Operators which are the difference of two projections

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Abstract

We study the set $\mathcal{D}$ of differences

$$\mathcal{D} = \{A = P - Q : P, Q \in \mathcal{P}\},$$

where $\mathcal{P}$ denotes the set of orthogonal projections in $\mathcal{H}$. We describe models and factorizations for elements in $\mathcal{D}$, which are related to the geometry of $\mathcal{P}$. The study of $\mathcal{D}$ throws new light on the geodesic structure of $\mathcal{P}$ (we show that two projections in generic position are joined by a unique minimal geodesic). The topology of $\mathcal{D}$ is examined, particularly its connected components are studied, as well as the components of the subsets $\mathcal{D}_c \subset \mathcal{D}_F$,

$$\mathcal{D}_c = \{A \in \mathcal{D} : A \text{ is compact}\}$$

and

$$\mathcal{D}_F = \{A = P - Q : (P, Q) \text{ is a Fredholm pair}\}.$$

((P, Q) is a Fredholm pair if $QP|_{R(P)} : R(P) \to R(Q)$ is a Fredholm operator.)

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1 Introduction

We study bounded linear operators in a Hilbert space $\mathcal{H}$ which are the difference of two orthogonal projections:

$$A = P - Q.$$

Apparently, such operators A are selfadjoint contractions. Also, straightforward computations show that

$$N(A) = (N(P) \cap N(Q)) \oplus (R(P) \cap R(Q)), \quad N(A-1) = R(P) \cap N(Q)$$

and

$$N(A+1) = N(P) \cap R(Q).$$

Note that $N(A)$, $N(A-1)$, $N(A+1)$, and the orthogonal complement $\mathcal{H}_0$ of the sum of these, reduce P, Q and A. These subspaces depend on A and not on the projections P and Q. The space $\mathcal{H}_0$ is usually called the generic part of P and Q. We shall call it, we guess more appropriately, the generic part of $A = P - Q$. It is the generic part that is of interest, as A acts trivially on the non generic part. Namely, denote by $A_0 = A|_{\mathcal{H}_0}$ the generic part of A, acting in $\mathcal{H}_0$. Apparently, in the decomposition

$$\mathcal{H} = N(A) \oplus N(A-1) \oplus N(A+1) \oplus \mathcal{H}_0$$

A is given by

$$A = 0 \oplus 1 \oplus -1 \oplus A_0.$$

There is an extensive bibliography on pairs of projections. There is also a very good survey paper on the subject by A. Böttcher and I.M. Spitkowsky [4], and we refer the reader to the references therein. We shall base our remarks on two classic papers on the subject, by P. Halmos [9] and C. Davis [6]. The first of these papers provides a simple 2×2 matrix model for a given pair of projections P, Q, which we describe below. One of the many consequences is that the generic parts P_0 and Q_0 acting in $\mathcal{H}_0$ are unitarily equivalent, with an explicitely constructed unitary operator implementing this equivalence. The second paper characterizes the operators A which are a difference of projections: their generic parts are selfadjoint contractions A_0 which anticommute with a symmetry V (a symmetry is a selfadjoint unitary operator: $V^* = V = V^{-1}$).

We regard the present paper as an incomplete comment on these two papers. Given our interest in the differential geometry of the space $\mathcal{P}$ of projections in $\mathcal{H}$ [5], we relate Halmos and Davis results to the question of the existence and uniqueness of geodesics in $\mathcal{P}$.

The contents of the paper are the following. In Section 2 we recall the results by Halmos [9] and Davis [6], as well as certain facts from the geometry of $\mathcal{P}$ [5]. Section 3 contains consequences of Davis characterization of differences of projections A, particularly, that symmetries V which anticommute with A_0 parametrize all pairs P, Q such that $A = P - Q$. In Section 3 we show how each geodesic of $\mathcal{P}$ joining P and Q provides a factorization $A = e^{iZ}\sigma$, where A, $Z = Z^*$ and $\sigma = \sigma^*$ anticommute (in contrast to the polar decomposition $A = sgn(A)|A|$, where all data commute). In a previous work [2], it was shown that the projections P_0 and Q_0 in generic position can be joined by a (minimal) geodesic of $\mathcal{P}$. Using the ideas here we show that such geodesic is unique. In section 5 we obtain descriptions for operators $A = P - Q$ and anticommuting symmetries V, decomposing $\mathcal{H}$ in cyclic subspaces, as in the classic spectral theorem. In Section 6 we examine the topology of the space $\mathcal{D}$ of differences of projections. We study connected components and characterize the interior set of $\mathcal{D}$: it consists of operators A such that A_0 is non trivial. In section 7, using results from [3] (also [1]), we study operators $A = P - Q$ such that (P, Q) is a Fredholm pair. From the results obtained in [3] it is apparent that the property of being a Fredholm pair, depends on the difference A and not on the particular pair. So that an index for such (here called) Fredholm differences is defined, which coincides with $\dim(N(A-1)) - \dim(N(A+1))$. This allows us to characterize the connected components of the sets of the Fredholm differences and compact differences, as a consequence.

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2 Preliminaries

In [6], Chandler Davis proved the following result. Denote by

$$A_0 = A|_{\mathcal{H}_0} : \mathcal{H}_0 \to \mathcal{H}_0.$$

Then (Theorem 6.1 of [6]) the following are necessary and sufficient conditions on a selfadjoint operator A in order that it be the difference of two orthogonal projections:

$-1 \leq A \leq 1$ and there exists a unitary operator W of $\mathcal{H}_0$ such that $A_0 W = -W A_0$.

Moreover, Davis noted that in this case, if one denotes by $\mathcal{H}^+$ the closure of the range of A_0^+, the positive part of A_0, and $\mathcal{H}^-$ the closure of the range of the negative part, then W maps $\mathcal{H}^+$ onto $\mathcal{H}^-$ and viceversa, so that V defined as

$$V = W \text{ in } \mathcal{H}^+, \quad \text{and } V = W^* \text{ in } \mathcal{H}^-$$

is a symmetry (i.e. a selfadjoint unitary operator) in $\mathcal{H}_0$, which also verifies

$$VA_0 = -A_0 V.$$

Note that W uniquely determines V. In this case

$$P_V = \frac{1}{2}(1 + A_0 + V(1 - A_0^2)^{1/2}) \text{ and } Q_V = \frac{1}{2}(1 - A_0 + V(1 - A_0^2)^{1/2})$$

are orthogonal projections in $\mathcal{H}_0$ such that $A_0 = P_V - Q_V$.

In [9] P. Halmos proved that if P and Q are orthogonal projections, in the generic part $\mathcal{H}_0$ there exists an isometric isomorphism between $\mathcal{H}_0$ and a product Hilbert space $\mathcal{K} \times \mathcal{K}$, and positive contractions C, S acting in $\mathcal{K}$, with $C^2 + S^2 = 1_{\mathcal{K}}$, such that, via the isomorphism, the generic parts P_0 and Q_0 of P and Q are carried to

$$P_0 = \begin{pmatrix} 1_{\mathcal{K}} & 0 \\ 0 & 0 \end{pmatrix} \text{ and } Q_0 = \begin{pmatrix} C^2 & CS \\ CS & S^2 \end{pmatrix},$$

respectively. An elementary use of the functional calculus for selfadjoint operators shows that there exists a positive operator X in $\mathcal{K}$, $\|X\| \leq \pi/2$, such that $C = cos(X)$ and $S = sin(X)$.

We finish this section of preliminary facts, recalling the geometry of the space $\mathcal{P}$ of orthogonal projections in $\mathcal{H}$. Specifically, the linear connection introduced in [5], [11], and the properties of the geodesics of this connection (for instance see [2]).

Remark 2.1. The space $\mathcal{P}$ is a differentiable (C^∞) complemented submanifold of $\mathcal{B}(\mathcal{H})$. It carries a natural action of the unitary group $\mathcal{U}(\mathcal{H})$ of $\mathcal{H}$ (unitary conjugation). With this action, $\mathcal{P}$ becomes a homogeneous space with a natural reductive strucure. Instead of going into the details of this construction (which can be found in [5]), let us recall the form of the geodesics of the linear connection induced by the reductive structure:

1. [5] The geodesics starting at $P \in \mathcal{P}$ are of the form

$$\delta(t) = e^{itZ} P e^{-itZ},$$

where $Z^* = Z$ is a codiagonal matrix in terms of P, i.e. $PZP = (1-P)Z(1-P) = 0$.

