Inequalities related to Bourin and Heinz means with a complex parameter*

T. Bottazzi, R. Elencwajg, G. Larotonda and A. Varela †

Abstract

A conjecture posed by S. Hayajneh and F. Kittaneh claims that given A, B positive matrices, $0 \leq t \leq 1$, and any unitarily invariant norm it holds

$$|||A^t B^{1-t} + B^t A^{1-t}||| \leq |||A^t B^{1-t} + A^{1-t} B^t|||.$$

Recently, R. Bhatia proved the inequality for the case of the Frobenius norm and for $t \in [\frac{1}{4}; \frac{3}{4}]$. In this paper, using complex methods we extend this result to complex values of the parameter $t = z$ in the strip $\{z \in \mathbb{C} : \mathrm{Re}(z) \in [\frac{1}{4}; \frac{3}{4}]\}$. We give an elementary proof of the fact that equality holds for some z in the strip if and only if A and B commute. We also show a counterexample to the general conjecture by exhibiting a pair of positive matrices such that the claim does not hold for the uniform norm. Finally, we give a counterexample for a related singular value inequality given by $s_j(A^t B^{1-t} + B^t A^{1-t}) \leq s_j(A + B)$, answering in the negative a question made by K. Audenaert and F. Kittaneh. The methods of proof and examples can be adapted with no modifications to operator algebras (infinite dimensional setting), for instance it follows that the inequality above holds for Hilbert-Schmidt operators with their Banach algebra norm derived from the infinite trace of $B(H)$.[1]

1 Introduction

We begin this paper with some notations and definitions. The context here is the algebra of $n \times n$ complex entries matrices, but the proofs adapt well to other (infinite dimensional) settings in operator theory, so let us assume that $\mathcal{A}$ stands for an operator algebra with trace, for instance $\mathcal{A} = M_n(\mathbb{C})$ with its usual trace, or $\mathcal{A} = B_2(H)$, the Hilbert-Schmidt operators acting on a separable complex Hilbert space with the infinite trace, or $\mathcal{A} = (\mathcal{A}, Tr)$ a C^*-algebra with a finite faithful trace.

*2000 MSC. Primary 15A45, 47A30; Secondary 15A42, 47A63.

†All authors supported by Instituto Argentino de Matemática, CONICET and Universidad Nacional de General Sarmiento.

[1]**Keywords and phrases:** Frobenius norm, Heinz mean, matrix inequality, matrix power, positive matrix, trace inequality, unitarily invariant norm.

Definitions 1.1. *Let $|||\cdot|||$ denote an unitarily invariant norm on $\mathcal{A}$, which we assume is equivalent to a symmetric norm, that is*

$$|||XYZ|||| \leq \|X\|_\infty|||Y||||\|Z\|_\infty$$

whenever $Y \in \mathcal{A}$ (from now on $\|\cdot\|_\infty$ will denote the norm of the operator algebra).
For convenience we will use the notation $\tau(X) = Re\ Tr(X)$. Let $|X| = \sqrt{X^*X}$ stand for the modulus of the matrix or operator X, then the (right) polar decomposition of X is given by $X = U|X|$ where U is a unitary such that U maps $Ran|X|$ into $Ran(X)$ and is the identity on $Ran|X|^\perp = Ker(X)$. Note that $\|X\|_2^2 = Tr(X^*X) = Tr[|X|^2]$.

Consider the inequality

$$\tau(A^z B^z A^{1-z} B^{1-z}) \leq \tau(AB), \tag{1}$$

for positive invertible operators $A, B > 0$ in $\mathcal{A}$, and $z \in \mathbb{C}$. We introduce some notation regarding vertical strips in the complex plane: let

$$\mathcal{S}_0 = \{z \in \mathbb{C} : 0 \leq \text{Re}(z) \leq 1\}, \qquad \mathcal{S}_{1/4} = \{z \in \mathbb{C} : 1/4 \leq \text{Re}(z) \leq 3/4\};$$

we will study the validity of (1) in both S_0 and $S_{1/4}$.
Intimately related to the expression above are the inequalities

