# Weighted projections into closed subspaces

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**Abstract**

In this paper we study $A$-projections, i.e. operators of a Hilbert space $\mathcal{H}$ which act as projections when a seminorm is considered in $\mathcal{H}$. $A$-projections were introduced by Mitra and Rao [21] for finite dimensional spaces. We relate this concept to the theory of compatibility between positive operators and closed subspaces of $\mathcal{H}$. We also study the relationship between weighted least squares problems and compatibility.

## 1 Introduction

In 1974, S.K. Mitra and C.R. Rao [21] introduced the notion of projection into a subspace with respect to a seminorm. More precisely, given a positive

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(semidefinite) matrix $A \in \mathbb{C}^{n\times n}$ and a subspace $\mathcal{S}$ of $\mathbb{C}^n$, a matrix $T \in \mathbb{C}^{n\times n}$ is called an *A-projection into* $\mathcal{S}$ if $R(T) \subseteq \mathcal{S}$ and

$$\|y - Ty\|_A \leq \|y - s\|_A, \qquad \text{for all } \ y \in \mathbb{C}^n, s \in \mathcal{S},$$

where $\|z\|_A := \langle Az, z\rangle^{1/2} =: \langle z, z\rangle_A^{1/2}$. Notice that an $A$-projection $T$ need not to be an idempotent, but $AT^2 = AT$. This notion is related to very general least squares problems and Mitra and Rao have found several applications in statistics, in particular in linear models, see also [24, 28, 29].

In 1994, S. Hassi and K. Nordström [19] started the study of projections onto closed subspaces in Hilbert spaces, which are orthogonal with respect to an indefinite seminorm. Their paper suggested the notion of compatibility, proposed by G. Corach, A. Maestripieri and D. Stojanoff [8, 9, 10]. A closed subspace $\mathcal{S}$ of a Hilbert space $\mathcal{H}$ is said to be *compatible* with a positive (semidefinite bounded linear) operator $A$ on $\mathcal{H}$ if there exists a (bounded linear) projection $Q$ acting on $\mathcal{H}$ such that $\mathcal{S}$ is the image of $Q$ and $AQ = Q^*A$. This equality means that $Q$ is selfadjoint with respect to the semi-inner product defined by $A$. This notion has several applications in generalized contractions [5, 26, 27], Krein space operators [19, 20], frame theory [2], least squares problems [7], signal processing [14, 15] and so on. It should be noticed that non compatible pairs exist only if $\mathcal{H}$ has infinite dimension [10, 6.2]. Therefore, in order to study the relationship between Mitra-Rao's theory with the compatibility results, which is the main goal of this paper, it is necessary to extend that theory to the infinite dimensional case.

Sections 2 contains notations and preliminary results needed in the sequel, in particular the well-known Douglas factorization theorem [13, 17]. Section 3 contains a short resumé of definitions and the main results of compatibility theory with no proof. In particular, if $(A, \mathcal{S})$ is compatible then a description of the set

$$\mathcal{P}(A, \mathcal{S}) = \{Q \in L(\mathcal{H}) : Q^2 = Q, R(Q) = \mathcal{S}, AQ = Q^*A\}$$

is presented. Section 4 is devoted to develop the theory of $A$-projections in the context of infinite dimensional Hilbert spaces. We only include proofs if they are not similar to those for finite dimensional spaces provided by Mitra and Rao [21, 24]. The set $\Pi(A, \mathcal{S}) = \{T \in L(\mathcal{H}) : T$ is an $A$-projection into $\mathcal{S}\}$ is described and the precise relationship between $\mathcal{P}(A, \mathcal{S})$ and $\Pi(A, \mathcal{S})$ is presented, in the main result of the section, together with some minimality properties. Section 5 deals with least squares problems. An operator $G \in L(\mathcal{H})$ is called an *A-inverse* of a closed range operator $B$ if for each $y \in \mathcal{H}$,

$Gy$ is an $A$-LSS of $Bx = y$, i.e.

