Additivity properties of operator ranges

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Abstract

In the present paper we study different conditions regarding the sum of operator ranges. In particular, we find conditions on operators A, B which imply that $R(A+B) = R(A) + R(B)$ and derive some results concerning projections.

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1. Introduction

An operator range $\mathcal{M}$ in a Hilbert space $\mathcal{H}$ is a linear subspace of $\mathcal{H}$ such that there exists a bounded linear operator A on $\mathcal{H}$ with $R(A) = \mathcal{M}$ (where $R(A)$ denotes the range or image of A). This class of subspaces is much bigger than the Grassmannian of $\mathcal{H}$, which consists of all closed subspaces of $\mathcal{H}$, but the behavior of an operator range is closer to that of a closed subspace than that of an arbitrary linear subspace. The reader is referred to the treatises of J. Dixmier [10], [11], who called them "variétés de Julia",

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to the paper of C. Foias [14] who used the name "para-closed subspaces" and to the survey by P. Fillmore and J. P. Williams [13], with new proofs of many results of Dixmier. In this paper we shall pursue a few problems on operator ranges, most related with additive properties; for instance, describe for which $A, B \in \mathcal{L}(\mathcal{H})$ it holds $R(A+B) = R(A) + R(B)$, where $\mathcal{L}(\mathcal{H})$ denotes the algebra of bounded linear operators on the Hilbert space $\mathcal{H}$. We begin with positive operators. We say that $A \in \mathcal{L}(\mathcal{H})$ is positive if $A = A^*$ and $\langle A\xi, \xi\rangle \geq 0 \;\forall\, \xi \in \mathcal{H}$. We denote by $\mathcal{L}(\mathcal{H})^+$ the set of all positive operators on $\mathcal{H}$.

As mentioned by P. Fillmore and J. P. Williams [13, Corollary 3], as consequence of a theorem of T. Crimmins, if $A, B \in \mathcal{L}(\mathcal{H})^+$ and $R(A), R(B)$ are closed then $R(A+B)$ is closed if and only if $R(A) + R(B)$ is closed, and if these conditions hold then $R(A+B) = R(A) + R(B)$. These types of conditions have also been studied by M. Khadivi [18]. In this note we show that a lemma of A. Bikchentaev [4, Lemma 2] can be extended and used to get different situations of range additivity. In fact, Bikchentaev proved that the invertibility of $A+B$ forces the invertibility of $|A|^p + |B|^q$ for every $p, q \geq 0$. We notice that weaker properties than invertibility are also useful in this situation. Thus, we examine in different cases the properties "$R(A) + R(B)$ is closed", "$R(A) + R(B)$ is dense", "$R(A) + R(B) = \mathcal{H}$", and others. These matters are related to recent papers by O. Baksalary and G. Trenkler [2] and J. Benitez and V. Rakočević [3] (for matrices) which inspired our research, see also [7] (for operators). For results concerning sum of n operators see [20].

The contents of the paper are the following. Section 2 contains some notations and results which are used in the sequel. Among them, we describe Douglas' factorization theorem [12], Crimmins' theorem [13, Theorem 2.2], Bikchentaev's lemma [4, Lemma 2] and some properties on positive operators. Section 3 contains the main results. These are equivalent conditions on $A, B \in \mathcal{L}(\mathcal{H})$ for the properties "$R(A) + R(B)$ is dense (resp. is closed, resp. is equal to $\mathcal{H}$)". A particular consequence is that, for $A, B \in \mathcal{L}(\mathcal{H})^+$ it holds that $R(A+B)$ is closed if and only if $R(A)+R(B)$ is closed; this shows that the closedness of $R(A)$ and $R(B)$, required in the result by Fillmore and Williams, is unnecessary. Another corollary is that if $R(A) \cap R(B) = \{0\}$ then $R(A)+R(B)$ is closed if and only if $R(AA^*+BB^*)$ is closed or $R(AA^*-BB^*)$ is closed. It is also proven that if $R(A) \cap R(B) = \{0\}$ then $A - B$ is invertible if and only if $A + B$ is invertible; in such cases, $R(A)$ and $R(B)$ are automatically closed. A corollary of this result provides a generalization of a formula by T. Ando [1]. He proved that if $\mathcal{S}, \mathcal{T}$ are closed subspaces of $\mathcal{H}$ and $\mathcal{S} \dot{+} \mathcal{T} = \mathcal{H}$ then the projection $Q_{\mathcal{S}//\mathcal{T}}$ with range $\mathcal{S}$ and nullspace $\mathcal{T}$ is given by $Q_{\mathcal{S}//\mathcal{T}} = P_{\mathcal{S}}(P_{\mathcal{S}} + P_{\mathcal{T}})^{-1}$ where $P_{\mathcal{S}}, P_{\mathcal{T}}$ are the orthogonal projections onto $\mathcal{S}, \mathcal{T}$, respectively. Here, we prove that, if $A, B \in \mathcal{L}(\mathcal{H})^+$ satisfy $R(A) = \mathcal{S}$ and $R(B) = \mathcal{T}$ then $Q_{\mathcal{S}//\mathcal{T}} = A(A + B)^{-1}$. This result may have some computational advantages over Ando's formula. Finally, given an oblique projection E we study range properties of $E + E^*$, $E - E^*$ and $E^*E + EE^*$ following the lines of the previous results.

