SHARP EIGENVALUE ESTIMATES FOR RANK ONE PERTURBATIONS OF NONNEGATIVE OPERATORS IN KREIN SPACES

JUSSI BEHRNDT, LESLIE LEBEN, FRANCISCO MARTÍNEZ PERÍA, ROLAND MÖWS, AND CARSTEN TRUNK

ABSTRACT. Let A and B be selfadjoint operators in a Krein space and assume that the resolvent difference of A and B is of rank one. In the case that A is nonnegative and I is an open interval such that $\sigma(A) \cap I$ consists of isolated eigenvalues we prove sharp estimates on the numbers and multiplicities of eigenvalues of B in I. The general result is illustrated with eigenvalue estimates for singular left definite Sturm-Liouville problems.

1. INTRODUCTION

Rank one and finite rank perturbations of selfadjoint operators in Hilbert spaces have been considered in various papers and in many applications in theoretical physics, e.g. in the investigation of singular perturbations in quantum mechanics, see [1, 2, 3, 11, 12, 25, 30, 31, 34, 36, 50, 60]. It is well known that an n-dimensional selfadjoint perturbation of a selfadjoint operator preserves the essential spectrum and changes the spectral multiplicity by at most n, that is, for a bounded interval $I \subset \mathbb{R}$ and (in general unbounded) selfadjoint operators A, B in a Hilbert space $\mathscr{H}$ such that

$$(A-\lambda_0)^{-1} - (B-\lambda_0)^{-1} \tag{1.1}$$

is of rank n for some $\lambda_0 \in \rho(A) \cap \rho(B)$, the dimensions of the spectral subspaces of A and B corresponding to the interval I differ at most by n, and this estimate is sharp. In particular, if $I \subset \rho(A)$ then I contains at most n eigenvalues of B counted with multiplicities.

In the general non-selfadjoint case rank one and finite rank perturbations preserve the essential spectrum but precise results on the number and multiplicity of the discrete spectrum do not exist. Without further assumptions on the structure of the operators or the rank one perturbation the number of eigenvalues in a given interval can change arbitrarily, see [49, Theorem 1]. If the operators A and B under consideration are not selfadjoint in a Hilbert space but still selfadjoint in a Krein space, then several results on finite rank perturbations of different classes of operators exist; cf. [4, 5, 6, 8, 14, 24, 28, 38, 39, 40, 41]. However, these perturbation results are typically of qualitative nature and do not contain explicit bounds or estimates on the numbers and multiplicities of eigenvalues after the perturbation. In the matrix case we refer to [57, 58, 59] where so-called generic perturbations were investigated.

Our main objective in this paper is to obtain sharp bounds for the numbers and multiplicities of eigenvalues in the following Krein space perturbation problem: We assume that A and B are selfadjoint with respect to some indefinite inner product $[\cdot,\cdot]$, that A is nonnegative with respect to $[\cdot,\cdot]$, and that the perturbation (1.1) is of rank one. In that case

2010 *Mathematics Subject Classification.* Primary 47A55; Secondary 47B50, 81Q15.

Leslie Leben is founded by the Carl-Zeiss-Stiftung Jena.
Francisco Martínez Pería gratefully acknowledges the support from the grant PIP CONICET 0435.

B is either nonnegative ($\kappa_B = 0$) or the form $[B\cdot,\cdot]$ has one negative square ($\kappa_B = 1$). Let I be an open interval such that all spectral points of A in I are isolated eigenvalues and poles of the resolvent of A; here also eigenvalues of infinite multiplicity are allowed. In this setting our first main result (Theorem 3.5 below) states: The difference of the number $n_A(I)$ of distinct eigenvalues of A in I and the number $n_B(I)$ of distinct eigenvalues of B in I can be estimated by the number $n_{A,B}(I)$ of common eigenvalues of A and B in I, and a correction term which is at most 3. The correction term depends on the fact whether 0 is in the interval I and whether the operator B is nonnegative ($\kappa_B = 0$) or has one negative square ($\kappa_B = 1$):

(i) If $0 \notin I$ then

$$n_A(I) - n_{A,B}(I) - 1 \leq n_B(I) \leq n_A(I) + n_{A,B}(I) + \begin{cases} 1 & \text{if } \kappa_B = 0, \\ 3 & \text{if } \kappa_B = 1. \end{cases}$$

(ii) If $0 \in I$ then

$$n_A(I) - n_{A,B}(I) - 2 \leq n_B(I) \leq n_A(I) + n_{A,B}(I) + \begin{cases} 2 & \text{if } \kappa_B = 0, \\ 3 & \text{if } \kappa_B = 1. \end{cases}$$

It is remarkable that all the above estimates turn out to be sharp: There exist operators A and B (which are in fact matrices) such that the inequalities in (i) and (ii) become equalities. Moreover, we mention that the above estimates imply that the finiteness of the number of distinct eigenvalues of A in a gap of the essential spectrum is preserved under a one dimensional perturbation. This is a special case of a more general result from [14].

Our second main result are estimates of the total algebraic multiplicities $m_A(I)$ and $m_B(I)$ of the eigenvalues of A and B in I. This leads to the following estimates in Theorem 3.9 on the multiplicities of the eigenvalues which complement the results in Theorem 3.5 on the number of distinct eigenvalues:

(i) If $0 \notin I$ then

$$m_A(I) - 1 \leq m_B(I) \leq m_A(I) + \begin{cases} 1 & \text{if } \kappa_B = 0, \\ 3 & \text{if } \kappa_B = 1. \end{cases}$$

(ii) If $0 \in I$ and $0 \notin \sigma_p(A)$ then

$$m_A(I) - 2 \leq m_B(I) \leq m_A(I) + \begin{cases} 2 & \text{if } \kappa_B = 0, \\ 3 & \text{if } \kappa_B = 1. \end{cases}$$

(iii) If $0 \in I$ and $0 \in \sigma_p(A)$ then

$$m_A(I) - 4 \leq m_B(I) \leq m_A(I) + \begin{cases} 4 & \text{if } \kappa_B = 0, \\ 6 & \text{if } \kappa_B = 1. \end{cases}$$

Here, at the possible eigenvalue 0, Jordan chains of A and B may occur which makes the analysis more involved. In Theorem 3.8 we show that the dimension of the root subspaces of A and B at 0 differ at most by two, that is,

$$\big| m_A(\{0\}) - m_B(\{0\}) \big| \leq 2,$$

and that this estimate is sharp. We emphasize that the sharp estimates in Theorems 3.5, 3.8, and 3.9 are also new for the case of A and B being matrices.

The paper is organized as follows. After the introduction we recall some definitions in Section 2 and then provide a useful Krein type formula for the resolvent difference of two selfadjoint operators A and B in a Krein space which differ by a rank one operator. Here the resolvent difference is expressed in a rank one perturbation term with a scalar Weyl or Q-function M_A. Roughly speaking the poles (zeros) of M_A coincide with the isolated eigenvalues of A (B, respectively). In the rest of Section 2 we explore the connections between the sign types of the isolated spectral points of A and B, and the behaviour of the function M_A at its poles and zeros. In Section 3 the special case of a nonnegative operator A is investigated. This naturally leads to the function classes in Definition 3.2 studied by two of the authors in [15, 16]. After some preparations in Section 3.1, we state and prove the main results Theorem 3.5 and 3.9 and some special cases in Sections 3.2–3.4. The proof of Theorem 3.8 on the multiplicity of the eigenvalue 0 requires different techniques and is given in Section 3.5. Section 3.6 contains some simple matrix examples which illustrate the sharpness of the estimates in Theorem 3.5 and Theorem 3.9. In Section 4 we show how our general eigenvalue estimates can be applied to indefinite singular Sturm-Liouville problems. We consider the situation where the associated operator is nonnegative in an L^2-Krein space and, in this specific situation, the estimates from Section 3 can be slightly improved and lead to a generalization of [13, Theorem 4.1]. In particular, this also includes the so-called left definite Sturm-Liouville problems where the associated operator is uniformly positive in an L^2-Krein space; cf. [13, 17, 18, 19, 21, 22, 42, 44, 45, 46, 61] for related work on left definite problems.

2. Rank one perturbations and sign types of eigenvalues

2.1. **Preliminaries.** A complex linear space $\mathscr{K}$ with a nondegenerate hermitian sesquilinear form $[\cdot,\cdot]$ is called a *Krein space* if there exists a decomposition

$$\mathscr{K} = \mathscr{K}_+ \dot{+} \mathscr{K}_-$$

such that the subspaces $(\mathscr{K}_\pm, \pm[\cdot,\cdot])$ are Hilbert spaces and orthogonal to each other with respect to $[\cdot,\cdot]$. If $\mathscr{K}_-$ is finite dimensional then $(\mathscr{K},[\cdot,\cdot])$ is called a *Pontryagin space*. An element x in the Krein space $(\mathscr{K},[\cdot,\cdot])$ is *positive* (*negative, neutral*) if $[x,x] > 0$ ($[x,x] < 0$, $[x,x] = 0$, respectively). For the general theory of Krein spaces we refer the reader to the monographs [7, 20].

For a densely defined linear operator A in the Krein space $(\mathscr{K},[\cdot,\cdot])$ the adjoint with respect to the indefinite inner product $[\cdot,\cdot]$ is denoted by A^+. The operator A is called *selfadjoint* if $A = A^+$ and *symmetric* if $A \subset A^+$.

Let A be a selfadjoint operator in the Krein space $(\mathscr{K},[\cdot,\cdot])$. We denote the point spectrum by $\sigma_p(A)$, the spectrum by $\sigma(A)$ and the resolvent set by $\rho(A)$. The *root subspace* $\cup_{j=1}^{\infty} \ker(A-\lambda)^j$ at λ is denoted by $\mathscr{L}_\lambda(A)$. A *Jordan chain* of A at $\lambda \in \sigma_p(A)$ of length n is a finite ordered set of non-zero vectors $\{x_0,\dots,x_{n-1}\}$ contained in the root subspace $\mathscr{L}_0(A)$ such that $(A-\lambda)x_0 = 0$ and $(A-\lambda)x_i = x_{i-1}$, $i = 1,\dots,n-1$. The elements of a Jordan chain are linearly independent. The first $n-1$ elements of a Jordan chain of length n form a Jordan chain of length $n-1$. In the sequel the following simple observation will be used frequently: Let $\{x_0,x_1\}$ be a Jordan chain of a selfadjoint operator A at some real eigenvalue λ of length 2. Then we have

$$[x_0,x_0] = [x_0,(A-\lambda)x_1] = [(A-\lambda)x_0,x_1] = 0, \tag{2.1}$$

hence the eigenvector x_0 is a neutral vector in $(\mathscr{K},[\cdot,\cdot])$. A real isolated eigenvalue λ of A is called of *positive* (*negative*) *type* if all its corresponding eigenvectors are positive (negative, respectively). In this case we write $\lambda \in \sigma_{++}(A)$ ($\lambda \in \sigma_{--}(A)$, respectively). Observe (see (2.1)) that for an isolated eigenvalue of positive or negative type there is no Jordan chain of length greater than one, that is, $\mathscr{L}_\lambda(A) = \ker(A-\lambda)$, and the resolvent of A has a pole of order one in such a point. We mention that the notion of spectral points of positive and negative type can be extended to non-isolated eigenvalues and points in the continuous spectrum; cf. [37, 53].

2.2. **Rank one perturbations and sign types of isolated eigenvalues.** In the following let A and B be selfadjoint operators in the Krein space $(\mathcal{K},[\cdot,\cdot])$ such that $\rho(A)\cap\rho(B)\neq\emptyset$ and

$$\dim\operatorname{ran}\big((A-\lambda_0)^{-1}-(B-\lambda_0)^{-1}\big)=1 \tag{2.2}$$

holds for some (and hence for all) $\lambda_0\in\rho(A)\cap\rho(B)$. We express the difference of the resolvents of A and B with two scalar functions which can be viewed as Weyl functions or Q-functions corresponding to A and B, see [54] for the concept of Q-functions and, e.g., [8, 55] for similar considerations.

Proposition 2.1. *Let A and B be selfadjoint operators in the Krein space $(\mathcal{K},[\cdot,\cdot])$ which satisfy* (2.2). *Then there exist holomorphic functions $M_A:\rho(A)\to\mathbb{C}$, $M_B:\rho(B)\to\mathbb{C}$ symmetric with respect to the real line and vectors φ_A, φ_B in $\mathcal{K}$ such that the following holds.*

(i) *For $\gamma_A(\lambda):=(1+(\lambda-\lambda_0)(A-\lambda)^{-1})\varphi_A$, $\lambda\in\rho(A)$, we have*

$$M_A(\lambda)-M_A(\overline{\omega})=(\lambda-\overline{\omega})[\gamma_A(\lambda),\gamma_A(\omega)],\quad \lambda,\omega\in\rho(A).$$

(ii) *For $\gamma_B(\lambda):=(1+(\lambda-\lambda_0)(B-\lambda)^{-1})\varphi_B$, $\lambda\in\rho(B)$, we have*

$$M_B(\lambda)-M_B(\overline{\omega})=(\lambda-\overline{\omega})[\gamma_B(\lambda),\gamma_B(\omega)],\quad \lambda,\omega\in\rho(B).$$

(iii) *For $\lambda\in\rho(A)\cap\rho(B)$ we have $M_B(\lambda)=-\frac{1}{M_A(\lambda)}$ and*

$$(A-\lambda)^{-1}-(B-\lambda)^{-1}=\frac{1}{M_A(\lambda)}[\cdot,\gamma_A(\overline{\lambda})]\gamma_A(\lambda)=-\frac{1}{M_B(\lambda)}[\cdot,\gamma_B(\overline{\lambda})]\gamma_B(\lambda).$$

Proof. We make use of the theory of boundary triplets and their γ-fields and Weyl functions; cf. [23, 26, 27]. Consider $S=A\cap B$, which is a (possibly nondensely defined) closed symmetric operator in $(\mathcal{K},[\cdot,\cdot])$ of defect one. As in [12, Corollary 2.5] it follows that there exists a boundary triplet $\{\mathbb{C},\Gamma_0,\Gamma_1\}$ for the adjoint S^+ such that $A=S^+\upharpoonright\ker\Gamma_0$ and $B=S^+\upharpoonright\ker\Gamma_1$. Let γ and M be the corresponding γ-field and Weyl function, and define $\varphi_A:=\gamma(\lambda_0)$. From the property $\gamma(\lambda)=(1+(\lambda-\lambda_0)(A-\lambda)^{-1})\gamma(\lambda_0)$, $\lambda\in\rho(A)$, we see that $\gamma_A=\gamma$ holds. Moreover, $M_A:=M$ satisfies the formula in (i). Observe that $\{\mathbb{C},\Gamma_1,-\Gamma_0\}$ is also a boundary triplet for S^+. Let $\widetilde{\gamma}$ and $\widetilde{M}$ be the corresponding γ-field and Weyl function and define $\varphi_B:=\widetilde{\gamma}(\lambda_0)$. As above it follows that $\gamma_B=\widetilde{\gamma}$ and $M_B:=\widetilde{M}$ satisfy the assertion in (ii). By the definition of the Weyl function corresponding to a boundary triplet we have that $\widetilde{M}(\lambda)=-M(\lambda)^{-1}$, and hence $M_B(\lambda)=-M_A(\lambda)^{-1}$, $\lambda\in\rho(A)\cap\rho(B)$, as stated in (iii). Finally, the remaining resolvent formula in (iii) is a special case of [23, Theorem 2.1] (see also [27, Theorem 3.1]). □

Corollary 2.2. *Let A, B and M_A, M_B be as in Proposition 2.1. Then the following holds.*

(i) *For $\lambda\in\rho(A)$ we have $\lambda\in\sigma_p(B)$ if and only if $M_A(\lambda)=0$.*
(ii) *For $\lambda\in\rho(B)$ we have $\lambda\in\sigma_p(A)$ if and only if $M_B(\lambda)=0$.*

Proof. (i) Since the functions γ_A and M_A are holomorphic in a neighbourhood of $\lambda\in\rho(A)$, this follows from the resolvent formula in Proposition 2.1 (iii). Assertion (ii) follows in a similar way. □

From now on we will suppose that the following assumption holds.

Assumption (I). Let A and B be selfadjoint operators in the Krein space $(\mathscr{K}, [\cdot,\cdot])$ such that (2.2) holds for some (and hence for all) $\lambda_0 \in \rho(A)\cap\rho(B)$. Let $I \subset \mathbb{R}$ be an open interval and assume that $\rho(B)\cap I \neq \emptyset$ and that $\sigma(A)\cap I$ consists only of isolated eigenvalues which are poles of the resolvent of A.

Assumption (I) yields the following statements.

Theorem 2.3. *Let A, B and I be as in Assumption* (I).