$$\|BGy - y\|_A \leq \|Bx - y\|_A, \quad x \in \mathcal{H}.$$

We show that the existence of an $A$-inverse of an operator $B$ is equivalent to the compatibility of the pair $(A, R(B))$. Moreover the set of all $A$-inverses of $B$ is described. The second part of this section deals with restricted $A$-inverses of a certain $B$: $G \in L(\mathcal{H})$ is called an $A$-inverse of $B$ restricted to $\mathcal{M}$ if $R(G) \subseteq \mathcal{M}$ and

$$\|BGy - y\|_A \leq \|Bx - y\|_A, \quad \forall x \in \mathcal{M}.$$

This notion, also due to Rao and Mitra [24], is completely described in terms of some compatibility conditions. In particular, there exists such a $G$ if and only if $(A, B(\mathcal{M}))$ is compatible. The final part deals with the least squares solution of an equation like

$$Bx = y$$

where the vectors $x$'s are measured with the seminorm $\| \, \|_{A_1}$ defined by $A_1 \in L(\mathcal{H})^+$ and the vectors $y$'s are measured with $\| \, \|_{A_2}$, for another $A_2 \in L(\mathcal{H})^+$. Again, the situation is completely described by using certain compatibility conditions. Analogous problems have been considered in [7] and [18]. It should also be mentioned that L. Eldén [16] was the first to study this problem in finite dimensions.

## 2 Preliminaries

Throughout, $\mathcal{H}$ denotes a separable complex Hilbert space, $L(\mathcal{H})$ the algebra of linear bounded operators of $\mathcal{H}$ and $L(\mathcal{H})^+$ the cone of positive operators. Also, $\mathcal{Q}$ denotes the subset of $L(\mathcal{H})$ of oblique projections, i.e., $\mathcal{Q} = \{Q \in L(\mathcal{H}), \, Q^2 = Q\}$ and $\mathcal{P}$ the set of orthogonal projections, i.e. $\mathcal{P} = \{P \in L(\mathcal{H}) : P^2 = P = P^*\}$.

For every $A \in L(\mathcal{H})$, $R(A)$ denotes the range of $A$ and $N(A)$ its nullspace. Given $\mathcal{M}$ and $\mathcal{N}$ two closed subspaces of $\mathcal{H}$, then $\mathcal{M} \dot{+} \mathcal{N}$ denotes the direct sum of $\mathcal{M}$ and $\mathcal{N}$, $\mathcal{M} \oplus \mathcal{N}$ the orthogonal sum and $\mathcal{M} \ominus \mathcal{N} = \mathcal{M} \cap (\mathcal{M} \cap \mathcal{N})^\perp$. If $\mathcal{M} \dot{+} \mathcal{N} = \mathcal{H}$, denote by $P_{\mathcal{M}//\mathcal{N}}$ the oblique projection with range $\mathcal{M}$ and nullspace $\mathcal{N}$; in particular, $P_{\mathcal{M}} = P_{\mathcal{M}//\mathcal{M}^\perp}$.

Given a closed range operator $A$, $A^\dagger$ denotes the *Moore Penrose inverse* of $A$, i.e. $A^\dagger$ is the unique solution of the system

$$AXA = A, \quad XAX = X, \quad (AX)^* = AX, \quad (XA)^* = XA.$$

Given a closed subspace $\mathcal{S}$ of $\mathcal{H}$, then $P_{\mathcal{S}}$ induces a matrix decomposition for each $A \in L(\mathcal{H})$ as follows: if $P = P_{\mathcal{S}}$ then $A \in L(\mathcal{H})$ can be written as

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

where $a_{11} = PAP|_{\mathcal{S}} \in L(\mathcal{S})$, $a_{12} = PA(I-P)|_{\mathcal{S}^\perp} \in L(\mathcal{S}^\perp, \mathcal{S})$, $a_{21} = (I - P)AP|_{\mathcal{S}} \in L(\mathcal{S}, \mathcal{S}^\perp)$ and $a_{22} = (I-P)A(I-P)|_{\mathcal{S}^\perp} \in L(\mathcal{S}^\perp)$. If $A \in L(\mathcal{H})^+$, then

$$A = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}, \tag{1}$$

with $R(b) \subseteq R(a^{1/2})$, see [1]. Throughout this work, we will use the matrix representation of $A$ given by (1).

Given $A \in L(\mathcal{H})^+$, consider the following semi-inner product on $\mathcal{H}$:

$$\langle x, y \rangle_A = \langle Ax, y \rangle, \qquad x, y \in \mathcal{H}.$$

The seminorm associated is given by

$$\|x\|_A = \langle x, x \rangle_A^{1/2} = \|A^{1/2}x\|, \qquad x \in \mathcal{H}.$$

An operator $C \in L(\mathcal{H})$ is called $A$-*selfadjoint* if $\langle Cx, y \rangle_A = \langle x, Cy \rangle_A$ for all $x, y \in \mathcal{H}$, or equivalently $AC = C^*A$.