2. Preliminaries

Throughout this manuscript, $\mathcal{F}, \mathcal{H}, \mathcal{K}$ denote complex Hilbert spaces and $\mathcal{L}(\mathcal{H}, \mathcal{K})$ is the space of all bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$. The algebra $\mathcal{L}(\mathcal{H}, \mathcal{H})$ is abbreviated by $\mathcal{L}(\mathcal{H})$. By $\mathcal{L}(\mathcal{H})^+$ we denote the cone of all positive (semidefinite) operators of $\mathcal{L}(\mathcal{H})$, i.e., $\mathcal{L}(\mathcal{H})^+ = \{A \in \mathcal{L}(\mathcal{H}) : \langle A\xi, \xi\rangle \geq 0 \ \forall \xi \in \mathcal{H}\}$. For every $T \in \mathcal{L}(\mathcal{H})$ its range is denoted by $R(T)$, its nullspace by $N(T)$ and its adjoint by T^*. In addition, $|T|$ denotes the positive square root of T^*T. Recall that $R(T) = R(|T^*|)$. If $\mathcal{H}$ is decomposed as a direct sum $\mathcal{H} = \mathcal{S} \dot{+} \mathcal{T}$ where $\mathcal{S}$ and $\mathcal{T}$ are closed subspaces of $\mathcal{H}$ (the symbol $\dot{+}$ denotes that $\mathcal{S} \cap \mathcal{T} = \{0\}$), then the unique projection with range $\mathcal{S}$ and nullspace $\mathcal{T}$ is denoted by $Q_{\mathcal{S}//\mathcal{T}}$. If $\mathcal{H} = \mathcal{S} \oplus \mathcal{T}$ (the symbol $\oplus$ denotes that $\mathcal{S} \subseteq \mathcal{T}^{\perp}$), we simply write $P_{\mathcal{S}}$. In the sequel, $\mathcal{Q} = \{E \in \mathcal{L}(\mathcal{H}) : E^2 = E\}$ and $\mathcal{P} = \{P \in \mathcal{Q} : P = P^*\}$. The elements of $\mathcal{Q}$ will be called **oblique projections**. Notice that $E \in \mathcal{Q}$ if and only if $2E - I$ is a **reflection**, i.e, $(2E - I)^2 = I$. A relevant notion in this article is that of angle between subspaces. Recall that the **Friedrichs angle** between two closed subspaces $\mathcal{S}$ and $\mathcal{T}$ of $\mathcal{H}$ is the angle $\theta(\mathcal{S}, \mathcal{T}) \in [0, \frac{\pi}{2}]$ whose cosine is defined by

$$c(\mathcal{S}, \mathcal{T}) = \sup\{|\langle \xi, \eta\rangle| : \xi \in \mathcal{S} \cap (\mathcal{S} \cap \mathcal{T})^{\perp}, \eta \in \mathcal{T} \cap (\mathcal{S} \cap \mathcal{T})^{\perp}, \ \|\xi\| \leq 1, \ \|\eta\| \leq 1\}.$$

On the other hand, the **Dixmier angle** between $\mathcal{S}$ and $\mathcal{T}$ is the angle $\theta_0(\mathcal{S}, \mathcal{T}) \in [0, \frac{\pi}{2}]$ whose cosine is defined by

$$c_0(\mathcal{S}, \mathcal{T}) = \sup\{|\langle \xi, \eta\rangle| : \xi \in \mathcal{S}, \eta \in \mathcal{T} \text{ and } \|\xi\| \leq 1, \ \|\eta\| \leq 1\}.$$

Clearly, both angles coincide if $\mathcal{S} \cap \mathcal{T} = \{0\}$. In the next proposition we collect some properties about angles between subspaces. See [8] and [9] for their proofs.

Proposition 2.1. *Let $\mathcal{S}, \mathcal{T}$ be two closed subspaces of $\mathcal{H}$. Then,*

1. $\mathcal{S} + \mathcal{T}$ *is a closed subspace if and only if* $c(\mathcal{S}, \mathcal{T}) < 1$ *if and only if* $\mathcal{S} \cap (\mathcal{S} \cap \mathcal{T})^\perp + \mathcal{T} \cap (\mathcal{S} \cap \mathcal{T})^\perp$ *is closed.*
2. $c_0(\mathcal{S}, \mathcal{T}) = ||P_{\mathcal{S}} P_{\mathcal{T}}||$.
3. $\mathcal{S} \dot{+} \mathcal{T}$ *is a closed subspace if and only if* $c_0(\mathcal{S}, \mathcal{T}) < 1$.

We end this section recalling some results that we shall need along the article.

Theorem 2.2. *(Crimmins, [13, Theorem 2.2]) Let* $A, B \in \mathcal{L}(\mathcal{H})$ *then* $R(A) + R(B) = R((AA^* + BB^*)^{1/2})$.

Theorem 2.3. *[13, Theorem 2.3] Let* $A, B \in \mathcal{L}(\mathcal{H})$*. If* $R(A) \dot{+} R(B)$ *is closed then* $R(A)$ *and* $R(B)$ *are closed.*

Remark 2.4. Theorem 2.3 can be also derived from a much more general result proved by A. Taylor and D. Lay [22, Theorem 5.10]. In fact, they proved that for a closed operator A in a Banach space, if $R(A) \dot{+} \mathcal{S}$ is closed for some closed subspace $\mathcal{S}$ then $R(A)$ is automatically closed.

Theorem 2.5. *(Douglas, [12]) Let* $A \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ *and* $B \in \mathcal{L}(\mathcal{F}, \mathcal{K})$*. The following conditions are equivalent:*

1. $R(B) \subseteq R(A)$.
2. *There is a positive number* λ *such that* $BB^* \leq \lambda AA^*$.
3. *There exists* $C \in \mathcal{L}(\mathcal{F}, \mathcal{H})$ *such that* $AC = B$.

If one of these conditions holds then there is a unique operator $D \in \mathcal{L}(\mathcal{F}, \mathcal{H})$ *such that* $AD = B$ *and* $R(D) \subseteq N(A)^\perp$*. We shall call* D *the* ***reduced solution*** *of* $AX = B$*. Moreover,* $N(D) = N(B)$ *and* $\|D\| = \inf\{\lambda : BB^* \leq \lambda AA^*\}$.

