On normal operator logarithms

Eduardo Chiumiento *

Abstract

Let X, Y be normal bounded operators on a Hilbert space such that $e^X = e^Y$. If the spectra of X and Y are contained in the strip $\mathcal{S}$ of the complex plane defined by $|\mathrm{Im}(z)| \leq \pi$, we show that $|X| = |Y|$. If Y is only assumed to be bounded, then $|X|Y = Y|X|$. We give a formula for $X - Y$ in terms of spectral projections of X and Y provided that X, Y are normal and $e^X = e^Y$. If X is an unbounded self-adjoint operator, which does not have $(2k+1)\pi$, $k \in \mathbb{Z}$, as eigenvalues, and Y is normal with spectrum in $\mathcal{S}$ satisfying $e^{iX} = e^Y$, then $Y \in \{\, e^{iX} \,\}''$. We give alternative proofs and generalizations of results on normal operator exponentials proved by Ch. Schmoeger.

AMS classification: 47B15; 47A60.

Keywords: Exponential map, normal operator, spectral theorem.

1 Introduction

Solutions to the equation $e^X = e^Y$ were studied by E. Hille [1] in the general setting of unital Banach algebras. Under the assumption that the spectrum $\sigma(X)$ of X is incongruent (mod $2\pi i$), which means that $\sigma(X) \cap \sigma(X + 2k\pi i) = \emptyset$ for all $k = \pm 1, \pm 2, \ldots$, he proved that $XY = YX$ and there exist idempotents $E_1, E_2, \ldots, E_n$ commuting with X and Y such that

$$X - Y = 2\pi i \sum_{j=1}^{n} k_j E_j, \qquad \sum_{j=1}^{n} E_j = I, \qquad E_i E_j = \delta_{ij},$$

where $k_1, k_2, \ldots, k_n$ are different integers. If the hypothesis on the spectrum is removed, it is possible to find non commuting logarithms (see e.g. [1, 6]). In the setting of Hilbert spaces, when X is a normal operator, the above assumption on the spectrum can be weakened. In fact, Ch. Schmoeger [5] proved that X belongs to the double commutant of Y provided that $E_X(\sigma(X) \cap \sigma(X + 2k\pi i)) = 0$, $k = 1, 2, \ldots$, where E_X is the spectral measure of X. We also refer to [3] for a generalization of this result by F. C. Paliogiannis.

In this paper, we study the operator equation $e^X = e^Y$ in the setting of Hilbert spaces under the assumption that the spectra of X and Y belong to a non-injective domain of the complex exponential map. Our results include the relation between the modulus of X and Y (Theorem 3.1), a formula for the difference of two normal logarithms in terms of their spectral projections (Theorem 4.1) and commutation relations when X is a skew-adjoint unbounded operator (Theorem 5.1). The proofs of these results are elementary. In fact, they rely on the spectral theorem for normal operators. This approach allows us to give a generalization (Corollary 4.2) and an alternative proof (Corollary 3.2) of two results by Ch. Schmoeger (see [6]).

*Partially supported by Instituto Argentino de Matemática 'Alberto P. Calderón' and CONICET

2 Notation and preliminaries

Let $(\mathcal{H}, \langle \cdot\,, \cdot \rangle\,)$ be a complex Hilbert space and $\mathcal{B}(\mathcal{H})$ be the algebra of bounded operators on $\mathcal{H}$. The spectrum of an operator X is denoted by $\sigma(X)$, and the set of eigenvalues of X is denoted by $\sigma_p(X)$. The real part of $X \in \mathcal{B}(\mathcal{H})$ is $\mathrm{Re}(X) = \frac{1}{2}(X + X^*)$ and its imaginary part is $\mathrm{Im}(X) = \frac{1}{2}(X - X^*)$.

If X is a bounded or unbounded normal operator on $\mathcal{H}$, we denote by E_X the spectral measure of X. Recall that E_X is defined on the Borel subsets of $\sigma(X)$, but we may think that E_X is defined on all the Borel subsets of $\mathbb{C}$. Indeed, we can set $E_X(\Omega) = E_X(\Omega \cap \sigma(X))$ for every Borel set $\Omega \subseteq \mathbb{C}$. Our first lemma is a generalized version of [4, Ch. XII Ex. 25], where the normal operator can now be unbounded.

Lemma 2.1. *Let X be a (possibly unbounded) normal operator on $\mathcal{H}$ and f a bounded Borel function on $\sigma(X)$. Then*

$$E_{f(X)}(\Omega) = E_X(f^{-1}(\Omega)),$$

for every Borel set $\Omega \subseteq \mathbb{C}$.