$$|||b_t(A,B)|||| \leq |||h_t(A,B)||| \tag{2}$$

and

$$|||b_t(A,B)||| \leq |||A+B|||, \tag{3}$$

for positive matrices $A, B \geq 0$ in $\mathcal{A}$, where

$$b_t(A,B) = A^t B^{1-t} + B^t A^{1-t} \quad t \in [0,1];$$

the name b_t is due to Bourin, who conjectured inequality (3) for $n \times n$ matrices in [5], and

$$h_t(A,B) = A^t B^{1-t} + A^{1-t} B^t \quad t \in [0,1]$$

is named after Heinz, and the well-known [7] inequality

$$|||h_t(A,B)||| \leq |||A+B|||$$

carrying his name.

Recently, S. Hayajneh and F. Kittanneh proposed in [6] that the stronger (2) should also be valid in $M_n(\mathbb{C})$; however, numerical computations (see Section 3) show that, at least for *the uniform norm*, this is false.

If we focus on the case $|||X||| = \|X\|_2 = Tr(X^*X)^{1/2}$ (the Frobenius norm in the case of $n \times n$ matrices) and we write $h_t = h_t(A, B)$, $b_t = b_t(A, B)$, then

$$\begin{aligned} Tr|b_t|^2 &= \tau(b_t^* b_t) = \tau(B^{1-t}A^t + A^{1-t}B^t)(A^tB^{1-t} + B^tA^{1-t}) \\ &= \tau(B^{2(1-t)}A^{2t}) + \tau(A^{2(1-t)}B^{2t}) + 2\tau(A^tB^tA^{1-t}B^{1-t}) \end{aligned}$$

where we have repeatedly used the ciclicity of τ (i.e. $\tau(XY) = \tau(YX)$) and the fact that $\tau(Z^*) = \tau(Z)$. Likewise

$$Tr|h_t|^2 = \tau(B^{2(1-t)}A^{2t}) + \tau(A^{2(1-t)}B^{2t}) + 2\tau(AB).$$

Thus, proving that $\|b_t\|_2 \leq \|h_t\|_2$ amounts to prove that

$$\tau(A^tB^tA^{1-t}B^{1-t}) \leq \tau(AB), \tag{4}$$

and in fact, it is clear that both inequalities are equivalent -as remarked in [6]-.

2 Main results

We will divide the problem in regions of the plane (or the line), and then we will also consider the possiblity of attaining the equality; we will see that this is only possible in the trivial case, i.e. when A, B commute. We recall the generalized Hölder inequality, that we will use frequently: let $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$ for $p, q, r \geq 1$ and X, Y, Z in $\mathcal{A}$, then

$$Tr(XYZ) \leq \|XYZ\|_1 \leq \|X\|_p \|Y\|_q \|Z\|_r. \tag{5}$$

This is just a combination of the usual Hölder inequality together with

$$\|XY\|_s \leq \|X\|_p \|Y\|_q$$

provided $s \geq 1$ and $\frac{1}{p} + \frac{1}{q} = \frac{1}{s}$ (see [8], Theorem 2.8, for more details).

2.1 The inequality in the strip $\mathcal{S}_{1/4}$

We begin with an easy consequence of an inequality due to Araki-Lieb and Thirring.

Lemma 2.1. *If $A, B \geq 0$ and $r \geq 2$, then*

$$\|A^{1/r}B^{1/r}\|_r \leq Tr(AB)^{1/r}.$$

Proof. Note that

$$\|A^{1/r}B^{1/r}\|_r^r = Tr([A^{1/r}B^{1/r}B^{1/r}A^{1/r}]^{r/2}) = Tr([A^{1/r}B^{2/r}A^{1/r}]^{r/2})$$

which, by the inequality of Araki-Lieb and Thierring (see [2], and note that $r/2 \geq 1$) is less or equal than

$$Tr(A^{r/2r}B^{r2/2r}A^{r/2r}) = Tr(A^{1/2}BA^{1/2}),$$

which in turn equals $Tr(AB)$. □

Note that if we exchange the variables $z \mapsto 1-z$ and exchange the role of A, B, it suffices to consider half-strips or half-intervals around $\mathrm{Re}(z) = 1/2$.