Theorem 2.6. *(Bikchentaev, [4, Lemma 2]) Let* $A, B \in \mathcal{L}(\mathcal{H})$ *and* $p, q \geq 0$*. If* $A + B$ *is invertible then* $|A|^p + |B|^q$ *is also invertible.*

In the following result we collect several properties of positive operators. We add the proof for sake of completeness.

Proposition 2.7. *Let* $A, B \in \mathcal{L}(\mathcal{H})^+$*. Then,*

1. $R(A) \subseteq R(A^t)$ *for every* $t \in [0, 1]$.
2. $\overline{R(A)} = \overline{R(A^t)}$ *for all* $t > 0$.
3. $R(A) = R(A^{1/2})$ *if and only if* $R(A)$ *is closed if and only if* $R(A^{1/2})$ *is closed.*

4. *If* $||A|| \leq 1$ *then* $A \leq A^s$ *for all* $0 \leq s \leq 1$.
5. $\overline{R(A+B)} = \overline{\overline{R(A)} + \overline{R(B)}}$.
6. $N(A+B) = N(A) \cap N(B)$.

Proof. 1. Note that $A = A^t A^{1-t}$ for all $t \in (0,1]$. Hence, by Douglas' theorem, $R(A) \subseteq R(A^t)$ for every $t \in (0,1]$.

2. It is consequence of $N(A) = N(A^t)$ for all $t > 0$.
3. Suppose that $R(A) = R(A^{1/2})$. Then, by Corollary 1 of [13], there exists $W \in \mathcal{L}(\mathcal{H})$ invertible such that $A^{1/2} = AW$. Thus, $A^{1/2} P_{\overline{R(A^{1/2})}} = A^{1/2} = A^{1/2} A^{1/2} W$ i.e., $P_{\overline{R(A^{1/2})}}$ and $A^{1/2}W$ are both the reduced solution of the equation $A^{1/2} = A^{1/2}X$. Therefore, by the uniqueness of the reduced solution, we have that $P_{\overline{R(A^{1/2})}} = A^{1/2}W$ and so $\overline{R(A^{1/2})} = R(A^{1/2}W) = R(A^{1/2})$, i.e., $R(A) = R(A^{1/2})$ is closed. Now, if $R(A)$ is closed then, by the previous items, $R(A) \subseteq R(A^{1/2}) \subseteq \overline{R(A^{1/2})} = \overline{R(A)} = R(A)$, i.e., $R(A^{1/2}) = R(A)$ is closed. Finally, if $R(A^{1/2})$ is closed then $R(A^{1/2}) = R(A^{1/2}A^{1/2}) = R(A)$ and the equivalences are proved.
4. By item 1 it holds that $R(A^{1/2}) \subseteq R((A^{1/2})^s)$. Then, by Douglas' theorem the equation $(A^{1/2})^s X = A^{1/2}$ is solvable and $D = A^{(1-s)/2}$ is its reduced solution. Furthermore, by Cordes inequality [6], $\|D\| \leq 1$. Therefore, applying again Douglas' theorem, we get that $A \leq \|D\| A^s \leq A^s$.
5. Clearly $R(A+B) \subseteq \overline{R(A)} + \overline{R(B)}$ and so $\overline{R(A+B)} \subseteq \overline{\overline{R(A)} + \overline{R(B)}}$. Conversely, by Crimmins' theorem $R(A^{1/2}) + R(B^{1/2}) = R((A+B)^{1/2})$. Then both $R(A)$ and $R(B)$ are contained in $R((A+B)^{1/2})$. Hence, $\overline{R(A)} + \overline{R(B)} \subseteq \overline{R(A+B)}$ and we get the equality of their closures.
6. It follows from the above item and [8, Lemma 11].

$\square$

3. Main results

We begin this section with an elementary result of linear algebra.

Lemma 3.1. *Let* $A, B \in \mathcal{L}(\mathcal{H})$. *Then,*

$$R(A) + R(B) = R(A-B) + R(A+B).$$

Proof. Let $z = Ax + By \in R(A) + R(B)$ then $z = (A - B)(x - y)/2 + (A + B)(x + y)/2 \in R(A - B) + R(A + B)$. The other inclusion is trivial. □

As an immediate consequence of Lemma 3.1, we obtain the following result

Corollary 3.2. *Let $A, B \in \mathcal{L}(\mathcal{H})$. Then, $R(A + B) = R(A) + R(B)$ if and only if $R(A - B) \subseteq R(A + B)$.*

Theorem 3.3. *Let $A, B \in \mathcal{L}(\mathcal{H})^+$.*

1. *The following equivalences hold:*
   (a) *$R(A) + R(B)$ is dense;*
   (b) *$A + B$ is injective;*
   (c) *$A^p + B^q$ is injective for all $p, q \geq 0$.*
2. *The following equivalences hold:*
   (a) *$R(A) + R(B)$ is closed;*
   (b) *$A + B$ has closed range;*
   (c) *$A^p + B^q$ has closed range for all $p, q \geq 0$.*

   Moreover, if the above conditions hold then $R(A)+R(B) = R(A+B) = R(A^p + B^q)$ for all $p, q > 0$.
3. *The following equivalences hold:*
   (a) *$R(A) + R(B) = \mathcal{H}$;*
   (b) *$A + B$ is invertible;*
   (c) *$A^p + B^q$ is invertible for all $p, q \geq 0$.*

Proof.
1. $(a) \Leftrightarrow (b)$. If $R(A) + R(B)$ is dense then $\mathcal{H} = \overline{R(A) + R(B)} \subseteq \overline{\overline{R(A)} + \overline{R(B)}} = \overline{R(A + B)}$, where the last equality holds by Proposition 2.7. Hence, $\mathcal{H} = \overline{R(A + B)}$ and so $A + B$ is injective. Conversely, if $A + B$ is injective then it has dense range. Then $\mathcal{H} = \overline{R(A + B)} \subseteq \overline{R(A) + R(B)}$.