Proof. We define a spectral measure by $E'(\Omega) = E_X(f^{-1}(\Omega))$, where Ω is any Borel subset of $\mathbb{C}$. We are going to show that $E' = E_{f(X)}$. Since f is bounded, it follows that $f(X) \in \mathcal{B}(\mathcal{H})$. Moreover, the operator $f(X)$ is given by

$$\langle f(X)\xi, \eta\rangle = \int_{\mathbb{C}} f(z)\, dE_{X\,\xi,\eta}(z)\,,$$

where $\xi, \eta \in \mathcal{H}$ and $E_{X\,\xi,\eta}$ is the complex measure defined by $E_{X\,\xi,\eta}(\Omega) = \langle E_X(\Omega)\xi, \eta\rangle$ (see [4, Theorem 12.21]). By the change of measure principle ([4, Theorem 13.28]), we have

$$\int_{\mathbb{C}} z\, dE'_{\xi,\eta}(z) = \int_{\mathbb{C}} f(z)\, dE_{X\,\xi,\eta}(z).$$

Therefore E' satisfies the equation $\int_{\mathbb{C}} z\, dE'_{\xi,\eta}(z) = \langle f(X)\xi, \eta\rangle$, which uniquely determines the spectral measure of $f(X)$ (see [4, Theorem 12.23]). Hence $E' = E_{f(X)}$. □

The following lemma was first proved in [6, Corollary 2]. See also [3, Corollary 3] for another proof. We give below a proof for the sake of completeness, which does not depend on further results of these articles.

Lemma 2.2. *Let X and Y be normal operators in $\mathcal{B}(\mathcal{H})$. If $e^X = e^Y$, then $\mathrm{Re}(X) = \mathrm{Re}(Y)$.*

Proof. The following computation was done in [6]:

$$e^{X+X^*} = e^X e^{X^*} = e^X (e^X)^* = e^Y (e^Y)^* = e^Y e^{Y^*} = e^{Y+Y^*},$$

where the first and last equalities hold because X and Y are normal. Now we may finish the proof in a different fashion: note that the exponential map, restricted to real axis, has an inverse $\log : \mathbb{R}_+ \to \mathbb{R}$. Since $\sigma(X+X^*) \subseteq \mathbb{R}$ and $\sigma(e^{X+X^*}) \subseteq \mathbb{R}_+$, we can use the continuous functional calculus to get $X + X^* = \log(e^{X+X^*}) = \log(e^{Y+Y^*}) = Y + Y^*$. □

Throughout this paper, we use the following notation for subsets of the complex plane:

- $\Omega_1 + i\Omega_2 = \{\, x + iy \,:\, x \in \Omega_1,\, y \in \Omega_2 \,\}$, where Ω_i, $i = 1, 2$, are subsets of $\mathbb{R}$.
- For short, we write $\mathbb{R} + ia$ for the set $\mathbb{R} + i\{\, a \,\}$.
- We write $\mathcal{S}$ for the complex strip $\{\, z \in \mathbb{C} \,:\, -\pi \le \mathrm{Im}(z) \le \pi \,\}$, and $\mathcal{S}^\circ$ for the interior of $\mathcal{S}$.

Lemma 2.3. *Let X, Y be normal operators in $\mathcal{B}(\mathcal{H})$ such that $\sigma(X) \subseteq \mathcal{S}$ and $\sigma(Y) \subseteq \mathcal{S}$. Then $e^X = e^Y$ if and only if the following conditions hold:*

i) $E_X(\Omega) = E_Y(\Omega)$ *for all Borel subsets Ω of $\mathcal{S}^\circ$.*

ii) $\mathrm{Re}(X) = \mathrm{Re}(Y)$.

Proof. Suppose that $e^X = e^Y$. Let Ω be a Borel measurable subset of $\mathcal{S}^\circ$. By the spectral mapping theorem,

$$\sigma(e^X) = \{\, e^\lambda \,:\, \lambda \in \sigma(X)\,\} = \{\, e^\mu \,:\, \mu \in \sigma(Y)\,\} = \sigma(e^Y).$$

It is well-known that the restriction of the complex exponential map $\exp|_{\mathcal{S}^\circ}$ is bijective. Therefore we have $\sigma(X) \cap \Omega = \sigma(Y) \cap \Omega$, and by Lemma 2.1,

$$\begin{aligned} E_X(\Omega) &= E_X(\Omega \cap \sigma(X)) = E_X(\,\exp^{-1}(\exp(\Omega \cap \sigma(X)))\,) \\ &= E_{e^X}(\exp(\Omega \cap \sigma(X))) = E_{e^Y}(\exp(\Omega \cap \sigma(Y))) = E_Y(\Omega), \end{aligned}$$

which proves *i)*. On the other hand, *ii)* is proved in Lemma 2.2.

