Products of projections and positive operators

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Abstract

This article is devoted to the study of the set $\mathcal{T}$ of all products PA with P an orthogonal projection and A a positive (semidefinite) operator. We describe this set and study optimal factorizations. We also relate this factorization with the notion of compatibility and explore the polar decomposition of the operators in $\mathcal{T}$.

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1. Introduction

Given two classes of operators $\mathcal{M}$ and $\mathcal{B}$ in $\mathcal{L}(\mathcal{H})$ ($\mathcal{H}$ a Hilbert space), a problem which naturally arises is that of characterizing the set $\mathcal{M} \cdot \mathcal{B}$ of all products AB, $A \in \mathcal{M}, B \in \mathcal{B}$. These problems are as old as matrix theory and they form now an interesting part of factorization theory for matrices and operators. In 1958 Chandler Davis ([8], Theorem 6.3) proved that, if

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$\mathcal{I}$ denotes the set of Hermitian involutions (i.e., $T = T^* = T^{-1}$) then $\mathcal{I} \cdot \mathcal{I}$ coincides with all unitaries T such that T is similar to T^{-1}. H. Radjavi and J.P. Williams [21] proved later that $\mathcal{I} \cdot \mathcal{L}^h$, where $\mathcal{L}^h$ denotes the set of Hermitian operators on $\mathcal{H}$, is the set of all $T \in \mathcal{L}(\mathcal{H})$ such that T is unitarily equivalent to T^{-1}. Their paper also contains a characterization of $\mathcal{P} \cdot \mathcal{P}$ due to T. Crimmins and a characterization of $\mathcal{P} \cdot \mathcal{L}^h$ (here, $\mathcal{P}$ denotes the set of all orthogonal projectors of $\mathcal{L}(\mathcal{H})$). Other characterizations of $\mathcal{P} \cdot \mathcal{P}$ have been found by S. Nelson and M. Neumann [17], A. Arias and S. Gudder [1], T. Oikhberg [18] and the second author and A. Maestripieri [6]. In a series of papers, J.R. Holub [14],[15], [16] (see also Fujii and Furuta [12]) studied, as an approach to general Wiener-Hopf or Toeplitz operators, some properties of the class $\mathcal{P} \cdot \mathcal{G}^+ = \{PA : P \in \mathcal{P} \text{ and } A \in \mathcal{L}^+ \text{ is invertible}\}$, where $\mathcal{L}^+$ denotes the cone of positive semidefinite operators in $\mathcal{L}(\mathcal{H})$. They observed that the set $\mathcal{Q}$ of oblique (i.e., not necessarily orthogonal) projections in $\mathcal{L}(\mathcal{H})$ is contained in $\mathcal{P} \cdot \mathcal{G}^+$.

In this paper, we characterize operators in $\mathcal{T} := \mathcal{P} \cdot \mathcal{L}^+$. We extend several results on $\mathcal{P} \cdot \mathcal{P}$ and Holub's theorem that $\mathcal{Q}$ is contained in $\mathcal{P} \cdot \mathcal{G}^+$. It should be noticed that $\mathcal{Q}$ is not contained in $\mathcal{P} \cdot \mathcal{P}$, but it is contained in $(\mathcal{P} \cdot \mathcal{P})^\dagger$, the set of all Moore-Penrose inverses of products PQ, $P, Q \in \mathcal{P}$. This is an old result by Penrose [20] and Greville [13] which has been extended to the infinite dimensional case in [5] and [4]. The paper [21] by H. Radjavi and J. Williams and the survey [25] by P.Y.Wu contain many characterizations of classes of the type $\mathcal{M} \cdot \mathcal{B}$.

One of the main features of the class $\mathcal{P} \cdot \mathcal{L}^+$ is that their elements admit a particular polar decomposition where the partial isometry is an orthogonal projection. In fact, for $T \in \mathcal{T}$, any factorization $T = PA$, with $P \in \mathcal{P}$ and $A \in \mathcal{L}^+$ provides one such polar decomposition. Among all these expressions, we find one (the optimal factorization) with some relevant minimal properties. The main characterization of $\mathcal{T}$ is based on a result of Z. Sebestyén [22]. We include a proof, which is completely different from the original one, because it illustrates how the classical majorization theorem of R. G. Douglas [10], [11] can be used to provide special solutions of some operator equations. In fact, if $T \in \mathcal{T}$ and P is the orthogonal projection onto the closure of the image of T, then the positive solutions of the equation $PX = T$ play a natural role in this paper.

The contents of the paper are the following. Section 2 contains notations and the statements of some theorems by Crimmins ([11] Theorem 2.2), Douglas ([10], Theorem 1) and Sebestyén [22]. We include a proof of the last one based on Douglas' theorem. Section 3 is devoted to several properties of the set $\mathcal{T}$ and different characterizations of its elements. Just to mention two of them, $T \in \mathcal{L}(\mathcal{H})$ belongs to $\mathcal{T}$ if and only if there exists $\lambda \geq 0$ such that $(T^*T)^2 \leq \lambda T^*T^2$ (Theorem 3.2). If $R(T)$ is closed then $T \in \mathcal{T}$ if and only if $R(T) \dot{+} N(T) = \mathcal{H}$ and $TP \in \mathcal{L}^+$, where $P = P_{R(T)}$ (Theorem 3.3). A formula for the oblique projection onto $R(T)$ with nullspace $N(T)$ is exhibited at Section 4, where a particular factorization of $T \in \mathcal{T}$ is shown to have several optimal properties. For instance, if $T \in \mathcal{T}$ then there exist $P_T \in \mathcal{P}$ and $A_T \in \mathcal{L}^+$ such that $T = P_T A_T$ and $P_T \leq P$ and $A_T \leq A$ for all $P \in \mathcal{P}, A \in \mathcal{L}^+$ such that $T = PA$. The last result of section 4 is the characterization of the fiber of $T \in \mathcal{T}$ by the map $(P, A) \to PA$, i.e., we find all pairs $(P, A) \in \mathcal{P} \times \mathcal{L}^+$ such that $PA = T$. In Section 5 we relate the different factorizations of $T \in \mathcal{T}$ with the notions of compatibility and quasi-compatibility between positive operators and closed subspaces. It turns out that, if $T \in \mathcal{T}$ and $T = PA$ for some $P = P_{\mathcal{S}} \in \mathcal{P}$ and $A \in \mathcal{L}^+$, then the pair $(A, \mathcal{S})$ is compatible if and only if $\mathcal{H} = \overline{R(T)} \dot{+} N(T)$. The last section studies some properties of the standard polar decomposition of $T \in \mathcal{T}$.