The following result, due to R. G. Douglas, characterizes the operator range inclusion.

**Theorem 2.1.** *(Douglas). Consider Hilbert spaces $\mathcal{H}, \mathcal{K}, \mathcal{G}$ and operators $A \in L(\mathcal{H}, \mathcal{G}), B \in L(\mathcal{K}, \mathcal{G})$. The following conditions are equivalent:*

1. $R(B) \subseteq R(A)$,
2. $BB^* \leq \lambda AA^*$, *for some* $\lambda > 0$,
3. *the equation* $AX = B$ *has a solution in* $L(\mathcal{K}, \mathcal{H})$.

   *In this case, there exists a unique* $D \in L(\mathcal{K}, \mathcal{H})$ *such that* $AD = B$ *and* $R(D) \subseteq \overline{R(A^*)}$*; moreover,* $\|D\|^2 = \inf\{\lambda > 0 : \ BB^* \leq \lambda AA^*\}$ *and* $N(D) = N(B)$*. This solution is called the reduced solution of* $AX = B$.

The reader is referred to [13, Theorem 1] and [17, Theorem 2.1] for the proof of Douglas' theorem.

# 3 Compatibility

Given $A \in L(\mathcal{H})^+$ and $\mathcal{S}$ a closed subspace of $\mathcal{H}$, consider the following set

$$\mathcal{P}(A, \mathcal{S}) = \{Q \in \mathcal{Q} : R(Q) = \mathcal{S}, AQ = Q^*A\}.$$

The pair $(A, \mathcal{S})$ is called *compatible* if the set $\mathcal{P}(A, \mathcal{S})$ is not empty, or equivalently, if there exists a projection $Q \in \mathcal{Q}$ with range $\mathcal{S}$ such that $AQ = Q^*A$.

The following proposition collects some results about compatibility that can be found in [9, 12].

**Proposition 3.1.** *Consider $A \in L(\mathcal{H})^+$ with matrix form given by equation (1) and $\mathcal{S}$ a closed subspace of $\mathcal{H}$.*

1. *If the pair $(A, \mathcal{S})$ is compatible, then $\mathcal{S} + N(A)$ is closed.*
2. *If $A \in L(\mathcal{H})^+$ has closed range and $\mathcal{S} + N(A)$ is closed, then $(A, \mathcal{S})$ is compatible.*
3. *The pair $(A, \mathcal{S})$ is compatible if and only if $\mathcal{H} = \mathcal{S} + A(\mathcal{S})^\perp$.*
4. *The pair $(A, \mathcal{S})$ is compatible if and only if $R(b) \subseteq R(a)$.*

As a consequence of Douglas' theorem and item 4 of the above proposition, we obtain the following characterization of the set $\mathcal{P}(A, \mathcal{S})$, see [8] for details.

**Corollary 3.2.** *Consider $(A, \mathcal{S})$ compatible, then*

$$\mathcal{P}(A, \mathcal{S}) = \{ \begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix} : x \in L(\mathcal{S}^\perp, \mathcal{S}) \text{ and } ax = b\}.$$

If the pair $(A, \mathcal{S})$ is compatible, there is a distinguished element $P_{A,\mathcal{S}} \in \mathcal{P}(A, \mathcal{S})$, namely the unique projection onto $\mathcal{S}$ with kernel $A(\mathcal{S})^\perp \ominus \mathcal{N}$, where $\mathcal{N} = A(\mathcal{S})^\perp \cap \mathcal{S} = N(A) \cap \mathcal{S}$. By [10, Proposition 4.1], $P_{A,\mathcal{S}} = P_{A,\mathcal{S}\ominus\mathcal{N}} + P_{\mathcal{N}}$ and $P_{A,\mathcal{S}\ominus\mathcal{N}} = P_{\mathcal{S}\ominus\mathcal{N}//A(\mathcal{S})^\perp}$. Then the matrix decomposition of $P_{A,\mathcal{S}}$ induced by $P_{\mathcal{S}}$ is given by

$$P_{A,\mathcal{S}} = \begin{pmatrix} 1 & d \\ 0 & 0 \end{pmatrix},$$

where $d \in L(\mathcal{S}^\perp, \mathcal{S})$ is the reduced solution of $ax = b$.