To prove the converse assertion, we first note that

$$\begin{aligned} E_X(\mathbb{R} - i\pi) + E_X(\mathbb{R} + i\pi) &= I - E_X(\mathcal{S}^\circ) = I - E_Y(\mathcal{S}^\circ) \\ &= E_Y(\mathbb{R} - i\pi) + E_Y(\mathbb{R} + i\pi), \end{aligned}$$

since $\sigma(X) \subseteq \mathcal{S}$, $\sigma(Y) \subseteq \mathcal{S}$ and $E_X(\mathcal{S}^\circ) = E_Y(\mathcal{S}^\circ)$. Due to the fact that E_X and E_Y coincide on Borel subsets of $\mathcal{S}^\circ$, we find that

$$\int_{\mathcal{S}^\circ} e^z \, dE_X(z) = \int_{\mathcal{S}^\circ} e^z \, dE_Y(z).$$

Hence we get

$$\begin{aligned} e^X &= \int_{\mathcal{S}} e^z \, dE_X(z) = -\int_{\mathbb{R}+i\pi} e^{\mathrm{Re}(z)} \, dE_X(z) - \int_{\mathbb{R}-i\pi} e^{\mathrm{Re}(z)} \, dE_X(z) + \int_{\mathcal{S}^\circ} e^z \, dE_X(z) \\ &= -e^{\mathrm{Re}(X)}(\,E_X(\mathbb{R} + i\pi) + E_X(\mathbb{R} - i\pi)\,) + \int_{\mathcal{S}^\circ} e^z \, dE_X(z) \\ &= -e^{\mathrm{Re}(Y)}(\,E_Y(\mathbb{R} + i\pi) + E_Y(\mathbb{R} - i\pi)\,) + \int_{\mathcal{S}^\circ} e^z \, dE_Y(z) = e^Y. \end{aligned}$$

□

Remark 2.4. We have shown that $E_X(\mathbb{R} - i\pi) + E_X(\mathbb{R} + i\pi) = E_Y(\mathbb{R} - i\pi) + E_Y(\mathbb{R} + i\pi)$, whenever X, Y are normal bounded operators such that $\sigma(X) \subseteq \mathcal{S}$, $\sigma(Y) \subseteq \mathcal{S}$ and $e^X = e^Y$.

Theorem 2.5. *(S. Kurepa [2]) Let $X \in \mathcal{B}(\mathcal{H})$ such that $e^X = N$ is a normal operator. Then*

$$X = N_0 + 2\pi i W,$$

where $N_0 = \log(N)$ and $\log$ is the principal (or any) branch of the logarithm function. The bounded operator W commutes with N_0 and there exists a bounded and regular, positive definite self-adjoint operator Q such that $W_0 = Q^{-1} W Q$ is a self-adjoint operator the spectrum of which belongs to the set of all integers.

3 Modulus and square of logarithms

Now we show the relation between the modulus of two normal logarithms with spectra contained in $\mathcal{S}$.

Theorem 3.1. *Let X be a normal operator in $\mathcal{B}(\mathcal{H})$. Assume that $\sigma(X) \subseteq \mathcal{S}$ and $e^X = e^Y$.*

i) If Y is normal in $\mathcal{B}(\mathcal{H})$ and $\sigma(Y) \subseteq \mathcal{S}$, then $|X| = |Y|$.

ii) If $Y \in \mathcal{B}(\mathcal{H})$, then $|X|Y = Y|X|$.