2. [11] These geodesics have a nice description in terms of symmetries. A projection P induces the symmetry (selfadjoint unitary) $2P - 1$. The exponent Z is codiagonal with respect to P if and only if it anticommutes with $2P - 1$. Thus the geodesic curve of symmetries has the form

$$2\delta(t) - 1 = e^{iZ}(2P - 1)e^{-iZ} = e^{2iZ}(2P - 1) = (2P - 1)e^{-2iZ}.$$

3. [11] If one measures tangent vectors (which are selfadjoint operators) using the usual norm of $\mathcal{B}(\mathcal{H})$, these geodesics have minimal length for time $|t| \le \frac{\pi}{2\|Z\|}$.

4. [2] Two projections P, Q can be joined by a geodesic (which can be chosen of minimal length) if and only if
$$\dim(N(A-1)) = \dim(N(A+1)),$$
where $A = P - Q$. In particular, if they are in generic position, they are joined by a minimal geodesic.

3 Davis characterization

Remark 3.1. There is a one to one correspondence between pairs of projections P, Q such that $A = P - Q$ and symmetries V of $\mathcal{H}_0$ which anticommute with A_0 (i.e. $VA_0 = -A_0V$) such that $P_0 = P_V$ and $Q_0 = Q_V$ in the generic part $\mathcal{H}_0$ of A.

Indeed, let V, V' be two symmetries which anticommute with A_0 such that $P_V = P_{V'}$. Then
$$V(1-A_0^2)^{1/2} = V'(1-A_0^2)^{1/2}.$$
Note that $N(1-A_0^2) = N(A_0-1) \oplus N(A_0+1)$, therefore $1-A_0^2$ (and therefore also $(1-A_0^2)^{1/2}$) has trivial nullspace in $\mathcal{H}_0$. Thus $R((1-A_0^2)^{1/2})$ is dense in $\mathcal{H}_0$ and then $V = V'$. Conversely, note that any pair P, Q of orthogonal projections in $\mathcal{H}_0$ is of the form $P = P_V$, $Q = Q_V$ for V a symmetry in $\mathcal{H}_0$ which anticommutes with A_0. In [6], Davis proves that if $D = 1 - A_0^2$ (which has trival nullspace), then
$$V = D^{-1/2}(P+Q-1)$$
is a (bounded) symmetry. A straightforward computation shows that $P_V = P$ and $Q_V = Q$.

Let us state the following applications of Davis characterization. The first is that any selfadjoint contraction in $\mathcal{H}$ can be dilated to a difference of orthogonal projections.

Proposition 3.2. *Let $B^* = B \in \mathcal{B}(\mathcal{H})$ with $\|B\| \le 1$. Then there exist orthogonal projections P, Q in $\mathcal{H} \times \mathcal{H}$ such that $\mathcal{H} \times 0$ is invariant for $P - Q$, and $P - Q$ regarded as an operator in $\mathcal{H} \times 0$ coincides with B.*

Proof. Consider the selfadjoint contraction $A : \mathcal{H} \times \mathcal{H} \to \mathcal{H} \times \mathcal{H}$, given by
$$A(\xi, \eta) = (B\xi, -B\eta).$$
Clearly $\mathcal{H} \times 0$ is invariant for A. Consider the symmetry V
$$V(\xi, \eta) = (\eta, \xi).$$
Then apparently $VA = -AV$. Thus $A = P - Q$, and $P - Q$ restricted to $\mathcal{H} \times 0$ coincides with B. □

Also the compression of a difference of projections by a spectral projection corresponding to a subset of the spectrum which is symmetric with respect to the origin, is itself a difference of projections:

Proposition 3.3. *Let $A = P - Q$, and $E = E_\Omega(A)$ a spectral projection of A, corresponding to a Borel subset $\Omega \subset \sigma(A)$ which is symmetric with respect to the origin (i.e. $t \in \Omega$ implies $-t \in \Omega$). Then $EA = P' - Q'$, where P', Q' are orthogonal projections.*

Proof. Clearly $EA = AE$ is a self-adjoint contraction. Let A_0 be, as before, the restriction of A to its non-generic part $\mathcal{H}_0$. Clearly the restriction of E to $\mathcal{H}_0$ is also a spectral projection of A_0, and corresponds to the symmetric set $\Omega_0 = \Omega - \{-1, 0, 1\}$. By Davis result, there exists a symmetry V in $\mathcal{H}_0$ such that $VA_0 = -A_0V$. That is,

$$VA_0V = VA_0V^* = -A_0.$$

Thus, for any bounded Borel function g in $\mathbb{R}$, $Vg(A_0)V = g(-A_0)$. Consider $g(t) = t\chi_{\Omega_0}(t)$, where χ_{Ω_0} is the characteristic function of the set Ω_0. Note that, since Ω_0 is symmetric, $g(-t) = -g(t)$. Then

$$VEA_0V = Vg(A_0)V = g(-A_0).$$

Note that, for any $\xi, \eta \in \mathcal{H}_0$,

$$< g(-A_0)\xi, \eta >= \int_{\mathbb{R}} g(-t) du_{\xi,\eta}(t) = -\int_{\mathbb{R}} g(t) du_{\xi,\eta}(t) = - < g(A_0)\xi, \eta >,$$

where $\mu_{\xi,\eta}$ denotes the scalar spectral measure of A_0 corresponding to the vectors ξ, η. Then

$$VEA_0V = -EA_0,$$

i.e. V anticommutes with EA_0. □

4 The codiagonal factorization

If T is a positive operator in the Hilbert space $\mathcal{L}$ with $\|T\| \leq 1$, then the operator

$$\sigma_T = \begin{pmatrix} 0 & T \\ T & 0 \end{pmatrix}$$

acting in $\mathcal{L} \times \mathcal{L}$ is a difference of two projections. Indeed, the symmetry

$$V(\xi, \eta) = (\xi, -\eta)$$

anticommutes with σ_T. Let us see that any $A = P - Q$ is of this form under a suitable change of coordinates.

Remark 4.1. Let A_0 be the generic part of $A = P - Q$ as above. Then, with the current notations (as in Halmos result cited above), there are two natural factorizations for A_0, which can be described as 2×2 matrices in terms of P_0:

1. The polar decomposition:

$$A_0 = \begin{pmatrix} -S^2 & CS \\ CS & S^2 \end{pmatrix} = \begin{pmatrix} -S & C \\ C & S \end{pmatrix} \begin{pmatrix} S & 0 \\ 0 & S \end{pmatrix} = sgn(A_0)|A_0|.$$

where $sgn(A_0)$ is the sign function at A_0. Indeed, a straightforward matrix computation (using that C and S commute, and that $C^2 + S^2 = 1_{\mathcal{K}}$) shows that

$$A_0^2 = \begin{pmatrix} S^2 & 0 \\ 0 & S^2 \end{pmatrix},$$

and then, since $S \geq 0$,

$$|A_0| = \begin{pmatrix} S & 0 \\ 0 & S \end{pmatrix}.$$

Since $N(A_0)$ is trivial, the partial isometry in the polar decomposition of A_0 is a symmetry, and is given by $sgn(A_0)$. Clearly it corresponds with the right hand matrix in the decomposition of A_0 above. Note also that the polar factorization commutes.

2. Another factorization of A_0 is given by

$$A_0 = \begin{pmatrix} C & -S \\ S & C \end{pmatrix}\begin{pmatrix} 0 & S \\ S & 0 \end{pmatrix} = exp(i\begin{pmatrix} 0 & iX \\ -iX & 0 \end{pmatrix})\begin{pmatrix} 0 & sin(X) \\ sin(X) & 0 \end{pmatrix}$$

$$= exp(\frac{i}{2}\begin{pmatrix} 0 & iX \\ -iX & 0 \end{pmatrix})\begin{pmatrix} 0 & sin(X) \\ sin(X) & 0 \end{pmatrix} exp(-\frac{i}{2}\begin{pmatrix} 0 & iX \\ -iX & 0 \end{pmatrix})$$

$$= e^{iZ/2}\sigma e^{-iZ/2} = e^{iZ}\sigma.$$

Note that

(a) Z is a selfadjoint operator which anticommutes with σ.