   $(b) \Leftrightarrow (c)$. It is sufficient to note that $N(A^p + B^q) = N(A + B)$ for all $p, q \geq 0$. Indeed, as $A, B \in \mathcal{L}(\mathcal{H})^+$ then, by Proposition 2.7, $N(A^p + B^q) = N(A^p) \cap N(B^q) = N(A) \cap N(B) = N(A + B)$ for all $p, q \geq 0$.
2. $(a) \Leftrightarrow (b)$. If $R(A) + R(B)$ is closed then $R(A) + R(B) \subseteq R(A^{1/2}) + R(B^{1/2}) = R((A+B)^{1/2}) \subseteq \overline{R(A + B)} \subseteq \overline{R(A) + R(B)} = R(A)+R(B)$. Then $R((A + B)^{1/2}) = R(A) + R(B)$ is closed. So that, $R(A + B)$ is closed. Conversely, suppose $R(A + B)$ is closed. Then $R(A + B) \subseteq$

$R(A)+R(B) \subseteq \overline{\overline{R(A)}+\overline{R(B)}} = \overline{R(A+B)} = R(A+B)$. So that, $R(A)+R(B) = R(A+B)$ is closed.
$(b) \Leftrightarrow (c)$. Suppose that $A+B$ has closed range. Thus, by the equivalence between (a) and (b), $R(A)+R(B)$ is closed and then, by Crimmins' result, $R(A^2+B^2) = R(A)+R(B)$ is a closed subspace. Hence, $R(A)+R(B) = R(A^2+B^2) \subseteq R(A^2)+R(B^2) \subseteq R(A)+R(B)$, i.e., $R(A^2+B^2) = R(A^2)+R(B^2)$ is a closed subspace. In this way, $R(A^{2^n}+B^{2^n}) = R(A^{2^n})+R(B^{2^n})$ is a closed subspace for all $n \in \mathbb{N}$. Furthermore, by Proposition 2.7, this implies $\overline{R(A^{2^n})}+\overline{R(B^{2^n})} = R(A^{2^n})+R(B^{2^n})$. Now take $n \in \mathbb{N}$ such that $\max\{p,q\} \leq 2^n$ and suppose, without loss of generality, $\|A\| \leq 1$ and $\|B\| \leq 1$. Then, by Proposition 2.7, it holds that $A^{2^n}+B^{2^n} \leq (A^{2^n})^{p/2^n}+(B^{2^n})^{q/2^n} = A^p+B^q$. Thus, $R(A^{2^n})+R(B^{2^n}) = R(A^{2^n}+B^{2^n}) = R((A^{2^n}+B^{2^n})^{1/2}) \subseteq R((A^p+B^q)^{1/2})$ where the inclusion follows by Douglas' theorem. So, $R(A^{2^n})+R(B^{2^n}) \subseteq R((A^p+B^q)^{1/2}) \subseteq \overline{R(A^p+B^q)} = \overline{\overline{R(A^p)}+\overline{R(B^q)}} = \overline{\overline{R(A^{2^n})}+\overline{R(B^{2^n})}} = \overline{R(A^{2^n})+R(B^{2^n})} = R(A^{2^n})+R(B^{2^n})$. Therefore, $R((A^p+B^q)^{1/2}) = R(A^{2^n})+R(B^{2^n})$ is closed and so $R(A^p+B^q)$ is a closed subspace.
On the other hand, if any of the above conditions holds then, by the proof of $(a) \Leftrightarrow (b)$, we get that $R(A+B) = R(A)+R(B)$ is closed. Thus, by Crimmins' result, we obtain that $R((A^2+B^2)^{1/2})$ is closed, i.e., $R((A^2+B^2)^{1/2}) = R(A^2+B^2)$ is closed and so $R(A^2)+R(B^2) = R(A^2+B^2) = R(A)+R(B)$. Iteratively, $R(A^{2^n})+R(B^{2^n}) = R(A^{2^n}+B^{2^n}) = R(A)+R(B)$ for all $n \in \mathbb{N}$. Now, given $p,q>0$ take $n \in \mathbb{N}$ such that $\max\{p,q\} \leq 2^n$. Therefore, following the proof of $(b) \Rightarrow (c)$ we obtain that $R(A^{2^n})+R(B^{2^n}) = R(A^p+B^q)$. Hence, $R(A)+R(B) = R(A+B) = R(A^p+B^q)$ for all $p,q>0$.

3. It is consequence of the previous items.

$\square$

Remark 3.4. The equivalence "$R(A)+R(B)$ closed if and only if $R(A+B)$ is closed" was proven in [13, Corollary 3] under the hypotheses that $R(A)$ and $R(B)$ are closed. Item 2 of Theorem 3.3 shows that these hypotheses are unnecessary.

Remark 3.5. As it was said in the introduction, the technique of the proof of the equivalence between (b) and (c) in item 2 is due to Bikchentaev [4, Lemma 2].

Corollary 3.6. *Let $A, B \in \mathcal{L}(\mathcal{H})$.*

1. *The following conditions are equivalent:*
   (a) $R(A) + R(B)$ *is a dense subspace of* $\mathcal{H}$*;*
   (b) $|A^*| + |B^*|$ *is injective;*
   (c) $|A^*|^p + |B^*|^q$ *is injective for all* $p, q \geq 0$*.*
2. *The following conditions are equivalent:*
   (a) $R(A) + R(B)$ *is a closed subspace;*
   (b) $|A^*| + |B^*|$ *has closed range;*
   (c) $|A^*|^p + |B^*|^q$ *has closed range for all* $p, q \geq 0$*;*
   (d) $R(AA^* + BB^*) = R(A) + R(B)$*;*
   (e) $R(AA^* + BB^*) = \overline{R(A)} + \overline{R(B)}$*.*
3. *The following assertions are equivalent:*
   (a) $R(A) + R(B) = \mathcal{H}$*;*
   (b) $|A^*| + |B^*|$ *is invertible;*
   (c) $|A^*|^p + |B^*|^q$ *is invertible for all* $p, q \geq 0$*.*