(i) *Any eigenvalue of infinite algebraic multiplicity of A in I is also an eigenvalue of infinite algebraic multiplicity of B.*
(ii) *The set $\sigma(B)\cap I$ consists of eigenvalues which may only accumulate to the eigenvalues of infinite algebraic multiplicity or to the boundary of I.*
(iii) *If $\mu \in \rho(A)\cap I$ then either $\mu \in \rho(B)$ or $\mu \in \sigma_p(B)$ with* $\dim\ker(B-\mu) = 1$. *If, in addition, $\mu \in \sigma_{\pm\pm}(B)$ then $\mathscr{L}_\mu(B) = \ker(B-\mu)$.*
(iv) *If $\mu \in \rho(B)\cap I$ then either $\mu \in \rho(A)$ or $\mu \in \sigma_p(A)$ with* $\dim\ker(A-\mu) = 1$. *If, in addition, $\mu \in \sigma_{\pm\pm}(A)$ then $\mathscr{L}_\mu(A) = \ker(A-\mu)$.*

Proof. Due to Assumption (I) an eigenvalue $\mu \in I$ of A is a pole of the resolvent. Hence $A-\mu$ is either a Fredholm operator or $\dim\ker(A-\mu) = \infty$, see [43, Theorem IV.5.28]. Due to (2.2), the dimension of $\ker(A-\mu)$ and $\ker(B-\mu)$ differ at most by one, which implies (i). Assertion (ii) follows from general perturbation results in [32, 43]. In order to verify (iii) assume $\dim\ker(B-\mu) \geq 2$. As the operator $A\cap B$ is a one dimensional restriction of B we obtain $\dim\ker(A\cap B-\mu) \geq 1$ and, hence, $\dim\ker(A-\mu) \geq 1$, a contradiction to $\mu \in \rho(A)$. Eigenvectors with a Jordan chain of length greater than one are neutral (cf. (2.1)) and, hence, (iii) is shown. Statement (iv) is proved analogously. □

In the next lemma we relate sign type properties of eigenvalues of B in $\rho(A)$ with the local behaviour of the function M_A from Proposition 2.1, see also [56, Theorem 3.3].

Lemma 2.4. *Let A, B and I be as in Assumption* (I). *Assume $M_A(\mu) = 0$ for some $\mu \in \rho(A)\cap I$. Then $\mu \in \sigma_p(B)$ and* $\dim\ker(B-\mu) = 1$. *Moreover, the following assertions hold.*

(i) $\mu \in \sigma_{++}(B)$ *if and only if* $M_A'(\mu) > 0$. *In this case* $\mathscr{L}_\mu(B) = \ker(B-\mu)$.
(ii) $\mu \in \sigma_{--}(B)$ *if and only if* $M_A'(\mu) < 0$. *In this case* $\mathscr{L}_\mu(B) = \ker(B-\mu)$.
(iii) $\mu \in \sigma_p(B)$ *has a neutral eigenvector if and only if* $M_A'(\mu) = 0$. *In this case* $\mathscr{L}_\mu(B) \neq \ker(B-\mu)$ *and there exist nonzero elements* $x_0 \in \ker(B-\mu)$, $x_1 \in \mathscr{L}_\mu(B)$ *with* $(B-\mu)x_1 = x_0$ *and* $(B-\mu)x_0 = 0$ *such that*

$$[x_0,x_0] = M_A'(\mu) = 0 \quad \textit{and} \quad [x_1,x_0] = \frac{1}{2}M_A''(\mu). \tag{2.3}$$

Moreover, in this case, $(\mathscr{L}_\mu(B), [\cdot,\cdot])$ is a Krein space with at least one positive and one negative element.

Proof. By Corollary 2.2 $M_A(\mu) = 0$ implies $\mu \in \sigma_p(B)$ and $\dim\ker(B-\mu) = 1$ follows from Theorem 2.3. In order to show (i)–(iii) we start with the following observation.

For M_A, φ_B, γ_B as in Proposition 2.1 and $\lambda\in\rho(A)\cap\rho(B)$ we conclude from Proposition 2.1 (iii):

$$\begin{aligned}
M_A(\lambda)\gamma_B(\lambda) &= M_A(\lambda)\big(1+(\lambda-\lambda_0)(B-\lambda)^{-1}\big)\varphi_B\\
&= M_A(\lambda)\left(\varphi_B+(\lambda-\lambda_0)\left((A-\lambda)^{-1}\varphi_B-\frac{1}{M_A(\lambda)}[\varphi_B,\gamma_A(\overline{\lambda})]\gamma_A(\lambda)\right)\right)\\
&= (\lambda_0-\lambda)[\varphi_B,\gamma_A(\overline{\lambda})]\gamma_A(\lambda)+M_A(\lambda)\left(1+(\lambda-\lambda_0)(A-\lambda)^{-1}\right)\varphi_B. \qquad (2.4)
\end{aligned}$$

Then $M_A(\mu)=0$ and $\mu\in\rho(A)\cap\mathbb{R}$ imply the existence of

$$x_0 := \lim_{\lambda\to\mu} M_A(\lambda)\gamma_B(\lambda) = (\lambda_0-\mu)[\varphi_B,\gamma_A(\mu)]\gamma_A(\mu).$$

The vector x_0 is nonzero. Indeed, for $\omega\in\rho(A)\cap\rho(B)$, $\overline{\omega}\neq\mu$, it follows from Proposition 2.1 that

$$\begin{aligned}
[x_0,\gamma_B(\omega)] &= \lim_{\lambda\to\mu}[M_A(\lambda)\gamma_B(\lambda),\gamma_B(\omega)] = \lim_{\lambda\to\mu} M_A(\lambda)\frac{M_B(\lambda)-M_B(\overline{\omega})}{\lambda-\overline{\omega}}\\
&= \lim_{\lambda\to\mu} M_A(\lambda)\frac{-\frac{1}{M_A(\lambda)}+\frac{1}{M_A(\overline{\omega})}}{\lambda-\overline{\omega}} = \lim_{\lambda\to\mu}\frac{-1+\frac{M_A(\lambda)}{M_A(\overline{\omega})}}{\lambda-\overline{\omega}} = \frac{-1}{\mu-\overline{\omega}}\neq 0.
\end{aligned}$$

Furthermore $x_0\in\ker(B-\mu)$, since for $\omega\in\rho(B)$ we have

$$\begin{aligned}
(B-\omega)^{-1}x_0 &= \lim_{\lambda\to\mu}(B-\omega)^{-1}M_A(\lambda)\gamma_B(\lambda)\\
&= \lim_{\lambda\to\mu}(B-\omega)^{-1}M_A(\lambda)\big(1+(\lambda-\lambda_0)(B-\lambda)^{-1}\big)\varphi_B\\
&= \lim_{\lambda\to\mu}\frac{M_A(\lambda)}{\lambda-\omega}\big((\lambda-\omega)(B-\omega)^{-1}+(\lambda-\lambda_0)(B-\lambda)^{-1}\\
&\qquad\qquad -(\lambda-\lambda_0)(B-\omega)^{-1}\big)\varphi_B \qquad (2.5)\\
&= \lim_{\lambda\to\mu}\frac{M_A(\lambda)}{\lambda-\omega}\big((\lambda-\lambda_0)(B-\lambda)^{-1}-(\omega-\lambda_0)(B-\omega)^{-1}\big)\varphi_B\\
&= \lim_{\lambda\to\mu}\frac{M_A(\lambda)}{\lambda-\omega}\left(\gamma_B(\lambda)-\gamma_B(\omega)\right) = \frac{1}{\mu-\omega}x_0.
\end{aligned}$$

Moreover, Proposition 2.1 (ii) and (iii) imply

$$\begin{aligned}
[x_0,x_0] &= \lim_{\lambda,\omega\to\mu} M_A(\lambda)\overline{M_A(\omega)}[\gamma_B(\lambda),\gamma_B(\omega)] = \lim_{\lambda,\omega\to\mu} M_A(\lambda)M_A(\overline{\omega})\frac{-\frac{1}{M_A(\lambda)}+\frac{1}{M_A(\overline{\omega})}}{\lambda-\overline{\omega}}\\
&= \lim_{\lambda,\omega\to\mu}\frac{M_A(\lambda)-M_A(\overline{\omega})}{\lambda-\overline{\omega}} = \lim_{\lambda\to\mu}\frac{M_A(\lambda)-M_A(\mu)}{\lambda-\mu} = M_A'(\mu).
\end{aligned}$$

This yields (i), (ii) and the first statement in (iii). In order to show the remaining statements of (iii) assume $M_A(\mu)=M_A'(\mu)=0$. Relation (2.4) implies the existence of

$$\begin{aligned}
x_1 &:= \lim_{\lambda\to\mu}\left(M_A(\lambda)\gamma_B(\lambda)\right)'\\
&= -[\varphi_B,\gamma_A(\overline{\mu})]\gamma_A(\mu)+(\lambda_0-\mu)[\varphi_B,\gamma_A'(\overline{\mu})]\gamma_A(\mu)+(\lambda_0-\mu)[\varphi_B,\gamma_A(\overline{\mu})]\gamma_A'(\mu).
\end{aligned}$$

We obtain

$$\begin{aligned}(B-\omega)^{-1}x_1 &= \lim_{\lambda\to\mu}(B-\omega)^{-1}\left(M_A(\lambda)\gamma_B(\lambda)\right)' \\ &= \lim_{\lambda\to\mu}\left((B-\omega)^{-1}M_A'(\lambda)\gamma_B(\lambda)+(B-\omega)^{-1}M_A(\lambda)\gamma_B'(\lambda)\right).\end{aligned} \tag{2.6}$$

As in (2.5) one verifies

$$(B-\omega)^{-1}M_A'(\lambda)\gamma_B(\lambda)=\frac{M_A'(\lambda)}{\lambda-\omega}\left(\gamma_B(\lambda)-\gamma_B(\omega)\right)$$

and we have from Proposition 2.1 (ii) $\gamma_B'(\lambda)=(B-\lambda)^{-1}\gamma_B(\lambda)$. Hence (2.6) takes the form

$$(B-\omega)^{-1}x_1=\lim_{\lambda\to\mu}\left(\frac{M_A'(\lambda)}{\lambda-\omega}\left(\gamma_B(\lambda)-\gamma_B(\omega)\right)+(B-\omega)^{-1}M_A(\lambda)(B-\lambda)^{-1}\gamma_B(\lambda)\right)$$

and with $M_A'(\mu)=0$ we conclude

$$\begin{aligned}(B-\omega)^{-1}x_1 &= \lim_{\lambda\to\mu}\left(\frac{M_A'(\lambda)\gamma_B(\lambda)}{\lambda-\omega}+\frac{M_A(\lambda)\gamma_B'(\lambda)}{\lambda-\omega}-(B-\omega)^{-1}\frac{M_A(\lambda)\gamma_B(\lambda)}{\lambda-\omega}\right)\\ &= \lim_{\lambda\to\mu}\left(\frac{(M_A(\lambda)\gamma_B(\lambda))'}{\lambda-\omega}-(B-\omega)^{-1}\frac{M_A(\lambda)\gamma_B(\lambda)}{\lambda-\omega}\right)\\ &= \frac{x_1}{\mu-\omega}-(B-\omega)^{-1}\frac{x_0}{\mu-\omega}=\frac{x_1}{\mu-\omega}-\frac{x_0}{(\mu-\omega)^2}.\end{aligned}$$

This yields $(B-\mu)x_1=x_0$. Moreover, Proposition 2.1 (ii) and (iii) imply

$$\begin{aligned}[x_1,x_0] &= \lim_{\lambda,\omega\to\mu}[(M_A(\lambda)\gamma_B(\lambda))',M_A(\omega)\gamma_B(\omega)]=\lim_{\lambda,\omega\to\mu}\frac{d}{d\lambda}[M_A(\lambda)\gamma_B(\lambda),M_A(\omega)\gamma_B(\omega)]\\ &= \lim_{\lambda,\omega\to\mu}\frac{d}{d\lambda}\left(M_A(\lambda)M_A(\overline{\omega})\frac{-\frac{1}{M_A(\lambda)}+\frac{1}{M_A(\overline{\omega})}}{\lambda-\overline{\omega}}\right)=\lim_{\lambda,\omega\to\mu}\frac{d}{d\lambda}\left(\frac{M_A(\lambda)-M_A(\overline{\omega})}{\lambda-\overline{\omega}}\right)\\ &= \lim_{\lambda\to\mu}\frac{d}{d\lambda}\left(\frac{M_A(\lambda)}{\lambda-\mu}\right)=\lim_{\lambda\to\mu}\left(\frac{M_A'(\lambda)(\lambda-\mu)-M_A(\lambda)}{(\lambda-\mu)^2}\right)=\frac{1}{2}M_A''(\mu),\end{aligned}$$

where the last equality follows from the power series expansion of M_A in μ and $M_A(\mu)=M_A'(\mu)=0$. By [51, Proposition I.3.2 and Theorem I.5.2] the space $(\mathscr{L}_\mu(B),[\cdot,\cdot])$ is a Krein space and (iii) is shown. □

Lemma 2.5. *Let A, B and I be as in Assumption* (I) *and let* $\mu\in I\cap\sigma_{++}(A)$ *(*$\mu\in I\cap\sigma_{--}(A)$*) with* $\mu\in\rho(B)$*. Then the function* M_A *has a pole at* μ *of order one with*

$$\lim_{\lambda\nearrow\mu}M_A(\lambda)=+\infty,\quad \lim_{\lambda\searrow\mu}M_A(\lambda)=-\infty$$

$$\left(\lim_{\lambda\nearrow\mu}M_A(\lambda)=-\infty,\quad \lim_{\lambda\searrow\mu}M_A(\lambda)=+\infty,\ \textit{respectively}\right).$$

Proof. According to Theorem 2.3 $\mathscr{L}_\mu(A)=\ker(A-\mu)$ is a one dimensional subspace. The corresponding Riesz-Dunford projection onto $\ker(A-\mu)$ will be denoted by E. By Proposition 2.1 (i) we have $\gamma_A(\lambda_0)=\varphi_A$ and

$$\begin{aligned}M_A(\lambda) &= M_A(\bar\lambda_0)+(\lambda-\bar\lambda_0)\left[(1+(\lambda-\lambda_0)(A-\lambda)^{-1})\varphi_A,\varphi_A\right]\\ &= M_A(\bar\lambda_0)+(\lambda-\bar\lambda_0)[\varphi_A,\varphi_A]+(\lambda-\bar\lambda_0)(\lambda-\lambda_0)\left[(A-\lambda)^{-1}\varphi_A,\varphi_A\right]\end{aligned}$$

holds for all $\lambda \in \rho(A)$. Since $[E\varphi_A,(I-E)\varphi_A] = 0$ and the function

$$\lambda \mapsto \big[(A-\lambda)^{-1}(I-E)\varphi_A,(I-E)\varphi_A\big]$$

is holomorphic in a neighbourhood of the isolated eigenvalue μ we conclude that M_A can be written in the form

$$\begin{aligned} M_A(\lambda) &= h(\lambda) + (\lambda-\bar\lambda_0)(\lambda-\lambda_0)\big[(A-\lambda)^{-1}E\varphi_A,E\varphi_A\big] \\ &= h(\lambda) + \frac{(\lambda-\bar\lambda_0)(\lambda-\lambda_0)}{\mu-\lambda}\big[E\varphi_A,E\varphi_A\big], \end{aligned} \tag{2.7}$$

where h is holomorphic in a neighbourhood of the point μ. Here we have also used $(A-\lambda)^{-1}E\varphi_A = (\mu-\lambda)^{-1}E\varphi_A$ in the last equality.

Since by assumption $\mu \in \rho(B)$ we conclude from Corollary 2.2 (ii) that the function $M_B = -M_A^{-1}$ has a zero at the point μ, that is, M_A has a pole at μ. As h is holomorphic we obtain $[E\varphi_A,E\varphi_A] \neq 0$ from (2.7). Assume now that $\mu \in \sigma_{++}(A)$ ($\mu \in \sigma_{--}(A)$). Then $[E\varphi_A,E\varphi_A] > 0$ ($[E\varphi_A,E\varphi_A] < 0$, respectively) and the statements in Lemma 2.5 follow from the representation (2.7). □

The preceding Lemmas 2.4 and 2.5 lead to the following interlacing of eigenvalues of A and of B.

Proposition 2.6. *Let A, B and I be as in Assumption* (I). *Let $\mu_1,\mu_2 \in \rho(B)\cap I$ such that $(\mu_1,\mu_2) \subset \rho(A)$.*

(i) *If $\mu_1,\mu_2 \in \sigma_{++}(A)$, then there exists $\mu \in (\mu_1,\mu_2)$ with $\mu \in \sigma_p(B)\setminus\sigma_{--}(B)$.*
(ii) *If $\mu_1,\mu_2 \in \sigma_{--}(A)$, then there exists $\mu \in (\mu_1,\mu_2)$ with $\mu \in \sigma_p(B)\setminus\sigma_{++}(B)$.*

Proof. (i) The function M_A has poles of order one at μ_1, μ_2 and its behaviour near these poles is given by Lemma 2.5. Therefore, as M_A is a holomorphic function on $\rho(A)$, it is continuous on $(\mu_1,\mu_2) \subset \rho(A)$ and there exists $\mu \in (\mu_1,\mu_2)$ with $M_A(\mu) = 0$ and $M_A'(\mu) \geq 0$, hence (i) follows from Lemma 2.4. Statement (ii) is shown analogously. □

Corollary 2.2 (ii) states the following: If μ is an eigenvalue of A in $\rho(B)$ then the function M_A has a pole in μ. In the next proposition we prove the same conclusion under a slightly different assumption: If μ is an eigenvalue of A of positive or of negative type and μ is no eigenvalue of the symmetric operator $S = A\cap B$, then M_A has a pole in μ (and, moreover, μ belongs to the resolvent set of B).