It is easy to see that the pair $(A, \mathcal{S})$ is compatible if and only if the pair $(A, \mathcal{S} \ominus \mathcal{N})$ is compatible.

# 4 Weighted projections

Along this work $A$ is a positive bounded operator, i.e. $A \in L(\mathcal{H})^+$ and $\mathcal{S}$ is a closed subspace of $\mathcal{H}$.

The following definition is due to Mitra and Rao for operators acting on finite dimensional Hilbert spaces, see [21].

**Definition 4.1.** An operator $T \in L(\mathcal{H})$ is called an *$A$-projection into $\mathcal{S}$* if $R(T) \subseteq \mathcal{S}$ and

$$(2) \qquad \|y - Ty\|_A \leq \|y - s\|_A, \quad \textit{for all } y \in \mathcal{H}, \quad \textit{for all } s \in \mathcal{S}.$$

$T$ is called an *$A$-projection* if $T$ is an $A$-projection into $\overline{R(T)}$.

An $A$-projection into $\mathcal{S}$ is also called an *$A$-weighted least squares process*, see [6, 25].

**Remark 4.2.** It is not difficult to see that inequality (2) alone does not imply the boundedness of $T$. Indeed, if $A$ has infinite dimensional nullspace it is enought to consider $T = T_1 P_{N(A)} + P_{\overline{R(A)}}$, with $T_1 : N(A) \to N(A)$ unbounded. Similarly, it can be proved that the range of an $A$-projection is not necessarily closed.

**Definition 4.3.** The operator $T \in L(\mathcal{H})$ is an *$A$-idempotent* if $AT^2 = AT$.

Observe that the definition of $A$-idempotent only depends on $N(A)$ in the sense that if $A, B \in L(\mathcal{H})$ are such that $N(A) = N(B)$ then $T$ is $A$-idempotent if and only if $T$ is $B$-idempotent.

The next two propositions generalize some of the results in [21]. The proofs for infinite dimensional Hilbert spaces follow essentially the same steps.

**Proposition 4.4.** *Let $T \in L(\mathcal{H})$. The following statements are equivalent:*

1. *$T$ is an $A$-projection,*
2. *$T^*AT = AT$,*
3. *$AT = T^*A$ and $AT^2 = AT$; or equivalently, $T$ is an $A$-selfadjoint and also $A$-idempotent.*

*Proof.* See [21, Lemma 2.1 and Lemma 2.2]. □

**Proposition 4.5.** *Consider $T \in L(\mathcal{H})$ such that $R(T) \subseteq \mathcal{S}$. The following conditions are equivalent:*

1. *$T$ is an $A$-projection into $\mathcal{S}$,*
2. *$AT = T^*A$ and $ATP_{\mathcal{S}} = AP_{\mathcal{S}}$,*
3. *$P_{\mathcal{S}}AT = P_{\mathcal{S}}A$.*

*Proof.* *1* $\rightarrow$ *2*: Let $T$ be an $A$-projection into $\mathcal{S}$. In particular $T$ is an $A$-projection. Then by Proposition 4.4, $AT = T^*A$. On the other hand, for each $y \in \mathcal{H}$ it holds that $\|y - Ty\|_A = \|A^{1/2}y - A^{1/2}Ty\| \leq \|y - s\|_A$ for all $s \in \mathcal{S}$. In particular, given $x \in \mathcal{H}$, then $\|A^{1/2}P_{\mathcal{S}}x - A^{1/2}TP_{\mathcal{S}}x\| \leq \|P_{\mathcal{S}}x - s\|_A$ for all $s \in \mathcal{S}$. Therefore, $A^{1/2}P_{\mathcal{S}} = A^{1/2}TP_{\mathcal{S}}$ , so that $AP_{\mathcal{S}} = ATP_{\mathcal{S}}$.

*2* $\rightarrow$ *3*: If $AT = T^*A$ and $ATP_{\mathcal{S}} = AP_{\mathcal{S}}$ then $P_{\mathcal{S}}A = P_{\mathcal{S}}T^*A = P_{\mathcal{S}}AT$, so that $P_{\mathcal{S}}A = P_{\mathcal{S}}AT$.