Proof. Equivalences $(a) \Leftrightarrow (b) \Leftrightarrow (c)$ of item 1,2 and 3 follow from Theorem 3.3 using the fact that $R(T) = R(|T^*|)$ for all $T \in \mathcal{L}(\mathcal{H})$.
2. $(a) \Leftrightarrow (d)$. It follows from Crimmins' theorem.
$(a) \Leftrightarrow (e)$. Assume that $R(A) + R(B)$ is closed. Then, $\overline{R(A)} + \overline{R(B)} \subseteq \overline{R(A) + R(B)} = R(A) + R(B) = R(AA^* + BB^*) \subseteq \overline{R(A)} + \overline{R(B)}$ and item (e) is obtained. Conversely, if $\overline{R(A)} + \overline{R(B)} = R(AA^* + BB^*)$ then $\overline{R(A)} + \overline{R(B)} \subseteq R((AA^* + BB^*)^{1/2}) = R(A) + R(B) \subseteq \overline{R(A)} + \overline{R(B)}$. Hence, $R(A) + R(B) = R((AA^* + BB^*)^{1/2}) = R(AA^* + BB^*)$ is closed. □

Remark 3.7. It should be noticed that the hypothesis $R(A) + R(B) = \mathcal{H}$ does not force $R(A)$ and $R(B)$ to be closed. For instance, let $K \in \mathcal{L}(\mathcal{H})$ be a compact operator such that $R(K)$ is infinite-dimensional and consider $A, B \in \mathcal{L}(\mathcal{H} \oplus \mathcal{H})$ defined as $A = I \oplus K$ and $B = K \oplus I$.

As a consequence of the previous result we obtain another extension of Bikchentaev's lemma.

Corollary 3.8. *Let $A, B \in \mathcal{L}(\mathcal{H})$. If $A + B$ is injective with closed range then $|A|^p + |B|^q$ is invertible for all $p, q \geq 0$.*

Proof. If $A + B$ is injective and $R(A + B)$ is closed then $R(A^* + B^*) = N(A + B)^{\perp} = \mathcal{H}$. Thus, $R(A^*) + R(B^*) = \mathcal{H}$ and so the result follows by Corollary 3.6 item 3. □

Corollary 3.9. *Let $A, B \in \mathcal{L}(\mathcal{H})^+$ be such that $R(A) \cap R(B) = \{0\}$. Then, $R(A) + R(B)$ is closed if and only if $R(A)$ and $R(B)$ are closed and $R(A) + R(B) = R(A + B)$.*

Proof. If $R(A) \dot{+} R(B)$ is closed then the assertion follows from Theorem 2.3 and Theorem 3.3. Conversely, if $R(A)$ and $R(B)$ are closed and $R(A) + R(B) = R(A + B)$ then $R(A + B) = R(A^{1/2}) + R(B^{1/2}) = R((A + B)^{1/2})$, i.e., $R(A) + R(B) = R(A + B)$ is closed. $\square$

The above corollary can be also found in [18, Theorem 5], but in a different context.
In the next theorem we study the range condition "$\overline{R(A)} \dot{+} \overline{R(B)}$ is a closed subspace" for $A, B \in \mathcal{L}(\mathcal{H})$. It should be noticed that the condition "$\overline{R(A)} \cap \overline{R(B)} = \{0\}$" is drammatically stronger than "$R(A) \cap R(B) = \{0\}$". The reader is referred to a nice example by Fillmore and Williams [13, §4] of two positive injective operators A, B such that $R(A) \cap R(B) = \{0\}$, $R(A^{1/2}) = R(B^{1/2})$ and, then, $\overline{R(A)} \cap \overline{R(B)} = \mathcal{H}$.

Theorem 3.10. *Let $A, B \in \mathcal{L}(\mathcal{H})$. The following conditions are equivalent:*

- (a) *$\overline{R(A)} \dot{+} \overline{R(B)}$ is a closed subspace.*
- (b) *there exists a reflection W such that $W(A + B) = A - B$.*
- (c) *there exists a reflection W such that $W(T_1 + T_2) = T_1 - T_2$, for every $T_1, T_2 \in \mathcal{L}(\mathcal{H})$ such that $\overline{R(T_1)} = \overline{R(A)}$, $\overline{R(T_2)} = \overline{R(B)}$.*
- (d) *$\|P_{\overline{R(A)}} P_{\overline{R(B)}}\| < 1$.*

Moreover, if one of the previous conditions holds then $R(A^* + B^*) = R(A^* - B^*) = R(A^*) + R(B^*)$.

Proof.
$(a) \Leftrightarrow (b)$ This equivalence is scattered in [15] for the case $A, B \in \mathcal{L}(\mathcal{H})^+$. The proof follows the same ideas that Corollary 4.6 in [15]. Suppose that $\overline{R(A)} \dot{+} \overline{R(B)}$ is closed. Since $\mathcal{H} = \overline{R(A)} \dot{+} (\overline{R(B)} \oplus N(A^*) \cap N(B^*))$ then take $E = Q_{\overline{R(A)}//\overline{R(B)} \oplus N(A^*) \cap N(B^*)}$ and define $W = 2E - I$. Then $W^2 = I$ and $W(A + B) = A - B$. Conversely, suppose that there exists a reflection W such that $W(A + B) = A - B$. Define $Q = (W + I)/2$ and note that $Q^2 = Q$. Observe that $A = Q(A - B)$ and $B = (Q - I)(A - B)$. Therefore, it holds that $\overline{R(A)} \subseteq R(Q)$ and $\overline{R(B)} \subseteq R(Q - I) = N(Q)$. In consequence, $c_0(\overline{R(A)}, \overline{R(B)}) \leq c_0(R(Q), N(Q))) < 1$, because of Proposition 2.1. Therefore, $\overline{R(A)} \dot{+} \overline{R(B)}$ is closed because of Proposition 2.1.