Proposition 2.7. *Let A, B and I be as in Assumption* (I), *let $S = A\cap B$ and let $\mu \in I$. Then the following holds.*

(i) *If $\mu \in \sigma_{\pm\pm}(A)\backslash\sigma_p(S)$ then M_A has a pole of order one in μ and $\mu \in \rho(B)$.*
(ii) *If $\mu \in \sigma_{\pm\pm}(B)\backslash\sigma_p(S)$ then M_B has a pole of order one in μ and $\mu \in \rho(A)$.*

Proof. We verify assertion (i). The adjoint S^+ of $S = A\cap B$ is a closed linear relation with one dimensional multivalued part if $\operatorname{dom} S$ is not dense, or an operator otherwise. In both cases S^+ is a one dimensional extension of A and B, and in both cases we regard S^+ as a linear relation and denote the elements in S^+ in the form $\{f,f'\}$ where $f \in \operatorname{dom} S^+$ and $f' \in \operatorname{ran} S^+$. Let λ_0 be as in (2.2) and let $\varphi_A \in \mathscr{K}$ be as in Proposition 2.1 (i). By Proposition 2.1 (iii) we have for $y \in \mathscr{K}$

$$(A-\overline{\lambda}_0)^{-1}y - (B-\overline{\lambda}_0)^{-1}y = \frac{1}{M_A(\overline{\lambda}_0)}[y,\varphi_A]\gamma_A(\overline{\lambda}_0)$$

and the left hand side (and, hence the right hand side) is zero if and only if $y \in \operatorname{ran}(S - \overline{\lambda}_0)$. Thus $\varphi_A \in (\operatorname{ran}(S - \overline{\lambda}_0))^{[\perp]} = \ker(S^+ - \lambda_0)$ and we have the direct sum decomposition

$$S^+ = A \dotplus \{\alpha\{\varphi_A, \lambda_0\varphi_A\} : \alpha \in \mathbb{C}\}.$$

Accordingly we write $\{f, f'\} = \{f_A + \alpha\varphi_A, Af_A + \alpha\lambda_0\varphi_A\} \in S^+$ for some $f_A \in \operatorname{dom} A$. Suppose now that μ is an eigenvalue of positive or negative type of A such that $\mu \notin \sigma_p(S)$, let $g_\mu \in \ker(A - \mu)$ be nonzero and denote the orthogonal projection in $(\mathcal{K}, [\cdot,\cdot])$ onto the Hilbert (or anti-Hilbert) space $(\ker(A-\mu), [\cdot,\cdot])$ by P_μ. Since A is selfadjoint we obtain

$$\begin{aligned}
[f', g_\mu] - [f, Ag_\mu] &= [Af_A + \alpha\lambda_0\varphi_A, g_\mu] - [f_A + \alpha\varphi_A, Ag_\mu] \\
&= [\alpha\lambda_0\varphi_A, g_\mu] - [\alpha\varphi_A, \mu g_\mu] = \alpha(\lambda_0 - \mu)[P_\mu\varphi_A, g_\mu].
\end{aligned}$$

Hence

$$P_\mu\varphi_A \neq 0 \tag{2.8}$$

as otherwise $\{g_\mu, Ag_\mu\} \in S^{++} = S$ and $g_\mu \in \operatorname{dom} S$ and $Sg_\mu = \mu g_\mu$ which is impossible by $\mu \notin \sigma_p(S)$. On the other hand (see, e.g., [29, Proof of Theorem 1.1]), it follows for $\lambda \in \rho(A)$ from Proposition 2.1 (i)

$$\begin{aligned}
[(A-\lambda)^{-1}\varphi_A, \varphi_A] &= \frac{[\gamma_A(\lambda), \gamma_A(\lambda_0)] - [\varphi_A, \varphi_A]}{\lambda - \lambda_0} \\
&= \frac{M_A(\lambda)}{(\lambda-\lambda_0)(\lambda-\overline{\lambda}_0)} - \frac{M_A(\overline{\lambda}_0)}{(\lambda-\lambda_0)(\lambda-\overline{\lambda}_0)} - \frac{[\varphi_A, \varphi_A]}{\lambda - \lambda_0}.
\end{aligned}$$

Thus, if the function M_A admits an analytic continuation into the point μ, then by the above formula also the function $\lambda \mapsto [(A-\lambda)^{-1}\varphi_A, \varphi_A]$ admits an analytic continuation into μ and

$$[P_\mu\varphi_A, \varphi_A] = -\frac{1}{2\pi i}\int_{\mathscr{C}_\mu} [(A-\lambda)^{-1}\varphi_A, \varphi_A]\, d\lambda = 0,$$

where the above contour integral is along a sufficiently small circle $\mathscr{C}_\mu$ containing μ. As $(\ker(A-\mu), [\cdot,\cdot])$ is a Hilbert (or anti-Hilbert) space this implies $P_\mu\varphi_A = 0$; a contradiction to (2.8). Thus M_A can not be continued analytically into μ. As $\mu \in \sigma_{\pm\pm}(A)$, this pole is of order one.

The same reasoning applies to the first assertion in (ii). Hence every eigenvalue of positive or negative type of B which is not an eigenvalue of S is a pole of first order of M_B.

In order to complete the proof of (i) we have to show $\mu \in \rho(B)$. As $\mu \notin \sigma_p(S)$ the dimension of $\ker(B-\mu)$ is at most one. By the above reasoning M_A has a pole at μ, hence $M_B = -M_A^{-1}$ has a zero at μ. It then follows from the first assertion in (ii) that $\mu \notin \sigma_{\pm\pm}(B)$. Thus it remains to exclude the possibility of a neutral eigenvector of B corresponding to μ. In fact, if there is a neutral eigenvector there exists a Jordan chain of length greater than one which results in a pole of at least second order of the resolvent of B at μ. But as $\mu \in \sigma_{\pm\pm}(A)$ the resolvent of A, γ_A and, as shown above, also M_A have poles of first order at μ. Therefore by Proposition 2.1 (iii) the resolvent of B has a pole of at most first order at μ; a contradiction. We have shown $\mu \in \rho(B)$. □

3. Rank one perturbations of nonnegative operators and eigenvalue estimates

3.1. **Nonnegative operators, operators with one negative square, and related classes of functions.**

In this section we assume, in addition to (2.2), that A is nonnegative in the

Krein space $(\mathcal{K},[\cdot,\cdot])$, i.e.

$$[Ax,x] \geq 0, \qquad x \in \operatorname{dom} A.$$

This implies, in particular, that $\sigma(A) \subset \mathbb{R}$. From the fact that $A \cap B$ is a symmetric operator which is a one dimensional restriction of A and B it follows that B is nonnegative or B has one negative square, which is equivalent to $[Bx,x] < 0$ for some $x \neq 0$ in this setting. We shall write $\kappa_B = 0$ if B is nonnegative and $\kappa_B = 1$ if B has one negative square. Clearly, if $\kappa_B = 0$ then $\sigma(B) \subset \mathbb{R}$. If $\kappa_B = 1$ then the nonreal spectrum of B consists of at most one pair of isolated eigenvalues symmetric to the real line; cf. [16, 21].

The following proposition provides additional information on the sign types of the (isolated) spectral points of A and B; it is a special case of [16, Theorem 3.1], see also [51]. We remark that the assertions extend to all positive and negative spectral points when sign types are defined for points in the continuous spectrum as in [37, 53].

Proposition 3.1. *Let A, B be selfadjoint operators in* $(\mathcal{K},[\cdot,\cdot])$ *which satisfy* (2.2) *and assume that A is nonnegative. Then the following holds.*

- (i) *The isolated positive (negative) eigenvalues of A belong to* $\sigma_{++}(A)$ *(*$\sigma_{--}(A)$*, respectively).*
- (ii) *If* $\kappa_B = 0$ *then the isolated positive (negative) eigenvalues of B belong to* $\sigma_{++}(B)$ *(*$\sigma_{--}(B)$*, respectively).*
- (iii) *If* $\kappa_B = 1$ *then there is at most one isolated eigenvalue* $\mu \in \mathbb{R}$*,* $\mu \neq 0$*, such that* $\mu \notin \sigma_{++}(B) \cap \mathbb{R}^+$ *and* $\mu \notin \sigma_{--}(B) \cap \mathbb{R}^-$*.*

In the present situation the functions M_A and M_B in Proposition 2.1 belong to special classes of functions introduced and studied in [15, 16] and hence admit particular representations in terms of Nevanlinna and generalized Nevanlinna functions with one negative square. Recall first that a complex valued function N piecewise meromorphic in $\mathbb{C} \setminus \mathbb{R}$ and symmetric with respect to the real axis belongs to the class of generalized Nevanlinna functions $\mathcal{N}_\kappa$ with $\kappa \in \mathbb{N}_0$ negative squares if the kernel

$$\frac{N(z_i) - N(\bar{z}_j)}{z_i - \bar{z}_j}$$

has κ negative squares; cf. [47]. The class $\mathcal{N}_0$ is the class of Nevanlinna functions.

The following definition is taken from [15], see also [15, Theorem 2].

Definition 3.2. A complex valued function M meromorphic in $\mathbb{C} \setminus \mathbb{R}$ and symmetric with respect to the real axis belongs to the class $\mathcal{D}_\kappa$ if for some, and hence for every, z in the domain of holomorphy of M, there exists a generalized Nevanlinna function $N \in \mathcal{N}_\kappa$ holomorphic in z and a rational function g holomorphic in $\overline{\mathbb{C}} \setminus \{z, \bar{z}\}$ such that

$$\frac{\lambda}{(\lambda - z)(\lambda - \bar{z})} M(\lambda) = N(\lambda) + g(\lambda) \tag{3.1}$$

holds for all points λ where M, N and g are holomorphic. Here $\overline{\mathbb{C}}$ denotes the extended complex plane, $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

Proposition 3.3. *Let A, B be selfadjoint operators in the Krein space* $(\mathcal{K},[\cdot,\cdot])$ *which satisfy* (2.2)*, assume that A is nonnegative, and let* M_A *and* M_B *be as in Proposition 2.1. Then*

$$M_A \in \mathcal{D}_0 \qquad \text{and} \qquad M_B \in \mathcal{D}_0 \cup \mathcal{D}_1. \tag{3.2}$$

Furthermore, the following holds.

(i) *If $M_B \in \mathscr{D}_0$ then all positive (negative) zeros μ of M_A satisfy $M_A'(\mu) > 0$ ($M_A'(\mu) < 0$, respectively).*

(ii) *If $M_B \in \mathscr{D}_1$ then with the possible exception of at most one point μ_0 all positive zeros μ of M_A satisfy $M_A'(\mu) > 0$ and all negative zeros μ of M_A satisfy $M_A'(\mu) < 0$. If this exceptional zero μ_0 is in $\mathbb{R} \setminus \{0\}$, then it is a zero of M_A of at most order three. If it is a zero of order three then $M_A'''(\mu_0) > 0$ for $\mu_0 \in \mathbb{R}^+$ and $M_A'''(\mu_0) < 0$ for $\mu_0 \in \mathbb{R}^-$.*

(iii) *If there is a positive (negative) zero μ of M_A such that $M_A'(\mu) \leq 0$ ($M_A'(\mu) \geq 0$, respectively) then $M_B \in \mathscr{D}_1$.*

Proof. We obtain the assertions in (3.2) as a consequence of [15, Lemma 7] and the proof of Proposition 2.1. Since

$$M_B = -\frac{1}{M_A} \quad \text{on} \quad \rho(A) \cap \rho(B)$$

and $\rho(A) \cap \rho(B)$ is a dense subset in $\mathbb{C}$ (since $\sigma(A) \subset \mathbb{R}$ and with the possible exception of at most two points also $\sigma(B) \subset \mathbb{R}$) the zeros of M_A correspond to the poles of M_B and vice versa. The order of a zero of M_A is equal to the order of the corresponding pole of M_B. Moreover, if M_B has a pole of first order at μ then the residue at μ of M_B coincides with $\frac{-1}{M_A'(\mu)}$ and μ is a zero of first order of M_A.

By [15, Theorem 2 (iii)] all poles of $M_B \in \mathscr{D}_0$ in $\mathbb{R}^+$ ($\mathbb{R}^-$) are of first order with negative (positive, respectively) residue and (i) is shown. Assertion (ii) follows in the same way when taking into account that a function $M_B \in \mathscr{D}_1$ may have at most one pole which is not of first order with negative (positive) residue in $\mathbb{R}^+$ ($\mathbb{R}^-$, respectively), see [15, Theorem 2 (iii)]. Moreover, it also follows from [15, Theorem 2 (iii)] that this exceptional pole μ_0 is of at most order three and that the limit

$$\lim_{\lambda \to \mu_0} (\lambda - \mu_0)^3 M_B(\lambda)$$

exists and is nonpositive (nonnegative) if μ_0 is in $\mathbb{R}^+$ ($\mathbb{R}^-$, respectively). This shows (ii). Finally, if μ is a positive (negative) zero of M_A with $M_A'(\mu) \leq 0$ ($M_A'(\mu) \geq 0$, respectively) then M_B has a pole in μ which is not of first order with a negative (positive, respectively) residue in $\mathbb{R}^+$ ($\mathbb{R}^-$, respectively). As $M_B \in \mathscr{D}_0 \cup \mathscr{D}_1$ by (3.2) we conclude $M_B \in \mathscr{D}_1$ from [15, Theorem 2 (iii)]. □

The next lemma provides some more properties of the function M_A at the point 0.

Lemma 3.4. *Let the assumptions be as in Proposition 3.3. Then the following holds.*

(i) *If 0 is a pole of M_A then 0 is a pole of first or of second order. If 0 is a pole of second order then*

$$\lim_{\lambda \nearrow 0} M_A(\lambda) = \lim_{\lambda \searrow 0} M_A(\lambda) = -\infty.$$

(ii) *If $M_B \in \mathscr{D}_1$ and M_A is holomorphic in 0 then*

$$M_A(0) > 0.$$

(iii) *Assume that M_A is holomorphic in 0 and let 0 be a zero of M_A. Then 0 is a zero of at most second order and in this case we have*

$$M_A''(0) > 0.$$

Proof. (i) Let 0 be a pole of M_A. As $M_A \in \mathscr{D}_0$ it follows from [15, Definition 3 and Theorem 2 (iii)] that 0 is either a point of holomorphy or a pole of first order with a negative residue at 0 of the function $\lambda \mapsto \lambda M_A(\lambda)$. Therefore 0 is a pole of at most order two of M_A and, if 0 is a pole of second order of M_A, it satisfies

$$-\infty < \lim_{\lambda \to 0} \lambda^2 M_A(\lambda) < 0$$

and (i) is proved.

(ii) If $M_B \in \mathscr{D}_1$ then [16, Theorem 2.4] implies that 0 is not a generalized zero of nonpositive type of $\lambda \mapsto \lambda M_A(\lambda)$. For the notion of a generalized zero of nonpositive type we refer to [48, 52], see also [15, Section 3.1]. Under the assumption that M_A is holomorphic in 0, this is equivalent to (ii), see, e.g., [15, Section 3.1] and [48, 52].

(iii) Consider (3.1) with $z = 0$,

$$\lambda^{-1} M_A(\lambda) = N_A(\lambda) + g_A(\lambda), \tag{3.3}$$

where N_A is a Nevanlinna function holomorphic in 0 and g_A is a rational function holomorphic in the extended complex plane with a possible pole in 0. Assume

$$M_A(0) = M_A'(0) = 0. \tag{3.4}$$

Then the left hand side of (3.3) is holomorphic in 0 and hence g_A is equal to a real constant c, and (3.3) becomes

$$M_A(\lambda) = \lambda\,(N_A(\lambda) + c)\,. \tag{3.5}$$

We have $M_A'(\lambda) = N_A(\lambda) + c + \lambda N_A'(\lambda)$ and $M_A''(\lambda) = 2N_A'(\lambda) + \lambda N_A''(\lambda)$. In particular

$$M_A'(0) = N_A(0) + c \quad \text{and} \quad M_A''(0) = 2N_A'(0).$$

It follows from (3.4) that the function $N_A + c$ vanishes at 0. It is well-known that nonconstant Nevanlinna functions have a positive derivative in real points of holomorphy. Here, $N_A + c$ is not identically zero, as this would, by (3.5), imply that $M_A \equiv 0$, which is a contradiction to Proposition 2.1 (iii). We conclude

$$M_A''(0) = 2(N_A + c)'(0) > 0,$$

and hence 0 is a zero of at most second order of M_A. □

3.2. **Main results: Eigenvalue estimates.** For an interval $I \subset \mathbb{R}$ we denote the numbers of distinct eigenvalues of A and B in I by $n_A(I)$ and $n_B(I)$, respectively,

$$n_A(I) = \sharp\{\lambda : \lambda \in I \cap \sigma_p(A)\} \quad \text{and} \quad n_B(I) = \sharp\{\lambda : \lambda \in I \cap \sigma_p(B)\},$$

and we set

$$n_{A,B}(I) = \sharp\{\lambda : \lambda \in I \cap \sigma_p(A) \cap \sigma_p(B)\}.$$

Here, multiplicities of eigenvalues are not counted.

The next theorem provides sharp estimates from below and above on the number of distinct eigenvalues of B in terms of the number of distinct eigenvalues of A. The last assertion on the infinite number of distinct eigenvalues of A and B in I can be viewed as a special case of [14, Theorem 4.3].

