal curves of matrices $\{\beta_n\}$, but β does not have a compact minimal lifting, although each β_n has at least one minimal matricial lifting.

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