Proof. *i)* We will prove that the spectral measures of $|\mathrm{Im}(X)|$ and $|\mathrm{Im}(Y)|$ coincide. Let us set $A = \mathrm{Im}(X)$ and $B = \mathrm{Im}(Y)$. Given $\Omega \subseteq [0, \pi)$, put $\Omega' = \{ x \in \mathbb{R} : |x| \in \Omega \}$. Note that $\mathbb{R} + i\Omega' \subseteq \mathcal{S}^\circ$. As an application of Lemma 2.1 and Lemma 2.3, we see that

$$E_{|A|}(\Omega) = E_A(\Omega') = E_X(\mathbb{R} + i\Omega') = E_Y(\mathbb{R} + i\Omega') = E_B(\Omega') = E_{|B|}(\Omega).$$

By Remark 2.4, we have

$$\begin{aligned} E_{|A|}(\{ \pi \}) = E_A(\{ -\pi, \pi \}) &= E_X(\mathbb{R} - i\pi) + E_X(\mathbb{R} + i\pi) \\ &= E_Y(\mathbb{R} - i\pi) + E_Y(\mathbb{R} + i\pi) = E_{|B|}(\{ \pi \}). \end{aligned}$$

Thus, we have proved $E_{|A|} = E_{|B|}$, which implies that $|A| = |B|$. On the other hand, by Lemma 2.2, we know that $\mathrm{Re}(X) = \mathrm{Re}(Y)$. Therefore

$$|X|^2 = \mathrm{Re}(X)^2 + |A|^2 = \mathrm{Re}(Y)^2 + |B|^2 = |Y|^2.$$

Hence $|X| = |Y|$, and the proof is complete.

ii) Since X is a normal operator, $e^X = e^Y$ is also a normal operator. Then by a result by S. Kurepa (see Theorem 2.5), there exist operators N_0 and W such that N_0 is normal, $e^X = e^{N_0}$, W commutes with N_0 and $Y = N_0 + 2\pi i W$. In fact, N_0 can be defined using the Borel functional calculus by $N_0 = \log(e^X)$, where log is the principal branch of the logarithm. In particular, this implies that $\sigma(N_0) \subseteq \mathcal{S}$. Now we can apply *i)* to find that $|N_0| = |X|$. Since $N_0 W = W N_0$, we have $|N_0|W = W|N_0|$, and this gives $W|X| = |X|W$. Hence $|X|Y = Y|X|$. □

Following similar arguments, we can give an alternative proof of a result by Ch. Schmoeger ([6, Theorem 3]). This result was originally proved using inner derivations. Note that a minor improvement on the assumption on $\sigma(X)$ over the boundary $\partial\mathcal{S}$ of the strip $\mathcal{S}$ can now be done. Given a set $\Omega \subseteq \mathbb{C}$, we denote by $\bar{\Omega}$ the set $\{ x - iy : x + iy \in \Omega \}$.

Corollary 3.2. *Let X be a normal operator in $\mathcal{B}(\mathcal{H})$, $\sigma(X) \subseteq \mathcal{S}$, $Y \in \mathcal{B}(\mathcal{H})$ and $e^X = e^Y$. Suppose that for every Borel subset $\Omega \subseteq \partial\mathcal{S} \setminus \{ -i\pi, i\pi \}$, it holds that $E_X(\bar{\Omega}) = 0$, whenever $E_X(\Omega) \neq 0$. Then $X^2 Y = Y X^2$.*

Proof. We will show that $E_{X^2}(\Omega_0)$ commutes with Y for every Borel subset $\Omega_0 \subseteq \sigma(X^2)$. From the equation $e^X = e^Y$, we have $e^X Y = Y e^X$, and thus, $E_{e^X}(\Omega)Y = Y E_{e^X}(\Omega)$ for any Borel set Ω. Since the set Ω is arbitrary, by Lemma 2.1 we get

1. $E_X(\Omega')Y = Y E_X(\Omega')$ for every subset $\Omega' \subseteq \mathcal{S}^\circ$.
2. $(E_X(\Omega') + E_X(\bar{\Omega}'))Y = Y(E_X(\Omega') + E_X(\bar{\Omega}'))$, whenever $\Omega' \subseteq \partial\mathcal{S}$.

On the other hand, the image of $\mathcal{S}$ by the analytic map $f(z) = z^2$ is given by

$$f(\mathcal{S}) = \{ u \pm i2t\sqrt{u + t^2} : u \in [-\pi^2, \infty),\ u + t^2 \geq 0 \}.$$

Let us write $f^{-1}(\Omega_0) = \Omega_- \cup \Omega_+$, where $\Omega_- = f^{-1}(\Omega_0) \cap \{ z \in \mathbb{C} : \mathrm{Re}(z) < 0 \}$ and $\Omega_+ = f^{-1}(\Omega_0) \cap \{ z \in \mathbb{C} : \mathrm{Re}(z) \geq 0 \}$. We point out that $E_{X^2}(\Omega_0) = E_X(\Omega_+) + E_X(\Omega_-)$.