2. Preliminaries

Throughout $\mathcal{F}$, $\mathcal{H}$ and $\mathcal{K}$ denote separable complex Hilbert spaces. By $\mathcal{L}(\mathcal{H}, \mathcal{K})$ we denote the space of all bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$. The algebra $\mathcal{L}(\mathcal{H}, \mathcal{H})$ is abbreviated by $\mathcal{L}(\mathcal{H})$. By $\mathcal{L}(H)^+$ we denote the cone of positive (semidefinite) operators of $\mathcal{L}(\mathcal{H})$ i.e., $T \in \mathcal{L}(\mathcal{H})^+$ if and only if $\langle Tx, x\rangle \geq 0$ for all $x \in \mathcal{H}$. Furthermore, $\mathcal{G}(\mathcal{H})$ denotes the group of invertible operators on $\mathcal{H}$ and $\mathcal{CR}(\mathcal{H})$ the set of closed range operators on $\mathcal{H}$. When no confusion can arise, we omit the Hilbert space and we write it simply $\mathcal{L}^+, \mathcal{G}$ and $\mathcal{CR}$ respectively. Moreover, we denote $\mathcal{G}^+ = \mathcal{G} \cap \mathcal{L}^+$.We shall denote by $\mathcal{Q} = \{Q \in \mathcal{L}(\mathcal{H}) : Q = Q^2\}$ and $\mathcal{P} = \{P \in \mathcal{Q} : P = P^*\}$. Given $T \in \mathcal{L}(\mathcal{H}, \mathcal{K})$, $R(T)$ denotes the range or image of T, $N(T)$ the nullspace of T, T^* the adjoint of T and $T^\dagger$ the Moore-Penrose inverse of T. Recall that $T^\dagger \in \mathcal{L}(\mathcal{K}, \mathcal{H})$ if and only if $R(T)$ is closed. Moreover, fixed a closed subspace $\mathcal{S}$, $P_{\mathcal{S}}$ stands for the orthogonal projection onto $\mathcal{S}$. In the sequel we denote by $\mathcal{S} \dot{+} \mathcal{W}$ the direct sum of the subspaces $\mathcal{S}$ and $\mathcal{W}$. In particular, if $\mathcal{S} \subseteq \mathcal{W}^\perp$ we denote $\mathcal{S} \oplus \mathcal{W}$.

We end this section by stating three important results that we will frequently use along this article.

Theorem 2.1. *([11], Theorem 2.2) If $A, B \in \mathcal{L}(\mathcal{H})$ then $R(A) + R(B) = R((AA^* + BB^*)^{1/2})$.*

Theorem 2.2. *(Douglas, [10]) Let $A \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ and $B \in \mathcal{L}(\mathcal{F}, \mathcal{K})$. The following conditions are equivalent:*

1. *$R(B) \subseteq R(A)$.*
2. *There is a positive number λ such that $BB^* \leq \lambda AA^*$.*
3. *There exists $C \in \mathcal{L}(\mathcal{F}, \mathcal{H})$ such that $AC = B$.*

If one of these conditions holds then there is a unique operator $D \in \mathcal{L}(\mathcal{F}, \mathcal{H})$ such that $AD = B$ and $R(D) \subseteq N(A)^{\perp}$. We shall call D the ***reduced solution*** *of $AX = B$. Moreover, $N(D) = N(B)$.*

The following result due to Sebestyén will be crucial along this article. Here, we present a different proof by means of Douglas' theorem.

Theorem 2.3. *([22]) Let $A, B \in \mathcal{L}(\mathcal{H}, \mathcal{K})$. The equation $AX = B$ has a positive solution if and only if $BB^* \leq \lambda AB^*$ for some $\lambda \geq 0$.*

Proof. Let Y be a positive solution of $AX = B$. Since $R(AY) \subseteq R(AY^{1/2})$ we obtain, by Douglas' theorem, that $BB^* = AYYA^* \leq \lambda AY^{1/2}Y^{1/2}A^* = \lambda AYA^* = \lambda AB^*$ for some $\lambda \geq 0$.

Conversely, if $BB^* \leq \lambda AB^*$ for some $\lambda \geq 0$ then, by Douglas' theorem, there exists $D \in \mathcal{L}(\mathcal{H})$ such that $(AB^*)^{1/2}D = B$, $R(D) \subseteq N((AB^*)^{1/2})^{\perp}$ and $N(D) = N(B)$. Then,

$$(AB^*)^{1/2}DA^* = BA^* = (AB^*)^{1/2}(AB^*)^{1/2}. \tag{1}$$

Therefore, DA^* and $(AB^*)^{1/2}$ are both the reduced solution of $(AB^*)^{1/2}X = BA^*$. Thus, by the uniqueness of the reduced solution, we get that $DA^* = (AB^*)^{1/2}$. So $AD^*D = (AB^*)^{1/2}D = B$, i.e, $Y = D^*D \in \mathcal{L}^+$ is solution of $AX = B$ and the result is obtained. □

Corollary 2.4. *If the operator equation $AX = B$ has a positive solution then there exists $Y \in \mathcal{L}(\mathcal{H})^+$ such that $AY = B$ and $N(Y) = N(B)$.*

Proof. Let D be the reduced solution of $(AB^*)^{1/2}X = B$. Then, by the proof of Theorem 2.3, $Y = D^*D$ is a positive solution of $AX = B$ with $N(Y) = N(B)$. □

3. The set $\mathcal{T}$

This section is devoted to the study of the set.

$$\mathcal{T} := \mathcal{P} \cdot \mathcal{L}^{+} = \{T \in \mathcal{L}(\mathcal{H}) : T = PA \text{ with } P \in \mathcal{P} \text{ and } A \in \mathcal{L}^{+}\}.$$

As we mentioned, the subclass $\mathcal{P} \cdot \mathcal{P}$ has been studied in [6] where several properties of this set have been provided. However, it must be noted that many properties of $\mathcal{P} \cdot \mathcal{P}$ are not longer valid in $\mathcal{T}$. For instance, given $T \in \mathcal{P} \cdot \mathcal{P}$, it holds that $T \in \mathcal{CR}$ if and only if $\mathcal{H} = \overline{R(T)} + N(T)$ (see [6] Theorem 3.2). Now, this characterization is not true if $T \in \mathcal{T}$. Indeed, consider $T \in \mathcal{L}^{+}$ with non-closed range then $T \in \mathcal{T}$ and $\mathcal{H} = \overline{R(T)} + N(T)$. Moreover, both sets have different topological properties. For example, $\mathcal{P} \cdot \mathcal{P}$ is closed but $\mathcal{T}$ is not. In fact, $T_n = \begin{bmatrix} 1/n & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/n & 1 \\ 1 & n \end{bmatrix} \in \mathcal{T}$. However, $\lim_{n\to\infty} T_n = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \notin \mathcal{T}$ (see Theorem 3.2).

In what follows, given $T \in \mathcal{T}$ we denote

$$\mathcal{T}_T^{+} := \{A \in \mathcal{L}(\mathcal{H})^{+} : \exists P \in \mathcal{P} \text{ such that } T = PA\}$$

and

$$\mathcal{T}_T^{\mathcal{P}} := \{P \in \mathcal{P} : \exists A \in \mathcal{L}(\mathcal{H})^{+} \text{ such that } T = PA\}.$$

In the following lemma we collect some properties of $\mathcal{T}$.

Lemma 3.1. *Let $T \in \mathcal{T}$. Then, the following conditions hold:*

1. $P_{\overline{R(T)}} \in \mathcal{T}_T^{\mathcal{P}}$.
2. *The spectrum of T, $\sigma(T)$, is positive.*
3. $T^n \in \mathcal{T}$ *for all* $n \in \mathbb{N}$.
4. $T \in \mathcal{G}$ *if and only if* $T \in \mathcal{G}^{+}$.
5. *if* $TT^* = T^*T$ *then* $T \in \mathcal{L}^{+}$.
6. $\overline{R(T)} \cap N(T) = \{0\}$, *i.e.,* $N(T^*) + \overline{R(T^*)}$ *is a dense subspace of* $\mathcal{H}$.
7. $\overline{R(T^*)} \cap N(T^*) = \{0\}$ *but, in general,* $\overline{R(T^*)} \cap N(T^*) \neq \{0\}$ *(i.e.,* $\overline{R(T)} + N(T)$ *is not dense, in general). As a consequence, in general,* $T^* \notin \mathcal{T}$.