Proposition 2.2. *If* $0 < A, B$ *and* $z \in \mathcal{S}_{1/4}$, *then*

$$|Tr(A^z B^z A^{1-z} B^{1-z})| \leq Tr(AB).$$

Proof. Let $z = 1/2 + iy$, $y \in \mathbb{R}$ denote any point in vertical line of the complex plane passing through $x = 1/2$. Then

$$\begin{aligned} |Tr(A^z B^z A^{1-z} B^{1-z})| &= |Tr(A^{iy} A^{1/2} B^{1/2} B^{iy} A^{-iy} A^{1/2} B^{1/2} B^{-iy})| \\ &\leq Tr|A^{iy} A^{1/2} B^{1/2} B^{iy} A^{-iy} A^{1/2} B^{1/2} B^{-iy}| \\ &\leq \|A^{iy} A^{1/2} B^{1/2} B^{iy} A^{-iy}\|_2 \|A^{1/2} B^{1/2} B^{-iy}\|_2 = \|A^{1/2} B^{1/2}\|_2^2 \end{aligned}$$

by the Cauchy-Schwarz inequality and the fact that A^{iy}, B^{iy} are unitary operators. Then by the previous lemma,

$$|Tr(A^z B^z A^{1-z} B^{1-z})| \leq Tr(AB)^{2/2} = Tr(AB).$$

Now consider $z = 1/4 + iy$, $y \in \mathbb{R}$, a generic point in the vertical line over $x = 1/4$, then noting that $\frac{1}{4} + \frac{1}{4} + \frac{1}{2} = 1$,

$$\begin{aligned} |Tr(A^z B^z A^{1-z} B^{1-z})| &= |Tr(B^{1/4} A^{1/4} A^{iy} B^{iy} B^{1/4} A^{1/4} A^{-iy} A^{1/2} B^{1/2} B^{-iy})| \\ &\leq \|B^{1/4} A^{1/4}\|_4^2 \|B^{1/2} A^{1/2}\|_2 \leq Tr(AB)^{2/4+1/2} = Tr(AB), \end{aligned}$$

where we used again the previous Lemma and the generalized Hölder's inequality (5).
By Hadamard's three-lines theorem, the bound $\tau(AB)$ is valid in the vertical strip $1/4 \leq \mathrm{Re}(z) \leq 1/2$, since it holds in the frontier of the strip. Invoking the symmetry $z \mapsto 1-z$ and exchanging the roles of A, B gives the desired bound on the full strip $\mathcal{S}_{1/4} = \{1/4 \leq \mathrm{Re}(z) \leq 3/4\}$. □

Regarding the inequalities conjectured by Bourin et al., note that we can assume $A, B > 0$: replacing A with $A_\varepsilon = A + \varepsilon$ (and likewise with B), if the inequality (1) is valid for $A_\varepsilon, B_\varepsilon$ then making $\varepsilon \to 0^+$ gives the general result: the following result that we state as corollary was recently obtained by R. Bhatia in [4] and we should also point the reader to the paper by T. Ando, F. Hiai, K. Okubo [1].

Corollary 2.3. *For any* $A, B \geq 0$ *and any* $t \in [1/4, 3/4]$,

$$\|A^t B^{1-t} + B^t A^{1-t}\|_2 \leq \|A^t B^{1-t} + A^{1-t} B^t\|_2 \leq \|A + B\|_2.$$

2.2 Inequality becomes equality

Let us consider the special case when the inequality above becomes an equality. We begin with a lemma that we will use in several ocasions, and will be useful when we drop the assumption on nonsingularity of A, B. Note that

$$Tr(A^{1/2}B^{1/2}A^{1/2}B^{1/2}) = Tr((B^{1/4}A^{1/2}B^{1/4})^2) \geq 0.$$

Lemma 2.4. *Let $A, B \geq 0$, and assume*

$$Tr(A^{1/2}B^{1/2}A^{1/2}B^{1/2}) = Tr(AB),$$

or

$$\|A^{1/4}B^{1/4}\|_4 = Tr(AB)^{1/4}.$$

In either case, A commutes with B.