Next we need to consider three cases. In the case in which $\Omega_0 \subseteq f(\mathcal{S})^\circ$, then $\Omega_+ \subseteq \mathcal{S}^\circ$ and $\Omega_- \subseteq \mathcal{S}^\circ$. By the item 1. above we have $E_{X^2}(\Omega_0)Y = YE_{X^2}(\Omega_0)$. In the case where $\Omega_0 \subseteq \partial f(\mathcal{S}) \setminus \{ -\pi^2 \}$, we have that $\Omega_+ \subseteq \partial\mathcal{S} \cap \{ z \in \mathbb{C} : \mathrm{Re}(z) > 0 \}$. It follows that either $E_X(\Omega_+) = 0$ or $E_X(\bar{\Omega}_+) = 0$ by our assumption on the spectral measure of X. Similarly, it must be either $E_X(\Omega_-) = 0$ or $E_X(\bar{\Omega}_-) = 0$. Therefore item 2. above reduces to the desired conclusion, i.e. $E_{X^2}(\Omega_0)Y = YE_{X^2}(\Omega_0)$. Finally, if $\Omega_0 = \{-\pi^2\}$, then $E_{X^2}(\Omega_0) = E_X(\{ -i\pi \}) + E_X(\{ i\pi \})$ commutes with Y by item 2., and this concludes the proof. □

4 Difference of logarithms

Let X, Y be normal operators and $k \in \mathbb{Z}$. In order to avoid lengthly formulas, let us fix a notation for some special spectral projections of these operators:

- $P_{2k+1} = E_X(\mathbb{R} + i(\,(2k-1)\pi, (2k+1)\pi)\,)$;
- $Q_{2k+1} = E_Y(\mathbb{R} + i(\,(2k-1)\pi, (2k+1)\pi)\,)$;
- $E_{2k+1} = E_X(\mathbb{R} + i(2k+1)\pi\,)$;
- $F_{2k+1} = E_Y(\mathbb{R} + i(2k+1)\pi\,)$.

As we have pointed out in the introduction, E. Hille showed that the difference between two logarithms in Banach algebras may be expressed as the sum of multiples of projections (see [1, Theorem 4]). In order to prove that result, the spectrum of one of the logarithms must be incongruent (mod $2\pi i$). In the case where X and Y are both normal logarithms on a Hilbert space, the spectral theorem can be used to provide a more general formula.

Theorem 4.1. *Let X and Y be normal operators in $\mathcal{B}(\mathcal{H})$ such that $e^X = e^Y$. If $\sigma(X)$ and $\sigma(Y)$ are contained in $\mathbb{R} + i[\,(2k_0+1)\pi, (2k_1+1)\pi\,]$ for some $k_0, k_1 \in \mathbb{Z}$, then*

$$X - Y = \sum_{k=k_0}^{k_1} \big(2k\pi i\,(P_{2k+1} - Q_{2k+1}) + (2k+1)\pi i\,(E_{2k+1} - F_{2k+1})\big).$$

Proof. We first suppose that $\sigma(X)$ and $\sigma(Y)$ are contained in the strip $\mathcal{S}$. Then we have $\mathrm{Im}(X) = \mathrm{Im}(X)(E_X(\mathcal{S}^\circ) + E_X(\mathbb{R} + i\pi) + E_X(\mathbb{R} - i\pi)\,) = \mathrm{Im}(X)P_1 + \pi E_1 - \pi E_{-1}$. Analogously, $\mathrm{Im}(Y) = \mathrm{Im}(Y)Q_1 + \pi F_1 - \pi F_{-1}$. By Lemma 2.3, we know that $\mathrm{Re}(X) = \mathrm{Re}(Y)$ and $E_X(\Omega) = E_Y(\Omega)$ for every Borel subset Ω of $\mathcal{S}^\circ$. It follows that

$$\mathrm{Im}(X)P_1 = \int_{\mathcal{S}^\circ} \mathrm{Im}(z)\, dE_X(z) = \int_{\mathcal{S}^\circ} \mathrm{Im}(z)\, dE_Y(z) = \mathrm{Im}(Y)Q_1,$$

which implies

$$X - Y = \pi i(E_1 - F_1) - \pi i(E_{-1} - F_{-1}). \tag{1}$$

Thus, we have proved the formula in this case. For the general case, without restrictions on spectrum of X and Y, we need to consider the following Borel measurable function

$$f(t) = \sum_{k=k_0-1}^{k_1} (t - 2k\pi)\, \chi_{\,((2k-1)\pi,(2k+1)\pi]}\,(t),$$

where $\chi_I(t)$ is the characteristic function of the interval I. Set $A = \mathrm{Im}(X)$ and $B = \mathrm{Im}(Y)$. By Lemma 2.2, $\mathrm{Re}(X) = \mathrm{Re}(Y)$, and since the real and imaginary part of X and Y commute because X and Y are normal, $e^{iA} = e^X e^{-\mathrm{Re}(X)} = e^Y e^{-\mathrm{Re}(Y)} = e^{iB}$. The function f satisfies