Proof. 1. Since $T \in \mathcal{T}$, then $T = P_{\mathcal{S}}A$ for some $P_{\mathcal{S}} \in \mathcal{P}$ and $A \in \mathcal{L}^{+}$. Therefore, $R(T) \subseteq \mathcal{S}$ and as $\mathcal{S}$ is a closed subspace, then $\overline{R(T)} \subseteq \mathcal{S}$. Hence, $T = P_{\overline{R(T)}}T = P_{\overline{R(T)}}P_{\mathcal{S}}A = P_{\overline{R(T)}}A$ and so $P_{\overline{R(T)}} \in \mathcal{T}_T^{\mathcal{P}}$.

2. Let $T = PA$ then $\sigma(T) = \sigma(PA) = \sigma(A^{1/2}PA^{1/2}) \geq 0$.
3. Let $T = PA \in \mathcal{T}$ and $k \in \mathbb{N}$. Then, $T^{2k} = (PA)^{2k} = P(AP)^k(PA)^k = P(T^*)^kT^k \in \mathcal{T}$. On the other side, $T^{2k+1} = TT^{2k} = PAP(T^*)^kT^k = P(T^*)^{k+1}T^k = P(T^*)^kAPT^k = P((T^*)^kAT^k) \in \mathcal{T}$. Then the assertion follows.
4. If $T \in \mathcal{G}$ then, by item 1, $I \in \mathcal{T}_T^{\mathcal{P}}$ and so $T \in \mathcal{G}^+$.
5. Applying item 2, we have that T is a normal operator with $\sigma(T) \geq 0$, then $T \in \mathcal{L}^+$.
6. Let $T = PA$ and $x \in \overline{R(T)} \cap N(T)$. Since $R(T) \subseteq R(P)$ then $PAPx = 0$, i.e, $A^{1/2}Px = 0$ and so $APx = T^*x = 0$. Thus, $x \in \overline{R(T)} \cap N(T^*) = \{0\}$ and the result is obtained.
7. Let $T = PA$ and $z \in R(T^*) \cap N(T^*)$. Hence, $z = APx$ for some $x \in \mathcal{H}$ and $APz = 0$. Thus, $APAPx = 0$, and so $PAPAPx = 0$. Hence, $PAPx = 0$ and so $A^{1/2}Px = 0$. Therefore, $APx = z = 0$.
   For the second part, consider $A \in \mathcal{L}^+$ with non closed range and $x \in \overline{R(A)} \setminus R(A)$. Define $\mathcal{S} = \overline{\text{span}\{x\}^\perp}$ and $T = P_{\mathcal{S}}A$. Clearly, $T \in \mathcal{T}$ and $N(T) = N(A)$. Thus, $\overline{R(T^*)} = \overline{R(A)}$ and $\{0\} \neq \text{span}\{x\} = \mathcal{S}^\perp \cap \overline{R(A)} = \mathcal{S}^\perp \cap \overline{R(T^*)} \subseteq N(T^*) \cap \overline{R(T^*)}$.

□

In [21], it is proven that $T \in \mathcal{P} \cdot \mathcal{L}^h$ if and only if T^*T^2 is selfadjoint. In particular, this shows that $\mathcal{P} \cdot \mathcal{L}^h$ is closed; recall that $\mathcal{T} = \mathcal{P} \cdot \mathcal{L}^+$ is not. It is natural to ask if a necessary and sufficient condition for $T \in \mathcal{T} = \mathcal{P} \cdot \mathcal{L}^+$ is that T^*T^2 be positive. The next result proves that the answer is negative, and that a stronger condition is needed.

Theorem 3.2. *Let $T \in \mathcal{L}(\mathcal{H})$ and $P = P_{\overline{R(T)}}$. The following conditions are equivalent:*

1. $T \in \mathcal{T}$;
2. $TT^* \leq \lambda TP$ *for some* $\lambda \geq 0$;
3. $TP \in \mathcal{L}^+$ *and* $R(T(I-P)) \subseteq R((TP)^{1/2})$;
4. $(T^*T)^2 \leq \lambda T^*T^2$ *for some* $\lambda \geq 0$.

Proof. 1⇒ 2. If $T \in \mathcal{T}$ then the equation $T = PX$ has a positive solution and so, by Theorem 2.3, $TT^* \leq \lambda TP$ for some $\lambda \geq 0$.

2⇒ 3. If $TT^* \leq \lambda TP$ for some $\lambda \geq 0$ then $TP \in \mathcal{L}^+$. In addition, $T(I-P)T^* = TT^* - TPT^* \leq TT^* \leq \lambda TP$. Therefore, by Douglas' theorem, $R(T(I-P)) \subseteq R((TP)^{1/2})$.

3 ⇒4. Suppose $TP \in \mathcal{L}^+$ and $R(T(I-P)) \subseteq R((TP)^{1/2})$. Then, by Douglas' theorem, $TT^* - TPT^* \leq \alpha TP$ for some $\alpha > 0$. As $R(TP) \subseteq R((TP)^{1/2})$, using again Douglas' theorem, we get that $(TP)^2 \leq \beta TP$ for some $\beta > 0$. Hence $TT^* \leq (\alpha+\beta)TP$. Now, the assertion follows multiplying with T^*, T.

4⇒ 1. Assume that item 4 holds. By Theorem 2.3, there exists $X_0 \in \mathcal{L}^+$ such that $T^*T = T^*X_0$, and so $T^*T = T^*X_0 = T^*PX_0$. Thus, T and PX_0 are both solutions of the operator equation $T^*X = T^*T$. Moreover, $R(T), R(PX_0) \subseteq N(T^*)^\perp = \overline{R(T)}$, i.e., T and PX_0 are both the reduced solution of $T^*X = T^*T$. Hence, by the uniqueness of this solution, we obtain that $T = PX_0$ and so $T \in \mathcal{T}$. □

Theorem 3.3. *Let $T \in \mathcal{CR}$ and $P = P_{R(T)}$. The following conditions are equivalent:*

1. $T \in \mathcal{T}$;
2. *there exists $A \in \mathcal{G}^+$ such that $T = PA$;*
3. $R(T) \dot{+} N(T) = \mathcal{H}$ *and* $TP \in \mathcal{L}^+$.

Proof. 1⇒ 2. Let $T \in \mathcal{T}$. Hence, there exists $B \in \mathcal{L}^+$ such that $PB = T$. Therefore, $R(B) + R(T)^\perp = \mathcal{H}$. Define $A := B + P_{R(T)^\perp}$. Hence, $R(A^{1/2}) = R(B^{1/2}) + R(T)^\perp \supset \mathcal{H}$ and so $A \in \mathcal{G}^+$. Now, as $PA = PB = T$, the result is obtained.

2⇒ 3. Suppose that $T = PA$ with $A \in \mathcal{G}^+$. Since $A \in \mathcal{G}^+$, then $\langle x, y\rangle_A := \langle Ax, y\rangle$ defines a inner product equivalent to $\langle , \rangle$. Now, since $N(T) = A^{-1}(R(T)^\perp) = R(T)^{\perp_A}$ then $R(T) \dot{+} N(T) = \mathcal{H}$. In addition, $TP = PAP \in \mathcal{L}^+$.

3⇒ 1. Assume that $R(T) \dot{+} N(T) = \mathcal{H}$ and $TP \in \mathcal{L}^+$. Let us define $A := T^*(TP)^\dagger T$. Note that since $R(TP) = R(T)$ (because $R(T) \dot{+} N(T) = \mathcal{H}$) then TP has closed range and so $(TP)^\dagger \in \mathcal{L}^+$. Thus, $A \in \mathcal{L}^+$ and $T = PA$, i.e, $T \in \mathcal{T}$.

