A note on a question by T. Ando

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Abstract

In 1979 T. Ando posed the following question: suppose E and F are two projection valued measures defined on an algebra Σ of subsets of Ω, which verify

$$\|E(\Delta) - F(\Delta)\| \leq 1 - \delta, \ \Delta \in \Sigma,$$

for some $\delta > 0$. Does there exist a unitary operator u such that $u^*E(\Delta)u = F(\Delta)$ for all $\Delta \in \Sigma$? He knew that the answer was affirmative if both measures were strongly σ-additive and maximal (i.e. E and F have ciclic vectors). In this note, we show that the answer is also affirmative if both measures take values in a common finite von Neumann algebra.

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1 Introduction

In 1979, at the Proceedings of the Fourth Conference on Operator Theory in Timisoara [1], Prof. T. Ando posed the following question:

Suppose that E and F are two projection-valued spectral measures in $\mathcal{B}(\mathcal{H})$, defined on a common σ-algebra Σ of subsets of Ω, such that there exists $\delta > 0$ verifying that

$$\|E(\Delta) - F(\Delta)\| \leq 1 - \delta,$$

for all $\Delta \in \Sigma$.

Question: does there exist a unitary operator u in $\mathcal{B}(\mathcal{H})$ such that

$$F(\Delta) = u^*E(\Delta)u,$$

for all $\Delta \in \Sigma$?

The purpose of this note is to renew the atention to this question, which as far as we know, remains unanswered. In [1], Prof. T. Ando knew that the answer is positive if both measures are strongly σ-additive and *maximal*. A spectral measure E is maximal if any projection $p \in \mathcal{B}(\mathcal{H})$ which satisifes that $pE(\Delta) = E(\Delta)p$ for all $\Delta \in \Sigma$, is a spectral projection: $p = E(\Delta_p)$ for some $\Delta_p \in \Sigma$. Back in 1975 he also knew that the answer is positive if $\gamma \geq \frac{1}{\sqrt{2}}$ [2]. In [3] it was shown a weaker result, namely that the assertion is true for $\gamma = \frac{1}{2}$.

In this note we give an affirmative answer for this question, in an elementary fashion, in other particular cases. For instance, when both measures E and F take values in a common finite von Neumann.

Let $\mathcal{M}$ denote a von Neumann algebra with a finite normal and faithful tracial state τ. Let $\mathcal{H} = L^2(\mathcal{M}, \tau)$. We shall consider that $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$, and will denote by $\| \ \|_2$ the norm of $\mathcal{H}$. Note that any $a \in \mathcal{M}$ can be regarded as a vector in $\mathcal{H}$, and in that case $\|a\|_2 = \tau(a^*a)^{1/2}$. A spectral measure E in Ω is a map

$$E : \Sigma \to \mathcal{B}(\mathcal{H}),$$

such that $E(\Delta)$ is an orthogonal projection in $\mathcal{H}$, $E(\emptyset) = 0$, $E(\Omega) = 1$,

$$E(\Delta \cap \Delta') = E(\Delta)E(\Delta') = E(\Delta')E(\Delta),$$

and E is additive: if $\Delta, \Delta' \in \Sigma$ are disjoint, then

$$E(\Delta \cup \Delta) = E(\Delta) + E(\Delta').$$

We say that E is strongly σ-additive, if for any countable family Δ_k, $k \geq 1$ of disjoint elements in Σ, then for each $\xi \in \mathcal{H}$,

$$E(\cup_{k\geq 1}\Delta_k)\xi = \sum_{k\geq 1} E(\Delta_k)\xi.$$

Finally, we say that E is $\mathcal{M}$-valued if $E(\Delta) \in \mathcal{M}$ for all $\Delta \in \Sigma$.