$e^{if(t)} = e^{it}$, which implies that $e^{if(A)} = e^{iA} = e^{iB} = e^{if(B)}$. Since $\sigma(f(A))$ and $\sigma(f(B))$ are contained in $[-\pi, \pi]$, we can replace in equation (1) to find that

$$\begin{aligned} f(A) - f(B) &= \pi(E_{f(A)}(\{\pi\}) - E_{f(B)}(\{\pi\})) \\ &= \pi \sum_{k=k_0-1}^{k_1} \big(E_A(\{(2k+1)\pi\}) - E_B(\{(2k+1)\pi\})\big) \\ &= \pi \sum_{k=k_0}^{k_1} \big(E_{2k+1} - F_{2k+1}\big). \end{aligned} \tag{2}$$

Here we have used Lemma 2.1 to express $E_{f(A)}, E_A$ and $E_{f(B)}, E_B$ in terms of E_X and E_Y respectively. In particular, note that $E_{f(A)}(\{-\pi\}) = E_{f(B)}(\{-\pi\}) = 0$. On the other hand, we have

1. $f(A) = \displaystyle\sum_{k=k_0-1}^{k_1} (A - 2k\pi)\, \chi_{((2k-1)\pi,(2k+1)\pi]}(A) = A - \sum_{k=k_0}^{k_1} 2k\pi(P_{2k+1} + E_{2k+1}),$
2. $f(B) = B - \displaystyle\sum_{k=k_0}^{k_1} 2k\pi(Q_{2k+1} + F_{2k+1}).$

Therefore

$$\begin{aligned} X - Y &= i(A - B) \\ &= i(f(A) - f(B)) + \sum_{k=k_0}^{k_1} \big(2k\pi i(P_{2k+1} - Q_{2k+1}) + 2k\pi i(E_{2k+1} - F_{2k+1})\big). \end{aligned}$$

Combining this with the expression in (2), we get the desired formula. ☐

Below we give a generalization of another result due to Ch. Schmoeger (see [6, Theorem 5]). The assumptions on the spectrum of X and Y were more restrictive in [6]: $\|X\| \le \pi$, $\|Y\| \le \pi$ and either $-i\pi$ or $i\pi$ does not belong to the point spectrum of one of these operators. However, these hypothesis were necessary to conclude that $X - Y$ is a multiple of a projection; meanwhile $XY = YX$ can be obtained under more general assumptions (see [6, Theorem 3], [5, Theorem 1.4] and [3, Theorem 9]).

Corollary 4.2. *Let X, Y be normal operators in $\mathcal{B}(\mathcal{H})$. Assume that $\sigma(X) \subseteq \mathcal{S}$, $\sigma(Y) \subseteq \mathcal{S}$ and $e^X = e^Y$. The following assertions hold:*

i) If $E_1 = 0$, then $XY = YX$ and $X - Y = -2\pi i\, F_1$.

ii) If $E_{-1} = 0$, then $XY = YX$ and $X - Y = 2\pi i\, F_{-1}$.

iii) If $E_1 = E_{-1} = 0$, then $X = Y$.

Proof. i) Under these assumptions on the spectra of X and Y, we have established that $E_{-1} + E_1 = F_{-1} + F_1$ in Remark 2.4. On the other hand, by equation (1) in the proof of Theorem 4.1, we know that $X - Y = \pi i(E_1 - F_1) - \pi i(E_{-1} - F_{-1})$. Since $E_1 = 0$, we have $E_{-1} = F_1 + F_{-1}$. It follows that $X = -2\pi i F_1 + Y$. Hence X and Y commute. We can similarly conclude that ii) holds true. To prove iii), note that $E_1 = E_{-1} = 0$ implies that $F_1 + F_{-1} = 0$, and consequently, $F_1 = F_{-1} = 0$. Hence we get $X = Y$. ☐