(b) Z is codiagonal with respect to both P_0 and Q_0. This is apparent in the case of P_0, for Q_0 it follows from general considerations considering geodesics in $\mathcal{P}$, or by a direct computation:

$$ZQ_0 = \begin{pmatrix} 0 & iX \\ -iX & 0 \end{pmatrix}\begin{pmatrix} C^2 & CS \\ CS & S^2 \end{pmatrix} = \begin{pmatrix} iXCS & iXS^2 \\ -iXC^2 & -iXCS \end{pmatrix}$$

$$= \begin{pmatrix} -iXCS & -iXC^2 \\ iXS^2 & -iXC \end{pmatrix} + \begin{pmatrix} 0 & iX \\ -iX & 0 \end{pmatrix} = -Q_0Z + Z = (1 - Q_0)Z.$$

Thus $Q_0ZQ_0 = (1-Q_0)Z(1-Q_0) = 0$. These facts imply that A_0 and Z anticommute. Indeed, that Z is codiagonal with respect to a given projection P_0, means that it anticommutes with the symmetry $2P_0 - 1$:

$$A_0Z = \frac{1}{2}(2P_0 - 1 - (2Q_0 - 1))Z = -\frac{1}{2}Z(2P_0 - 1 - (2Q_0 - 1)) = -ZA_0.$$

We may regard this factorization as a codiagonal model σ for A_0 (via the unitary change of coordinates $e^{iZ/2}$

It is known that two projections P_0 and Q_0 in generic position can be joined by a geodesic in $\mathcal{P}$(see for instance [2]).

Proposition 4.2. *Let $A = P - Q$ and $A_0 = P_0 - Q_0$ be its generic part. Then any geodesic δ in $\mathcal{P}$, with $\delta(0) = P_0$, and $\delta(1) = Q_0$ provides a factorization*

$$A_0 = e^{iZ}\sigma = e^{iZ/2}\sigma e^{-Z/2}$$

as in the above remark:

1. *$Z^* = Z$, $\|Z\| \leq \pi/2$, Z anticommutes with σ, and is codiagonal with respect both to P_0 and Q_0 (and thus also anticommutes with A_0).*

2. *σ is codiagonal with respect to P_0, and $\sigma^2 = A_0^2$.*

Proof. Such a geodesic of $\mathcal{P}$ is given by a selfadjoint operator Z, which is codiagonal with respect to P_0,

$$\delta(t) = e^{itZ}P_0e^{-itZ}.$$

Since $\delta(1) = e^{iZ}P_0e^{-iZ} = Q_0$, it follows that Z is also codiagonal with respect to Q_0. Indeed, $\gamma(t) = \delta(1-t)$ is also a geodesic of $\mathcal{P}$, and

$$\gamma(t) = e^{-itZ}e^{iZ}Q_0e^{-iZ}e^{itZ} = e^{it(-Z)}Q_0e^{-it(-Z)},$$

which implies that $-Z$ is codiagonal with respect to Q_0. Then (using the same trick as above, and the fact that $Z(2P_0-1) = -(2P_0-1)Z)$),

$$A_0 = P_0 - e^{iZ}P_0e^{-iZ} = \frac{1}{2}(2P_0 - 1 - e^{iZ}(2P_0-1)e^{-iZ}) = \frac{1}{2}e^{iZ}(e^{-iZ}(2P_0-1) - (2P_0-1)e^{-iZ})$$

$$= \frac{1}{2}e^{iZ}(e^{-iZ/2}(2P_0-1)e^{iZ/2} - e^{iZ/2}(2P_0-1)e^{-iZ/2}) = e^{iZ}(e^{-iZ/2}P_0e^{iZ/2} - e^{iZ/2}P_0e^{-iZ/2}) = e^{iZ}\sigma.$$

Note that Z and $\sigma = e^{-iZ/2}P_0e^{iZ/2} - e^{iZ/2}P_0e^{-iZ/2}$ verify conditions 1 and 2. Z is codiagonal with respect to P_0 and Q_0. Also

$$Z\sigma = \frac{1}{2}Z(e^{-iZ}(2P_0-1) - (2P_0-1)e^{-iZ})\frac{1}{2}((e^{-iZ}Z(2P_0-1) - Z(2P_0-1)e^{-iZ})$$

$$= -\frac{1}{2}((e^{-iZ}(2P_0-1) - (2P_0-1)e^{-iZ})Z = -\sigma Z.$$

In particular, this implies that $e^{iZ}\sigma = \sigma e^{-iZ}$, and thus

$$A_0^2 = (e^{iZ}\sigma)^2 = (\sigma e^{-iZ})(e^{iZ}\sigma) = \sigma^2.$$

Finally, using that $(2P_0-1)^2 = 1$,

$$\sigma(2P_0-1) = \frac{1}{2}(e^{-iZ/2}(2P_0-1)e^{iZ/2} - e^{iZ/2}(2P_0-1)e^{-iZ/2})(2P_0-1) = \frac{1}{2}(e^{-iZ} - e^{iZ}),$$

and analogously

$$(2P_0-1)\sigma = \frac{1}{2}(e^{iZ} - e^{-iZ}), \tag{1}$$

i.e. $\sigma(2P_0-1) = -(2P_0-1)\sigma$ □

Therefore a geodesic joining P_0 and Q_0 determines a codiagonal factorization for $A_0 = P_0 - Q_0$. It is known that if $\|A_0\| < 1$ such geodesic is unique. As a consequence of the above result, we will show that also if $\|P_0 - Q_0\| = 1$, there exists a unique geodesic joining them. Indeed, let us prove that the unitary operator e^{iZ} is determined by P_0 and Q_0. That its square e^{2iZ} is determined by P_0 and Q_0 is apparent: $e^{2iZ}(2P_0 - 1) = (2Q_0 - 1)$, ant thus

$$e^{2iZ} = (2Q_0 - 1)(2P_0 - 1).$$

Proposition 4.3. *Let $A_0 = P_0 - Q_0$ be the generic part of A, and $Z^* = Z$ the velocity vector of a geodesic curve of $\mathcal{P}$, which joins P_0 and Q_0, with $\|Z\| \leq \pi/2$. Then*

$$V = e^{iZ}(2P_0 - 1)$$

is the symmetry which anticommutes with A_0, and is induced by the decomposition $A_0 = P_0 - Q_0$, i.e. in the notation of Davis [6] (as in Remark 3.1):

$$P_V = P_0 \ , \quad Q_V = Q_0.$$

In particular, the unitary operator e^{iZ} is determined by P_0 and Q_0.

Proof. Recall that Z anticommutes with $2P_0 - 1$ and with $2Q_0 - 1$, and that $e^{iZ}P_0 = Q_0 e^{iZ}$. Note that

$$V^2 = e^{iZ}(2P_0 - 1)e^{iZ}(2P_0 - 1) = e^{iZ}(2P_0 - 1)(2P_0 - 1)e^{-iZ} = 1.$$

Also $V^* = (2P_0 - 1)e^{-iZ} = e^{iZ}(2P_0 - 1) = V$. Next, using the previous proposition,

$$VA_0 = e^{iZ}(2P_0 - 1)e^{iZ}\sigma = (2P_0 - 1)\sigma.$$

and (using that Z and σ anticommute),

$$A_0 V = e^{iZ}\sigma e^{iZ}(2P_0 - 1) = \sigma(2P_0 - 1).$$

Since σ is codiagonal with respect to P_0, these operators also anticommute.

Finally, note that $P_V = P_0$. This assertion is equivalent to

$$A_0 + V(1 - A_0^2)^{1/2} = 2P_0 - 1.$$

Since $\sigma^2 = A_0^2$, multiplying by $V = e^{iZ}(2P_0 - 1) = (2P_0 - 1)e^{-iZ}$, this equality is equivalent to

$$(2P_0 - 1)\sigma + (1 - \sigma^2)^{1/2} = e^{iZ}.$$

In the last part of the proof of the previous proposition, it was shown (1) that: $(2P_0 - 1)\sigma = \frac{1}{2}(e^{iZ} - e^{-iZ})$. Then the above equation is equivalent to

$$(1 - \sigma^2)^{1/2} = \frac{1}{2}(e^{iZ} + e^{-iZ}) = \cos(Z).$$

Note that $\|Z\| \leq \pi/2$, and thus $\cos(Z) \geq 0$. To prove our assertion, it suffices to show that $\sigma^2 = (\sin(Z))^2$. Indeed, using again (1):

$$\sigma^2 = \sigma(2P_0 - 1)(2P_0 - 1)\sigma = ((2P_0 - 1)\sigma)^*(2P_0 - 1)\sigma = \frac{1}{4}((e^{iZ} - e^{-iZ})^*(e^{iZ} - e^{-iZ})$$

$$= -\frac{1}{4}(e^{iZ} - e^{-iZ})^2 = (\sin(Z))^2.$$

It follows that also $Q_V = Q_0$. □

Corollary 4.4. *If P_0 and Q_0 are two projections in generic position, then there is a unique minimal geodesic in $\mathcal{P}$ joining them.*