Proof. Name $X = A^{1/2}B^{1/2}$, and considering the inner product induced by τ, $\langle X, Y\rangle = \tau(XY^*)$,

$$\langle X, X^*\rangle = \tau(X^2) = \tau(A^{1/2}B^{1/2}A^{1/2}B^{1/2}) = \tau(AB) = \tau(X^*X) = \|X\|_2^2 = \|X\|_2\|X^*\|_2.$$

But Cauchy-Schwarz inequality becomes an equality if and only if $X = \lambda X^*$ for some $\lambda > 0$, and since both operators have equal norm ($= \|A^{1/2}B^{1/2}\|_2$), then $X = X^*$. This means

$$A^{1/2}B^{1/2} = B^{1/2}A^{1/2},$$

and this implies that A commutes with B. On the other hand,

$$\|A^{1/4}B^{1/4}\|_4^4 = Tr((B^{1/4}A^{1/2}B^{1/4})^2) = Tr(A^{1/2}B^{1/2}A^{1/2}B^{1/2}),$$

so what we have is just another way of writing the first equality condition. □

Proposition 2.5. *Let $A, B > 0$ and assume that there is $z_0 \in \mathcal{S}_{1/4}$ such that*

$$|Tr(A^{z_0}B^{z_0}A^{1-z_0}B^{1-z_0})| = Tr(AB).$$

Then A commutes with B and $Tr(A^zB^zA^{1-z}B^{1-z}) = Tr(AB)$ for any $z \in \mathbb{C}$.

Proof. First consider the case when equality is reached in an interior point of the strip $\mathcal{S}_{1/4}$. Note that by the maximum modulus principle, this would mean that the function

$$f(z) = Tr(A^zB^zA^{1-z}B^{1-z})$$

is constant in the strip $\mathcal{S}_{1/4}$, in particular equality holds at $z_0 = 1/2$, and by the previous Lemma, A commutes with B.

Now suppose equality is attained in the frontier, for instance at $z_0 = 1/4 + iy$ for some $y \in \mathbb{R}$. Let $X = B^{1/4}A^{1/4}A^{iy}B^{iy}B^{1/4}A^{1/4}$, $Y = B^{1/2}B^{iy}A^{iy}A^{1/2}$. Then, if we go through the proof of Proposition 2.2 again, assuming equality

$$\begin{aligned} \tau(AB) &= \tau(XY^*) = \langle X, Y \rangle \leq \|X\|_2 \|Y\|_2 \\ &\leq \|B^{1/4}A^{1/4}\|_4^2 \|A^{1/2}B^{1/2}\|_2 \leq \tau(AB). \end{aligned} \tag{6}$$

Arguing as in the previous Lemma, there exists $\lambda > 0$ such that $X = \lambda Y$,

$$B^{1/4}A^{1/4}A^{iy}B^{iy}B^{1/4}A^{1/4} = \lambda B^{1/2}B^{iy}A^{iy}A^{1/2}.$$

Cancelling $B^{1/4}$ on the left and $A^{1/4}$ on the right we obtain

$$A^{1/4}A^{iy}B^{iy}B^{1/4} = \lambda B^{1/4}B^{iy}A^{iy}A^{1/4},$$

but now both elements have the same norm and this shows that $\lambda = 1$; then

$$A^{1/4+iy}B^{1/4+iy} = B^{1/4+iy}A^{1/4+iy},$$

and since $A, B > 0$, the existence of analytic logarithms shows that again A commutes with B. By symmetry, the same argument applies for any $z_0 = 3/4 + iy$ in the other border of the strip. □

Corollary 2.6. *If A does not commute with B, the inequality is strict:*

$$|Tr(A^z B^t A^{1-z} B^{1-z})| < Tr(AB),$$

in some open set $\Omega \subset \mathbb{C}$ containing the closed strip $\mathcal{S}_{1/4}$.

If we allow A, B to be non invertible, holomorphy is lost, but nevertheless in the same spirit we have the following result.