5 Unbounded logarithms

Let X be a self-adjoint unbounded operator on $\mathcal{H}$. As before, E_X denotes the spectral measure of X. In item i) of our next result, we will give a version of [5, Theorem 1.4] for unbounded operators (see also [3, Theorem 9]). To this end, we extend the definition given in [5] for bounded operators: a self-adjoint unbounded operator X is *generalized* 2π*-congruence-free* if

$$E_X(\sigma(X) \cap \sigma(X + 2k\pi)) = 0, \quad k = \pm 1, \pm 2, \dots .$$

Given $Y \in \mathcal{B}(\mathcal{H})$, the commutant of Y is the set

$$\{ Y \}' = \{ Z \in \mathcal{B}(\mathcal{H}) \, : \, ZY = YZ \}.$$

The double commutant of Y is defined by

$$\{ Y \}'' = \{ W \in \mathcal{B}(\mathcal{H}) \, : \, WZ = ZW, \text{ for all } Z \in \{ Y \}' \}.$$

If X is a self-adjoint unbounded operator and $Y \in \mathcal{B}(\mathcal{H})$, recall that $XY = YX$, that is X commutes with Y, if $YE_X(\Omega) = E_X(\Omega)Y$ for every Borel subset $\Omega \subseteq \mathbb{R}$. Recall that the exponential e^{iX} of a self-adjoint unbounded operator X is a unitary operator, which can be defined via the Borel functional calculus (see e.g. [4]).

Theorem 5.1. *Let X be a self-adjoint operator on $\mathcal{H}$ and $Y \in \mathcal{B}(\mathcal{H})$ such that $e^{iX} = e^Y$.*

i) If X is generalized 2π-congruence-free, then $E_X(\Omega) \in \{ Y \}''$ for all Borel subsets Ω of $\mathbb{R}$. In particular, $XY = YX$.

ii) If $\{ (2k+1)\pi \, : \, k \in \mathbb{Z} \} \cap \sigma_p(X)$ has at most one element and Y is normal in $\mathcal{B}(\mathcal{H})$ such that $\sigma(Y) \subseteq \mathcal{S}$, then $XY = YX$.

iii) If $(2k+1)\pi \notin \sigma_p(X)$ for all $k \in \mathbb{Z}$ and Y is normal in $\mathcal{B}(\mathcal{H})$ such that $\sigma(Y) \subseteq \mathcal{S}$, then $Y \in \{ e^{iX} \}''$.

Proof. i) Let $Z \in \mathcal{B}(\mathcal{H})$ such that $ZY = YZ$. It follows that $Ze^Y = e^Y Z$. Then we have $Ze^{iX} = e^{iX}Z$, and by Lemma 2.1, $ZE_X(\exp^{-1}(\Omega)) = E_X(\exp^{-1}(\Omega))Z$ for every $\Omega \subseteq \mathbb{T}$. If $\Omega' = \exp^{-1}(\Omega) \cap [-\pi, \pi]$, then

$$E_X(\exp^{-1}(\Omega)) = \sum_{k \in \mathbb{Z}} E_X(\Omega' + 2k\pi),$$

where this series converges in the strong operator topology. Suppose now that there is some $k \in \mathbb{Z}$ such that $E_X(\Omega' + 2k\pi) \neq 0$. It follows that $\sigma(X) \cap (\Omega' + 2k\pi) \neq \emptyset$, and $(\Omega' + 2l\pi) \cap \sigma(X) \subseteq \sigma(X) \cap \sigma(X + 2(l-k)\pi)$ for all $l \in \mathbb{Z}$. By the assumption on the spectral measure of X, $E_X(\Omega' + 2l\pi) \leq E_X(\sigma(X) \cap \sigma(X + 2(l-k)\pi)) = 0$ for $l \neq k$. Therefore for each Ω, the above series reduces to only one spectral projection corresponding to a set of the form $\Omega' + 2k\pi$. Hence Z commutes with all the spectral projections of X.

ii) We need to consider the Borel measurable function f defined in the proof of Theorem 4.1. Since $e^{iX} = e^Y$, we have that $e^{if(X)} = e^Y$. Recall that $E_X(\{ (2k+1)\pi \}) \neq 0$ if and only if $(2k+1)\pi \in \sigma_p(X)$ ([4, Theorem 12.19]). By the hypothesis on the eigenvalues of X, there is at most one $n_0 \in \mathbb{Z}$ such that $E_X(\{ (2n_0+1)\pi \}) \neq 0$. According to Lemma 2.1, we get

$$E_{f(X)}(\{ \pi \}) = \sum_{k \in \mathbb{Z}} E_X(\{ (2k+1)\pi \}) = E_X(\{ (2n_0+1)\pi \}).$$