□

Observe that as an immediate consequence of Theorem 3.3 we obtain that $\mathcal{Q} \subseteq \mathcal{T}$.

Remark 3.4. Taking into account Lemma 3.1 and Theorem 3.3, a natural question is if $\overline{R(T)} \cap N(T) = \{0\}$ and $TP_{\overline{R(T)}} \in \mathcal{L}^+$ imply $T \in \mathcal{T}$. However, this is false in general. In fact, consider a Hilbert space decomposition $\mathcal{H} =$

$\mathcal{S} \oplus \mathcal{S}^{\perp}$ and define $T = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$ with a a positive injective operator with no closed range and b such that $R(b) \not\subseteq R(a^{1/2})$. Then, $\overline{R(T)} = \overline{R(a)} = \mathcal{S}$ and so, by the injectivity of a, we have that $\overline{R(T)} \cap N(T) = \{0\}$. Moreover, $TP_{\overline{R(T)}} = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \in \mathcal{L}^+$. However, since $R(b) \not\subseteq R(a^{1/2})$, then there does not exist $A \in \mathcal{L}^+$ such that $T = P_{\overline{R(T)}}A$, i.e., $T \notin \mathcal{T}$ (see [23] and [19]).

In the sequel, we abbreviate

$$\mathcal{T}_{cr} = \mathcal{T} \cap \mathcal{CR}.$$

Note that, by Theorem 3.3, $\mathcal{T}_{cr} = \mathcal{P} \cdot \mathcal{G}^+$.

Proposition 3.5. *It holds* $\mathcal{T}_{cr}^{\dagger} = \mathcal{T}_{cr}$.

Proof. Let $T \in \mathcal{T}_{cr,}$ then by Theorem 3.3, $T = PA$ with $A \in \mathcal{G}^+$ and $P = P_{R(T)}$. Now, define $C = P_{R(AP)}A^{-1}$. Observe that $TC = P$ and $R(C) = N(T)^{\perp}$ then, by Theorem 3.1 in [2], $T^{\dagger} = C = P_{R(AP)}A^{-1}$ and so $T^{\dagger} \in \mathcal{T}_{cr}$. The converse follows from the fact that $(T^{\dagger})^{\dagger} = T$. □

4. Optimal factorization

In this section, given $T \in \mathcal{T}$, we describe all factors $A \in \mathcal{T}_T^+$ and $P \in \mathcal{T}_T^{\mathcal{P}}$ such that $T = PA$. In particular we show that T admits an optimal factorization.

Proposition 4.1. *Let $T \in \mathcal{T}$. Then, there exists $A \in \mathcal{T}_T^+$ with $N(A) = N(T)$. Moreover, there exists a unique $A \in \mathcal{T}_T^+$ with $N(A) = N(T)$ if and only if $\overline{R(T^*)} \cap N(T^*) = \{0\}$.*

Proof. Let $P = P_{\overline{R(T)}}$. As $T \in \mathcal{T}$ then $PX = T$ has a positive solution. Now, by Corollary 2.4, there exists $A \in \mathcal{L}^+$ such that $PA = T$ and $N(A) = N(T)$.

On the other hand, suppose that $\mathcal{S} = \overline{R(T^*)} \cap N(T^*) \neq \{0\}$ and let $A \in \mathcal{T}_T^+$ with $N(A) = N(T)$. Define $Y := A + P_{\mathcal{S}}$. Observe that $Y \in \mathcal{T}_T^+$ and so $N(Y) \subseteq N(T)$. Now, let $x \in N(T) \subseteq \mathcal{S}^{\perp}$. Then $Yx = Ax + P_{\mathcal{S}}x = 0$. Thus, $N(T) \subseteq N(Y)$, i.e., $N(T) = N(Y)$ and so the uniqueness does not hold. Conversely, suppose that there exist $A_1, A_2 \in \mathcal{T}_T^+$ with $N(A_1) = N(A_2) = N(T)$. Then $A_1 - A_2$ is a selfadjoint solution of the equation $PX = 0$. Then, by Lemma 2.8 in [3], $A_1 - A_2 = (I - P)(A_1 - A_2)(I - P)$. Therefore $R(A_1 - A_2) \subseteq \overline{R(T^*)} \cap R(T)^{\perp} = \overline{R(T^*)} \cap N(T^*) = \{0\}$. So that $A_1 = A_2$. □

Remark 4.2. In the sequel, given $T \in \mathcal{T}$ we shall denote by

$$A_T := (((TP)^{1/2})^\dagger T)^*((TP)^{1/2})^\dagger T,$$

where $P = P_{\overline{R(T)}}$. Note that:

1. $A_T \in \mathcal{T}_T^+$.
2. $N(A_T) = N(T)$.
3. $T = PA_T$.
4. If $T \in \mathcal{T}_{cr}$ then $R(A_T) = R(T^*)$.

Indeed, as $T \in \mathcal{T}$, then the equation $PX = T$ has a positive solution. Therefore, items 1 and 2 follows by the proof of Corollary 2.4. Moreover, since $A_T \in \mathcal{T}_T^+$ then there exists $P_\mathcal{S} \in \mathcal{P}$ such that $T = P_\mathcal{S} A_T$. Then, as $\overline{R(T)} \subseteq \mathcal{S}$, it holds that $T = P_{\overline{R(T)}} P_\mathcal{S} A_T = P_{\overline{R(T)}} A_T$. On the other hand, if $T \in \mathcal{T}_{cr}$ then, by Theorem 3.3, $\mathcal{H} = R(T) \dot{+} N(T)$. Hence, $R(TP) = R(T)$, and $R(((TP)^{1/2})^\dagger T) = R(((TP)^{1/2})^\dagger TP) = R((TP)^{1/2}) = R(TP) = R(T)$. Then $R(A_T)$ is closed and so, by item 2, $R(A_T) = R(T^*)$.

Observe that, by Theorem 3.3, given $T \in \mathcal{T}_{cr}$, it holds that $\mathcal{H} = R(T) \dot{+} N(T)$. Thus, the projection $Q_{R(T)//N(T)}$ with range $R(T)$ and nullspace $N(T)$ is well-defined. In the next proposition we show that this projection can also be factorized in terms of the factors of $T \in \mathcal{T}$.

Proposition 4.3. *Let $T \in \mathcal{T}_{cr}$ and $P = P_{R(T)}$. Then,*

$$Q_{R(T)//N(T)} = P(A_T P)^\dagger A_T.$$

Proof. It is easy to check that $P(A_T P)^\dagger A_T$ is an idempotent operator with $R(P(A_T P)^\dagger A_T) \subseteq R(T)$. Thus, let us show that $N(P(A_T P)^\dagger A_T) = N(T)$. Now, let $x \in N(P(A_T P)^\dagger A_T)$. Then $(A_T P)^\dagger A_T x \in R(T)^\perp \cap N(A_T P)^\perp = R(T)^\perp \cap R(T) = \{0\}$. So that $A_T x \in R(T^*) \cap N((A_T P)^\dagger) = R(T^*) \cap N(PA_T) = R(T^*) \cap N(T) = \{0\}$. Hence $x \in N(A_T) = N(T)$ and so $N(P(A_T P)^\dagger A_T) \subseteq N(T)$. The other inclusion is trivial. In consequence $P(A_T P)^\dagger A_T = Q_{R(T)//N(T)}$. □

Our next result fully describes the set $\mathcal{T}_T^+$.

Proposition 4.4. *Let $T \in \mathcal{T}$ and $P = P_{\overline{R(T)}}$. Then*

$$\mathcal{T}_T^+ = \{A_T + (I - P)C(I - P) : C \in \mathcal{L}^+\}.$$

In particular, $\mathcal{T}_T^+$ is a closed convex set.