Proposition 2.7. *For given $A, B \geq 0$, there exists $\delta = \delta(A, B) > 0$ such that*

$$|Tr(A^t B^t A^{1-t} B^{1-t})| \leq Tr(AB)$$

holds in the interval $[1/4 - \delta, 3/4 + \delta]$. If A does not commute with B, the inequality is strict in the whole $(1/4 - \delta, 3/4 + \delta)$.

Proof. If A commutes with B, then the assertion is trivial. If not, arguing as in the last part of the proof of the previous proposition, we must have strict inequality

$$|Tr(A^t B^t A^{1-t} B^{1-t})| < Tr(AB)$$

for $t = 1/4$, $t = 3/4$, and then by continuity the inequality extends a bit out of the closed interval $[1/4, 3/4]$.

Consider $t \in (1/4, 1/2)$ and put $X = B^{1/4}A^{1/4}A^{t-1/4}B^{t-1/4}$, $Y = B^{1/4}A^{1/4}A^{3/4-t}B^{3/4-t}$. Note that $\frac{1}{t}, \frac{1}{1-t} \geq 1$ and define $1/p = t - 1/4 \in (0, 1/4)$, $1/q = 3/4 - t \in (1/4, 1/2)$, note also that $1/p + 1/4 = t$, $1/q + 1/4 = 1 - t$. By reiterated use of Hölder's inequality compute

$$\begin{aligned} |Tr(A^t B^t A^{1-t} B^{1-t})| &\leq \|XY\|_1 \leq \|X\|_{t^{-1}} \|Y\|_{(1-t)^{-1}} \\ &\leq \|B^{1/4}A^{1/4}\|_4 \|A^{1/p}B^{1/p}\|_p \|B^{1/q}A^{1/q}\|_q \|A^{1/4}B^{1/4}\|_4. \end{aligned}$$

Now apply Lemma 2.1 to each of the four terms (note that $p > 4$ and $q > 2$), and we have[2]

$$|Tr(A^t B^t A^{1-t} B^{1-t})| \leq \|B^{1/4}A^{1/4}\|_4 \|A^{1/p}B^{1/p}\|_p \|B^{1/q}A^{1/q}\|_q \|A^{1/4}B^{1/4}\|_4 \leq Tr(AB).$$

If we assume equality of the traces, then

$$Tr(AB) = \|B^{1/4}A^{1/4}\|_4 \|A^{1/p}B^{1/p}\|_p \|B^{1/q}A^{1/q}\|_q \|A^{1/4}B^{1/4}\|_4$$

and in particular, it must be that $\|A^{1/4}B^{1/4}\|_4 = Tr(AB)^{1/4}$, and from Lemma 2.4 we can deduce that A commutes with B. By the symmetry ($t \mapsto 1 - t$) the argument extends to $(1/2, 3/4)$, and again by Lemma 2.4 we already know that A commutes with B if equality is attained at $t = 1/2$. This finishes the proof of the assertion that the inequality is strict in $[1/4, 3/4]$ unless A commutes with B. $\square$

Remark 2.8. *The inequalities in the previous proof give in fact*

$$Tr|B^{\frac{1}{4}} A^t B^t A^{1-t} B^{\frac{3}{4}-t}| \leq Tr(AB)$$

for any $t \in [\frac{1}{4}, \frac{3}{4}]$*; this is a particular instance of [1, Theorem 2.10].*