2 $\mathcal{M}$-valued measures

Let π be a (finite) partition of Ω, i.e. $\pi = \{\Delta_1, \dots, \Delta_n\}$, where $\Delta_i \in \Sigma$, $\Delta_i \cap \Delta_j = \emptyset$ if $i \neq j$, and $\cup_{i=1}^n \Delta_i = \Omega$. Let E, F be two spectral measures, and denote by $g_\pi^{E,F}$, or shortly $g_\pi \in \mathcal{B}(\mathcal{H})$ the element

$$g_\pi = \sum_{i=1}^n E(\Delta_i)F(\Delta_i).$$

The following result is well-known. We include the proof, which is elementary.

Lemma 2.1. *If $\|E(\Delta) - F(\Delta)\| < 1$ for all $\Delta \in \Sigma$, then g_π is invertible.*

Proof. Note that

$$g_\pi g_\pi^* = \sum_{i=1}^n E(\Delta_i)F(\Delta_i) \sum_{j=1}^n F(\Delta_j)E(\Delta_j) = \sum_{i=1}^n E(\Delta_i)F(\Delta_i)E(\Delta_i).$$

Each operator $E(\Delta_i)F(\Delta_i)E(\Delta_i)$ can be regarded as an operator in $R(E(\Delta_i))$, the range of $E(\Delta_i)$. Note that it is an invertible operator there:

$$\|E(\Delta_i)F(\Delta_i)E(\Delta_i) - E(\Delta_i)\| = \|E(\Delta_i)(F(\Delta_i) - E(\Delta_i))E(\Delta_i)\| \leq \|F(\Delta_i) - E(\Delta_i)\| < 1.$$

Since $E(\Delta_i)$ is the identity operator in $R(E(\Delta_i))$, our claim follows. Note that the spaces $R(E(\Delta_i))$ are mutually orthogonal, and their sum is $\mathcal{H}$. It follows that $g_\pi g_\pi^*$ is invertible. Analogously $g_\pi^* g_\pi$ is also invertible. It follows that g_π is invertible. □

Given E and F, we can regard g_π as a net of elements in $\mathcal{B}(\mathcal{H})$, indexed in the set Π of finite partitions of Ω. A natural partial order in Π is given by $\pi' \geq \pi$ if every subset Δ' in π' is contained in a subset Δ in π.

Remark 2.2. If there exists $\delta > 0$ such that $\|E(\Delta) - F(\Delta)\| \leq 1 - \delta$ for all $\Delta \in \Sigma$, then, with the same computation as in the above lemma,

$$\|g_\pi g_\pi^* - 1\| \leq 1 - \delta, \quad \|g_\pi^* g_\pi - 1\| \leq 1 - \delta.$$

Indeed,

$$g_\pi g_\pi^* - 1 = \sum_{i=1}^{n} E(\Delta_i)[F(\Delta_i) - E(\Delta_i)]E(\Delta_i),$$

so that

$$\|g_\pi g_\pi^* - 1\| = \max_{1 \leq i \leq n} \|F(\Delta_i) - E(\Delta_i)\| \leq 1 - \delta,$$

and similarly for $g_\pi^* g_\pi$.

It is apparent that

$$\|g_\pi\| = \|g_\pi^* g_\pi\|^{1/2} = \|\sum_{i=1}^{n} F(\Delta_i)E(\Delta_i)F(\Delta_i)\|^{1/2} = \max_{1 \leq i \leq n} \|F(\Delta_i)E(\Delta_i)F(\Delta_i)\|^{1/2} \leq 1.$$

Therefore,

Lemma 2.3. *There exists a directed set Π_0 of partitions of Ω such that the subnets $\{g_\pi\}_{\pi \in \Pi_0}$, $\{g_\pi g_\pi^*\}_{\pi \in \Pi_0}$ and $\{g_\pi^* g_\pi\}_{\pi \in \Pi_0}$ converge, respectively, to elements a, b and c in $\mathcal{B}(\mathcal{H})$, in the weak operator topology. If E and F are $\mathcal{M}$-valued, then $a, b, c \in \mathcal{M}$. Moreover, $\|a\|, \|b\|, \|c\| \leq 1$. If additionally there exists $0 < \delta < 1$ such that*