Theorem 3.5. *Let A, B and I be as in Assumption* (I) *and assume, in addition, that A is nonnegative. Then B is nonnegative or has one negative square and if $n_A(I) < \infty$ then the following estimates hold.*

(i) *If* $0 \notin I$ *then*

$$n_A(I) - n_{A,B}(I) - 1 \le n_B(I) \le n_A(I) + n_{A,B}(I) + \begin{cases} 1 & \text{if } \kappa_B = 0, \\ 3 & \text{if } \kappa_B = 1. \end{cases}$$

(ii) *If* $0 \in I$ *then*

$$n_A(I) - n_{A,B}(I) - 2 \le n_B(I) \le n_A(I) + n_{A,B}(I) + \begin{cases} 2 & \text{if } \kappa_B = 0, \\ 3 & \text{if } \kappa_B = 1. \end{cases}$$

Each of the estimates in (i) *and* (ii) *is sharp. Moreover,* $n_A(I) = \infty$ *if and only if* $n_B(I) = \infty$.

The upper and lower estimates in the next corollary follow from $n_{A,B}(I) \le n_A(I)$ and $-n_B(I) \le -n_{A,B}(I)$, respectively.

Corollary 3.6. *Let the assumptions be as in Theorem 3.5. Then the following estimates hold.*

(i) *If* $0 \notin I$ *then*

$$\frac{n_A(I) - 1}{2} \le n_B(I) \le 2n_A(I) + \begin{cases} 1 & \text{if } \kappa_B = 0, \\ 3 & \text{if } \kappa_B = 1. \end{cases}$$

(ii) *If* $0 \in I$ *then*

$$\frac{n_A(I) - 2}{2} \le n_B(I) \le 2n_A(I) + \begin{cases} 2 & \text{if } \kappa_B = 0, \\ 3 & \text{if } \kappa_B = 1. \end{cases}$$

Each of the estimates in (i) *and* (ii) *is sharp.*

The next corollary treats the case $n_{A,B}(I) = 0$ and will play an important role in the proof of Theorem 3.9.

Corollary 3.7. *Let the assumptions be as in Theorem 3.5 and assume, in addition, that* $I \cap \sigma_p(A) \cap \sigma_p(B) = \emptyset$*. Then the following estimates hold.*

(i) *If* $0 \notin I$ *then*

$$n_A(I) - 1 \le n_B(I) \le n_A(I) + \begin{cases} 1 & \text{if } \kappa_B = 0, \\ 3 & \text{if } \kappa_B = 1. \end{cases}$$

(ii) *If* $0 \in I$ *then*

$$n_A(I) - 2 \le n_B(I) \le n_A(I) + \begin{cases} 2 & \text{if } \kappa_B = 0, \\ 3 & \text{if } \kappa_B = 1. \end{cases}$$

Each of the estimates in (i) *and* (ii) *is sharp.*

In the following we provide in Theorem 3.9 a variant of Theorem 3.5, where the total multiplicity $m_B(I)$ of the eigenvalues of B in I is estimated by the total multiplicity $m_A(I)$ of the eigenvalues of A in I. We start by stating a theorem which focuses on the total multiplicity of the eigenvalue 0.

Theorem 3.8. *Let A, B and I be as in Assumption* (I) *and assume, in addition, that A is nonnegative,* $0 \in I$ *and that* $m_A(\{0\}) < \infty$*. Then*

$$|m_A(\{0\}) - m_B(\{0\})| \le 2$$

and the estimate is sharp.

The sharp estimate in Theorem 3.8 will be used in the proof of the next theorem.

Theorem 3.9. *Let A, B and I be as in Assumption* (I) *and assume, in addition, that A is nonnegative and that* $m_A(I) < \infty$. *Then the following estimates hold.*

(i) *If* $0 \notin I$ *then*

$$m_A(I) - 1 \leq m_B(I) \leq m_A(I) + \begin{cases} 1 & \text{if } \kappa_B = 0, \\ 3 & \text{if } \kappa_B = 1. \end{cases}$$

(ii) *If* $0 \in I$ *and* $0 \notin \sigma_p(A)$ *then*

$$m_A(I) - 2 \leq m_B(I) \leq m_A(I) + \begin{cases} 2 & \text{if } \kappa_B = 0, \\ 3 & \text{if } \kappa_B = 1. \end{cases}$$

(iii) *If* $0 \in I$ *and* $0 \in \sigma_p(A)$ *then*

$$m_A(I) - 4 \leq m_B(I) \leq m_A(I) + \begin{cases} 4 & \text{if } \kappa_B = 0, \\ 6 & \text{if } \kappa_B = 1. \end{cases}$$

Moreover, $m_A(I) = \infty$ *if and only if* $m_B(I) = \infty$.

Remark 3.10. It follows immediately from Corollary 3.7 that the estimates in Theorem 3.9 (i) and (ii) are sharp. It is not clear if estimate (iii) is sharp as well.

In the following subsections the proofs of Theorems 3.5, 3.8 and 3.9 will be given. The proofs of Theorems 3.5 and 3.9 make use of similar techniques and are related; they are presented in Sections 3.3 and 3.4. The proof of Theorem 3.8 is independent from the proofs of Theorems 3.5 and 3.9, and therefore postponed to Section 3.5.

3.3. **Proof of Theorem 3.5.** Theorem 3.5 is proved in eight separate steps, the proof of Theorem 3.9 is given afterwards. In Steps 1 and 2 the lower estimates are shown and in Steps 3 - 5 the upper estimates are verified. The sharpness of the estimates is shown in Steps 6 and 7 for two particularly interesting situations; from the construction it is clear how the sharpness of the remaining estimates follows. Finally, in Step 8 we verify the assertion on the infiniteness of the eigenvalues.

Step 1. Lower estimate in (i). We verify the estimate

$$n_A(I) - n_{A,B}(I) - 1 \leq n_B(I). \tag{3.6}$$

By assumption $0 \notin I$ and we have $I \subset \mathbb{R}^+$ or $I \subset \mathbb{R}^-$. We discuss the case $I \subset \mathbb{R}^+$ only; the simple modifications for the case $I \subset \mathbb{R}^-$ are left to the reader. Then, as A is nonnegative, all eigenvalues of A in I are of positive type, that is $\sigma(A) \cap I \subset \sigma_{++}(A)$; cf. Proposition 3.1 (i). As $n_A(I) < \infty$ we have $n_{A,B}(I) < \infty$. If $n_A(I) - 1 - n_{A,B}(I) \leq n_{A,B}(I)$ then the estimate (3.6) holds since $n_{A,B}(I) \leq n_B(I)$. If $n_A(I) - 1 - n_{A,B}(I) > n_{A,B}(I)$ then there exist at least $n_A(I) - 1 - 2n_{A,B}(I)$ pairs of eigenvalues in $\sigma_{++}(A) \cap \rho(B)$ to which Proposition 2.6 (i) can be applied. This leads to $n_A(I) - 1 - 2n_{A,B}(I)$ eigenvalues of B in $\rho(A) \cap I$ and since there are also $n_{A,B}(I)$ eigenvalues of B in $\sigma(A) \cap I$ we obtain the estimate (3.6).

Step 2. Lower estimate in (ii). Let $0 \in I$ and set $I_\pm = I \cap \mathbb{R}^\pm$. In order to show the estimate

$$n_A(I) - n_{A,B}(I) - 2 \le n_B(I) \tag{3.7}$$

observe that by Step 1 the estimates

$$n_A(I_\pm) - n_{A,B}(I_\pm) - 1 \le n_B(I_\pm) \tag{3.8}$$

hold. Clearly,

$$n_A(I_+) + n_A(I_-) = \begin{cases} n_A(I) & \text{if } 0 \notin \sigma_p(A), \\ n_A(I) - 1 & \text{if } 0 \in \sigma_p(A) \end{cases}$$

and

$$n_{A,B}(I_+) + n_{A,B}(I_-) = \begin{cases} n_{A,B}(I) & \text{if } 0 \notin \sigma_p(A) \cap \sigma_p(B), \\ n_{A,B}(I) - 1 & \text{if } 0 \in \sigma_p(A) \cap \sigma_p(B). \end{cases}$$

Together with (3.8) this yields

$$\begin{aligned} n_B(I) &= \begin{cases} n_B(I_+) + n_B(I_-) & \text{if } 0 \notin \sigma_p(B), \\ n_B(I_+) + n_B(I_-) + 1 & \text{if } 0 \in \sigma_p(B), \end{cases} \\ &\ge \begin{cases} n_A(I_+) - n_{A,B}(I_+) + n_A(I_-) - n_{A,B}(I_-) - 2 & \text{if } 0 \notin \sigma_p(B), \\ n_A(I_+) - n_{A,B}(I_+) + n_A(I_-) - n_{A,B}(I_-) - 1 & \text{if } 0 \in \sigma_p(B), \end{cases} \\ &= \begin{cases} n_A(I) - n_{A,B}(I) - 2 & \text{if } 0 \notin \sigma_p(B),\ 0 \notin \sigma_p(A), \\ n_A(I) - n_{A,B}(I) - 3 & \text{if } 0 \notin \sigma_p(B),\ 0 \in \sigma_p(A), \\ n_A(I) - n_{A,B}(I) - 1 & \text{if } 0 \in \sigma_p(B),\ 0 \notin \sigma_p(A), \\ n_A(I) - n_{A,B}(I) - 1 & \text{if } 0 \in \sigma_p(B),\ 0 \in \sigma_p(A). \end{cases} \end{aligned}$$

It remains to show estimate (3.7) in the case $0 \in \sigma_p(A)$ and $0 \notin \sigma_p(B)$. Assume first that $I_- \cap \sigma(A)$ is empty. Then $n_B(I_-) \ge 0$, $n_A(I_+) = n_A(I) - 1$, and (3.8) yield

$$n_B(I) \ge n_B(I_+) \ge n_A(I_+) - n_{A,B}(I_+) - 1 = n_A(I) - n_{A,B}(I) - 2,$$

that is, (3.7) holds. A similar reasoning implies (3.7) for the case that $I_+ \cap \sigma(A)$ is empty. Now we assume $I_\pm \cap \sigma(A) \neq \emptyset$. Denote by λ_- the largest eigenvalue of A in I_- and by λ_+ the smallest eigenvalue of A in I_+. Assume first $\lambda_- \in \sigma_p(B)$ and apply the lower estimate from Step 1 to the intervals $I_{\lambda_-} := (-\infty, \lambda_-) \cap I_-$ and I_+:

$$\begin{aligned} n_B(I) &= n_B(I_{\lambda_-}) + n_B([\lambda_-, 0]) + n_B(I_+) \\ &\ge n_A(I_{\lambda_-}) - n_{A,B}(I_{\lambda_-}) - 1 + n_B([\lambda_-, 0]) + n_A(I_+) - n_{A,B}(I_+) - 1 \\ &= n_A(I_{\lambda_-}) + n_A(I_+) - \big(n_{A,B}(I_{\lambda_-}) + n_{A,B}(I_+)\big) + n_B([\lambda_-, 0]) - 2. \end{aligned}$$

In the present situation we have

$$\begin{aligned} n_A(I) &= n_A(I_{\lambda_-}) + n_A([\lambda_-, 0]) + n_A(I_+) = n_A(I_{\lambda_-}) + 2 + n_A(I_+) \\ n_{A,B}(I) &= n_{A,B}(I_{\lambda_-}) + n_{A,B}([\lambda_-, 0]) + n_{A,B}(I_+) = n_{A,B}(I_{\lambda_-}) + 1 + n_{A,B}(I_+) \end{aligned}$$

and hence we obtain

$$\begin{aligned} n_B(I) &\ge n_A(I) - 2 - \big(n_{A,B}(I) - 1\big) + n_B([\lambda_-, 0]) - 2 \\ &= n_A(I) - n_{A,B}(I) + n_B([\lambda_-, 0]) - 3. \end{aligned}$$

Together with $n_B([\lambda_-, 0]) \ge 1$ we conclude (3.7). In a similar way the estimate (3.7) follows if $\lambda_+ \in \sigma_p(B)$. Thus it remains to show (3.7) for $0 \in \sigma_p(A)$, $0 \notin \sigma_p(B)$, and $\lambda_\pm \notin \sigma_p(B)$. For this we consider the function $M_A : \rho(A) \to \mathbb{C}$ from Proposition 2.1 which is continuous and real valued on $\rho(A)\cap\mathbb{R}$. By Corollary 2.2 (ii) the point 0 is a pole of M_A and by Lemma 3.4 (i) it is of first or of second order. If 0 is a pole of first order we conclude from $\lambda_-\in\sigma_{--}(A)$, $\lambda_+\in\sigma_{++}(A)$, and Lemma 2.5 that M_A has a zero either in $(\lambda_-,0)$ or in $(0,\lambda_+)$, and hence an eigenvalue of B; cf. Corollary 2.2 (i). If 0 is a pole of second order, then M_A has zeros (and, hence, eigenvalues of B) in both intervals $(\lambda_-,0)$ and $(0,\lambda_+)$; cf. Lemma 3.4 (i), Corollary 2.2 (i), and Lemma 2.5. Thus in both cases there is at least one eigenvalue of B in the interval (λ_-,λ_+). Therefore, for $\varepsilon>0$ sufficiently small we conclude

$$n_B([\lambda_-+\varepsilon,\lambda_+-\varepsilon])\geq 1,\qquad \lambda_-+\varepsilon<0<\lambda_+-\varepsilon. \tag{3.9}$$

Let us apply the lower estimate from Step 1 to $I_{\lambda_-+\varepsilon}=(-\infty,\lambda_-+\varepsilon)\cap I_-$ and $I_{\lambda_+-\varepsilon}=(\lambda_+-\varepsilon,\infty)\cap I_+$. Then, with (3.9) we obtain

$$\begin{aligned}
n_B(I) &= n_B(I_{\lambda_-+\varepsilon})+n_B([\lambda_-+\varepsilon,\lambda_+-\varepsilon])+n_B(I_{\lambda_+-\varepsilon})\\
&\geq n_A(I_{\lambda_-+\varepsilon})-n_{A,B}(I_{\lambda_-+\varepsilon})-1+n_B([\lambda_-+\varepsilon,\lambda_+-\varepsilon]\\
&\quad +n_A(I_{\lambda_+-\varepsilon})-n_{A,B}(I_{\lambda_+-\varepsilon})-1\\
&\geq n_A(I_{\lambda_-+\varepsilon})-n_{A,B}(I_{\lambda_-+\varepsilon})+n_A(I_{\lambda_+-\varepsilon})-n_{A,B}(I_{\lambda_+-\varepsilon})-1\\
&= n_A(I)-n_A([\lambda_-+\varepsilon,\lambda_+-\varepsilon])-(n_{A,B}(I)-n_{A,B}([\lambda_-+\varepsilon,\lambda_+-\varepsilon]))-1.
\end{aligned}$$

In the present setting we have $n_A([\lambda_-+\varepsilon,\lambda_+-\varepsilon])=1$ and $n_{A,B}([\lambda_-+\varepsilon,\lambda_+-\varepsilon])=0$. This implies the estimate (3.7).

Step 3. Upper estimate in (i) *and* (ii) *if* $\kappa_B=0$. If B is nonnegative these two estimates follow immediately from (3.6) and (3.7) by interchanging the roles of A and B.

Step 4. Upper estimate in (i) *if* $\kappa_B=1$. We show that the inequality

$$n_B(I)\leq n_A(I)+n_{A,B}(I)+3 \tag{3.10}$$

holds if $0\notin I$ and B has one negative square. Let us again discuss the case $I\subset\mathbb{R}^+$ only; the simple modifications for the case $I\subset\mathbb{R}^-$ are left to the reader. Since $I\cap\sigma(A)$ consists of $n_A(I)$ distinct eigenvalues the set $I\cap\rho(A)$ consists of $n_A(I)+1$ open subintervals I_k, $1\leq k\leq n_A(I)+1$. We use that M_A is continuous and real valued on each subinterval I_k, and that by Corollary 2.2 (i) the zeros of M_A in I_k coincide with the eigenvalues of B in I_k. As $\kappa_B=1$ there is at most one point $\nu\in\sigma_p(B)\cap I$ with $\nu\notin\sigma_{++}(B)$ by Proposition 3.1 (iii). If $\nu\in\sigma_p(A)$ then $I_k\cap\sigma(B)$, $1\leq k\leq n_A(I)+1$, is contained in $\sigma_{++}(B)$ according to Proposition 3.1 (iii) and each zero μ in I_k of M_A satisfies $M_A'(\mu)>0$ by Lemma 2.4 (i). Thus in each subinterval I_k, $1\leq k\leq n_A(I)+1$, there is at most one eigenvalue of B so that the set $I\cap\rho(A)$ contains at most $n_A(I)+1$ eigenvalues of B. Clearly, the set $I\cap\sigma(A)$ contains $n_{A,B}(I)$ eigenvalues of B and hence $n_B(I)\leq n_A(I)+n_{A,B}(I)+1$. In particular, (3.10) follows in the case $\nu\in\sigma_p(A)$. It remains to show estimate (3.10) in the case $\nu\in\rho(A)$. Then ν belongs to some subinterval I_j for some j with $1\leq j\leq n_A(I)+1$ and the function M_A satisfies $M_A'(\nu)\leq 0$ by Lemma 2.4 (i). Since all other eigenvalues μ of B in $I\cap\rho(A)$ belong to $\sigma_{++}(B)$ it follows from Lemma 2.4 (i) that $M_A'(\mu)>0$. Hence in I_j there are at most three eigenvalues of B and in each of the subintervals I_k, $1\leq k\leq n_A(I)+1$, $k\neq j$, there is at most one eigenvalue of B. Summing up it follows that the set $I\cap\rho(A)$ contains at most $n_A(I)+3$ eigenvalues and, as $I\cap\sigma(A)$ contains $n_{A,B}(I)$ eigenvalues of B, (3.10) is shown.