$(a) \Leftrightarrow (c)$ Notice that $W = 2E - I$ with $E = Q_{\overline{R(A)}//\overline{R(B)}\oplus N(A^*)\cap N(B^*)}$ satisfies $W(T_1 + T_2) = T_1 - T_2$ for all $T_1, T_2 \in \mathcal{L}(\mathcal{H})$ such that $\overline{R(T_1)} = \overline{R(A)}$, $\overline{R(T_2)} = \overline{R(B)}$.
$(a) \Leftrightarrow (d)$ It follows from Proposition 2.1.
Finally, if $\overline{R(A)} \dot{+} \overline{R(B)}$ is a closed subspace then, by item 2, $R(A^* + B^*) = R(A^* - B^*)$ and, by Corollary 3.2, $R(A^*) + R(B^*) = R(A^* + B^*)$.

□

Remark 3.11. We have proved in Corollary 3.2 that $R(A + B) = R(A) + R(B)$ if and only if $R(A - B) \subseteq R(A + B)$. The last theorem shows that if $\overline{R(A^*)} \dot{+} \overline{R(B^*)}$ is a closed subspace then $R(A) + R(B) = R(A + B) = R(A - B)$. Compare this with Remark 3.4.

Corollary 3.12. *Let $A, B \in \mathcal{L}(\mathcal{H})$ be such that $R(A) \cap R(B) = \{0\}$. The following conditions are equivalent:*

(a) *$R(A) + R(B)$ is closed;*
(b) *$R(AA^* + BB^*)$ is closed;*
(c) *$R(AA^* - BB^*)$ is closed.*

Moreover, if one of these conditions holds then $R(AA^* - BB^*) = R(AA^* + BB^*) = R(A) + R(B)$.

Proof. $(a) \Leftrightarrow (b)$ It follows by Crimmins' result.
$(a) \Leftrightarrow (c)$ Assume that $R(A) \dot{+} R(B)$ is closed. Then, by Theorem 2.3, $R(A)$ and $R(B)$ are closed and so $R(AA^*) \dot{+} R(BB^*)$ is closed. Therefore, by Theorem 3.6, there exists a reflection W such that $W(AA^* + BB^*) = AA^* - BB^*$. Hence, as $AA^* + BB^*$ has closed range because of $(a) \Leftrightarrow (b)$, we get that $AA^* - BB^*$ has closed range.
Conversely, suppose that $R(A) \cap R(B) = \{0\}$ and $R(AA^* - BB^*)$ is closed. From $R(A) \cap R(B) = \{0\}$ we obtain that $N(AA^* - BB^*) = N(A^*) \cap N(B^*)$. Therefore, $R(A) \dot{+} R(B) \supseteq R(AA^* - BB^*) = N(AA^* - BB^*)^\perp = (N(A^*) \cap N(B^*))^\perp \supseteq \overline{R(A) \dot{+} R(B)}$. Thus, $R(A) \dot{+} R(B)$ is closed.
Finally, suppose that $R(A) \dot{+} R(B)$ is closed. Then by Crimmins' result it holds that $R(AA^* + BB^*) = R(A) + R(B)$. In addition, the equality $R(AA^* - BB^*) = R(A) + R(B)$ follows of the proof of $(c) \Rightarrow (a)$. □

Remarks 3.13. 1. Notice that $R(A) \dot{+} R(B)$ closed does not imply, in general, $R(A) \dot{+} R(B) = R(A+B)$. For example, consider $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$.

2. The closedness of $R(AA^* - BB^*)$ does not imply, in general, the closedness of $R(AA^* + BB^*)$. For example, consider a normal operator A with non-closed range and $B = A^*$.

As immediate consequence of the previous corollary we obtain the following result, which can be seen as an extension of Buckholtz theorem [5]: given $P, Q \in \mathcal{P}$ it holds $R(P) \dot{+} R(Q) = \mathcal{H}$ if and only if $P - Q$ is invertible.

Corollary 3.14. *Let $A, B \in \mathcal{L}(\mathcal{H})$. The following conditions are equivalent:*

1. $R(A) \dot{+} R(B) = \mathcal{H}$;
2. $R(A) \cap R(B) = \{0\}$ *and* $AA^* + BB^*$ *is invertible;*
3. $R(A) \cap R(B) = \{0\}$ *and* $AA^* - BB^*$ *is invertible.*

Given $A, B \in \mathcal{L}(\mathcal{H})^+$ Bikchentaev [4] proved that $A+B$ is invertible if $A-B$ is invertible. Note that this assertion can be derived from Theorem 3.3. In fact, if $A - B$ is invertible then $\mathcal{H} = R(A - B) \subseteq R(A) + R(B)$. In the next proposition we study the simultaneous invertibility of $A + B$ and $A - B$ for $A, B \in \mathcal{L}(\mathcal{H})$.

Proposition 3.15. *Let $A, B \in \mathcal{L}(\mathcal{H})$ such that $R(A) \cap R(B) = \{0\}$. Then, $A - B$ is invertible if and only if $A + B$ is invertible. Moreover, if one of the previous conditions holds then $R(A)$ and $R(B)$ are closed.*

Proof. If $A - B$ is invertible then $\mathcal{H} = R(A - B) \subseteq R(A) + R(B)$. So that $R(A) \dot{+} R(B) = \mathcal{H}$ and, by Theorem 2.3, both $R(A)$ and $R(B)$ are closed. Take $Q = Q_{R(A)//R(B)}$ and $W = 2Q - I$. The closedness of $R(A)$ and $R(B)$ imply that Q and W are bounded. Note that $W^2 = I$ and $W(A-B) = A+B$. So that $A+B$ is invertible because $A-B$ and W are invertible. The converse is similar. $\square$

4. Projections

We start with two extensions of a nice formula of T. Ando [1, Theorem 3.4], who proved that, for closed subspaces $\mathcal{S}$ and $\mathcal{T}$ of $\mathcal{H}$ such that $\mathcal{S} \dot{+} \mathcal{T} = \mathcal{H}$, it holds that $Q_{\mathcal{S}//\mathcal{T}} = P_{\mathcal{S}}(P_{\mathcal{S}} + P_{\mathcal{T}})^{-1}$.