3 Counterexamples

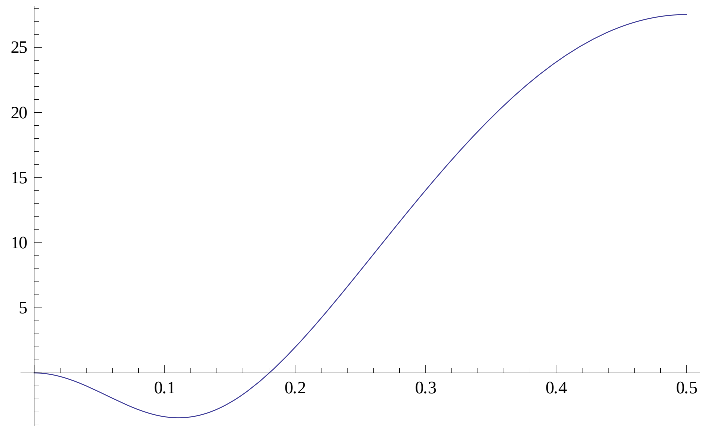
In this section we exhibit specific cases of different kind. In Example 3.1 we choose A, B such that $\|b_t(A,B)\|_\infty > \|h_t(A,B)\|_\infty$, while in Example 3.2, it is shown that the j^{th} singular value of $A + B$ is not always greater than the j^{th} singular value of $b_t(A, B)$. This provides negative answers to [6, Conjecture 1.2] and [3, Problem 4] respectively.

Example 3.1. *Consider the following positive definite matrices*

$$A = \begin{pmatrix} 1141 & 0 & 0 \\ 0 & 204 & 0 \\ 0 & 0 & 1/8 \end{pmatrix} \qquad \textit{and} \qquad B = \begin{pmatrix} 39 & 90 & 43 \\ 90 & 418 & 370 \\ 43 & 370 & 426 \end{pmatrix}.$$

[2]Note that this is another proof of the inequality for real $t \in [\frac{1}{4}, \frac{3}{4}]$.

The following is the graph of $f(t) = -\|b_t(A,B)\|_\infty + \|h_t(A,B)\|_\infty$ *for* $t \in [0, \frac{1}{2}]$*:*



For these matrices $-\|b_t(A,B)\|_\infty + \|h_t(A,B)\|_\infty \simeq -2.3$ *at* $t = .15$.

In [3, Problem 4] K. Audenaert and F. Kittaneh asked if $s_j(b_t(A,B)) \leq s_j(A+B)$ for every j and $0 < t < 1$ (where $s_j(M)$, $j = 1 \ldots n$ denote the singular values of the matrix M arranged in non-increasing order).

Example 3.2. *Consider the following positive definite matrices*

$$A = \begin{pmatrix} 6317 & 0 & 0 \\ 0 & 474 & 0 \\ 0 & 0 & 6 \end{pmatrix} \quad \textit{and} \quad B = \begin{pmatrix} 2078 & 2362 & 2199 \\ 2362 & 3267 & 2585 \\ 2199 & 2585 & 2492 \end{pmatrix}.$$

Then, for $t = \frac{1}{2}$ *we have*

$$s(b_{\frac{1}{2}}(A,B)) = (6826.57,\ 878.499,\ 591.716)$$

and

$$s(A+B) = (10561.4,\ 3629.62,\ 443.017).$$

In particular, $s_3(b_{\frac{1}{2}}(A,B)) > s_3(A+B)$.

References

[1] T. Ando, F. Hiai, K. Okubo. *Trace inequalities for multiple products of two matrices.* Math. Inequal. Appl. 3 (2000), no. 3, 307-318.

[2] H. Araki. *On an inequality of Lieb and Thirring*, Lett. Math. Phys. 19 (1990), pp. 167-170.

[3] K. Audenaert, F. Kittaneh. *Problems and Conjectures in Matrix and Operator Inequalities*, eprint arXiv:1201.5232v3 [math.FA]

[4] R. Bhatia. *Trace inequalities for products of positive definite matrices*, J. Math. Phys. 55 (2014).

[5] J.C. Bourin. *Matrix subadditivity inequalities and block-matrices.* Internat. J. Math. 20 (2009), no. 6, 679–691.

[6] S. Hayajneh, F. Kittaneh. *Lieb-Thirring trace inequalities and a question of Bourin.* J. Math. Phys. 54 (2013), no. 3, 033504, 8 pp.

[7] E. Heinz. *Beiträge zur Störungstheorie der Spektralzerlegung.* (German) Math. Ann. 123, (1951). 415-438.

[8] B. Simon. *Trace ideals and their applications.* Second edition. Mathematical Surveys and Monographs, 120. American Mathematical Society, Providence, RI, 2005.