Step 5. Upper estimate in (ii) *if* $\kappa_B = 1$. In this step we discuss the case $0 \in I$ and B has one negative square. We verify the inequality

$$n_B(I) \le n_A(I) + n_{A,B}(I) + 3. \tag{3.11}$$

In order to show this we consider again the open subintervals I_k, $1 \le k \le n_A(I)+1$, as in Step 4. Assume that $0 \in \sigma_p(A)$. Then the arguments used in the proof of Step 4 remain valid and it follows that in at most one interval I_j there might be at most three zeros of M_A, in all other intervals I_k there is at most one zero. This implies (3.11) if $0 \in \sigma_p(A)$. Let us now discuss the case $0 \in \rho(A)$ so that $0 \in I_j$ for some j. If M_A has two or three zeros in one of the other subintervals I_k, $k \neq j$, then according to Lemma 2.4 (i)-(ii) one of these zeros is an eigenvalue μ of B which does not belong to $\sigma_{++}(B)$ $(\sigma_{--}(B))$ if $I_k \subset \mathbb{R}^+$ $(I_k \subset \mathbb{R}^-$, respectively). Moreover, by Proposition 3.3 (iii) the function M_B belongs to the class $\mathscr{D}_1$ and by Lemma 3.4 (ii) we have $M_A(0) > 0$. But this implies that there are no zeros of M_A in I_j as otherwise $M_A'(\mu_-) \ge 0$ for some $\mu_- < 0$ in I_j or $M_A'(\mu_+) \le 0$ for some $\mu_+ > 0$ in I_j which is impossible by Proposition 3.3 (ii). Hence if $0 \in I_j$ and M_A has two or three zeros in one of the other subintervals I_k then (3.11) is valid. It remains to discuss the case $0 \in I_j$ and M_A has at most one zero in each of the other subintervals I_k, $k \neq j$. Suppose that $M_A(0) > 0$. By Proposition 3.3 (i) and (ii) there are at most two zeros of M_A in I_j and (3.11) is true for $M_A(0) > 0$. In the case $M_A(0) = 0$ three other zeros in I_j would imply $M_B \in \mathscr{D}_1$ by Proposition 3.3 (iii) and hence $M_A(0) > 0$ by Lemma 3.4 (ii). Thus only two zeros in $I_j \backslash \{0\}$ may exist and (3.11) holds also in the case $M_A(0) = 0$. Finally, if $M_A(0) < 0$ then again three zeros in I_j would imply $M_B \in \mathscr{D}_1$ by Proposition 3.3 (iii) and hence $M_A(0) > 0$ by Lemma 3.4 (ii). Thus also in this case there are at most two zeros of M_A in I_j. We have proved (3.11).

Step 6. Sharpness of the upper estimate in (i) *if* $\kappa_B = 1$. We discuss the case $0 \notin I$. Our aim is to show that the estimate

$$n_B(I) \le n_A(I) + n_{A,B}(I) + 3 \tag{3.12}$$

is sharp. For this we show that there exist matrices A, B and an open interval I such that Assumption (I) is satisfied and equality holds in (3.12). Here we give an idea how to construct specific examples fitting to a given eigenvalue distribution. For explicit examples, see Section 3.6. Let $0 < \lambda_0 < \lambda_1 < \cdots < \lambda_n < \lambda_{n+1}$ for some $n \in \mathbb{N}$ and define $I := (\lambda_0, \lambda_{n+1})$. Choose a rational function M symmetric with respect to the real axis such that:

- M has poles of first order in 0 and in each λ_i. These are the only poles of M and M is monotonously increasing in every interval $(\lambda_1, \lambda_2), \ldots, (\lambda_n, \lambda_{n+1})$.
- M has three zeros $\mu_1 < \mu_2 < \mu_3$ in the interval (λ_0, λ_1) such that $M'(\mu_1) > 0$, $M'(\mu_2) < 0$, and $M'(\mu_3) > 0$.
- $\lim_{x \to \pm\infty} M(x) \in \mathbb{R} \backslash \{0\}$.
- $M \in \mathscr{D}_0$ and the function $\lambda \mapsto -\frac{1}{M(\lambda)}$ belongs to $\mathscr{D}_1$.

We leave it to the reader to verify that such functions exist. An example for $n = 0$ is the function M_1 in Figure 1 in Section 3.6.

Then M belongs to the class of generalized Nevanlinna functions and according to [10, Corollary 3.5] there exists a Pontryagin space $(\mathscr{K}, [\cdot,\cdot])$, a (possibly nondensely defined) symmetric operator S with defect one and a boundary triplet $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ for the adjoint S^+ such that the corresponding Weyl function coincides with M. Let $A := S^+ \upharpoonright \ker \Gamma_0$. The operator S and the boundary triplet $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ can be chosen in such a way that $\mathscr{K}$ is finite dimensional, $\sigma(A)$ coincides with the poles of M and, in particular, A has no multivalued part as M has no pole at $\pm\infty$, see also [35, 47]. It is important to note that $\sigma(A) \cap I$ consists of the n distinct eigenvalues $\lambda_1,\dots,\lambda_n$. As in the proof of Proposition 2.1 we make use of the fact that $\{\mathbb{C},\Gamma_1,-\Gamma_0\}$ is a boundary triple for S^+ with Weyl function $-M^{-1}$. Let $B := S^+ \upharpoonright \ker\Gamma_1$. Then B is a selfadjoint matrix with $\kappa_B = 1$ (see, e.g. [15, Lemma 7]). As both A and B are selfadjoint extensions of the symmetric (nondensely defined) matrix S with defect one the difference of A and B and of their resolvents is a rank one operator, so that Assumption (I) is satisfied. Moreover, the zeros of M in I coincide with $\sigma(B)\cap I$. Hence B has 3 eigenvalues in the interval (λ_0,λ_1) and one eigenvalue in each of the n intervals $(\lambda_1,\lambda_2),\dots,(\lambda_n,\lambda_{n+1})$, that is, $n_B(I) = n+3$ and equality in (3.12) is shown for the case $n_{A,B}(I) = 0$. In order to obtain a sharp estimate in the remaining cases add orthogonally to A and B a nonnegative matrix C such that $\sigma_p(C) \subset \sigma_p(A)$. Then,

$$\begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix} \tag{3.13}$$

differ by a rank one matrix and have $n_C(I)$ common eigenvalues in the interval I. This shows that (3.12) is sharp.

Step 7. Sharpness of the lower estimate in (ii). In order to show that for $0 \in I$ the estimate

$$n_A(I) - n_{A,B}(I) - 2 \le n_B(I) \tag{3.14}$$

is sharp let $\lambda_0 < 0 < \lambda_1 < \cdots < \lambda_n$ with $n \in \mathbb{N}$ and consider a rational function M such that:

- M has poles of first order in each λ_i. These are the only poles of M and M is monotonously increasing in every interval $(\lambda_1,\lambda_2),\dots,(\lambda_{n-1},\lambda_n)$.
- M is positive in the interval (λ_0,λ_1).
- $\lim_{x\to\pm\infty} M(x) \in \mathbb{R}\backslash\{0\}$ and $M \in \mathscr{D}_0$.

An example for such a function in the case $n = 2$ is given by $M(\lambda) := M_2(\lambda)+2$, where M_2 is the function in Figure 2 in Section 3.6.

The zeros of M in $(\lambda_j,\lambda_{j+1})$, $j = 1,\dots,n-1$, are denoted by μ_j. As above it follows that there exists a Pontryagin space and selfadjoint matrices A and B which differ by a rank one matrix such that λ_i, $i = 0,\dots,n$, are eigenvalues of A and μ_j, $j = 1,\dots,n-1$, are eigenvalues of B. Hence for $\varepsilon > 0$ sufficiently small A has $n+1$ distinct eigenvalues in the interval $I = (\lambda_0-\varepsilon,\lambda_n+\varepsilon)$ and B has $n-1$ eigenvalues in I, that is, (3.14) is sharp if $n_{A,B}(I) = 0$. In the case $n_{A,B}(I) > 0$ one obtains that (3.14) is sharp by adding orthogonally a suitable nonnegative matrix C as in (3.13).

Step 8. Proof of $n_A(I) = \infty$ if and only if $n_B(I) = \infty$. If $n_{A,B}(I) = \infty$ then $n_B(I) = \infty = n_A(I)$ and the assertion is true. If $n_A(I) = \infty$ and $n_{A,B}(I) < \infty$ then there are infinitely many pairs of eigenvalues in $\sigma_{++}(A)$ or $\sigma_{--}(A)$ to which Proposition 2.6 (i) or (ii) can be applied. This yields $n_B(I) = \infty$. Conversely, if $n_B(I) = \infty$ then the same reasoning implies $n_A(I) = \infty$ and the assertion is proved.

3.4. **Proof of Theorem 3.9.** The proof of Theorem 3.9 uses Corollary 3.7 and is done in eleven steps. We decompose the space $\mathscr{K}$ into the spectral subspace related to the common eigenvalues of A and B and its $[\cdot,\cdot]$-orthogonal companion. Then Corollary 3.7 can be applied to the restrictions of A and B to this $[\cdot,\cdot]$-orthogonal companion and we prove the estimates in (i), (ii) and (iii).

Step 1. Decomposition of $\mathscr{K}$ *for* $0 \notin I$. Let us assume that $I \subset \mathbb{R}^+$. The spectral subspace of A corresponding to I is an $m_A(I)$-dimensional Hilbert space by Proposition 3.1 (i). The subspace $\mathscr{E}_+$ spanned by the eigenvectors of the (possibly nondensely defined) symmetric operator $S = A \cap B$ in I is invariant for S, and hence for A and B. As $\mathscr{E}_+$ is a subset of the spectral subspace of A corresponding to I, the space $(\mathscr{E}_+, [\cdot,\cdot])$ is a (finite dimensional) Hilbert space. Denote the restriction of S to $\mathscr{E}_+$ by S_+. With respect to the decomposition $\mathscr{K} = \mathscr{E}_+ [\dot{+}] \mathscr{E}_+^{[\perp]}$ we have

$$S = \begin{pmatrix} S_+ & 0 \\ 0 & S' \end{pmatrix}, \quad A = \begin{pmatrix} S_+ & 0 \\ 0 & A' \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} S_+ & 0 \\ 0 & B' \end{pmatrix},$$

with S' symmetric, $\sigma_p(S') \cap I = \emptyset$, and A' and B' selfadjoint in the Krein space $(\mathscr{E}_+^{[\perp]}, [\cdot,\cdot])$. Therefore

$$m_A(I) = m_{S_+}(I) + m_{A'}(I) \quad \text{and} \quad m_B(I) = m_{S_+}(I) + m_{B'}(I). \tag{3.15}$$

We claim that A' and B' satisfy the assumptions in Corollary 3.7. Indeed, it is easy to see that A', B' and I satisfy Assumption (I) and since A is nonnegative in the Krein space $\mathscr{K}$ the operator A' is nonnegative in the Krein space $\mathscr{E}_+^{[\perp]}$. Furthermore, as $\sigma_p(S') \cap I = \emptyset$ and all eigenvalues of A' in I are in $\sigma_{++}(A')$ by Proposition 3.1 (i), we conclude from Proposition 2.7 (i) that

$$\sigma_p(A') \cap \sigma_p(B') \cap I = \emptyset. \tag{3.16}$$

Step 2. Lower estimate in (i). As $I \subset \mathbb{R}^+$, all eigenvalues of the nonnegative operator A' in I are of positive type and belong to $\rho(B')$. According to Theorem 2.3 (iv) each of these eigenvalues is of multiplicity one and therefore

$$n_{A'}(I) = m_{A'}(I). \tag{3.17}$$

As $n_{B'}(I) \le m_{B'}(I)$, Corollary 3.7 (i) together with (3.15) imply the estimate

$$m_A(I) - 1 \le m_B(I). \tag{3.18}$$

Step 3. Upper estimate in (i) *if* $\kappa_B = 0$. The estimate follows immediately from (3.18) by interchanging the roles of A and B.

Step 4. Upper estimate in (i) *if* $\kappa_B = 1$. In this case $\kappa_{B'} = 1$ and by Proposition 3.1 (iii) there is at most one eigenvalue μ of B' in I which is not of positive type. If μ is of negative type it has multiplicity one; cf. Theorem 2.3 (iii). All other eigenvalues of B' in I are of positive type, belong to $\rho(A')$ and hence have multiplicity one according to Proposition 3.1 (iii) and Theorem 2.3 (iii). Therefore $n_{B'}(I) = m_{B'}(I)$ and as $n_{A'}(I) \le m_{A'}(I)$, Corollary 3.7 (i) together with (3.15) imply the estimate

$$m_B(I) \le m_A(I) + 3. \tag{3.19}$$

It remains to show (3.19) in the case that $\mu \in \sigma_p(B') \cap I$ is not of positive and not of negative type, that is, there exists a neutral eigenvector x_0. Then by Lemma 2.4 $\dim\ker(B' - \mu) = 1$ and the multiplicity of μ is larger than one. On the other hand it follows from [51] (see also [16, Theorem 3.1 (ii)]) that the multiplicity of μ is at most 3. We discuss the cases $\dim \mathscr{L}_\mu(B') = 2$ and $\dim \mathscr{L}_\mu(B') = 3$ separately.

If $\dim \mathscr{L}_\mu(B') = 3$ then there exists a Jordan chain $\{x_0, x_1, x_2\}$ of B' at μ of length 3, and (2.3) implies $M'_{A'}(\mu) = 0$ and

$$M''_{A'}(\mu) = 2[x_1, x_0] = 2[(B' - \mu)x_2, x_0] = 2[x_2, (B' - \mu)x_0] = 0. \tag{3.20}$$

By Proposition 3.3 (iii) we have $M_{B'} \in \mathscr{D}_1$ and Proposition 3.3 (ii) yields

$$M'''_{A'}(\mu) > 0. \tag{3.21}$$

As in Step 4 in the proof of Theorem 3.5 the set $I \cap \rho(A')$ consists of $n_{A'}(I)+1 = m_{A'}(I)+1$ open subintervals I_k. We have $\mu \in \rho(A')$ (see (3.16)) and hence $\mu \in I_j$ for some j with $1 \le j \le m_{A'}(I)+1$. Since all other eigenvalues of B' in $I \cap \rho(A')$ belong to $\sigma_{++}(B')$ it follows from Lemma 2.4 (i) that the derivative of $M_{A'}$ in such an eigenvalue is positive. This together with (3.21) shows that except for μ there is no other eigenvalue of B' in I_j. Moreover in each of the subintervals I_k, $1 \le k \le m_{A'}(I)+1$, $k \neq j$, there is at most one eigenvalue of B'. Summing up we have

$$m_{B'}(I) = n_{B'}(I) + 2 \quad \text{and} \quad n_{B'}(I) \le n_{A'}(I) + 1.$$

Together with (3.15) and (3.17) the estimate (3.19) follows if the multiplicity of μ is 3.

It remains to consider the case $\dim \mathscr{L}_\mu(B') = 2$. Relation (2.3) implies $M'_{A'}(\mu) = [x_0, x_0] = 0$. If $M''_{A'}(\mu) = 0$ then a similar reasoning as above implies (3.21) and the estimate (3.19) follows in the same way. If $M''_{A'}(\mu) \neq 0$ then we consider again the open subintervals I_k from above, $1 \le k \le m_{A'}(I)+1$, and for some subinterval I_j with $1 \le j \le m_{A'}(I)+1$ we have $\mu \in I_j$. Again, by Lemma 2.4 (i), the derivative of $M_{A'}$ is positive in all eigenvalues except in μ. Hence in each I_k, $k \neq j$, there is at most one eigenvalue of B'. In I_j the eigenvalue μ has multiplicity 2 and Lemma 2.5 yields that there is precisely one more eigenvalue of B' (with multiplicity one) in I_j. This implies

$$m_{B'}(I) = n_{B'}(I) + 1 \quad \text{and} \quad n_{B'}(I) \le n_{A'}(I) + 2.$$

With (3.15) and (3.17) the upper estimate in (i) with $\kappa_B = 1$ follows.

Step 5. Lower estimate in (ii) *and* (iii). If $0 \in I$ we apply the lower estimate in (i) to the intervals $I_+ = I \cap \mathbb{R}^+$ and $I_- = I \cap \mathbb{R}^-$ separately. Taking into account the assumption $0 \notin \sigma_p(A)$ we obtain the lower estimate in (ii). If $0 \in \sigma_p(A)$ we obtain

$$\begin{aligned} m_A(I) - 2 &= m_A(I_+) - 1 + m_A(I_-) - 1 + m_A(\{0\}) \\ &\le m_B(I_+) + m_B(I_-) + m_B(\{0\}) - m_B(\{0\}) + m_A(\{0\}) \\ &\le m_B(I) + |m_A(\{0\}) - m_B(\{0\})| \end{aligned}$$

and the lower estimate in (iii) follows from Theorem 3.8.