Theorem 4.1. *Let $A, B \in \mathcal{L}(\mathcal{H})^+$ such that $R(A) \dot{+} R(B) = \mathcal{H}$. Then*

$$Q_{R(A)//R(B)} = A(A + B)^{-1} = A(A - B)^{-1}.$$

Proof. If $R(A) \dot{+} R(B) = \mathcal{H}$ then, by Theorem 3.3, Theorem 2.3 and Proposition 3.15, it holds that $A+B$, $A-B$ are invertible and $R(A)$, $R(B)$ are closed. So $Q = Q_{R(A)//R(B)} \in \mathcal{L}(\mathcal{H})$ and, by the definition of Q, it holds that $Q(A+B) = A$ and $Q(A-B) = A$; this means that $Q_{R(A)//R(B)} = A(A+B)^{-1} = A(A-B)^{-1}$. □

Corollary 4.2. *Let $A, B \in \mathcal{L}(\mathcal{H})$ such that $R(A) \dot{+} R(B) = \mathcal{H}$. Then,*

$$Q_{R(A)//R(B)} = AA^*(AA^* \pm BB^*)^{-1}.$$

The next result about orthogonal projections is due to J. Koliha and V. Rakočević [19, Lemma 2.4]; here we present a different proof, which does not use spectral properties of P, Q, for $P, Q \in \mathcal{P}$.

Proposition 4.3. *Let $P, Q \in \mathcal{P}$. Then, $R(P+Q)$ is closed if and only if $R(P-Q)$ is closed.*

Proof. Assume that $R(P+Q)$ is closed. Then, by Theorem 3.3, $R(P+Q) = R(P) + R(Q)$. Now, define $\mathcal{S} := R(P) \cap R(Q)$ and $\mathcal{T} = R(Q) \cap \mathcal{S}^\perp$. Note that $R(P) + R(Q) = R(P) \dot{+} \mathcal{T}$ and $R(Q) = \mathcal{S} \oplus \mathcal{T}$. Therefore, $Q = P_{\mathcal{S}} + P_{\mathcal{T}}$ and $P - P_{\mathcal{S}} \in \mathcal{P}$. In addition, as $R(P) = \mathcal{S} \oplus R(P) \cap \mathcal{S}^\perp$ then $R(P - P_{\mathcal{S}}) = R(P) \cap \mathcal{S}^\perp$. As a consequence, $R(P - P_{\mathcal{S}}) \cap R(P_{\mathcal{T}}) = \{0\}$. Furthermore, since $R(P) + R(Q)$ is closed then, by Proposition 2.1, $R(P - P_{\mathcal{S}}) \dot{+} R(P_{\mathcal{T}})$ is closed. Now, applying Proposition 3.10, we obtain that $R(P - P_{\mathcal{S}}) + R(P_{\mathcal{T}}) = R(P - P_{\mathcal{S}} - P_{\mathcal{T}}) = R(P-Q)$, i.e., $P-Q$ has closed range.
Conversely, suppose that $R(P-Q)$ is closed. Therefore, $R(P-Q) = R((P-Q)^2) \subseteq R(P+Q) + R(PQ+QP) \subseteq R(P+Q)$ where the last inclusion holds because $PQ + QP = (P+Q)(Q+P-I)$. Now, applying Corollary 3.2, we get that $R(P+Q) = R(P) + R(Q) = R((P+Q)^{1/2})$ i.e., $P+Q$ has closed range. □

Observe that the previous result is false, in general, for oblique projections. For instance, consider the Hilbert space decomposition $\mathcal{H} = \mathcal{S} \oplus \mathcal{S}^\perp$ and the projections onto $\mathcal{S}$, $P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $E = \begin{bmatrix} 1 & x \\ 0 & 0 \end{bmatrix}$ with x a non-closed range operator. Then $R(P+E)$ is closed (in fact, $R(P+E) = \mathcal{S}$), but $R(P-E)$ is not closed because $R(P-E) = R(x)$.
The next proposition has the same spirit that the result of Koliha and Rakočević but, to a pair E, E^* for any $E \in \mathcal{Q}$.

Proposition 4.4. *Let $E \in \mathcal{Q}$ and $P = P_{R(E)}$.*

1. *The following conditions are equivalent:*
   (a) $R(E - E^*)$ *is a dense subspace of* $\mathcal{H}$.
   (b) $R(E + E^*)$ *is a dense subspace of* $\mathcal{H}$ *and* $R(E) \cap R(E^*) = \{0\}$.
   (c) $R(E) \dot{+} R(E^*)$ *is a dense subspace of* $\mathcal{H}$.
2. *The following conditions are equivalent:*
   (a) $R(E - E^*)$ *is closed.*
   (b) $R(E - P)$ *is closed.*
   (c) $R(E) + R(E^*)$ *is closed.*
   (d) $R(EE^* + E^*E)$ *is closed.*
   (e) $R(E + E^*)$ *is closed.*

   Moreover, if some of these conditions holds then $R(E) + R(E^*) = R(EE^* + E^*E) = R(E + E^*)$.
3. *The following conditions are equivalent:*
   (a) $R(E - E^*) = \mathcal{H}$.
   (b) $R(E + E^*) = \mathcal{H}$ *and* $R(E) \cap R(E^*) = \{0\}$.
   (c) $R(E) \dot{+} R(E^*) = \mathcal{H}$.