$$\|E(\Delta) - F(\Delta)\| \leq 1 - \delta$$

for all $\Delta \in \Sigma$, then b and c are positive and invertible

Proof. Since the unit ball of $\mathcal{B}(\mathcal{H})$ is compact in the weak operator topology, there exists a convergent subnet of g_π. There exists a subnet of this subnet such that $g_\pi^* g_\pi$ converges, etc. Clearly, the limits a,b and c belong to the unit ball of $\mathcal{M}$. If there exists δ with the above mentioned property, by the remark above, the net $g_\pi g_\pi^* - 1$ converges to the element $b - 1$, which belongs to the ball of radius $1 - \delta$, and therefore b is invertible. Similarly for c. □

Remark 2.4. Suppose that E and F are $\mathcal{M}$-valued. Note that

$$\tau(g_\pi g_\pi^*) = \tau(g_\pi^* g_\pi) = \sum_{i=1}^{n} \tau(E(\Delta_i)F(\Delta_i)E(\Delta_i)) = \sum_{i=1}^{n} \tau(E(\Delta_i)F(\Delta_i)) = \tau(g_\pi).$$

Moreover, if the net $\{g_\pi\}$ is convergent in the weak operator topology, then the net $\{\tau(g_\pi)\}$ is convergent. Indeed, if we denote by 1 the unit element of $\mathcal{M}$ regarded as a vector in $\mathcal{H}$:

$$\tau(g_\pi) = \langle g_\pi 1, 1 \rangle \to \langle a1, 1 \rangle = \tau(a),$$

and similarly for $g_\pi^* g_\pi$ and $g_\pi g_\pi^*$. Therefore, for any limit elements a, b, c obtained as above, one has $\tau(a) = \tau(b) = \tau(c) > 0$.

In what follows, we fix the directed set Π_0, such that all three subnets $\{g_\pi\}_{\pi \in \Pi_0}$, $\{g_\pi g_\pi^*\}_{\pi \in \Pi_0}$ and $\{g_\pi^* g_\pi\}_{\pi \in \Pi_0}$ converge to a, b and c, in the weak operator topology.

Lemma 2.5. *If E and F are $\mathcal{M}$-valued, then $\|g_\pi - a\|_2 \to 0$.*

Proof. Suppose that $\pi = \{\Delta_i\}$ and $\pi' = \{\Delta'_j\}$ are two partitions in Π_0 such that $\pi' \geq \pi$. Then

$$\tau(g_\pi^* g_{\pi'}) = \sum_{i,j} \tau(F(\Delta_i)E(\Delta_i)E(\Delta'_j)F(\Delta'_j))$$

$$= \sum_i \left(\sum_{j:\Delta'_j \subset \Delta_i} \tau(F(\Delta'_j)E(\Delta'_j)) \right) = \tau(g_{\pi'}^*) = \tau(g_{\pi'}).$$

Given $\epsilon > 0$, let π_ϵ be a partition in Π_0 such that for all $\pi \geq \pi_\epsilon$, $|\tau(g_\pi) - \tau(g_{\pi_\epsilon})| < \epsilon$. Then, if π, π' are two partitions finer than π_ϵ,

$$\|g_\pi - g_{\pi'}\|_2 \leq \|g_\pi - g_{\pi_\epsilon}\|_2 + \|g_{\pi_\epsilon} - g_{\pi'}\|_2,$$

and, by the above computation

$$\|g_\pi - g_{\pi_\epsilon}\|_2^2 = \tau(g_\pi^* g_\pi) + \tau(g_{\pi_\epsilon}^* g_{\pi_\epsilon}) - \tau(g_\pi^* g_{\pi_\epsilon}) - \tau(g_{\pi_\epsilon}^* g_\pi) = \tau(g_\pi) - \tau(g_{\pi_\epsilon}) < \epsilon.$$

Analogously for the other term. Then for $\pi, \pi' \geq \pi_\epsilon$,

$$\|g_\pi - g_{\pi'}\|_2 \leq 2\sqrt{\epsilon}.$$

Thus $\{g_\pi\}$ is a Cauchy net in $\mathcal{H}$, and converges to an element in $\mathcal{H}$. On the other hand this net converges in the weak operator topology to a. □

Note that also $\|g_\pi^* - a^*\|_2 \to 0$.