Step 6. Decomposition of $\mathscr{K}$ if $0 \in I$. As in Step 1 the spectral subspace of A corresponding to $I_+ = I \cap \mathbb{R}^+$ ($I_- = I \cap \mathbb{R}^-$) is a Hilbert space (anti-Hilbert space, respectively); cf. Proposition 3.1 (i). The subspace $\mathscr{E}_+$ ($\mathscr{E}_-$) spanned by the eigenvectors of $S = A \cap B$ in I_+ (I_-) is a subset of the spectral subspace of A corresponding to I_+ (I_-, respectively), and the space $\mathscr{E} := \mathscr{E}_+ [\dot{+}] \mathscr{E}_-$ is a Krein space. Denote the restriction of S to $\mathscr{E}$ by $S_{\mathscr{E}}$. With respect to the decomposition $\mathscr{K} = \mathscr{E} [\dot{+}] \mathscr{E}^{[\perp]}$ we have

$$S = \begin{pmatrix} S_{\mathscr{E}} & 0 \\ 0 & S' \end{pmatrix}, \quad A = \begin{pmatrix} S_{\mathscr{E}} & 0 \\ 0 & A' \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} S_{\mathscr{E}} & 0 \\ 0 & B' \end{pmatrix},$$

with S' symmetric, $\sigma_p(S') \cap I \subset \{0\}$, A' nonnegative, and B' selfadjoint in the Krein space $(\mathscr{E}^{[\perp]}, [\cdot,\cdot])$. Again A', B' and I satisfy Assumption (I) and, as in (3.15), we have

$$m_A(I) = m_{S_{\mathscr{E}}}(I) + m_{A'}(I) \quad \text{and} \quad m_B(I) = m_{S_{\mathscr{E}}}(I) + m_{B'}(I). \tag{3.22}$$

If $0 \notin \sigma_p(A)$ then $0 \notin \sigma_p(A')$ and we conclude from Proposition 2.7 (i) in the same way as in Step 1 that

$$\sigma_p(A') \cap \sigma_p(B') \cap I = \emptyset. \tag{3.23}$$

Step 7. Upper estimate in (ii) *if* $\kappa_B = 0$. In the case $0 \notin \sigma_p(B)$ the upper estimate in (ii) for $\kappa_B = 0$ follows immediately from the lower estimate in Step 5 by interchanging the roles of A and B.

Hence we consider the case $0 \in \sigma_p(B)$. Then we also have $0 \in \sigma_p(B')$. As $0 \notin \sigma_p(A')$ Theorem 2.3 (iv) implies $n_{A'}(I) = m_{A'}(I)$ also for an interval which contains 0. The set $I \cap \rho(A')$ consists of $n_{A'}(I)+1 = m_{A'}(I)+1$ open subintervals I_k. We have $0 \in \rho(A')$ and hence $0 \in I_j$ for some j with $1 \le j \le m_{A'}(I)+1$. As B and B' are nonnegative operators all eigenvalues of B' in I_+ (I_-) belong to $\sigma_{++}(B')$ ($\sigma_{--}(B')$, respectively). It follows from Lemma 2.4 (i)–(ii) and (3.23) that the derivative of $M_{A'}$ in eigenvalues of B' in I_+ (I_-) is positive (negative, respectively) and the multiplicity of these eigenvalues is one. We estimate the multiplicity of the eigenvalues of B' in I_j. Since $0 \in \sigma_p(B') \cap \rho(A')$ we have $M_{A'}(0) = 0$ and by Lemma 3.4 (iii) the point 0 is a zero of $M_{A'}$ of at most order two. If it is of order two, Lemma 3.4 (iii) and the above reasoning imply that 0 is the only zero in I_j. As B' is a nonnegative operator, the (algebraic) multiplicity of the eigenvalue 0 is at most two. If 0 is a zero of $M_{A'}$ of order one then the sign properties of $M'_{A'}$ at the other zeros yield that there is at most one more eigenvalue of B' in I_j. As a consequence of Lemma 2.4 (i)–(ii) the multiplicities of these two eigenvalues in I_j are both one. Therefore in both cases we have

$$m_{B'}(I) \le m_{A'}(I) + 2.$$

Together with (3.22) the upper estimate in (ii) in the case $\kappa_B = 0$ is shown.

Step 8. Upper estimate in (ii) *if* $\kappa_B = 1$. We again make use of the open subintervals I_k from Step 7 such that $0 \in I_j$. We proceed in a similar way as in Step 5 of the proof of Theorem 3.5. By Proposition 3.3 the function $M_{A'}$ has at most one zero $\mu \in I_{k_0}$ in a subinterval I_{k_0}, $k_0 \ne j$, with $M'_{A'}(\mu) \le 0$ if $\mu > 0$ or $M'_{A'}(\mu) \ge 0$ if $\mu < 0$. If $M_{A'}$ has such an exceptional zero, then by Proposition 3.3 (iii) $M_{B'} \in \mathscr{D}_1$ and, hence, $M_{A'}(0) > 0$ by Lemma 3.4 (ii). Thus $M_{A'}$ has no zero in I_j and therefore B' has no eigenvalue in I_j. As in Step 4 of the proof of Theorem 3.5 it follows that the total multiplicity of the eigenvalues of B' in I_{k_0} is at most three. Moreover, in the other subintervals I_k, $k \ne k_0$, $k \ne j$, B' has at most one eigenvalue of multiplicity one. This yields the upper estimate in (ii).

It remains to discuss the case that $M_{A'}$ has at most one zero in each of the subintervals I_k, $k \ne j$, with positive (negative) derivative at these zeros if they are in $I_k \subset \mathbb{R}^+$ ($I_k \subset \mathbb{R}^-$, respectively). We distinguish in this situation the cases $M_{A'}(0) > 0$, $M_{A'}(0) = 0$, and $M_{A'}(0) < 0$.

Observe that in the first case there is no zero of $M_{A'}$ of third order in I_j (Proposition 3.3 (ii)) and there may appear either one zero of $M_{A'}$ of second order or two zeros of order one in I_j; cf. Proposition 3.3. Hence we have either one eigenvalue of B' of multiplicity two (cf. (3.20) in Step 4) or two eigenvalues of multiplicity one. If $M_{A'}(0) = 0$ then $M_{B'} \in \mathscr{D}_0$ by Lemma 3.4 (ii) and 0 is a zero of at most second order by Lemma 3.4 (iii). If 0 is a zero of second order then $M''_{A'}(0) > 0$, there are no other zeros of $M_{A'}$ in I_j (Proposition 3.3 (i)), and therefore 0 is an eigenvalue of B' of multiplicity two (cf. (3.20) in Step 4). If 0 is a zero of first order there is at most one other zero in I_j of multiplicity one (Proposition 3.3 (i)); thus the total multiplicity of the eigenvalues of B' in I_j is at most two. If $M_{A'}(0) < 0$ then again $M_{B'} \in \mathscr{D}_0$ by Lemma 3.4 (ii) and it follows from Proposition 3.3 (i) that $M_{A'}$ has at most two zeros of first order in I_j. Again, the total multiplicity of the eigenvalues of B' in I_j is at most two and the upper estimate in (ii) follows.

Step 9. Upper estimate in (iii) *if* $\kappa_B = 0$. The upper estimate in (iii) for $\kappa_B = 0$ follows from Theorem 3.8 and from the upper estimate in (i) applied to the intervals $I_+ = I \cap \mathbb{R}^+$ and $I_- = I \cap \mathbb{R}^-$ separately.

Step 10. Upper estimate in (iii) *if* $\kappa_B = 1$. From Proposition 2.7 (i) we conclude

$$\sigma_p(A') \cap \sigma_p(B') \cap (I_- \cup I_+) = \emptyset$$

and Theorem 2.3 (iv) implies

$$n_{A'}(I_- \cup I_+) = m_{A'}(I_- \cup I_+).$$

By Proposition 3.3 (ii) the function $M_{A'}$ has at most one zero μ in I_+ (I_-) with $M'_{A'}(\mu) \le 0$ ($M'_{A'}(\mu) \ge 0$, respectively). For simplicity, we assume that $M'_{A'}$ has such an exceptional zero μ in I_-. As in Step 4 of the proof of Theorem 3.5 it follows that the total multiplicity of the eigenvalues of B' in I_- exceeds the total multiplicity of the eigenvalues of A' in I_- by at most 3, whereas in I_+ it exceeds by at most 1, hence

$$m_{B'}(I_- \cup I_+) \le m_{A'}(I_- \cup I_+) + 4.$$

Together with Theorem 3.8 we obtain

$$m_{B'}(I) = m_{B'}(I_- \cup I_+) + m_{B'}(\{0\}) \le m_{A'}(I_- \cup I_+) + 4 + m_{A'}(\{0\}) + 2 = m_{A'}(I) + 6$$

and, together with (3.22) the upper estimate in (iii) is shown.

Step 11. Proof of $m_A(I) = \infty$ *if and only if* $m_B(I) = \infty$. If $m_A(I) = \infty$ then either $n_A(I) = \infty$ in which case the assertion follows from Theorem 3.5, or $n_A(I) < \infty$ in which case there exists at least one eigenvalue of A with infinite multiplicity and the assertion follows from Theorem 2.3 (i). Conversely, if $m_B(I) = \infty$ then the same reasoning implies $m_A(I) = \infty$.

3.5. **Proof of Theorem 3.8.** The proof of Theorem 3.8 is a consequence of four lemmas which are also of independent interest. From now on let A and B be as in the assumptions of Theorem 3.8. As A is nonnegative we have

$$[Ax, x] = 0 \quad \Longrightarrow \quad x \in \ker A \tag{3.24}$$

for every $x \in \operatorname{dom} A$. Indeed, the application of the Cauchy-Bunyakowski inequality to the semi-definite inner product $[A\cdot, \cdot]$ gives $|[Ax, y]|^2 \le [Ax, x][Ay, y]$ for all $x, y \in \operatorname{dom} A$, and (3.24) follows. Moreover, from Proposition 2.1 we find that

$$(B - \overline{\lambda}_0)^{-1} - (A - \overline{\lambda}_0)^{-1} = \frac{1}{M_A(\overline{\lambda}_0)} [\cdot, \varphi_A] \gamma_A(\overline{\lambda}_0).$$

Observe that $(B - \overline{\lambda}_0)^{-1}$ and $(A - \overline{\lambda}_0)^{-1}$ coincide on $\{\varphi_A\}^{[\perp]}$ and define

$$M := (A - \overline{\lambda}_0)^{-1} \{\varphi_A\}^{[\perp]} = (B - \overline{\lambda}_0)^{-1} \{\varphi_A\}^{[\perp]}.$$

Hence, $M \subset \operatorname{dom} A \cap \operatorname{dom} B$. For $y \in M$ there exists $x \in \{\varphi_A\}^{[\perp]}$ such that $y = (A - \overline{\lambda}_0)^{-1} x = (B - \overline{\lambda}_0)^{-1} x$ and hence

$$Ay = x + \overline{\lambda}_0 (A - \overline{\lambda}_0)^{-1} x = x + \overline{\lambda}_0 (B - \overline{\lambda}_0)^{-1} x = By.$$

Thus, A and B coincide on M and their domains decompose as

$$\operatorname{dom} A = (A - \overline{\lambda}_0)^{-1} \mathscr{K} = (A - \overline{\lambda}_0)^{-1} \big(\{\varphi_A\}^{[\perp]} \oplus \operatorname{span} \{J\varphi_A\} \big) = M \dot{+} \operatorname{span} \{f_A\},$$
$$\operatorname{dom} B = (B - \overline{\lambda}_0)^{-1} \mathscr{K} = (B - \overline{\lambda}_0)^{-1} \big(\{\varphi_A\}^{[\perp]} \oplus \operatorname{span} \{J\varphi_A\} \big) = M \dot{+} \operatorname{span} \{f_B\},$$

where J is a fundamental symmetry in the Krein space $\mathscr{K}$ and $f_A := (A - \overline{\lambda}_0)^{-1} J\varphi_A \neq 0$ and $f_B := (B - \lambda_0)^{-1} J\varphi_A \neq 0$. It follows, in particular, that M has codimension 1 in $\operatorname{dom} A$

and $\operatorname{dom} B$. Hence for $x,y \in \operatorname{dom} A$ (or $x,y \in \operatorname{dom} B$) with $y \notin M$ there exists $\alpha \in \mathbb{C}$ such that

$$x - \alpha y \in M.$$

This observation will be used frequently in the following considerations.

Lemma 3.11. *Let A and B be as in Theorem 3.8. Then the following assertions hold.*

(i) *A has Jordan chains at* 0 *of length at most* 2.
(ii) *B has Jordan chains at* 0 *of length at most* 4.
(iii) *If B has a Jordan chain at* 0 *of length* 3 *or* 4 *then* $\ker B \subseteq \ker A$.

Proof. Assertion (i) is well known, see [51, Proposition II.2.1]. In order to show (ii) assume that B has a Jordan chain $\{x_0,\dots,x_4\}$ at 0 of length 5. Then

$$[x_2,x_1] = [B^2x_4,x_1] = [x_4,B^2x_1] = [x_4,0] = 0$$

and, analogously, $[x_0,x_0] = [x_0,x_1] = [x_0,x_2] = [x_1,x_1] = 0$. If $x_2 \in M$ then

$$0 = [x_1,x_2] = [Bx_2,x_2] = [Ax_2,x_2],$$

which, by (3.24), implies that $x_2 \in \ker A \cap M \subseteq \ker B$; a contradiction to $Bx_2 = x_1 \neq 0$. Hence, $x_2 \notin M$ and there exists $\alpha \in \mathbb{C}$ such that $x_1 - \alpha x_2 \in M$ and

$$0 = [x_0 - \alpha x_1, x_1 - \alpha x_2] = [B(x_1 - \alpha x_2), x_1 - \alpha x_2] = [A(x_1 - \alpha x_2), x_1 - \alpha x_2].$$

Again (3.24) implies $x_1 - \alpha x_2 \in \ker A \cap M \subseteq \ker B$; a contradiction to $B(x_1 - \alpha x_2) = x_0 - \alpha x_1 \neq 0$ and (ii) follows.

It remains to check (iii). Assume that $\{x_0,x_1,x_2\}$ is a Jordan chain of B at 0 of length 3 (the proof for a Jordan chain of length 4 is the same), let $y \in \ker B$ and assume $y \notin \ker A$. Then $y \notin M$ and there exists $\alpha \in \mathbb{C}$ such that $x_1 - \alpha y \in M$ and

$$[A(x_1 - \alpha y), x_1 - \alpha y] = [B(x_1 - \alpha y), x_1 - \alpha y] = [x_0, x_1 - \alpha y] = -[Bx_1, \alpha y] = 0.$$

Here we have used that $[x_0,x_1] = [B^2x_2,x_1] = [x_2,B^2x_1] = 0$. From (3.24) we then conclude $x_1 - \alpha y \in \ker A \cap M \subseteq \ker B$, but $B(x_1 - \alpha y) = x_0 \neq 0$; a contradiction. Thus we have $\ker B \subseteq \ker A$. □

In the following lemma we collect some results on the dimensions of the kernel of B (and its powers) compared with the corresponding dimensions of the kernel of A.

Lemma 3.12. *Let A and B be as in Theorem 3.8. Then the following assertions hold.*

(i) $|\dim \ker A - \dim \ker B| \leq 1$*;*
(ii) $|\dim \ker A^2 - \dim \ker B^2| \leq 2$*;*
(iii) $|\dim \left(\ker A^2/\ker A\right) - \dim \left(\ker B^2/\ker B\right)| \leq 1$*;*
(iv) $\dim \left(\ker B^3/\ker B^2\right) \leq 1$*, that is, B has no two (linearly independent) Jordan chains at* 0 *of length* 3.

Proof. In order to show (i) assume that $\dim \ker B > \dim \ker A + 1$. Then there exist $n := \dim \ker A + 2$ linearly independent vectors $\{x_1,\dots,x_n\}$ in $\ker B$. If $x_j \in M$ for all $j = 1,\dots,n$ then $Ax_j = Bx_j = 0$ and $x_j \in \ker A$, a contradiction. Hence there exists a vector $x_{k_0} \in \ker B \setminus M$, $1 \leq k_0 \leq n$. After reordering we can assume $k_0 = n$. Then there exist $\alpha_k \in \mathbb{C}$ such that

$$z_k := x_k - \alpha_k x_n \in M, \qquad k = 1,\dots,n-1.$$

Thus $Az_k = Bz_k = 0$, $k = 1,\dots,n-1$, and we conclude that $\{z_1,\dots,z_{n-1}\}$ is a linearly independent set in $\ker A$; a contradiction. Therefore, $\dim \ker B \leq \dim \ker A + 1$. The same considerations with A replaced by B show $\dim \ker A - 1 \leq \dim \ker B$ and hence (i) follows.