Proof. 1. $(a) \Rightarrow (b)$. By [19, Lemma 2.2], $N(E - E^*) = N(E) \cap N(E^*) \dot{+} R(E) \cap R(E^*)$. Hence, if $R(E - E^*)$ is a dense subspace or, equivalently $N(E - E^*) = \{0\}$ then $N(E) \cap N(E^*) = \{0\}$ and $R(E) \cap R(E^*) = \{0\}$. From this, $N(E + E^*) = N(E) \cap N(E^*) = \{0\}$ and so $R(E + E^*)$ is a dense subspace of $\mathcal{H}$ and $R(E) \cap R(E^*) = \{0\}$.
$(b) \Rightarrow (c)$. It is elementary.
$(c) \Rightarrow (a)$. Assume that $R(E) \dot{+} R(E^*)$ is a dense subspace of $\mathcal{H}$. Hence, $R(E) \cap R(E^*) = \{0\}$ and $N(E) \cap N(E^*) = \{0\}$. Therefore, $N(E - E^*) = N(E) \cap N(E^*) \dot{+} R(E) \cap R(E^*) = \{0\}$, i.e., $\overline{R(E - E^*)} = N(E - E^*)^\perp = \mathcal{H}$.

2. $(a) \Leftrightarrow (b)$. An easy computation shows that $R(E - E^*) = R(E - P) \oplus R(E^* - P)$. Thus, if $R(E - E^*)$ is closed then, by Theorem 2.3, $R(E - P)$ is closed. Conversely, if $R(E - P)$ is closed then $R(E^* - P)$ is also closed and, since $c_0(R(E - P), R(E^* - P)) = 0 < 1$ then $R(E - E^*) = R(E - P) \oplus R(E^* - P)$ is closed.
$(b) \Leftrightarrow (c)$. As $R(E - P) = R(E(I - P))$ then the equivalence follows from [8, Theorem 22].
$(c) \Leftrightarrow (d)$. It follows by Corollary 3.6.
$(c) \Leftrightarrow (e)$. Assume that $R(E) + R(E^*)$ is closed. Hence, $R(E - E^*)$ is closed because of the equivalence between (a) and (c). Therefore,

$R(E-E^*) = R((E-E^*)(E^*-E)) \subseteq R(E+E^*) + R(EE^*+E^*E) \subseteq R(E+E^*)$ where the last inclusion follows from $EE^*+E^*E = (E+E^*)(E^*+E-I)$. Now, by Corollary 3.2, we obtain that $R(E+E^*) = R(E)+R(E^*)$, i.e., $R(E+E^*)$ is closed.
For the converse, let us prove first that $\overline{R(E+E^*)} = R(E) \oplus \overline{R(E^*-P)}$ or, equivalently $N(E+E^*) = N(E^*) \cap N(E-P)$. Indeed, if $x = x_1+x_2 \in N(E+E^*)$ with $x_1 \in R(E)$, $x_2 \in R(E)^\perp$ then $0 = (E+E^*)x = x_1 + Ex_2 + E^*x_1 = 2x_1 + (E-P)x_2 - (P-E^*)x_1$. Therefore, $(P-E^*)x_1 = 0$ and $2x_1 + (E-P)x_2 = 0$. Hence, $0 = (P-E^*)(2x_1+(E-P)x_2) = (P-E^*)(E-P)x_2$ and so $Ex_2 = 0$ and $x_1 + E^*x_1 = 0$. Hence, $E^*(x_1 + E^*x_1) = 0$ i.e., $x_1 \in N(E^*) \cap R(E) = \{0\}$. Therefore, $x = x_2 \in R(E)^\perp$ with $Ex_2 = 0$ or, equivalently $x \in N(E^*) \cap N(E-P)$. Conversely, given $x \in N(E^*) \cap N(E-P)$ we have that $E^*x = 0$ and $Ex = (E-P)x = 0$, i.e., $x \in N(E+E^*)$. Therefore, $\overline{R(E+E^*)} = R(E) \oplus \overline{R(E^*-P)}$ as claimed. Now, suppose that $R(E+E^*)$ is closed. Consequently, $R(E) \oplus \overline{R(E^*-P)} = \overline{R(E+E^*)} \subseteq R(E)+R(E^*)$. On the other side, since $R(E^*) \subseteq R(E) \oplus \overline{R(E^*-P)}$ we obtain that $R(E)+R(E^*) \subseteq R(E) \oplus \overline{R(E^*-P)}$. Therefore, $R(E)+R(E^*) = R(E+E^*)$ is closed which is the desired conclusion.

3. It follows from the above items.

$\square$

I. Spitkovsky [21, Theorem 13], characterized those $E \in \mathcal{Q}$ such that $||E^\dagger|| \neq ||(I-E)^\dagger||$ where $\dagger$ denotes the Moore-Penrose inverse. A. Galantai [16, Theorem 64] found a different proof for matrices. The previous result allows us to obtain a new equivalent condition for $||E^\dagger|| \neq ||(I-E)^\dagger||$.

Proposition 4.5. *Let $E \in \mathcal{Q}$. The following conditions are equivalent:*

1. $||E^\dagger|| \neq ||(I-E)^\dagger||$.
2. $R(E-E^*)$ *is closed and exactly one of the subspaces* $N(E) \cap N(E^*)$ *and* $R(E^*) \cap R(E)$ *is nonzero.*
3. *Exactly one of the following conditions holds:* $R(E)+R(E^*) = \mathcal{H}$ *with* $R(E) \cap R(E^*) \neq \{0\}$ *or* $N(E^*)+N(E) = \mathcal{H}$ *with* $N(E^*) \cap N(E) \neq \{0\}$.

Proof. $1 \Leftrightarrow 2$. See [21, Theorem 13].
$2 \Leftrightarrow 3$. It follows from Proposition 4.4. $\square$

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