Corollary 2.6. *With the current assumptions and notations, $aa^* = b$ and $a^*a = c$. In particular, $a \in \mathcal{M}$ is invertible.*

Proof. Note that

$$\|g_\pi g_\pi^* - aa^*\|_2 \leq \|g_\pi g_\pi^* - g_\pi a^*\|_2 + \|g_\pi a^* - aa^*\|_2 \leq \|g_\pi\| \|g_\pi^* - a^*\|_2 + \|g_\pi - a\|_2 \|a^*\|$$

$$\leq \|g_\pi^* - a^*\|_2 + \|g_\pi - a\|_2.$$

Thus $aa^* = b$. Analogously $a^*a = c$. □

Theorem 2.7. *Let E and F be two operator valued spectral measures, with values in a finite von Neumann algebra $\mathcal{M}$. Suppose that there exists $\delta > 0$ verifying that*

$$\|E(\Delta) - F(\Delta)\| \leq 1 - \delta,$$

for all $\Delta \in \Sigma$. Then there exists a unitary element $u \in \mathcal{M}$ such that

$$F(\Delta) = u^* E(\Delta) u,$$

for all $\Delta \in \Sigma$.

Proof. Pick $\Delta \in \Sigma$. Consider the partition $P = \{\Delta, \Omega \setminus \Delta\}$. There exists a partition $\pi_0 \in \Pi_0$ such that $\pi_0 \geq P$. Then

$$g_{\pi_0} F(\Delta) = \sum_i E(\Delta_i) F(\Delta_i) F(\Delta) = \sum_{\Delta_i \subset \Delta} E(\Delta_i) F(\Delta_i).$$

Analogously,

$$E(\Delta) g_{\pi_0} = \sum_{\Delta_i \subset \Delta} E(\Delta_i) F(\Delta_i).$$

Thus $g_{\pi_0} F(\Delta) = E(\Delta) g_{\pi_0}$. Apparently this also holds for any $\pi \in \Pi_0$ such that $\pi \geq \pi_0$ Taking limits, $aF(\Delta) = E(\Delta)a$. Thus $a^* E(\Delta) = F(\Delta) a^*$, and therefore $a^* a$ commutes with $F(\Delta)$. Note that by the previous result, a is invertible. Let u be the unitary part in the polar decomposition of a: $a = u|a|$. Note that $|a| = (a^* a)^{1/2}$ commutes with $F(\Delta)$ for all $\Delta \in \Sigma$. Then $u = a|a|^{-1} \in \mathcal{M}$ verifies

$$uF(\Delta) = a|a|^{-1} F(\Delta) = aF(\Delta)|a|^{-1} = E(\Delta) a|a|^{-1} = E(\Delta) u.$$

□

3 Non finite case

The above ideas can be used to prove a partial result in the general $\mathcal{B}(\mathcal{H})$-valued case. With the current notations and assumptions (namely: $\|E(\Delta) - F(\Delta)\| < 1 - \delta$ for some $\delta > 0$ and every $\Delta \in \Sigma$, and the directed set of partitions Π_0 is fixed such that for $\pi \in \Pi_0$, g_π, $g_\pi^* g_\pi$ and $g_\pi g_\pi^*$ converge to a, b and c in the weak operator topology). As in the proof of the above theorem, it suffices to show that a is invertible. Let us still denote with $\| \ \|_2$ the norm of $\mathcal{H}$. Note that for any vector $\xi \in \mathcal{H}$,

$$0 \leq \|g_\pi \xi - a\xi\|_2^2 = < g_\pi^* g_\pi \xi, \xi > + < a\xi, a\xi > - < g_\pi \xi, a\xi > - < a\xi, g_\pi \xi >$$

converges to $< b\xi, \xi > - < a^* a\xi, \xi >$. It follows that $a^* a \leq b$. Analogously, $aa^* \leq c$. The following result can be regarded as a generalization of the theorem in the previous section.