Observe that (ii) follows from (i) and (iii). In order to show (iii) assume

$$n := \dim\left(\ker B^2/\ker B\right) \geq \dim\left(\ker A^2/\ker A\right) + 2. \tag{3.25}$$

and choose linearly independent vectors $x_{1,1},\dots,x_{1,n}$ with

$$\ker B^2 = \ker B \dotplus \operatorname{span}\{x_{1,1},\dots,x_{1,n}\}.$$

Define for $1 \leq j \leq n$ elements in $\ker B$ via

$$x_{0,j} := Bx_{1,j}.$$

If $x_{0,j} \in M$ holds for all $1 \leq j \leq n$ then there exists $x_{1,n_0} \notin M$ for some n_0 with $1 \leq n_0 \leq n$ as otherwise $\{x_{0,1},x_{1,1}\},\dots,\{x_{0,n},x_{1,n}\}$ are n Jordan chains of A at 0 of length 2, a contradiction to (3.25). Hence there exists $\alpha_j \in \mathbb{C}$ with

$$x_{1,j} - \alpha_j x_{1,n_0} \in M \quad \text{for } 1 \leq j \leq n \text{ and } j \neq n_0.$$

Thus, $\{x_{0,j} - \alpha_j x_{0,n_0}, x_{1,j} - \alpha_j x_{1,n_0}\}$ is a Jordan chain of A at 0 of length 2 for all j with $1 \leq j \leq n$ and $j \neq n_0$ which contradicts (3.25). From this we conclude that at least one of the elements $x_{0,j}$ is not in M. We assume $x_{0,n} \notin M$. Then for $1 \leq j \leq n-1$ there exist $\beta_j,\gamma_j \in \mathbb{C}$ with

$$x_{0,j} - \beta_j x_{0,n} \in M \quad \text{and} \quad x_{1,j} - \beta_j x_{1,n} - \gamma_j x_{0,n} \in M$$

and $\{x_{0,j} - \beta_j x_{0,n}, x_{1,j} - \beta_j x_{1,n} - \gamma_j x_{0,n}\}$ is a Jordan chain of A at 0 of length 2 for all j with $1 \leq j \leq n-1$. Hence (3.25) is not valid, i.e. $\dim(\ker B^2/\ker B) \leq \dim(\ker A^2/\ker A) + 1$. The same considerations with A replaced by B show $\dim(\ker A^2/\ker A) \leq \dim(\ker B^2/\ker B) + 1$ and (iii) follows.

It remains to prove (iv). Assume that there are linearly independent vectors $x_2,y_2 \notin \ker B^2$ with

$$\ker B^3 = \ker B^2 \dotplus \operatorname{span}\{x_2,y_2\}.$$

Define elements in $\ker B^2$ and $\ker B$, respectively, via

$$x_1 := Bx_2; \quad x_0 := Bx_1; \quad y_1 := By_2 \quad \text{and} \quad y_0 := By_1.$$

By Lemma 3.11 (iii) we obtain $x_0,y_0 \in \ker A$. We find $\alpha \in \mathbb{C}$ such that $x_2 - \alpha y_2$ or $y_2 - \alpha x_2$ belongs to M. If x_1,y_1 belong to M, then $\{x_0 - \alpha y_0, x_1 - \alpha y_1, x_2 - \alpha y_2\}$ or $\{y_0 - \alpha x_0, y_1 - \alpha x_1, y_2 - \alpha x_2\}$ is a Jordan chain of A at 0 of length 3, a contradiction to Lemma 3.11 (i). Hence, at least one of the vectors x_1,y_1 does not belong to M. Let $y_1 \notin M$. Then there exist $\beta,\gamma \in \mathbb{C}$ with

$$x_1 - \beta y_1 \in M \quad \text{and} \quad x_2 - \beta y_2 - \gamma y_1 \in M$$

and $\{x_0 - \beta y_0, x_1 - \beta y_1 - \gamma y_0, x_2 - \beta y_2 - \gamma y_1\}$ is a Jordan chain of A at 0 of length 3, a contradiction to Lemma 3.11 (i) and Lemma 3.12 is shown. □

By Lemma 3.11 (i) and (ii) we see $\mathscr{L}_0(B) = \ker B^4$, $\mathscr{L}_0(A) = \ker A^2$, and with Lemma 3.12 (ii) we obtain

$$m_A(\{0\}) - 2 = \dim\ker A^2 - 2 \leq \dim\ker B^2 \leq \dim\ker B^4 = m_B(\{0\}). \tag{3.26}$$

For two special cases we prove the opposite bound in the next lemma.

Lemma 3.13. *Let A and B be as in Theorem 3.8. Then the following assertions hold.*

(i) *If $0 \in \rho(A)$ then*

$$|m_A(\{0\}) - m_B(\{0\})| = m_B(\{0\}) \leq 2.$$

(ii) *If* $\mathscr{L}_0(A) \subseteq \mathscr{L}_0(B)$ *and* $A|_{\mathscr{L}_0(A)} = B|_{\mathscr{L}_0(A)}$ *then*

$$|m_A(\{0\}) - m_B(\{0\})| \le 2.$$

Proof. By (3.26) we only need to prove that $m_B(\{0\}) \le m_A(\{0\}) + 2$.

(i) If $0 \in \rho(A)$ then B has Jordan chains at 0 of length at most 2. Indeed, assume that B has a Jordan chain $\{x_0, x_1, x_2\}$ at 0 of length 3. Then $[x_0, x_0] = [Bx_1, x_0] = 0$ and $[x_1, x_0] = [Bx_2, x_0] = 0$. If $x_0 \in M$ then $0 = Bx_0 = Ax_0$; a contradiction to $0 \in \rho(A)$. Consequently, $x_0 \notin M$. Then there exists $\alpha \in \mathbb{C}$ with $0 \ne x_1 - \alpha x_0 \in M$ and

$$0 = [x_0, x_1 - \alpha x_0] = [B(x_1 - \alpha x_0), x_1 - \alpha x_0] = [A(x_1 - \alpha x_0), x_1 - \alpha x_0].$$

Relation (3.24) implies that $x_1 - \alpha x_0 \in \ker A$; a contradiction to $0 \in \rho(A)$. Therefore we have $\mathscr{L}_0(B) = \ker B^2$ and the claim follows by Lemma 3.12 (ii).

(ii) Since 0 is an isolated point in $\sigma(A)$ we have $\mathscr{K} = \mathscr{L}_0(A)[\dot{+}]\mathscr{L}_0(A)^{[\perp]}$, where both $(\mathscr{L}_0(A), [\cdot,\cdot])$ and $(\mathscr{L}_0(A)^{[\perp]}, [\cdot,\cdot])$ are Krein spaces; cf. [7, Theorem II.2.20]. Since A and B coincide on $\mathscr{L}_0(A)$ this subspace is invariant under A and B, and according to the chosen decomposition of $\mathscr{K}$ we obtain

$$A = \begin{pmatrix} A_0 & 0 \\ 0 & A_1 \end{pmatrix}, \quad B = \begin{pmatrix} A_0 & 0 \\ 0 & B_1 \end{pmatrix},$$

where A_1 is nonnegative, $0 \in \rho(A_1)$, B_1 is selfadjoint and $(B_1 - \lambda_0)^{-1} - (A_1 - \lambda_0)^{-1}$ is a selfadjoint rank one operator in the Krein space $(\mathscr{L}_0(A)^{[\perp]}, [\cdot,\cdot])$. Applying (i) to B_1 and A_1, the claim follows. □

Lemma 3.14. *Let A and B be as in Theorem 3.8. If* $\{x_0, x_1, x_2\}$ *is a Jordan chain of B at* 0 *of length* 3 *and B has no Jordan chain at* 0 *of length* 4 *then there exists a basis b of* $\mathscr{L}_0(B)$ *containing* $\{x_0, x_1, x_2\}$ *with*

$$b \setminus \{x_1, x_2\} \subseteq \mathscr{L}_0(A).$$

If B has a Jordan chain $\{x_0, x_1, x_2, x_3\}$ *at* 0 *of length* 4 *then there exists a basis b of* $\mathscr{L}_0(B)$ *containing* $\{x_0, x_1, x_2, x_3\}$ *with*

$$b \setminus \{x_1, x_2, x_3\} \subseteq \mathscr{L}_0(A).$$

Proof. We consider the case that there is a Jordan chain $\{x_0, x_1, x_2\}$ of B at 0 of length 3 and none of length 4. In this case we have $[x_0, x_0] = [x_1, x_0] = 0$. We show $x_0 \in M$ and $x_1 \notin M$. If $x_0 \notin M$ then there exists $\alpha \in \mathbb{C}$ such that $x_1 - \alpha x_0 \in M$. Hence,

$$0 = [x_0, x_1 - \alpha x_0] = [B(x_1 - \alpha x_0), x_1 - \alpha x_0] = [A(x_1 - \alpha x_0), x_1 - \alpha x_0],$$

and (3.24) implies $x_1 - \alpha x_0 \in \ker A \cap M \subseteq \ker B$; a contradiction to $Bx_1 = x_0 \ne 0$. Thus $x_0 \in M$. If $x_1 \in M$ then $[Ax_1, x_1] = [Bx_1, x_1] = [x_0, x_1] = 0$. Hence by (3.24) $x_1 \in \ker A \cap M \subseteq \ker B$; a contradiction. Consequently, $x_1 \notin M$.

As $m_A(\{0\}) < \infty$ by assumption it follows from Lemma 3.12 and Lemma 3.11 (ii) that the dimension $m_B(\{0\})$ of the root subspace $\mathscr{L}_0(B)$ is finite as well. If $\mathscr{L}_0(B) = \operatorname{span}\{x_0, x_1, x_2\}$ then in view of Lemma 3.11 (iii) the assertion of Lemma 3.14 follows. Let $\{x_0, x_1, x_2, u_3, \ldots, u_n\}$ be a basis of $\mathscr{L}_0(B)$ for some $n \ge 3$. For $3 \le k \le n$ we define z_k in the following way: If $u_k \in \ker B$ then by Lemma 3.11 (iii) also $u_k \in \ker A$ and we set $z_k := u_k$. If $u_k \notin \ker B$ then by Lemma 3.12 (iv) we obtain $u_k \in \ker B^2$ and we set $y_k := Bu_k \ne 0$. As $x_1 \notin M$ there exist $\alpha_k \in \mathbb{C}$ such that $z_k := u_k - \alpha_k x_1 \in M$ and we have

$$Az_k = Bz_k = y_k - \alpha_k x_0 \in \ker B \subseteq \ker A \quad \text{and} \quad z_k \in \ker A^2 = \mathscr{L}_0(A).$$

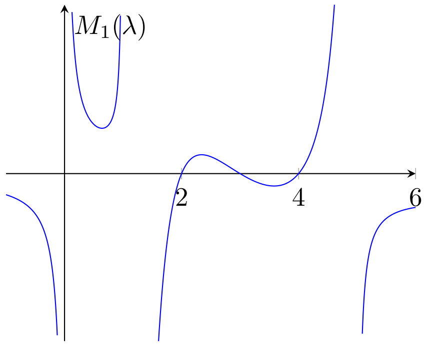

FIGURE 1. Schematic plot of the function $M_1(\lambda) = -\frac{(\lambda-2)(\lambda-3)(\lambda-4)}{\lambda(\lambda-1)(\lambda-5)}$.

The elements $x_0, x_1, x_2, z_3, \ldots, z_n$ are linearly independent. Moreover, $x_0 \in M \cap \ker B$ and hence $x_0 \in \ker A \subseteq \mathscr{L}_0(A)$. Thus $b := \{x_0, x_1, x_2, z_3, \ldots, z_n\}$ is a basis of $\mathscr{L}_0(B)$ with the desired properties.

The case of a Jordan chain at 0 of length 4 is proved analogously. □

Proof of Theorem 3.8. By Lemma 3.12 and Lemma 3.11 (ii) the root subspace $\mathscr{L}_0(B)$ is finite dimensional. In regard of (3.26) it remains to prove

$$m_B(\{0\}) \le m_A(\{0\}) + 2. \tag{3.27}$$

By Lemma 3.12 (iv), B cannot have two linearly independent Jordan chains at 0 of length 3, so that B has at most a single Jordan chain at 0 of length 3 or 4. Hence, if $\dim\ker B^2 \le \dim\ker A^2$ the claim follows. Therefore, assume that $\dim\ker B^2 > \dim\ker A^2$. If there is no Jordan chain of B at 0 of length 3 the estimate follows from Lemma 3.12 (ii). Now assume, that B has a Jordan chain $\{x_0, x_1, x_2\}$ at 0 of length 3 and none of length 4 (the case of a Jordan chain at 0 of length 4 is analogous). By Lemma 3.11 (iii) we have $\ker B \subseteq \ker A$ and because of Lemma 3.12 (i) there are only two possible cases:

(i) $\dim\ker B = \dim\ker A$: Hence, $\ker A = \ker B$. Then Lemma 3.12 (iii) and Lemma 3.11 imply that $\dim \mathscr{L}_0(A) = \dim\ker A^2 = \dim\ker B^2 - 1$. Let b be the basis of $\mathscr{L}_0(B)$ constructed in the proof of Lemma 3.14. Then $b \setminus \{x_2\}$ is a basis of $\ker B^2$. Moreover, $b \setminus \{x_1, x_2\}$ is contained in $\mathscr{L}_0(A)$. But $\dim \mathscr{L}_0(A) = \dim\ker B^2 - 1$ is the cardinality of $b \setminus \{x_1, x_2\}$. Thus $\mathscr{L}_0(A) = \operatorname{span}\{b \setminus \{x_1, x_2\}\}$. Recall that $b = \{x_0, x_1, x_2, z_3, \ldots, z_n\}$ and $z_k \in M$, $k = 3, \ldots, n$; cf. the proof of Lemma 3.14. Then $A|_{\mathscr{L}_0(A)} = B|_{\mathscr{L}_0(A)}$ and (3.27) is a consequence of Lemma 3.13 (ii).

(ii) $\dim\ker B = \dim\ker A - 1$: Since $\ker B \subseteq \ker A \subseteq \ker A^2$ we see

$$\dim\left(\ker B^2/\ker B\right) > \dim\left(\ker A^2/\ker B\right) = \dim\left(\ker A^2/\ker A\right) + 1$$

in contradiction to Lemma 3.12 (iii).

It remains to show the sharpness of (3.27). For this consider the space $\mathbb{C}^2$ with a fundamental symmetry J and operators A and B defined via

$$J := A := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

It is easily seen that A and B satisfy Assumption (I), $m_A(\{0\}) = 0$, and $m_B(\{0\}) = 2$. □

3.6. **Three examples.** Define the function M_1 by

$$M_1(\lambda) = -\frac{(\lambda-2)(\lambda-3)(\lambda-4)}{\lambda(\lambda-1)(\lambda-5)};$$

cf. Figure 1. By Definition 3.2 (see also [15, Theorem 2]) M_1 belongs to the class $\mathscr{D}_0$ and

$$M_1(\lambda) = \frac{24}{5\lambda} - \frac{3}{2(\lambda-1)} - \frac{3}{10(\lambda-5)} - 1.$$

From Proposition 3.3 (iii) we conclude that the function $\lambda \mapsto -\frac{1}{M_1(\lambda)}$ belongs to $\mathscr{D}_1$. The Pontryagin space and the selfadjoint matrices A and B from Step 6 in the proof of Theorem 3.5 can easily be computed with standard methods; cf. [33] and e.g. [9, Proof of Theorem 4.6]. Here we equip $\mathbb{C}^3$ with the indefinite inner product

$$[x,y] := -x_1\bar{y}_1 + x_2\bar{y}_2 + x_3\bar{y}_3, \qquad x = (x_1,x_2,x_3)^\top,\ y = (y_1,y_2,y_3)^\top, \tag{3.28}$$

and obtain the matrices

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} \frac{24}{5} & -\frac{6}{\sqrt{5}} & -\frac{6}{5} \\ \frac{6}{\sqrt{5}} & -\frac{1}{2} & -\frac{3}{\sqrt{20}} \\ \frac{6}{5} & -\frac{3}{\sqrt{20}} & \frac{47}{10} \end{pmatrix},$$

which are selfadjoint in the Pontryagin space $(\mathbb{C}^3, [\cdot,\cdot])$ and differ by a rank one matrix. Clearly $\sigma(A) = \{0,1,5\}$ coincides with the poles of M_1 and the zeros of M_1 coincide with $\sigma(B) = \{2,3,4\}$. We also mention that A is nonnegative and it can be checked that B has one negative square. Obviously the matrix B has three eigenvalues in the interval $(1,5)$ whereas A has no eigenvalues in $(1,5)$; cf. the upper estimate in Theorem 3.5 (i) with $\kappa_B = 1$. Moreover, in $(-1,2)$ are no eigenvalues of B whereas A has two eigenvalues there; cf. the lower estimate in Theorem 3.5 (ii). Similarly, any sufficiently small interval containing a positive pole of M_1 is an example for the lower estimate in Theorem 3.5 (i).

As a second example consider the function

$$M_2(\lambda) = -\frac{(\lambda+1)(\lambda-1)(\lambda-3)}{(\lambda+2)(\lambda-2)(\lambda-4)},$$

which belongs to $\mathscr{D}_0$; cf. Figure 2. Here the function $\lambda \mapsto -\frac{1}{M_2(\lambda)}$ belongs to $\mathscr{D}_0$ and we have

$$M_2(\lambda) = \frac{5}{8(\lambda+2)} - \frac{3}{8(\lambda-2)} - \frac{5}{4(\lambda-4)} - 1.$$

We equip $\mathbb{C}^3$ with the indefinite inner product (3.28) and obtain the selfadjoint matrices

$$A = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} -\frac{11}{8} & -\frac{\sqrt{15}}{8} & -\frac{5}{4\sqrt{2}} \\ \frac{\sqrt{15}}{8} & \frac{13}{8} & -\frac{1}{4}\sqrt{\frac{15}{2}} \\ \frac{5}{4\sqrt{2}} & -\frac{1}{4}\sqrt{\frac{15}{2}} & \frac{11}{4} \end{pmatrix}$$

as minimal realizations of the functions M_2 and $-M_2^{-1}$; cf. Step 6 in the proof of Theorem 3.5. It can be checked that in fact $A - B$ is a rank one matrix, $\kappa_B = 0$, and that $\sigma(A) = \{-2,2,4\}$ and $\sigma(B) = \{-1,1,3\}$ are the poles and zeros of M_2, respectively. The matrix B has two eigenvalues in the interval $(-2,2)$ whereas A has no eigenvalue in $(-2,2)$, which is the upper estimate in Theorem 3.5 (ii) with $\kappa_B = 0$. Similarly, any sufficiently small interval containing a zero of M_2 is an example for the upper estimate in

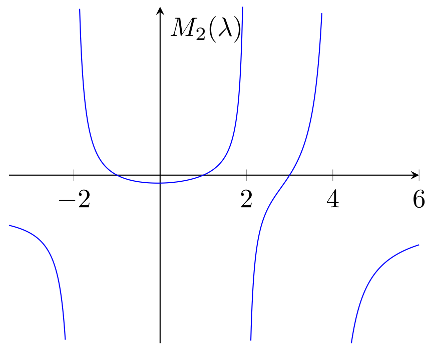

FIGURE 2. Schematic plot of the function $M_2(\lambda) = -\frac{(\lambda+1)(\lambda-1)(\lambda-3)}{(\lambda+2)(\lambda-2)(\lambda-4)}$.