Proof. By the above proposition, P_0 and Q_0 determine a unique symmetry V such that $V(P_0 - Q_0) = -(P_0 - Q_0)V$, $P_0 = P_V$ and $P_0 = Q_V$. If $V = e^{iZ}(2P_0 - 1)$ as above, then $e^{iZ} = (2P_0 - 1)V$, and since $\|Z\| \le \pi/2 < \pi$,

$$iZ = \log((2P_0 - 1)V),$$

the unique anti-hermitian logarithm of the unitary $(2P_0 - 1)V$. □

Remark 4.5. In [2] it was shown that two projections are joined by a geodesic of $\mathcal{P}$ if and only if $\dim(N(P-Q-1)) = \dim(N(P-Q+1))$. The above result shows that the non injectivity of the exponential map of $\mathcal{P}$ (as well the non surjectivity, as the cited result implies) depends on the (cardinal) numbers $\dim(N(P-Q-1))$ and $\dim(N(P-Q+1))$. One can show the existence of many geodesics joining $P_{N(P-Q-1)}$ and $P_{N(P-Q+1)}$ (when the dimensions coincide). Indeed, pick $W : N(P-Q-1) \to N(P-Q+1)$ an isometric isomorphism. Let U be the unitary operator of $\mathcal{H}' = N(P-Q-1) \oplus N(P-A+1)$ given by $U(\xi,\eta) = (W^*\eta, -W\xi)$, and put $Z = -i\frac{\pi}{2}U$. Then apparently

$$UP_{N(P-Q-1)}U^* = P_{N(P-Q+1)},$$

and $U = e^{iZ}$. This fact follows readily noting that $U^2 = -1$. Also it is clear that Z is self-adjoint, codiagonal in $\mathcal{H}'$, with respect both to $P_{N(P-Q-1)}$ and $P_{N(P-Q+1)}$, and $\|Z\| = \pi/2$. Therefore the exponent Z induces a geodesic joining $P_{N(P-Q-1)}$ and $P_{N(P-Q+1)}$ in $\mathcal{B}(\mathcal{H}')$. Different isomorphisms W induce different geodesics (infinitely many). Thus one obtains infinitely many geodesics joining P and Q. Indeed, in the decomposition $\mathcal{H} = N(P-Q) \oplus \mathcal{H}' \oplus \mathcal{H}_0$ the projections P and Q reduce to $0 \oplus P_{N(P-Q-1)} \oplus P_0$ and $0 \oplus P_{N(P-Q+1)} \oplus Q_0$ respectively.

Finally note any geodesic joining P and Q is reduced by this decomposition. The exponent Z of such a geodesic is codiagonal with respect to P and Q, i.e. $Z(R(E)) \subset N(E)$ and $Z(N(E)) \subset R(E)$ for $E = P, Q$. Therefore all geodesics between P and Q (always under the assumption $\dim(N(P-Q-1)) = \dim(N(P-Q+1))$ are of the above form. The multiple geodesics coincide in the generic part $\mathcal{H}_0$ (as well as they do, trivially, in $N(P-Q)$) .

Remark 4.6. The uniqueness property above, by no means implies the uniqueness of the codiagonal factorization. As it will be shown in examples in the next section, an operator A_0 which is a difference of two projections in generic position, may be decomposed in infinitely many ways as a difference of (generic) projections.

Next we show that the geodesic between P_0 and Q_0 allows one to construct another symmetry J, intertwining P_0 and $1 - P_0$.

Proposition 4.7. *Let $J = sgn(A_0)e^{iZ}$. Then J is a symmetry which satisfies*

$$JP_0J = 1 - P_0.$$

Proof. Since Z anticommutes with A_0, it commutes with any odd power of A_0, $ZA_0^{2k+1} = -A_0^{2k+1}Z$. It follows that Z anticommutes with any polynomial $p(t)$ containing only monomials of odd degree. There are polynomials $p_n(t)$ of this type which converge uniformly to the function $f_k(t) = t^{1/(2k+1)}$ uniformly in the interval $[-1, 1]$. Thus Z anticommutes with $A_0^{1/(2k+1)}$. If

$k \to \infty$, $A_0^{(1/2k+1)}\xi \to sgn(A_0)\xi$, for any $\xi \in \mathcal{H}_0$. Indeed, the functions f_k converge pointwise to $sgn(t)$, and are uniformly bounded in the interval $[-1,1]$. This implies that $sgn(A_0)Z = -Zsgn(A_0)$, and thus $sgn(A_0)e^{iZ} = e^{-iZ}sgn(A_0)$. Then

$$J^* = e^{-iZ}sgn(A_0) = J \text{ and } J^2 = e^{-iZ}sgn(A_0)sgn(A_0)e^{iZ} = 1.$$

In order to prove that J intertwines P_0 and $1-P_0$ we do the computations with 2×2 matrices in terms of P_0 (identifying $\mathcal{H}_0$ with $\mathcal{K}\times\mathcal{K}$ as in Halmos contruction). Note from Remark 4.1 that

$$J = sgn(A_0)e^{iZ} = \begin{pmatrix} -S & C \\ C & S \end{pmatrix}\begin{pmatrix} C & -S \\ S & C \end{pmatrix} = \begin{pmatrix} 0 & 1_{\mathcal{K}} \\ 1_{\mathcal{K}} & 0 \end{pmatrix}.$$

It is apparent then that J intertwines

$$P_0 = \begin{pmatrix} 1_{\mathcal{K}} & 0 \\ 0 & 0 \end{pmatrix} \text{ with } 1-P_0 = \begin{pmatrix} 0 & 0 \\ 0 & 1_{\mathcal{K}} \end{pmatrix}.$$

□

Remark 4.8. By the same procedure as in Remark 4.5, the symmetry J above induces a particular geodesic between P_0 and $1-P_0$. Indeed, note that

$$Z = -i\frac{\pi}{2}(P_0J - JP_0)$$

is codiagonal with respect to P_0, and satisfies

$$e^{iZ} = e^{\pi/2(P_0J-JP_0)} = J,$$

and thus $\delta(t) = e^{itZ}P_0e^{-itZ}$ is a minimal geodesic joining P_0 and $1-P_0$.

Remark 4.9. If $A_0 = P_0 - Q_0$ and Z as above, then

$$\|A_0\| \le \|[Z,P_0]\| = \|Z\|.$$

Indeed, one can construct two paths inside $\mathcal{B}_h(\mathcal{H})$ (in fact, consisting of differences of projections) joining A_0 and 0. Namely tA_0 and $P_0 - e^{itZ}P_0e^{-itZ}$, for $t\in[0,1]$. Clearly the first, being a straight line, is shorter. Its lengths are, respectively, $\|A_0\|$ and $\|[Z,P_0]\|$, which equals $\|Z\|$ because Z is P_0-codiagonal:

$$[Z,P_0] = ZP_0 - P_0Z = \frac{1}{2}(Z(2P_0-1)-(2P_0-1)Z) = Z(2P_0-1),$$

and thus $\|[Z,P_0]\| = \|Z(2P_0-1)\| = \|Z\|$.

5 Decompositions and cyclic vectors

One may obtain decompositions $A = P - Q$ (and in some cases parametrize them) by means of Davis' result, representing A_0 as a sum of multiplication operators. The idea is based on the following example.

Example 5.1. Let $A = M_t$ acting in $L^2(-1,1)$, i.e. $M_t f(t) = tf(t)$ for $f \in L^2(-1,1)$. Since A has no eigenvalues, $\mathcal{H}_0 = L^2(-1,1)$ and $A = A_0$. A symmetry which anticommutes with A is $V_0 f(t) = f(-t)$. Note that if V is another symmetry which anticommutes with A, then $V_0 V$ commutes with A. Since the commutant of A consists of multiplication operators, it follows that

$$V = V_0 M_\varphi,$$

where φ is a measurable function with $|\varphi(t)| = 1$ a.e. Indeed, $V_0 V$ is a multiplication unitary operator. Moreover, since $V^2 = 1$, for any $f \in L^2(-1,1)$

$$f(t) = V^2 f(t) = V_0 M_\varphi \varphi(-t) f(-t) = \varphi(-t)\varphi(t) f(t) \text{ a.e.}$$

which implies that $\varphi(-t) = \bar{\varphi}(t)$ a.e. That is, $\varphi(t) = e^{ig(t)}$ with g an odd real function ($g(-t) = -g(t) \in \mathbb{R}$ a.e.). Then $V_0 M_g = -M_g V_0$, and thus

$$V^* = M_{e^{-ig}} V_0 = V_0 M_{e^{ig}} = V.$$

Therefore all possible decompositions $A = P - Q$ are parametrized in this particular example.