Proof. Let us consider the orthogonal decomposition $\mathcal{H} = \overline{R(T)} \oplus R(T)^{\perp}$. Then, under this decomposition, $T = \begin{bmatrix} t_1 & t_2 \\ 0 & 0 \end{bmatrix}$ and $P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Thus, if $A \in \mathcal{T}_T^+$ then $T = PA$ and so, by [23] (see also [19]), $A = \begin{bmatrix} t_1 & t_2 \\ t_2^* & d^*d + f \end{bmatrix}$ with $d = (t_1^{1/2})^{\dagger} t_2$ and $f \in \mathcal{L}(R(T)^{\perp})^+$. Now, $A = \begin{bmatrix} t_1 & t_2 \\ t_2^* & d^*d \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & f \end{bmatrix} = A_T + (I-P)C(I-P)$ with $C \in \mathcal{L}^+$.

The other inclusion follows from Remark 4.2. □

Corollary 4.5. *Let $T \in \mathcal{T}_{cr}$ and $P = P_{R(T)}$. Then,*

$$\mathcal{T}_T^+ \cap \mathcal{G} \;=\; \{A_T + (I-P)C(I-P) :\; C \in \mathcal{G}^+\}$$

Proof. Let $A \in \mathcal{T}_T^+$ be an invertible operator. Then, by Proposition 4.4, $A = A_T + (I-P)S(I-P)$ with $S \in \mathcal{L}^+$. Now, let us define $C = S + TT^* \in \mathcal{L}^+$. Note that $R(C) = \mathcal{H}$. In fact, by Theorem 2.1 and Remark 4.2, $\mathcal{H} = R(A) = R(T^*) + R((I-P)S^{1/2})$. Now, by Theorem 3.3, $R(I-P) \dot{+} R(T^*) = N(T^*) \dot{+} R(T^*) = \mathcal{H}$. Hence, $R((I-P)S^{1/2}) = R(I-P)$ and so $\mathcal{H} = R(S^{1/2}) + R(T) = R(C)$. Thus, $C \in \mathcal{G}(\mathcal{H})^+$ and $A = A_T + (I-P)S(I-P) = A_T + (I-P)C(I-P)$.

For the converse, it is sufficient to note that given $C \in \mathcal{G}(\mathcal{H})^+$ then $R((A_T + (I-P)C(I-P))^{1/2}) = R((A_T)^{1/2}) + R((I-P)C^{1/2}) = R(T^*) + N(T^*) = \mathcal{H}$, where the last equality is consequence of Theorem 3.3. Therefore, $A_T + (I-P)C(I-P) \in \mathcal{G}^+$ and the result is proved. □

In the next proposition we describe the elements of $\mathcal{G}^+$ that factorize $\mathcal{CR}^+$ and $\mathcal{Q}$.

Proposition 4.6. *Let $A \in \mathcal{G}^+$. The following equivalence holds:*

1. *$PA \in \mathcal{CR}^+$ for some $P \in \mathcal{P}$ if and only if $A(\mathcal{S}) = \mathcal{S}$ for some closed subspace $\mathcal{S}$.*
2. *$PA \in \mathcal{Q}$ for some $P \in \mathcal{P}$ if and only if $A^{1/2}|_{\mathcal{S}}$ is an isometry for some closed subspace $\mathcal{S}$.*

Proof. 1. If $PA \in \mathcal{CR}^+$ for some $P \in \mathcal{P}$ with $R(P) = \mathcal{S}$ then $PA = AP$ and so $A\mathcal{S} = \mathcal{S}$. Conversely, if $A\mathcal{S} = \mathcal{S}$ then $P_{\mathcal{S}}AP_{\mathcal{S}} = AP_{\mathcal{S}}$ and so $AP_{\mathcal{S}} \in \mathcal{CR}^+$.

2. See Theorem 2 in [15].

$\square$

In the next proposition we show that A_T is optimal in $\mathcal{T}_T^+$ in two senses.

Proposition 4.7. *Let $T \in \mathcal{T}$. Then, $A_T = \min \mathcal{T}_T^+$. Moreover, $\|A_T\| = \min\{\|A\| : A \in \mathcal{T}_T^+\}$.*

Proof. If $A \in \mathcal{T}_T^+$ then, by Proposition 4.4, $A = A_T + C$ with $C \in \mathcal{L}^+$ and so $A_T \leq A$. Thus, the first equality is proved. For the second equality, as $0 \leq A_T \leq A$, then for all $x \in \mathcal{H}$ with $\|x\| = 1$, we have that $\langle A_T x, x\rangle \leq \langle Ax, x\rangle \leq \|A\|$. Thus, $\|A_T\| = \sup_{\|x\|=1} \langle A_T x, x\rangle \leq \|A\|$. $\square$

We now study the set $\mathcal{T}_T^{\mathcal{P}}$.

Proposition 4.8. *Let $T \in \mathcal{T}$. Then,*

$$\mathcal{T}_T^{\mathcal{P}} = \{P_{\mathcal{S}} \in \mathcal{P} : \mathcal{S} = \overline{R(T)} \oplus \mathcal{M} \text{ for some } \mathcal{M} \subseteq N(T)\}.$$

Moreover, fixed $A \in \mathcal{T}_T^+$ then

$$\{P_{\mathcal{S}} \in \mathcal{P} : T = P_{\mathcal{S}} A\} = \{P_{\mathcal{S}} \in \mathcal{P} : \mathcal{S} = \overline{R(T)} \oplus \mathcal{M} \text{ for some } \mathcal{M} \subseteq N(A)\}.$$

On the other hand, fixed $P \in \mathcal{T}_T^{\mathcal{P}}$,

$$\{A \in \mathcal{L}^+ : T = PA\} = \{A_T + (I - P)C(I - P) : C \in \mathcal{L}^+\}.$$

Proof. Let us prove the first equality. For this, if $T = P_{\mathcal{S}} A$ with $A \in \mathcal{L}^+$ then $\overline{R(T)} \subseteq \mathcal{S}$ and so $T = P_{\overline{R(T)}} A$. Furthermore, $\mathcal{M} := \mathcal{S} \ominus \overline{R(T)}$ is well-defined and $\mathcal{S} = \overline{R(T)} \oplus \mathcal{M}$. Therefore, $P_{\mathcal{S}} = P_{\overline{R(T)}} + P_{\mathcal{M}}$ and $P_{\overline{R(T)}} A = T = P_{\mathcal{S}} A = P_{\overline{R(T)}} A + P_{\mathcal{M}} A$. After cancellation, we get $P_{\mathcal{M}} A = 0$, i.e., $\mathcal{M} \subseteq N(A)$. Now, since $N(A) \subseteq N(T)$ we obtain the desired inclusion.

Conversely, let $\mathcal{S} = \overline{R(T)} \oplus \mathcal{M}$ with $\mathcal{M} \subseteq N(T)$. Since $T \in \mathcal{T}$, there exists $A \in \mathcal{L}^+$ with $N(A) = N(T)$ such that $T = P_{\overline{R(T)}} A$. Now, as $\mathcal{M} \subseteq N(T) = N(A)$ we obtain that $P_{\mathcal{S}} A = P_{\overline{R(T)}} A + P_{\mathcal{M}} A = T$. The equality is proved.

The second equality can be proved similarly.