On the other hand, $E_{f(X)}(\{ -\pi \}) = 0$ for all $k \in \mathbb{Z}$ by definition of the function f. According to Corollary 4.2 ii), it follows that $if(X) = Y + 2\pi i F_{-1}$. By Remark 2.4, we also know that

$E_X(\{\,(2n_0+1)\pi\,\}) = F_{-1}+F_1$. In order to show that Y commutes with all the spectral projections of X, we divide into two cases. If $\Omega \subseteq \mathbb{C}\setminus\{\,(2k+1)\pi\,:\,k\in\mathbb{Z}\,\}$, note that $E_X(\Omega)F_{-1}=0$ because $F_{-1}\leq E_X(\{\,(2n_0+1)\pi\,\})$. Hence we get

$$E_X(\Omega)Y = E_X(\Omega)(if(X)-2\pi i F_{-1}) = iE_X(\Omega)f(X) = if(X)E_X(\Omega) = YE_X(\Omega).$$

If $\Omega\subseteq\{\,(2k+1)\pi\,:\,k\in\mathbb{Z}\,\}$, we only need to prove that $E_X(\{\,(2n_0+1)\pi\,\})$ commutes with Y. This follows immediately, because $E_X(\{\,(2n_0+1)\pi\,\})$ is the sum of two spectral projections of Y.

iii) As in the proof of *ii*), we have $e^{if(X)}=e^Y$. Now by the assumption on the eigenvalues of X, it follows that

$$E_{f(X)}(\{\,-\pi,\,\pi\,\}) = \sum_{k\in\mathbb{Z}} E_X(\{\,(2k+1)\pi\,\}) = 0. \tag{3}$$

Applying Corollary 4.2 *iii*), we get $if(X)=Y$. Recall that $f(X)$ is a self-adjoint operator such that $\sigma(f(X))\subseteq[-\pi,\pi]$.

Let $Z\in\mathcal{B}(\mathcal{H})$ such that $Ze^{iX}=e^{iX}Z$. Then we have $ZE_{e^{iX}}(\Omega)=E_{e^{iX}}(\Omega)Z$ for every Borel set $\Omega\subseteq\mathbb{T}$. We are going to show that $ZE_{f(X)}(\Omega')=E_{f(X)}(\Omega')Z$ for every $\Omega'\subseteq[-\pi,\pi]$. We need to consider two cases. If $\Omega'\subseteq(-\pi,\pi)$, there exists a unique set $\Omega\subseteq\mathbb{T}\setminus\{\,-1\,\}$ such that $\exp^{-1}(\Omega)\cap[-\pi,\pi]=\Omega'$. Therefore

$$E_{f(X)}(\Omega') = \sum_{k\in\mathbb{Z}} E_X(\Omega'+2k\pi) = E_X(\exp^{-1}(\Omega)) = E_{e^{iX}}(\Omega).$$

If $\Omega'\subseteq\{\,-\pi,\,\pi\,\}$, by equation (3) we find that $E_{f(X)}(\Omega')=0$. Hence we obtain that Z commutes with every spectral projection of $f(X)$. The latter is equivalent to saying that Z commute with Y, and this concludes the proof. □

Acknowledgment

I would like to thank Esteban Andruchow and Gabriel Larotonda for suggesting me to prove Theorem 3.1 *i*). I am also grateful to them for several helpful conversations.

References

[1] E. Hille, On roots and logarithms of elements of a complex Banach algebra, Math. Ann. 136 (1958), 46-57.

[2] S. Kurepa, A note on logarithms of normal operators, Proc. Amer. Math. Soc. 13 (1962), 307-311.

[3] F. C. Paliogiannis, On commuting operator exponentials, Proc. Amer. Math. Soc. 131 (2003), no. 12, 3777-3781.

[4] W. Rudin, Functional Analysis, McGraw-Hill, New York (1974).

[5] Ch. Schmoeger, On normal operator exponentials, Proc. Amer. Math. Soc. 130 (2001), no. 3, 697-702.

[6] Ch. Schmoeger, On the operator equation $e^A=e^B$, Linear Algebra Appl. 359 (2003), 169-179.

Departamento de de Matemática, FCE-UNLP,Calles 50 y 115, (1900) La Plata, Argentina and Instituto Argentino de Matemática, 'Alberto P. Calderón', CONICET, Saavedra 15 3er. piso, (1083) Buenos Aires, Argentina.
e-mail: eduardo@mate.unlp.edu.ar