Let us denote by V_g the symmetry given by the a.e.-odd real function g:

$$V_g = V_0 M_{e^{ig}} = M_{e^{-ig}} V_0.$$

A straightforward computation shows that the projections P_{V_g} and Q_{V_g} are, respectively, the orthogonal projections onto the subspaces

$$R(P_{V_g}) = \{f \in L^2(-1,1) : e^{-ig(t)}(1-t)^{1/2} f(t) \text{ is (a.e.) even}\}$$

and

$$R(Q_{V_g}) = \{f \in L^2(-1,1) : e^{-ig(t)}(1+t)^{1/2} f(t) \text{ is (a.e.) even}\}.$$

The factorization $A = A_0 = e^{iZ_g}\sigma_g$ given in Proposition 4.2, arising from the decomposition $A = P_{V_g} - Q_{V_g}$ is

$$e^{iZ_g} f(t) = -te^{ig(t)} f(-t) + (1-t^2)^{1/2} f(t), \ f \in L^2(-1,1),$$

and

$$\sigma f(t) = e^{-iZ_g} A = -t^2 e^{-ig(t)} f(-t) + t(1-t^2)^{1/2} f(t).$$

Indeed, recall from the preceeding section that $e^{iZ_g} = V_g(2P_{V_g} - 1) = V_g M_t + M_{(1-t^2)^{1/2}}$.

In particular, the codiagonal factorization is non unique.

For $g = 0$ (i.e. $V = V_0$), one has $e^{iZ_0} f(t) = -tf(-t) + (1-t^2)^{1/2} f(t)$. This unitary operator has a nice description in the even-odd decomposition of $L^2(-1,1) = L^2_e \oplus L^2_o$ where L^2_e (resp. L^2_o) denotes the subspace of a.e. even (resp. odd) functions in $L^2(-1,1)$. In matrix form

$$e^{iZ_0} = \begin{pmatrix} M_{(1-t^2)^{1/2}} & M_t \\ -M_t & M_{(1-t^2)^{1/2}} \end{pmatrix} \begin{matrix} L^2_e \\ L^2_o \end{matrix},$$

and thus

$$Z_0 = \begin{pmatrix} 0 & -iM_{\arcsin(t)} \\ iM_{\arcsin(t)} & 0 \end{pmatrix} \begin{matrix} L^2_e \\ L^2_o \end{matrix}.$$

The above example motivates the following result:

Lemma 5.2. *Suppose that A_0 is a cyclic operator in $\mathcal{H}_0$, and let V be a symmetry which anticommutes with A_0. Then there exists a cyclic vector ξ_0 for A_0 in $\mathcal{H}_0$, such that $V\xi_0 = \xi_0$. If $p(t)$ is a polynomial, then*

$$Vp(A_0)\xi_0 = p(-A_0)\xi_0.$$

Proof. Let ξ be a cyclic vector for A_0. As above, let A^+ and A^- be the positive and negative parts of A_0, $\mathcal{H}^+ = \overline{R(A^+)}, \mathcal{H}^- = \overline{R(A^-)}$. Recall that, since $N(A_0) = \{0\}$, $\mathcal{H}^+ \oplus \mathcal{H}^- = \mathcal{H}_0$. Then

$$A^+ - A^- = A_0 = -VA_0V = -VA^+V + VA^-V.$$

Note that VA^+V, VA^-V are positive operators which commute with A_0, such that

$$VA^+V - VA^-V = VA_0V = -A_0,$$

and whose product is zero. It follows that

$$VA^+V = A^- \text{ and } VA^-V = A^+.$$

Denote by 1_+ the orthogonal projection onto $\mathcal{H}^+$ (which can be regarded as the identity operator of $\mathcal{H}^+$ as well), and similarly for 1_- and $\mathcal{H}_-$. Note that

$$V1_+ = 1_-V,$$

because V maps $\mathcal{H}^+$ onto $\mathcal{H}^-$ (and viceversa). Put $\xi = \xi_+ + \xi_-$ in this decomposition. If $p(t) = \sum_{j=0}^k a_j t^j$ is a polynomial, then

$$p(A_0) = 1_+ + \sum_{j=1}^k a_j (A^+)^j + 1_- + \sum_{j=1}^k a_j (A^-)^k,$$

so that

$$p(A_0)\xi = p(A^+)\xi_+ + p(A^-)\xi_-.$$

It follows that ξ_+ is a cyclic vector for A^+ in $\mathcal{H}_+$. Also since $VA^+ = A^-V$ and $V1_+ = 1_-V$,

$$Vp(A^+)\xi_+ = p(A^-)V\xi_+.$$

This implies that $V\xi_+$ is a cyclic vector for A^- in $\mathcal{H}_-$. Thus

$$\xi_0 = \xi_+ + V\xi_+$$

is a cyclic vector for A_0, which verifies that $V\xi_0 = \xi_0$. Then for any polynomial p, since $VA_0V = -A_0$, clearly $Vp(A_0)V = p(-A_0)$

$$Vp(A_0)\xi_0 = Vp(A_0)V\xi_0 = p(-A_0)\xi_0.$$

□

With the notations of the above Lemma, we have the following:

Lemma 5.3. *Let $\mu = \mu_{\xi_0,\xi_0}$ be the scalar spectral measure of A_0 associated to the cyclic vector ξ_0, and V a symmetry which anticommutes with A_0, such that $V\xi_0 = \xi_0$. Consider the unitary transformation given in the Spectral Theorem of A_0,*

$$U : \mathcal{H}_0 \to L^2(\sigma(A_0), d\mu), \quad Up(A)\xi_0 = [p],$$

for p a polynomial and $[p]$ the class of p in $L^2(\sigma(A_0), d\mu)$, which transforms A_0 in M_t (= multiplication by the variable). Then U transforms V in the operator

$$V_0 f(t) = f(-t), \quad f \in L^2(\sigma(A_0), d\mu).$$

Proof. Let p be a polynomial. Then

$$\int_{\sigma(A_0)} p(t)d\mu(t) =< p(A_0)\xi_0, \xi_0 >=< p(A_0)V\xi_0, V\xi_0 >=< Vp(A_0)V\xi_0, \xi_0 >=< p(-A_0)\xi_0, \xi_0 >$$

$$= \int_{\sigma(A_0)} p(-t)d\mu(t),$$

i.e. μ is invariant under the change of variables $t \mapsto -t$. It follows that, if we set U as in the Spectral Theorem for cyclic selfadjoint operators (as above), then V transfoms to

$$V_0 p(t) = UVU^{-1}p(t) = UVp(A_0)\xi_0 = UVp(A_0)V\xi_0 = Up(-A_0)\xi_0 = p(-t).$$

□

Proposition 5.4. *Let V be a symmetry which anticommutes with A_0. Then there exist a countable set $\mathbb{I}$, subspaces $\mathcal{H}_n$, $n \in \mathbb{I}$, and vectors $\xi_n \in \mathcal{H}_n$ such that*

1. *$\oplus_{n\in\mathbb{I}}\mathcal{H}_n = \mathcal{H}_0$.*
2. *$\mathcal{H}_n$ is invariant for A_0 and for V, and $A_0|_{\mathcal{H}_n}$ is a difference of projections in $\mathcal{H}_n$.*
3. *ξ_n is cyclic for $A_0|_{\mathcal{H}_n}$, and $V\xi_n = \xi_n$.*

Proof. Pick a non zero vector $\xi \in \mathcal{H}_0$ such that $V\xi = \xi$, and consider $\mathcal{H}' = \{p(A_0)\xi : p \text{ a polynomial}\}$. Clearly $\mathcal{H}'$ is invariant for A_0, and ξ is a cyclic vector for $A_0|_{\mathcal{H}'}$. Note that

$$Vp(A_0)\xi = p(-A_0)V\xi = p(-A_0)\xi \in \mathcal{H}'.$$

Thus $\mathcal{H}'$ is invariant for V. $V|_{\mathcal{H}'}$ is a symmetry which anticommutes with $A_0|_{\mathcal{H}'}$, thus $A_0|_{\mathcal{H}'}$ is a difference of projections in $\mathcal{H}'$.