Now, given $P \in \mathcal{T}_T^{\mathcal{P}}$, we know that $R(P) = \overline{R(T)} \oplus \mathcal{M}$ with $\mathcal{M} \subseteq N(T)$. Thus, $P = P_{\overline{R(T)}} + P_{\mathcal{M}}$. Note that as $N(T) = N(A_T)$ then we get that $PA_T = T$. Now, let $A \in \mathcal{L}^+$ such that $PA = T$. Hence, by Proposition

4.4, $A = A_T + (I - P_{\overline{R(T)}})C(I - P_{\overline{R(T)}})$ for some $C \in \mathcal{L}^+$. Hence, $T = PA = PA_T + (P - PP_{\overline{R(T)}})C(I - P_{\overline{R(T)}}) = T + (P - P_{\overline{R(T)}})C(I - P_{\overline{R(T)}})$. Thus, $(P - P_{\overline{R(T)}})C(I - P_{\overline{R(T)}}) = 0$ and so $(P - P_{\overline{R(T)}})C(I - P_{\overline{R(T)}})P = (P - P_{\overline{R(T)}})C(P - P_{\overline{R(T)}}) = 0$ Now, as $P - P_{\overline{R(T)}} = P_{\mathcal{M}}$, then $P_{\mathcal{M}}CP_{\mathcal{M}} = 0$, i.e., $R(C^{1/2}) \subseteq \mathcal{M}^\perp$. Finally, $(I - P_{\overline{R(T)}})C(I - P_{\overline{R(T)}}) = (I - (P - P_{\mathcal{M}}))C(I - (P - P_{\mathcal{M}})) = (I - P)C(I - P)$ and so $A = A_T + (I - P)C(I - P)$. The other inclusion follows from $PA_T = T$. □

As consequence of the previous results we obtain a characterization of the set $\{(P, A) : PA = T\}$ for a given $T \in \mathcal{T}$. Observe that Proposition 4.8 gives partial answers of this problem.

Theorem 4.9. *Let $T \in \mathcal{T}, P \in \mathcal{P}$ and $A \in \mathcal{L}^+$. Then, $T = PA$ if and only if there exists a closed subspace, $\mathcal{M}$, of $\mathcal{H}$ and $C \in \mathcal{L}^+$ such that*

1. $R(P) = \overline{R(T)} \oplus \mathcal{M}$;
2. $\mathcal{M} \subseteq N(T)$;
3. $A = A_T + (I - P)C(I - P)$.

Proof. It follows from Proposition 4.8. □

We prove now the minimality of $P_{\overline{R(T)}}$ in $\mathcal{T}_T^{\mathcal{P}}$.

Proposition 4.10. *Let $T \in \mathcal{T}$. Then, $P_{\overline{R(T)}} = \min \mathcal{T}_T^{\mathcal{P}}$.*

Proof. Let $P_{\mathcal{S}} \in \mathcal{T}_T^{\mathcal{P}}$. Then, $\overline{R(T)} \subseteq \mathcal{S}$, i.e., $P_{\overline{R(T)}} \leq P_{\mathcal{S}}$. □

Definition 4.11. *For $T \in \mathcal{T}$ the identity $T = P_{\overline{R(T)}}A_T$ is called the **optimal factorization** of T.*

Remark 4.12. In [6] it is proven that, for $T \in \mathcal{P} \cdot \mathcal{P}$, the identity $T = P_{\overline{R(T}}P_{N(T)^\perp}$, found by T. Crimmins (see [21] Theorem 8) has several minimality properties. We show now that it coincides with the optimal factorization of T, i.e., for $T \in \mathcal{P} \cdot \mathcal{P}$ it holds $A_T = P_{N(T)^\perp}$. In fact, $P_{N(T)^\perp}$ is a positive operator with nullspace $N(T)$, so, by Crimmins' result, $P_{N(T)^\perp} \in \mathcal{T}_T^+$. On the other hand, by [6] Theorem 3.2, $\overline{R(T^*)} \cap N(T^*) = \{0\}$. Then, by Proposition 4.1, we get $A_T = P_{N(T)^\perp}$.

5. Compatibility

The aim of this section is to relate the factors P, A of a given $T \in \mathcal{T}$ with compatibility. The notion of compatibility relates a closed subspace $\mathcal{S}$ of $\mathcal{H}$ and a positive operator $A \in \mathcal{L}(\mathcal{H})$. More precisely, the pair $(A, \mathcal{S})$ is called **compatible** if there exists $Q \in \mathcal{Q}$ with $R(Q) = \mathcal{S}$ such that $AQ = Q^*A$ (this means that Q is Hermitian respect to the semi-inner product induced by A). This notion can be also extended to unbounded projections, in which case the pair $(A, \mathcal{S})$ is called **quasi-compatible** if there exists a densely defined closed projection Q onto $\mathcal{S}$ such that AQ is symmetric. The quasi-compatibility (resp., compatibility) of a pair $(A, \mathcal{S})$ is equivalent to $\overline{\mathcal{S} + (A\mathcal{S})^\perp} = \mathcal{H}$ (resp. $\mathcal{S} + (A\mathcal{S})^\perp = \mathcal{H}$). In particular, the notion of compatibility is also equivalent to certain angle condition, more precisely, the Dixmier angle between $\mathcal{S}^\perp$ and $\overline{A\mathcal{S}}$ is non-zero. Recall that the Dixmier angle between two closed subspaces $\mathcal{S}_1, \mathcal{S}_2$ is that whose cosine is $c_0(\mathcal{S}_1, \mathcal{S}_2) = \sup\{| \langle x, y\rangle | : x \in \mathcal{S}_1, y \in S_2, ||x||, ||y|| \leq 1\}$. Therefore, it holds that $(A, \mathcal{S})$ is compatible if and only if $c_0(\mathcal{S}^\perp, \overline{A\mathcal{S}}) < 1$. For more results on the theory of compatibility see [7] and references therein. For details on quasi-compatibility see [4].

Given $T \in \mathcal{T}$, we study the quasi-compatibility (resp., compatibility) of the pairs $(A, \mathcal{S})$ such that $T = P_{\mathcal{S}}A$. We begin by showing that such compatibility is independent of the factors chosen, that is, it depends only on T and not on the particular $P_{\mathcal{S}}, A$.

Proposition 5.1. *Let $T = P_{\mathcal{S}}A \in \mathcal{T}$.*

1. *The following conditions are equivalent:*
   (a) *$(A, \mathcal{S})$ is quasi-compatible,*
   (b) *$\overline{R(T)} + N(T)$ is a dense subspace of $\mathcal{H}$;*
   (c) *$\overline{R(T^*)} \cap N(T^*) = \{0\}$.*
2. *The following conditions are equivalent:*
   (a) *$(A, \mathcal{S})$ is compatible;*
   (b) *$c_0(\mathcal{S}^\perp, \overline{A\mathcal{S}}) < 1$;*
   (c) *$\overline{R(T)} \dot{+} N(T) = \mathcal{H}$;*
   (d) *$\overline{R(T^*)} \dot{+} N(T^*) = \mathcal{H}$;*
   (e) *$c_0(\overline{R(T)}, N(T)) < 1$.*
   (f) *$c_0(\overline{R(T^*)}, N(T^*)) < 1$.*

Proof. By Proposition 4.8, $\mathcal{S} = \overline{R(T)} \oplus \mathcal{M}$ with $\mathcal{M} \subseteq N(A)$. Then, $(A\mathcal{S})^\perp = (A\overline{R(T)})^\perp = R(AP_{\overline{R(T)}})^\perp = N(P_{\overline{R(T)}}A) = N(T)$.