Lemma 3.1. *Suppose that there exists a trace class positive operator h, with zero kernel, which commutes with E (i.e. $hE(\Delta) = E(\Delta)h$ for all $\Delta \in \Sigma$). Then $aa^* = c$, and thus there exists a co-isometry v such that $v^* E(\Delta) v = F(\Delta)$.*

Proof. Denote with Tr the usual trace of $\mathcal{B}(\mathcal{H})$. We may normalize h so that $Tr(h) = 1$. Let $\varphi = Tr(h \cdot)$. Note that φ is a faithful normal state in $\mathcal{B}(\mathcal{H})$, and that the spectral measure E lies in the centralizer of φ,

$$\varphi(xE(\Delta)) = Tr(hxE(\Delta)) = Tr(E(\Delta)hx) = Tr(hE(\Delta)x) = \varphi(E(\Delta)x),$$

for all $x \in \mathcal{B}(\mathcal{H})$. Then an argument similar to the one in Lemma 2.5 can be carried out. Namely, if π' is finer than π,

$$\varphi(g_\pi g_{\pi'}^*) = \sum_{i,j} \varphi(E(\Delta_i) F(\Delta_i) F(\Delta'_j) E(\Delta'_j)) = \sum_i \left(\sum_{\Delta'_j \subset \Delta_i} \varphi(E(\Delta_i) F(\Delta_i) F(\Delta'_j) E(\Delta'_j)) \right)$$

$$= \sum_i \left(\sum_{\Delta'_j \subset \Delta_i} \varphi(E(\Delta'_j)F(\Delta'_j) \right) = \varphi(g_{\pi'}).$$

It follows that g_π is a Cauchy net for the norm $|x|_\varphi = \varphi(xx^*)^{1/2}$. Then $\varphi((g_\pi - a)(g_\pi - a)^*) \to 0$. On the other hand

$$\varphi((g_\pi - a)(g_\pi - a)^*) = \varphi(g_\pi g_\pi^*) + \varphi(aa^*) - \varphi(g_\pi a^*) - \varphi(a g_\pi^*).$$

Since φ is normal and the nets are uniformly bounded by 1, this expression tends to

$$\varphi(c) - \varphi(aa^*).$$

It follows that $\varphi(c) = \varphi(aa^*)$. By the above remark $aa^* \leq c$. Since φ is faithful, this implies that $aa^* = c$.

In particular, a is surjective. Let $c = (aa^*)^{1/2}v$ be the reversed polar decomposition of c. Then v is a co-isometry, and clearly, as above $v^*Ev = F$. □

Corollary 3.2. *Suppose that there exist positive trace class operators h_E and h_F with zero kernel, which commute, respectively with E and F. Then there exists a unitary operator u such that $u^*Eu = F$.*

Proof. By the above proposition, $aa^* = c$. Reasoning analogously with the state $\psi = Tr(h_F \cdot)$, it follows also that $a^*a = b$. Since b and c are invertible, then a is invertible. □

The existence of a positive trace class operator h with zero kernel which commutes with E is equivalent to the existence of a mutually orthogonal family $\{p_j\}_{j\geq 1}$ such that $\dim R(p_j) < \infty$, $\sum_{j\geq 1} p_j = 1$, and p_j commute with $E(\Delta)$, for all $\Delta \in \Sigma$. Indeed, the projections onto the eigenspaces of h provide such a family. Conversely, given a family of projections as above, take for instance $h = \sum_{j\geq 1} \frac{1}{2^j} p_j$.

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