Consider

$$\mathcal{F} = \{\{(\mathcal{J}_j, \xi_j)\}_{j\in J} : A_0(\mathcal{J}_j) \subset \mathcal{J}_j,\ V(\mathcal{J}_j) \subset \mathcal{J}_j,\ V\xi_j = \xi_j,\ \xi_j \text{ is cyclic for } A_0|_{\mathcal{J}_j}\}.$$

Consider the following order in $\mathcal{F}$: $\{(\mathcal{J}_j, \xi_j)\}_{j\in J} \le \{(\mathcal{J}_k, \xi_k)\}_{k\in K}$ if every $\mathcal{J}_j$ coincides with some $\mathcal{J}_{k(j)}$ and for such j, $\xi_j = \xi_{k(j)}$. By the above lines, $\mathcal{F}$ is non empty. Apparently, one can apply Zorn's Lemma to this ordered set. Let $\{(\mathcal{J}_m, \xi_m)\}_{m\in M}$ is a maximal element in $\mathcal{F}$. Then $\mathcal{J}_0 := \oplus_{m\in M}\mathcal{J}_m = \mathcal{H}_0$. Suppose that $\mathcal{J}_0^\perp \neq 0$. Let $0 \neq \eta \in \mathcal{J}_0^\perp$. Clearly $\mathcal{J}_0^\perp$ is invariant both for A_0 and V. As above, one can construct a non trivial subspace, invariant both for A_0 and V, with η as cyclic vector. Then using Lemma 5.2, one can find in this subspace another cyclic vector η_0 such that $V\eta_0 = \eta_0$. This contradicts the maximality of $\{(\mathcal{J}_m, \xi_m)\}_{m\in M}$. □

Using the usual construction of the multiplication operator model of a selfadjoint operator, one obtains the following:

Corollary 5.5. *Let A_0 be the generic part of $A = P - Q$, V a symmetry which anticommutes with A_0, and $\mathbb{I}$ the index set of the above decomposition of $\mathcal{H}$ in cyclic subspaces $\mathcal{H}_n$, $n \in \mathbb{I}$, with cyclic vectors ξ_n such that $V\xi_n = \xi_n$. There exists a finite Borel measure μ on the set*

$$M = \cup_{n\in\mathbb{I}}\ \sigma(A_0) \times \{n\}$$

and a the unitary isomorphism

$$U : \mathcal{H} = \oplus_{n\in\mathbb{I}}\mathcal{H}_n \to L^2(M, \mu),\ U(\sum_{n\in\mathbb{I}} p_n(A|_{\mathcal{H}_n})\xi_n) = \sum_{n\in\mathbb{I}} \mathbf{p}_n,$$

where $\mathbf{p}_n : \sigma(A_0) \times \{n\} \to \mathbb{C}$ is the function given by $\mathbf{p}_n(t, n) = [p_n](t)$, for any polynomial p_n, which satisfies

$$UA_0U^*f(t, n) = tf(t, n) \quad \text{and } UVU^*f(t, n) = f(-t, n).$$

Proof. Recall that $\sigma(A_0) \subset \mathbb{R}$ is symmetric with respect to the origin. The proof follows as in the classical spectral theorem, summing the cyclic subspaces $\mathcal{H}_n$ [12]. □

With these results one can show the non uniqueness of the decomposition $A_0 = P_0 - Q_0$ of the generic part of A:

Proposition 5.6. *Let A be a difference of projections, with non trivial generic part A_0. Then there exist infinitely many pairs P_0, Q_0 of projections such that $A_0 = P_0 - Q_0$.*

Proof. Fix a decomposition $A_0 = P_0 - Q_0$, which induces the isometry V_0 which anticommutes with A_0. Let $\xi_0 \in \mathcal{H}_0$ be a unit vector in the generic part $\mathcal{H}_0$ of $\mathcal{H}$, such that $V\xi_0 = \xi_0$. Let

$$\mathcal{H}_{\xi_0} = \{p(A_0)\xi_0 : p \text{ a polynomial}\}.$$

Then $B = A_0|_{\mathcal{H}_{\xi_0}}$ is also a difference of projections in generic position, with cyclic vector ξ_0 (note that $\mathcal{H}_{\xi_0}$ is non trivial because $N(A_0) = \{0\}$). Apparently, different decompositions of B produce different decompositions of A_0. Thus it suffices to consider the case when A_0 has a cyclic vector which is a fixed point for V_0, as in Lemmas 5.2 and 5.3. Then, via the unitary transformation of the spectral Theorem, we may suppose $A_0 = M_t$ in $L^2(\sigma(A_0))$, and $V_0f(t) = f(-t)$. Note that there are at least two points λ and $-\lambda$ in $\sigma(A_0)$ (otherwise, since the spectrum is symmetric with respect to the origin, it would be trivial). Then, as in Example 5.1, any operator

$$V_g = V_0M_{e^{ig}}$$

for g a real bounded Borel odd function in $\sigma(A_0)$, is a symmetry which anticommutes with A_0. Since there are at least two points in $\sigma(A_0)$, there are infinitely many different V_g. □

The non generic part A' of A also has many decompositions, though it has a canonic one, $A' = P_{N(A-1)} - P_{N(A+1)}$.

6 Connected components

In this section we examine the connected components of the set $\mathcal{D}$ of differences of projections:

$$\mathcal{D} = \{P - Q : P, Q \in \mathcal{P}\}.$$

By means of the following elementary lemmas, we show that any operator $A \in \mathcal{D}$ is connected (inside $\mathcal{D}$) to 0 or to plus or minus a projection of finite codimension.

Lemma 6.1. *Given $A \in \mathcal{D}$, there exists a continuous path $A(t)$, $t \in [0,1]$, $A(t) \in \mathcal{D}$, such that $A(1) = A$, $A(0) = E_{+1} - E_{-1}$, where E_{+1}, E_{-1} are mutually orthogonal selfadjoint projections.*

Proof. Consider the decomposition of A in its generic and non-generic parts,

$$A = 0 \oplus E_{+1} \oplus -E_{-1} \oplus A_0 \text{ in } \mathcal{H} = N(A) \oplus N(A-1) \oplus N(A+1) \oplus \mathcal{H}_0.$$

By Davis's [6] characterization of operators in $\mathcal{D}$, there exists a self-adjoint symmetry V in $\mathcal{H}_0$ such that $VA_0 = -A_0V$. Note that for $|t| \leq 1$, tA_0 is a selfadjoint contraction which also anticommutes with V. Therefore

$$A(t) = 0 \oplus E_{+1} \oplus -E_{-1} \oplus tA_0$$

is a path in $\mathcal{D}$, which is apparently continuous. Clearly $A(1) = A$ and $A(0) = E_{+1} - E_{-1}$ is a difference of mutually orthogonal projections. □

Lemma 6.2. *Suppose that $\dim \mathcal{H} = \infty$ and let E, F be mutually orthogonal projections in $\mathcal{H}$. Then there exists a continuous path $B(t) \in \mathcal{D}$, $t \in [0,1]$, such that $B(1) = E - F$ and $B(0)$ is either 0, P_0 or $-P_0$, where P_0 is a projection with finite codimension.*

Proof. Suppose first that $E+F$ has infinite codimension. Let $P = 1-(E+F)$, and $P = P_1 + P_2$ with P_1, P_2 mutually orthogonal of infinite rank. It follows that $E+P_1$ and $F+P_1$ are orthogonal projections (no longer mutually orthogonal), with infinite rank and co-rank. It is well known that infinite (rank and co-rank) projections are homotopic, i.e. there exists a continuous path $P(t) \in \mathcal{P}$ such that $P(0) = E + P_1$, $P(1) = F + P_1$. Then $B(t) = E + P_1 - P(t)$ is a continuous path in $\mathcal{D}$ such that $P(0) = 0$ and $P(1) = E + P_1 - (F + P_1) = E - F$.

Suppose now that $E+F$ has finite codimension. This implies that either E or F has infinite rank. If both E and F have infinite rank, since their are mutually orthogonal, this would imply that they have infinite co-rank. Thus they would be homotopical: in that case, let $E(t)$ be a continuous path in $\mathcal{P}$ with $E(0) = E$, $E(1) = F$. Then $B(t) = E - E(t)$ is a continuous path in $\mathcal{D}$ such that $B(0) = 0$ and $B(1) = E - F$.