1. If $(A,\mathcal{S})$ is quasi-compatible then $\mathcal{H} = \overline{\mathcal{S}+(A\mathcal{S})^{\perp}} = \overline{\mathcal{S}+N(T)} \subseteq \overline{\overline{R(T)}+\mathcal{M}+N(T)} = \overline{\overline{R(T)}+N(T)}$ because $\mathcal{M}\subseteq N(A)\subseteq N(T)$. The converse is similar. $b \Leftrightarrow c$ follows by taking orthogonal complements.
2. $a \Leftrightarrow b$ follows from Theorem 2.15 in [7]. Now, if item a) holds then $\mathcal{H} = \mathcal{S}+(A\mathcal{S})^{\perp}$. That is, $\mathcal{H} = \mathcal{S}+N(T) = \overline{R(T)}\oplus\mathcal{M}+N(T) = \overline{R(T)}+N(T)$, where the last equality follows from $\mathcal{M}\subseteq N(A)\subseteq N(T)$. Moreover, by Lemma 3.1, $\mathcal{H} = \overline{R(T)} \dot{+} N(T)$. Thus, $a \Rightarrow c$. $c \Leftrightarrow d$ is consequence of Lemma 2.11 in [9]. $d \Rightarrow e$ follows from Theorem 2.15 and Lemma 2.12 in [9]. $e \Rightarrow f$ is also consequence of Lemma 2.12 in [9].
   Finally, if item f) holds, as $\overline{R(T^*)} = \overline{A\mathcal{S}}$ and $\mathcal{S}^{\perp}\subseteq N(T^*)$, then $c_0(\overline{A\mathcal{S}},\mathcal{S}^{\perp}) < c_0(\overline{R(T^*)},N(T^*))$. Therefore, $f \Rightarrow a$ because of Theorem 2.15 in [7].

□

In the next result, given a positive operator A and a closed subspace $\mathcal{S}$, we characterize the quasi-compatibiity of $(A,\mathcal{S})$ in terms of the existence of certain operator in $\mathcal{T}$.

Proposition 5.2. *Let $A\in\mathcal{L}^+$ and $\mathcal{S}$ a closed subspace of $\mathcal{H}$. The pair $(A,\mathcal{S})$ is quasi-compatible if and only if there exists $T\in\mathcal{T}$ such that $\overline{R(T)} = \overline{A\mathcal{S}}$ and $N(T) = (\mathcal{S}\ominus(A\mathcal{S})^{\perp})^{\perp}$.*

Proof. If $(A,\mathcal{S})$ is quasi-compatible then, by [[4], Proposition 2.15] there exists $T\in\mathcal{L}(\mathcal{H})$ such that $TT^*T = T^2$, $\overline{R(T)} = \overline{A\mathcal{S}}$ and $N(T) = (\mathcal{S}\ominus(A\mathcal{S})^{\perp})^{\perp}$. Then, $(T^*T)^2 = T^*T^2$ and so, by Theorem 3.2, $T\in\mathcal{T}$. Conversely, if there exists $T\in\mathcal{T}$ such that $\overline{R(T)} = \overline{A\mathcal{S}}$ and $N(T) = (\mathcal{S}\ominus(A\mathcal{S})^{\perp})^{\perp}$ then, by Lemma 3.1, $\overline{A\mathcal{S}}\cap(\mathcal{S}\ominus(A\mathcal{S})^{\perp})^{\perp} = \{0\}$. So that $\mathcal{S}+(A\mathcal{S})^{\perp}$ is dense in $\mathcal{H}$. Therefore, $(A,\mathcal{S})$ is quasi-compatible. □

Given a quasi-compatible pair $(A,\mathcal{S})$ there exists a distinguished element with optimal properties among all densely defined idempotents Q with domain $\mathcal{S}+(A\mathcal{S})^{\perp}$, $R(Q)=\mathcal{S}$ and AQ symmetric, namely, $P_{A,\mathcal{S}} := Q_{\mathcal{S}//(A\mathcal{S})^{\perp}\ominus\mathcal{S}}$ (see [4]). If the pair $(A,\mathcal{S})$ is compatible then $P_{A,\mathcal{S}}$ is bounded.

Proposition 5.3. *Let $T\in\mathcal{T}$ be such that $\overline{R(T)} \dot{+} N(T) = \mathcal{H}$. Therefore, if $T = P_{\mathcal{S}}A$ then $(A,\mathcal{S})$ is compatible and*

$$P_{A,\mathcal{S}} = Q_{\overline{R(T)}//N(T)} + P_{\mathcal{M}},$$

where $\mathcal{S} = \overline{R(T)}\oplus\mathcal{M}$ and $\mathcal{M}\subseteq N(A)$.

Proof. The compatibility of the pair $(A,\mathcal{S})$ follows from Proposition 5.1. Moreover, by Proposition 4.8, $\mathcal{S} = \overline{R(T)} \oplus \mathcal{M}$ with $\mathcal{M} \subseteq N(A)$. Now, define $E = Q_{\overline{R(T)}//N(T)} + P_{\mathcal{M}}$. Since $\mathcal{M} \subseteq N(T)$ and $\mathcal{M} \perp \overline{R(T)}$ then $E^2 = E$. Furthermore, $AE = E^*A$. Indeed, since $N(T) = (A\overline{R(T)})^{\perp}$ then $AQ_{\overline{R(T)}//N(T)} = Q^*_{\overline{R(T)}//N(T)}A$. Now, since $AE = AQ_{\overline{R(T)}//N(T)}$ we get that $AE = E^*A$. In addition, it is clear that $R(E) \subseteq \mathcal{S}$. Hence, it remains to show that $N(E) = N(P_{A,\mathcal{S}})$. Observe that $N(P_{A,\mathcal{S}}) = (A\mathcal{S})^{\perp} \cap ((A\mathcal{S})^{\perp} \cap \mathcal{S})^{\perp} = N(T) \cap (N(T) \cap (\overline{R(T)} + \mathcal{M}))^{\perp} = N(T) \cap (N(T) \cap \mathcal{M})^{\perp} = N(T) \cap \mathcal{M}^{\perp}$. Now, we prove the equality $N(E) = N(T) \cap \mathcal{M}^{\perp}$. Clearly, $N(T) \cap \mathcal{M}^{\perp} \subseteq N(E)$. For the other inclusion, if $x \in N(E)$ then $Q_{\overline{R(T)}//N(T)}x = -P_{\mathcal{M}}x \in \overline{R(T)} \cap \mathcal{M} = \{0\}$. So that $x \in N(T) \cap \mathcal{M}^{\perp}$. Then $N(E) = N(P_{A,\mathcal{S}})$ and so $E = P_{A,\mathcal{S}}$. □

Remark 5.4. Given $T \in \mathcal{T}$ such that $\overline{R(T)} + N(T)$ is a dense subspace of $\mathcal{H}$ then, by Proposition 5.1, the pair $(A,\mathcal{S})$ is quasi-compatible for all $A, \mathcal{S}$ such that $T = P_{\mathcal{S}}A$. In this case,

$$P_{A,\mathcal{S}} = Q_{\overline{R(T)}//N(T)} + P_{\mathcal{M}}|_{\overline{R(T)}+N(T)}.$$

Corollary 5.5. *Let $T \in \mathcal{T}_{cr}$. Then,*

1. *if $T = P_{\mathcal{S}}A$ then $(A,\mathcal{S})$ is compatible;*
2. *$(A, R(T))$ is compatible for all $A \in \mathcal{T}_T^+$ and $P_{A,R(T)} = P(A_TP)^{\dagger}A_T$ where $P = P_{R(T)}$.*

Proof. It follows from Theorem 3.3 and Propositions 4.3 and 5.3. □

6. Polar decomposition

This section is devoted to the study of the polar decomposition of the operators in $\mathcal{T}$. For this, given $T \in \mathcal{L}(\mathcal{H})$ we shall denote by V_T the partial isometry of the polar decomposition of T, i.e, $V_T \in \mathcal{J} = \{V \in \mathcal{L}(\mathcal{H}) : VV^*V = V\}$ and $T = V_T|T|$ with $N(V_T) = N(T)$ and $|T| = (T^*T)^{1/2}$. In addition, given a class of operators $\mathcal{M}$, we denote

$$\mathcal{J}_{\mathcal{M}} = \{V \in \mathcal{J} : \text{there exists } T \in \mathcal{M} \text{ such that } V = V_T\}.$$

The sets $\mathcal{J}_{\mathcal{Q}}$ and $\mathcal{J}_{\mathcal{P}\cdot\mathcal{P}}$ have been studied in [5] and [6], respectively. Our goal in this section is to describe the set $\mathcal{J}_{\mathcal{T}}$.