Theorem 3.5 (i) with $\kappa_B = 0$.

Finally, in order to provide an example for the upper estimate in Theorem 3.5 (ii) with $\kappa_B = 1$, consider the function

$$M_3(\lambda) = -\frac{(\lambda+1)(\lambda-1)(\lambda-2)(\lambda-3)}{(\lambda+2)\lambda^2(\lambda-4)},$$

which is in $\mathscr{D}_0$ and $\lambda \mapsto -\frac{1}{M_3(\lambda)}$ is in $\mathscr{D}_1$; cf. Proposition 3.3 (iii). Here we have

$$M_3(\lambda) = \frac{5}{2(\lambda+2)} - \frac{3}{4\lambda^2} + \frac{13}{16\lambda} - \frac{5}{16(\lambda-4)} - 1$$

and if $\mathbb{C}^4$ is equipped with the indefinite inner product

$$[x,y] := x_1\bar{y}_1 + x_2\bar{y}_2 - x_3\bar{y}_3 - x_4\bar{y}_4, \qquad x = (x_1,x_2,x_3,x_4)^\top,\ y = (y_1,y_2,y_3,y_4)^\top,$$

then the selfadjoint matrices

$$A = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & \frac{12}{13} & \frac{12}{13} & 0 \\ 0 & -\frac{12}{13} & -\frac{12}{13} & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} \frac{59}{16} & 0 & -\frac{\sqrt{65}}{16} & \frac{5}{4\sqrt{2}} \\ 0 & \frac{12}{13} & \frac{12}{13} & 0 \\ \frac{\sqrt{65}}{16} & -\frac{12}{13} & -\frac{23}{208} & -\frac{1}{4}\sqrt{\frac{65}{2}} \\ -\frac{5}{4\sqrt{2}} & 0 & -\frac{1}{4}\sqrt{\frac{65}{2}} & \frac{1}{2} \end{pmatrix}$$

can be computed as minimal realizations of M_3 and $-M_3^{-1}$, respectively. Then $A-B$ is a rank one matrix, $\kappa_B = 1$ and $\sigma(A) = \{-2,0,4\}$ and $\sigma(B) = \{-1,1,2,3\}$ are the poles and zeros of M_3, respectively. In the interval $(-2,4)$ the matrix B has 4 eigenvalues whereas A has one eigenvalue there; cf. the upper estimate in Theorem 3.5 (ii) with $\kappa_B = 1$.

4. SINGULAR INDEFINITE STURM-LIOUVILLE PROBLEMS

In this section the general eigenvalue estimates are illustrated in a typical application from the theory of singular Sturm-Liouville problems with indefinite weight functions. The main result Theorem 4.1 can be regarded as a generalization of an estimate in [13, Theorem 4.1]. In contrast to [13] here we go beyond the so-called left-definite case, which was studied intensively from different points of view; cf. [17, 18, 19, 21, 22, 42, 44, 45, 46, 61].

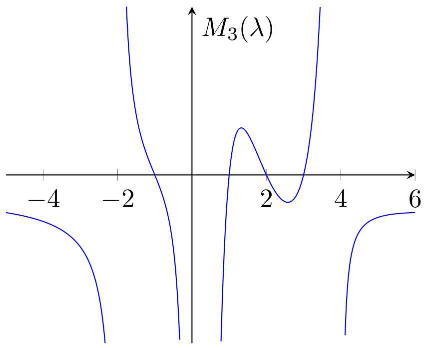

FIGURE 3. Schematic plot of the function $M_3(\lambda) = -\frac{(\lambda+1)(\lambda-1)(\lambda-2)(\lambda-3)}{(\lambda+2)\lambda^2(\lambda-4)}$.

We will fix some notation first. Let $r, p^{-1}, q \in L^1_{\mathrm{loc}}(\mathbb{R})$ be real valued, $p > 0$ and $r \neq 0$ a.e., and consider the differential expressions

$$\ell = \frac{1}{|r|}\left(-\frac{d}{dx}p\frac{d}{dx} + q\right) \quad \text{and} \quad \tau = \frac{1}{r}\left(-\frac{d}{dx}p\frac{d}{dx} + q\right).$$

We assume that ℓ is in the limit point case at $\pm\infty$ and that the weight function has one sign change at some point $c \in \mathbb{R}$ such that $r_+ = r \upharpoonright (c,\infty) > 0$ and $r_- = r \upharpoonright (-\infty, c) < 0$ a.e. Denote by $L^2(\mathbb{R}, |r|)$ the space of all equivalence classes of complex valued measurable functions f on $\mathbb{R}$ such that $\int_{\mathbb{R}} |f|^2|r| < \infty$. It is well known that ℓ gives rise to a selfadjoint operator

$$Tf = \ell(f) = \frac{1}{|r|}\big((-pf')' + qf\big) \qquad f \in \operatorname{dom} T, \tag{4.1}$$

in the Hilbert space $L^2(\mathbb{R}, |r|)$, where the domain $\operatorname{dom} T$ consists of all locally absolutely continuous functions $f \in L^2(\mathbb{R}, |r|)$ such that pf' is locally absolutely continuous and $\ell(f) \in L^2(\mathbb{R}, |r|)$. The limit point condition also implies that all eigenvalues of T are simple, that is, $n_T(I) = m_T(I)$ in the notation of Section 3, where $I \subset \mathbb{R}$ is an interval such that $\sigma_{\mathrm{ess}}(T) \cap I = \emptyset$. The *indefinite* Sturm-Liouville operator $B := \operatorname{sgn}(r)T$ which corresponds to $\tau = \operatorname{sgn}(r)\ell$ is defined by

$$Bf = \tau(f) = \frac{1}{r}\big((-pf')' + qf\big), \quad f \in \operatorname{dom} B = \operatorname{dom} T. \tag{4.2}$$

Note that B is selfadjoint in the Krein space $(L^2(\mathbb{R}, |r|), [\cdot,\cdot])$, where $[\cdot,\cdot]$ is given by

$$[f,g] = \int_{\mathbb{R}} f(x)\overline{g(x)}\, r(x)\, dx = (\operatorname{sgn}(r) f, g), \qquad f, g \in L^2(\mathbb{R}, |r|),$$

and $(\cdot,\cdot)$ stands for the usual inner product in the Hilbert space $L^2(\mathbb{R}, |r|)$.

The following theorem extends one of the main results in [13]. Instead of the left definite case, that is, T is uniformly positive, we consider the more general situation where T is only assumed to be nonnegative. A similar analysis as in [13] together with Theorem 3.9 then yields an eigenvalue estimate in a gap of the essential spectrum of the indefinite Sturm-Liouville operator B, which in the left definite case reduces to the estimate from [13, Theorem 4.1]. We again use the symbols $m_B(I)$ and $n_T(I)$ for the total multiplicity and the number of distinct eigenvalues of B and T in an interval I, respectively.

Theorem 4.1. *Assume that T in* (4.1) *is nonnegative in the Hilbert space $L^2(\mathbb{R},|r|)$ with $\eta = \min\sigma_{\text{ess}}(T) > 0$. Then the following assertions* (i)-(iii) *hold for the indefinite Sturm-Liouville operator B in* (4.2)*.*

(i) *B is nonnegative in the Krein space $(L^2(\mathbb{R},|r|),[\cdot,\cdot])$ and $\rho(B) \neq \emptyset$;*
(ii) *$(-\eta,\eta)\cap\sigma_{\text{ess}}(B) = \emptyset$;*
(iii) *$m_B((-\eta,\eta))$ is finite if and only if $n_T([0,\eta))$ is finite, in which case*

$$n_T([0,\eta)) - 3 \le m_B((-\eta,\eta)) \le n_T([0,\eta)) + 3. \tag{4.3}$$

Proof. (i) The fact $\rho(B) \neq \emptyset$ follows from [21, Corollary 1.4]. As T is assumed to be nonnegative in the Hilbert space $L^2(\mathbb{R},|r|)$ we have

$$[Bf,f] = (Tf,f) \ge 0, \qquad f \in \operatorname{dom} B = \operatorname{dom} T. \tag{4.4}$$

Hence B is a nonnegative operator in the Krein space $(L^2(\mathbb{R},|r|),[\cdot,\cdot])$.

(ii) & (iii) We will make use of the selfadjoint realizations T_+ and T_- of ℓ restricted to (c,∞) and $(-\infty,c)$, respectively, with Dirichlet boundary conditions at c in the Hilbert spaces $L^2((c,\infty),|r_+|)$ and $L^2((-\infty,c),|r_-|)$, respectively. Then the orthogonal sum $T_+\oplus T_-$ is a selfadjoint operator in the Hilbert space $L^2(\mathbb{R},|r|)$ and

$$\dim\big(\operatorname{ran}(T-\lambda)^{-1} - ((T_+\oplus T_-)-\lambda)^{-1}\big) = 1, \qquad \lambda \in \mathbb{C}\setminus\mathbb{R}.$$

As ℓ is in the limit point case at $\pm\infty$, all eigenvalues of T_+ and of T_- are simple and, hence, the multiplicity of each eigenvalue of $T_+\oplus T_-$ is at most two. Besides $T_+\oplus T_-$ also the selfadjoint operator

$$A := T_+ \oplus (-T_-)$$

will play an important role in the following. Note that A is also selfadjoint in the Krein space $(L^2(\mathbb{R},|r|),[\cdot,\cdot])$ with

$$\dim\big(\operatorname{ran}(B-\lambda)^{-1} - (A-\lambda)^{-1}\big) = 1, \qquad \lambda \in \mathbb{C}\setminus\mathbb{R},$$

and

$$[Af,f] = (\operatorname{sgn}(r)\big((T_+\oplus(-T_-))f,f\big) = \big((T_+\oplus T_-)f,f\big) \ge 0 \tag{4.5}$$

holds for all $f \in \operatorname{dom} A = \operatorname{dom} T_+ \oplus \operatorname{dom} T_-$. Hence A is a nonnegative operator in the Krein space $(L^2(\mathbb{R},|r|),[\cdot,\cdot])$.

It is not difficult to see that the following relations hold for the spectra of the operators T, $T_\pm$, A, and B; cf. [13, Lemma 2.2 and Proposition 2.3]. We mention that item (γ) below immediately yields assertion (ii) of the theorem.

(α) If $0 \in \sigma_p(T)$ then either $0 \in \rho(T_+)\cap\rho(T_-)$ or $0 \in \sigma_p(T_+)\cap\sigma_p(T_-)$;
(β) $0 \le \min\sigma(T) \le \min\sigma(T_\pm)$ and $\sigma_{\text{ess}}(T) = \sigma_{\text{ess}}(T_+\oplus T_-) = \sigma_{\text{ess}}(T_+)\cup\sigma_{\text{ess}}(T_-)$;
(γ) $\sigma_{\text{ess}}(B) = \sigma_{\text{ess}}(A) = \sigma_{\text{ess}}(T_+)\cup\sigma_{\text{ess}}(-T_-) \subset \mathbb{R}\setminus(-\eta,\eta)$;
(δ) $n_T([0,\eta))$ is finite if and only if $m_{T_+\oplus T_-}([0,\eta))$ is finite, in which case

$$\begin{aligned} n_T([0,\eta)) - 1 &\le m_{T_+\oplus T_-}([0,\eta)) \le n_T([0,\eta)) + 1,\\ n_T((0,\eta)) - 1 &\le m_{T_+\oplus T_-}((0,\eta)) \le n_T((0,\eta)) + 1; \end{aligned} \tag{4.6}$$

(ε) $m_{T_+\oplus T_-}([0,\eta)) = m_A((-\eta,\eta))$ and $m_{T_+\oplus T_-}((0,\eta)) = m_A((-\eta,0)) + m_A((0,\eta))$.

It remains to prove assertion (iii). Observe first that A, B and $I = (-\eta,\eta)$ satisfy the general Assumption (I) in Section 2. Then (δ) and (ε) together with Theorem 3.9 imply that $m_B((-\eta,\eta))$ is finite if and only if $n_T([0,\eta))$ is finite. In order to show the estimate

(4.3) let us first assume that $0 \notin \sigma_p(T)$. Then (β) implies that $0 \notin \sigma_p(T_+) \cup \sigma_p(T_-)$ and hence $0 \notin \sigma_p(A)$. According to Theorem 3.9 (ii) with $\kappa_B = 0$ we have

$$m_A((-\eta,\eta)) - 2 \leq m_B((-\eta,\eta)) \leq m_A((-\eta,\eta)) + 2$$

and hence the first estimate in (δ) together with (ε) imply (4.3).

Let us now consider the case $0 \in \sigma_p(T)$. Then either $0 \in \rho(T_+) \cap \rho(T_-)$ or $0 \in \sigma_p(T_+) \cap \sigma_p(T_-)$ by (α). In the first case we have $0 \notin \sigma_p(A)$ and again Theorem 3.9 (ii) with $\kappa_B = 0$ and (δ), (ε) yields (4.3).

In the second case 0 is an eigenvalue of (geometric) multiplicity 2 of $T_+ \oplus T_-$. As all eigenvalues of T are simple (ℓ is in the limit point case at $\pm\infty$) we have $m_T(\{0\}) = 1$. Moreover, every eigenvector of T at 0 is an eigenvector of B (and vice-versa) and we have

$$1 \leq m_B(\{0\}) \leq 2, \tag{4.7}$$

where the upper estimate in (4.7) follows from the fact that B is a nonnegative operator in the Krein space $(L^2(\mathbb{R},|r|),[\cdot,\cdot])$, see [51, Proposition II.2.1]. We obtain by (4.6), (ε), and Theorem 3.9 (i) with $\kappa_B = 0$ (applied to $(-\eta,0)$ and $(0,\eta)$)

$$\begin{aligned} n_T([0,\eta)) - 3 &= n_T((0,\eta)) - 2 \\ &\leq m_{T_+ \oplus T_-}((0,\eta)) - 1 \\ &= m_A((-\eta,0)) + m_A((0,\eta)) - 1 \\ &\leq m_B((-\eta,0)) + m_B((0,\eta)) + 1 \\ &\leq m_B((-\eta,0)) + m_B((0,\eta)) + m_B(\{0\}) = m_B((-\eta,\eta)), \end{aligned}$$

where we have used in the last estimate (4.7). Similarly, with the upper estimate in (4.7), with Theorem 3.9 (i), with (ε), with $m_{T_+ \oplus T_-}(\{0\}) = 2$ and by (4.6) we see

$$\begin{aligned} m_B((-\eta,\eta)) &= m_B((-\eta,0)) + m_B((0,\eta)) + m_B(\{0\}) \\ &\leq m_B((-\eta,0)) + m_B((0,\eta)) + 2 \\ &\leq m_A((-\eta,0)) + 1 + m_A((0,\eta)) + 1 + 2 \\ &= m_{T_+ \oplus T_-}((0,\eta)) + 4 \\ &= m_{T_+ \oplus T_-}([0,\eta)) + 2 \leq n_T([0,\eta)) + 3. \end{aligned}$$

This completes the proof of Theorem 4.1. □

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INSTITUT FÜR NUMERISCHE MATHEMATIK, TECHNISCHE UNIVERSITÄT GRAZ, STEYRERGASSE 30, 8010 GRAZ, AUSTRIA
E-mail address: behrndt@tugraz.at

INSTITUT FÜR MATHEMATIK, TECHNISCHE UNIVERSITÄT ILMENAU, POSTFACH 100565, D-98684 ILMENAU, GERMANY
E-mail address: leslie.leben@tu-ilmenau.de

DEPARTAMENTO DE MATEMÁTICA - FACULTAD DE CIENCIAS EXACTAS, UNIVERSIDAD NACIONAL DE LA PLATA, C.C. 172, (1900) LA PLATA, ARGENTINA AND INSTITUTO ARGENTINO DE MATEMÁTICA "ALBERTO P. CALDERÓN" (CONICET), SAAVEDRA 15 (1083) BUENOS AIRES, ARGENTINA
E-mail address: francisco@mate.unlp.edu.ar

INSTITUT FÜR MATHEMATIK, TECHNISCHE UNIVERSITÄT ILMENAU, POSTFACH 100565, D-98684 ILMENAU, GERMANY
E-mail address: roland.moews@tu-ilmenau.de

INSTITUT FÜR MATHEMATIK, TECHNISCHE UNIVERSITÄT ILMENAU, POSTFACH 100565, D-98684 ILMENAU, GERMANY
E-mail address: carsten.trunk@tu-ilmenau.de