Thus we are left in the case where either E or F has finite rank (and the other has infinite rank). If E has finite rank, then there exists a unitary operator U in $\mathcal{H}$ such that $F_0 = UEU^* \leq F$. Let $U(t)$ be a continuous path of unitaries such that $U(0) = 1$ and $U(1) = U$ (since that the unitary group is connected), and put $B(t) = U(t)EU(t)^* - F$. Then $B(0) = E - F$ and $B(1) = F_0 - F = -P_0$. Since $F_0 \leq F$, P_0 is a projection. Moreover, it is apparent that it has finite codimension.

If F has finite rank, a similar argument shows that $E - F$ can be joined to a sub-projection of E (which necessarily has finite co-rank). □

Proposition 6.3. *Suppose that* $\dim(\mathcal{H}) = \infty$*. Let* $A_1, A_2 \in \mathcal{D}$*, which are also compact. Then they lie in the component of* 0 *in* $\mathcal{D}$*.*

Proof. As in the beginning of the first lemma above, each A_i can be connected to $E_i - F_i$, where E_i and F_i are the projections onto the eigenspaces corresponding to $+1$ and -1, respectively. By the argument in the first paragraph of the proof of Lemma 6.2, since E_i and F_i have finite rank, $E_i - F_i$ can be connected to 0. □

Let $\mathcal{D}_c = \mathcal{D} \cap \mathcal{K}(\mathcal{H})$, the subset of compact differences of projections. The result above states that two elements in $\mathcal{D}_c$ can be connected with a curve inside $\mathcal{D}$. In the next section we examine the internal structure of the set $\mathcal{D}_c$. Below, consider the particular case of $\mathcal{D}_1 = \mathcal{D} \cap \mathcal{B}_1(\mathcal{H})$, the elements of $\mathcal{D}$ which are nuclear. Denote by Tr the usual trace. The following fact is well known, we include an elementary proof.

Lemma 6.4. *If* $A \in \mathcal{D}_1$*, then* $Tr(A)$ *is an integer.*

Proof. In the decomposition $N(A) \oplus N(A-1) \oplus N(A+1) \oplus \mathcal{H}_0$, A is given by

$$0 \oplus E_{+1} \oplus -E_{-1} \oplus A_0.$$

By Davis' characterization, the spectral decomposition of the generic part A_0 can be written

$$A_0 = \sum_{i \geq 1} \lambda_i (P_i^+ - P_i^-),$$

with P_i^+, P_j^- mutually orthogonal projections of finite rank, $\dim(R(P_i^+)) = \dim(R(P_i^-))$, and the sequence $\lambda_i > 0$ summable. Indeed, by Davis result there exists a unitary operator V in $\mathcal{H}_0$ which anticommutes with A_0. As seen in the previous section. this unitary operator intertwines the positive part A^+ and the negative part A^- of A_0: $VA^+V^* = A^-$. Therefore the spectrum of A_0 is symmetric with respect to the origin, and the multiplicity of each λ_i equals that of $-\lambda_i$. Clearly $A_0 \in \mathcal{B}_1(\mathcal{H}_0)$. Then $Tr(A_0) = 0$. Thus $Tr(A) = Tr(E_{+1}) - Tr(E_{-1}) \in \mathbb{Z}$ □

Proposition 6.5. *The connected components of* $\mathcal{D}_1$ *are parametrized by the integers. Namely, the connected components of* $\mathcal{D}_1$*, in the topology of the norm* $\| \ \|_1$ *of* $\mathcal{B}_1(\mathcal{H})$*, are*

$$\mathcal{D}_{1,m} = \{A \in \mathcal{D}_1 : Tr(A) = m\}.$$

Moreover,

$$\mathcal{D}_{1,k} = \{A \in \mathcal{D}_1 : A \textit{ can be connected in } \mathcal{D}_1 \textit{ with } E, \dim(R(E)) = k\},$$

$$\mathcal{D}_{1,0} = \{A \in \mathcal{D}_1 : A \textit{ can be connected in } \mathcal{D}_1 \textit{ with } 0\}$$

and

$$\mathcal{D}_{1,-k} = \{A \in \mathcal{D}_1 : A \textit{ can be connected in } \mathcal{D}_1 \textit{ with } -F, \dim(R(F)) = k\}.$$

Proof. Let $A \in \mathcal{D}_1$. As above, A is given by

$$0 \oplus E_{+1} \oplus -E_{-1} \oplus A_0.$$

As noted above,

$$A(t) = 0 \oplus E_{+1} \oplus -E_{-1} \oplus tA_0 \in \mathcal{D}.$$

Moreover, apparently $A(t) \in \mathcal{B}_1(\mathcal{H})$ is a continuous path joining A with $E_{+1}-E_{-1}$. Suppose first that $\dim(R(E_{+1})) > \dim(R(E_{-1}))$, and put $k = \dim(R(E_{+1})) - \dim(R(E_{-1}))$. The projection E_{-1} is unitarily equivalent to a subprojection of E_{+1},

$$UE_{-1}U^* \leq E_{+1}.$$

Let $U(t)$ be a continuous path of unitaries joining U and 1. Then $E_{+1} - U(t)E_{-1}U(t)^*$ is a continuous path in $\mathcal{D}_1$. Thus A is joined in $\mathcal{D}_1$ with the projection $E = E_{+1} - UE_{-1}U^*$ of rank k. The cases $\dim(R(E_{+1})) = \dim(R(E_{-1}))$ and $\dim(R(E_{+1})) < \dim(R(E_{-1}))$ are dealt similarly, and correspond to the situations $A \in \mathcal{D}_{1,0}$ or $A \in \mathcal{D}_{1,-k}$.

We need to show that a projection cannot be connected neither to a projection of different rank, nor to minus a projection. Let $A(t)$ be a curve in $\mathcal{D}_1$ starting at E. Then $Tr(A(t))$ is a continuous map, which by the above lemma, has integer values. It follows that it is constant, and the proof follows. □

Remark 6.6. If $\mathcal{H}$ is finite dimensional, the connected components of $\mathcal{D}$ are parametrized by the integers between $-\dim\mathcal{H}$ and $\dim\mathcal{H}$. Namely $A \in \mathcal{D}$ determines a unique integer d, such that there exists a continuous path in $\mathcal{D}$ linking A to an operator D, which is 0 or plus or minus a projection, with $\dim R(D) = |d|$.

We end this section noting that elements of $\mathcal{D}$ which are differences of mutually orthogonal projections, lie at the border of $\mathcal{D}$. In other words, that if one regards $\mathcal{D}$ with the relative topology given by the norm of $\mathcal{B}(\mathcal{H})$, then the interior points of $\mathcal{D}$ are the elements A which have non trivial generic part. These facts follow from the next elementary Lemma:

Lemma 6.7. *If $P_n - Q_n$ is a sequence of mutually orthogonal projections converging to $A \in \mathcal{D}$, then $A = P_{N(A-1)} - P_{N(A+1)}$ is also a difference of mutually orthogonal projections.*

Proof. Indeed, if $P_n - Q_n$ is norm convergent, then $(P_n - Q_n)^2 = P_n + Q_n$ is also norm convergent. It follows that both sequences P_n and Q_n are convergent. Clearly the space $\mathcal{P}$ of orthogonal projection is closed. Therefore $P_n \to P$ and $Q_n \to Q$, with P, Q mutually orthogonal projections. □

Corollary 6.8. *The interior of $\mathcal{D}$ consists of elements $A \in \mathcal{D}$ such that the generic part A_0 is non trivial. The border of $\mathcal{D}$ consists of differences of mutually orthogonal projections.*

Proof. Let $A \in \mathcal{D}$ such that $A_0 \neq 0$. Then there exists $r > 0$ such that if $B \in \mathcal{D}$ with $\|B - A\| < r$, then the generic part of B is non trivial. Otherwise, there would be a sequence $B_n \to A$, such that $B_n \in \mathcal{D}$ have trivial generic parts. This means that $B_n = P_{N(B_n-1)} - P_{N(B_n+1)}$. Thus, by the above lemma, B would be a difference of orthogonal projections, a contradiction. □

7 Differences of projections with a Fredholm index

Recall from [3] or [1] the notion of Fredholm pair (P, Q) of projections. A pair (P, Q) is called a Fredholm pair if the operator

$$QP|_{R(P)} : R(P) \to R(Q)$$

is a Fredholm operator, and in this case the index $i(P, Q)$ of the pair is the index of the above operator. In this section we shall study operators $A = P - Q$ such that (P, Q) is a Fredholm pair.