Proposition 6.1. *The following relations hold:*

1. $\mathcal{T} \cap \mathcal{J} = \mathcal{J}_{\mathcal{Q}}$;
2. $\mathcal{J}_{\mathcal{Q}} \subseteq \mathcal{J}_{\mathcal{P}\cdot\mathcal{P}} \subseteq \mathcal{J}_{\mathcal{T}}$.

Proof. 1. If follows from [5], Theorem 5.1.

2. The first inclusion can be deduced from [5], [6]. In fact, if $E \in \mathcal{Q}$ then $E^\dagger = P_{N(E)^\perp} P_{R(E)}$ by a result of Penrose [20] (see also Greville [13] or Vidav [24]): the reader can easily check that $X = P_{N(E)^\perp} P_{R(E)}$ satisfy the four Penrose conditions $EXE = E, XEX = X, (XE)^* = XE, (EX)^* = EX$. This shows that $\mathcal{Q}^\dagger \subseteq \mathcal{P} \cdot \mathcal{P}$ and, therefore, $\mathcal{J}_{\mathcal{Q}^\dagger} \subseteq \mathcal{J}_{\mathcal{P}\cdot\mathcal{P}}$. On the other side, for any class $\mathcal{M} \subseteq \mathcal{L}(\mathcal{H})$, the properties of the standard polar decomposition show that $\mathcal{J}_{\mathcal{M}^\dagger} = \mathcal{J}_{\mathcal{M}^*}$. Since Q is closed by the adjoint operation, we get $\mathcal{J}_{\mathcal{Q}} = \mathcal{J}_{\mathcal{Q}^\dagger} \subseteq \mathcal{J}_{\mathcal{P}\cdot\mathcal{P}}$ as claimed. The second inclusion is consequence of $\mathcal{P} \cdot \mathcal{P} \subset \mathcal{T}$.

□

Proposition 6.2. *Let $A \in \mathcal{G}^+$. Then, $PA \in \mathcal{J}_{\mathcal{Q}}$ for some $P \in \mathcal{P}$ if and only if $A|_{\mathcal{S}}$ is an isometry for some closed subspace $\mathcal{S}$.*

Proof. Suppose that $PA \in \mathcal{J}_{\mathcal{Q}}$ for some $P \in \mathcal{P}$. Then, $P = PA(PA)^* = PA^2P$ and so, $(PA^2)^2 = PA^2PA^2 = PA^2$, i.e, $PA^2 \in \mathcal{Q}$ and the result follows by Proposition 4.6. Conversely, if $A|_{\mathcal{S}}$ is an isometry for some closed subspace $\mathcal{S}$ then, by Proposition 4.6, $PA^2 \in \mathcal{Q}$. Moreover, $R(PA^2) = R(P)$ because $A \in \mathcal{G}$. Hence, $PA(PA)^*PA = PA^2PA = PA$, i.e, $PA \in \mathcal{J} \cap \mathcal{T} = \mathcal{J}_{\mathcal{Q}}$. □

Proposition 6.3. *Let $T \in \mathcal{T}$. Then $V_T^*A = |T|$ for all $A \in \mathcal{T}_T^+$ and $|T|V_T \in \mathcal{L}^+$.*

Proof. Let $T = P_{\overline{R(T)}}A = V_T|T|$, i.e, $V_TV_T^*A = V_T|T|$. Hence, $V_T^*A = V_T^*V_TV_T^*A = V_T^*V_T|T| = |T|$ and so $|T|V_T = V_T^*AV_T \in \mathcal{L}^+$.

□

Proposition 6.4.

$$\mathcal{J}_{\mathcal{T}} = \{V \in \mathcal{J} : \exists A \in \mathcal{L}^+ \text{ such that } AV \in \mathcal{L}^+ \text{ and } R(V) \cap N(A) = \{0\}\}.$$

Proof. Consider $T = V_T|T| \in \mathcal{T}$. Therefore, by Proposition 6.3, $AV_T = |T| \in \mathcal{L}^+$ for all $A \in \mathcal{T}_T^+$. In addition, $N(AV_T) = N(|T|) = N(T)$. Therefore, if $y \in R(V_T) \cap N(A)$ then $y = V_Tx$ for some $x \in \mathcal{H}$ and $Ay = AV_Tx = 0$, i.e.,

$x \in N(AV_T) = N(T) = N(V_T)$. So, $y = V_T x = 0$ and the first inclusion is proved.

Conversely, let $V \in \mathcal{J}$ such that $AV \in \mathcal{L}^+$ for some $A \in \mathcal{L}^+$ with $R(V) \cap N(A) = \{0\}$. Define $T := VV^*A \in \mathcal{T}$. Then, $T^*T = AVV^*VV^*A = AVV^*A = (V^*A)^2$, i.e., $|T| = V^*A$. Moreover, $N(T) = N(|T|) = N(V^*A) = N(AV) = N(V)$ where the last equality holds because $R(V) \cap N(A) = \{0\}$. Therefore VV^*A is the polar decomposition of $T \in \mathcal{T}$ and so $V \in \mathcal{J}_{\mathcal{T}}$. □

Given two operators $T, S \in \mathcal{L}(\mathcal{H})$ we write $T \sim_+ S$ if there exists $A \in \mathcal{G}^+$ such that $T = A^{-1}SA$.

Corollary 6.5.

$$\begin{aligned} \mathcal{J}_{\mathcal{T}_{cr}} &= \{V \in \mathcal{J} : \exists A \in \mathcal{G}^+ \text{ such that } AV \in \mathcal{L}^+\} \\ &= \{V \in \mathcal{J} : V \sim_+ B \text{ for some } B \in L(\mathcal{H})^+\}. \end{aligned}$$

Proof. Let us prove the first equality. Consider $T = V_T|T| \in \mathcal{T}_{cr}$. Then, by Theorem 3.3, there exists $A \in \mathcal{G}^+$ such that $T = V_T V_T^* A = V_T|T|$. So $V_T^* A = |T| \in L(\mathcal{H})^+$ with $A \in \mathcal{G}^+$.

For the other inclusion, let $V \in \mathcal{J}$ such that $AV \in L(\mathcal{H})^+$ for some $A \in \mathcal{G}^+$. Define $T := VV^*A$. Clearly, $T \in \mathcal{T}_{cr}$. Moreover, it is straightforward that $|T| = V^*A$ and $N(V) = N(T)$.

For the second equality, note that $AV \geq 0$ for some $A \in \mathcal{G}^+$ if and only if $A^{1/2}VA^{-1/2} = B \geq 0$, i.e., $V \sim_+ B$ with $B \geq 0$. □

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