As we have seen, there are many possible pairs whose difference is A. However the property of being a Fredholm pair is shared by all this pairs, and they have the same index. Thus the index of the pair should be more appropriately called the index of the difference.

Remark 7.1. In [3], Avron, Seiler and Simon proved the following fact (Prop. 3.1): (P, Q) is a Fredholm pair if and only if

1. 1 and -1 are isolated points of $\sigma(P - Q)$, and
2. $N(P - Q - 1)$ and $N(P - Q + 1)$ are finite dimensional.

In this case, $i(P, Q) = \dim N(P - Q - 1) - \dim N(P - Q + 1)$.

Thus, if $A = P - Q$, we define

$$i(A) = i(P, Q).$$

A special case of Fredholm pair occurs when $A = P - Q$ is compact. If furthermore $A \in \mathcal{D}_1$, then apparently $i(A) = Tr(A)$.

Let us denote by

$$\mathcal{D}_F = \{A = P - Q \in \mathcal{D} : (P, Q) \text{ is a Fredholm pair}\}.$$

It is known that the index of Fredholm pairs is locally constant (see for instance [2]).

Proposition 7.2. *The index is continuous in $\mathcal{D}_F$, i.e. it is locally constant.*

Proof. Consider the (closed) complemented linear subspace of $\mathcal{B}(\mathcal{H}) \times \mathcal{B}(\mathcal{H})$,

$$\Delta = \{(T, T) : T \in \mathcal{B}(\mathcal{H})\}.$$

Apparently, the difference map $d : \mathcal{B}(\mathcal{H}) \times \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$, $d(S, T) = S - T$ induces a linear isomorphism

$$\bar{d} : (\mathcal{B}(\mathcal{H}) \times \mathcal{B}(\mathcal{H}))/\Delta \to \mathcal{B}(\mathcal{H}), \quad \bar{d}([(S, T)]) = S - T.$$

This isomorphism maps the set

$$\{[(P, Q)] : P, Q \text{ projections such that } (P, Q) \text{ is a Fredholm pair}\}$$

onto $\mathcal{D}_F$. Note that the former set is indeed a subset of classes in $(\mathcal{B}(\mathcal{H}) \times \mathcal{B}(\mathcal{H}))/\Delta$, because the condition that (P, Q) be a Fredholm pair depends only on the difference (i.e. on the class of (P, Q)). On the other hand, the index map

$$i : \{(P, Q) : P, Q \text{ projections such that } (P, Q) \text{ is a Fredholm pair}\} \to \mathbb{Z}$$

is contimuous (see [3], [1]), and induces a continuous map on the classes. Thus the index, as a map defined in $\mathcal{D}_F$, is continuous. □

Corollary 7.3. *The connected components of $\mathcal{D}_F$ are parametrized by the index: two elements $A_1, A_2 \in \mathcal{D}_F$ can be joined by a continuous curve inside $\mathcal{D}_F$ if and only if*

$$i(A_1) = \dim N(A_1 - 1) - \dim N(A_1 + 1) = \dim N(A_2 - 1) - \dim N(A_2 + 1) = i(A_2).$$

Proof. By the above proposition, two elements in the the same connected component have the same index. Conversely, using the elementary techniques of Section 6, an element A of index $m \in \mathbb{Z}$ can be joined to $sgn(m)P$, where P is a projection of rank $|m|$. Note that the curve constructed remains inside $\mathcal{D}_F$. Indeed, the curve joining A to $sgn(m)P$ is obtained by shrinking the generic part A_0: if

$$A = 0 \oplus E_{+1} \oplus E_{-1} \oplus A_0,$$

put

$$A(t) = 0 \oplus E_{+1} \oplus E_{-1} \oplus tA_0.$$

Thus $\sigma(A(t)) = \{0, -1, 1\} \cup t\sigma(A_0)$, which implies that 1 and -1 are isolated points of the spectrum of $A(t)$, i.e. $A(t) \in \mathcal{D}_F$. Moreover, $N(A(t)-1) = N(A-1)$ and $N(A(t)+1) = N(A+1)$ are finite dimensional. Therefore $A(t) \in \mathcal{D}_F$ □

As remarked above, a particular (and proper, [2]) case of Fredholm pair (P, Q) occurs when $A = P - Q$ is compact. In other words, $\mathcal{D}_c \subset \mathcal{D}_F$.

Corollary 7.4. *The connected components of $\mathcal{D}_c$ are parametrized by the index: two elements $A_1, A_2 \in \mathcal{D}_c$ can be joined by a continuous curve inside $\mathcal{D}_c$ if and only if*

$$i(A_1) = \dim N(A_1 - 1) - \dim N(A_1 + 1) = \dim N(A_2 - 1) - \dim N(A_2 + 1) = i(A_2).$$

Proof. Note that the curve $A(t)$ remains inside $\mathcal{D}_c$, a fact which is apparent. □

In Propositon 6.3 it was shown that two elements in $\mathcal{D}_c$ can be joined by a continuous path in $\mathcal{D}$. The above corollary states that if their index is different, this path necesarily wanders outside $\mathcal{D}_c$.

Another consequence of the characterization of Fredholm pairs in terms of the spectrum of their difference, is the following.

Proposition 7.5. *$\mathcal{D}_F$ is open in $\mathcal{D}$: if $A \in \mathcal{D}_F$ there exists $r = r(A) > 0$ such that if $B \in \mathcal{D}$ and $\|B - A\| < r$, then $B \in \mathcal{D}_F$ and $i(B) = i(A)$.*

Proof. Since 1 is isolated in $\sigma(A)$, there exists $\delta > 0$ such that $\sigma(A) \cap (1 - 3\delta, 1 + 3\delta) = \{1\}$. By the semi-continuity of the spectrum [10], the set

$$\{T \in \mathcal{B}(\mathcal{H}) : T^* = T, \sigma(T) \subset (-2, 1 - 3\delta) \cup (1 - \delta, 1 + \delta)\}$$

is open in $\mathcal{B}_h(\mathcal{H}) := \{S \in \mathcal{B}(\mathcal{H}) : S^* = S\}$, and clearly contains A. Thus there exists r_1 such that if $B \in \mathcal{D}$ and $\|B - A\| < r_1$, then $\sigma(B) \subset (-2, 1 - 3\delta) \cup (1 - \delta, 1 + \delta)$. Consider the (well defined) selfadjoint projection $E_\delta(B)$ given by the Riesz integral

$$E_\delta(B) = \frac{1}{2\pi i} \int_{|z-1|=2\delta} (B - z.1)^{-1} dz.$$

As a map in B, defined in $\{B \in \mathcal{D} : \|B - A\| < r_1\}$, $E_\delta(B)$ is apparently continuous. Thus we may eventually further shrink r_1, in order that if $\|B - A\| < r_1$ also implies

$$\|E_\delta(B) - E_\delta(A)\| = \|E_\delta(B) - E_{+1}\| < 1,$$

where, as above, E_{+1} is the orthogonal projection onto the eigenspace of A corresponding to $+1$. It follows that $E_\delta(B)$ is unitarily equivalent to E_{+1}, therefore $E_\delta(B)$ also has finite rank, and $\sigma(B) \cap (1-\delta, 1+\delta)$ is finite. Thus 1 is isolated in $\sigma(B)$ (in the event that $1 \in \sigma(B)$) and the projection $E_{+1}(B)$ onto $N(B-1)$ verifies

$$E_{+1}(B) \leq E_\delta(B)$$

and therefore has finite rank.

Analogously there exists $r_2 > 0$ such that, -1 is isolated in $\sigma(B)$ if $\|B - A\| < r_2$, and E_{-1} has finite rank. Thus $B \in \mathcal{D}_F$. By the local continuity of the index in $\mathcal{D}_F$. it follows that $i(B) = i(A)$. □

Operators in $\mathcal{D}_F$ should not be confused with operators in $\mathcal{D}$ which are themselves Fredholm operators. The latter class consists of $A \in \mathcal{D}$ such that $\dim(N(A)) < \infty$ and $R(A)$ is closed. This class was characterized in [8]. Note that they have index zero. On the other hand, any value of the index can happen in the sense above (i.e. index of the pair of projections). Any $A = A_0$, equal to its generic part, belongs to $\mathcal{D}_F$, but need not be a Fredholm operator, for instance $A = M_t$ in $L^2(-1, 1)$.

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