*3* $\rightarrow$ *1*: Since $P_{\mathcal{S}}AT = P_{\mathcal{S}}A$, then $T^*AP_{\mathcal{S}} = AP_{\mathcal{S}}$ so that $T^*AT = AT = T^*A$ because $R(T) \subseteq \mathcal{S}$. Therefore, by Proposition 4.4, $T$ is an $A$-projection into $\overline{R(T)}$, then $\|y - Ty\|_A \leq \|y - Tx\|_A$ for all $x, y \in \mathcal{H}$. It remains to prove that $\|y - Ty\|_A \leq \|y - P_{\mathcal{S}}x\|_A$ for all $x, y \in \mathcal{H}$. Since $ATP_{\mathcal{S}} = AP_{\mathcal{S}}$, then $A^{1/2}TP_{\mathcal{S}} = A^{1/2}P_{\mathcal{S}}$, so that $\|y - Ty\|_A \leq \|y - TP_{\mathcal{S}}x\|_A = \|y - P_{\mathcal{S}}x\|_A$ for all $x, y \in \mathcal{H}$. □

**Remark 4.6.** By Proposition 4.4 and Proposition 4.5, given $T \in L(\mathcal{H})$ such that $R(T) \subseteq \mathcal{S}$ it holds that $T$ is an $A$-projection into $\mathcal{S}$ if and only if $T$ is an $A$-projection and $ATP_{\mathcal{S}} = AP_{\mathcal{S}}$.

**Lemma 4.7.** *If $T \in L(\mathcal{H})$ is an $A$-idempotent ($A$-projection) then $I - T$ is an $A$-idempotent ($A$-projection).*

*Proof.* If $T \in L(\mathcal{H})$ is an $A$-idempotent, then $A(I-T)^2 = A(I-2T+T^2) = A(I - T)$, i.e. $I - T$ is an $A$-idempotent. If $T$ is an $A$-projection then, by Proposition 4.4, $T$ is $A$-idempotent and $A$-selfadjoint. Hence $I - T$ is $A$-idempotent and $A(I - T) = A - T^*A = (I - T)^*A$. Again, by Proposition 4.4, $I - T$ is an $A$-projection. □

The following result characterizes $A$-projections in terms of oblique projections.

**Lemma 4.8.** *Let $T \in L(\mathcal{H})$. Then $T$ is an $A$-projection if and only if $P_{\overline{R(A)}}T \in \mathcal{Q}$ and it is $A$-selfadjoint.*

*Proof.* Let $T$ be an $A$-projection and denote $P = P_{\overline{R(A)}}$. By Proposition 4.4, it holds that $AT = T^*A$ and $AT = AT^2$, then $(PT)^2 = A^\dagger ATPT = A^\dagger T^*APT = A^\dagger T^*AT = A^\dagger AT = PT$ and $(PT)^*A = T^*PA = T^*A = AT = APT$. Conversely, if $PT \in \mathcal{Q}$ and it is $A$-selfadjoint then $AT = APT = (PT)^*A = T^*A$ so that $T$ is $A$-selfadjoint. Also, $AT^2 = APT^2 = (PT)^*AT = (PT)^*APT = A(PT)^2 = APT = AT$ so that $T$ is $A$-idempotent. By Proposition 4.4, $T$ is an $A$-projection. □

The next result shows that $A$-projections behave like orthogonal projections, under the seminorm induced by $A$, in the sense that for an $A$-idempotent, the condition of being $A$-selfadjoint is equivalent to being an $A$-contraction, or $A$-positive. For $A$-contractions see for example [5] and [26].

**Proposition 4.9.** *Consider $T \in L(\mathcal{H})$ such that $T$ is an $A$-idempotent. Then the following statements are equivalent:*

1. *$T$ is $A$-selfadjoint (so that $T$ is an $A$-projection),*
2. *$R(I - T) \subseteq R(AT)^\perp$,*
3. *$T$ is an $A$-contraction, i.e. $T^*AT \leq A$.*

*Proof.* *1 → 2*: Suppose that $AT = T^*A$. Consider $y \in R(I - T)$ and $z \in \mathcal{H}$ such that $y = z - Tz$. Then, for $x \in \mathcal{H}$

$$\langle ATx, y\rangle = \langle x, ATy\rangle = \langle x, AT(z - Tz)\rangle = 0,$$

because $AT^2 = AT$. Therefore, $y \in R(AT)^\perp$.

*2 → 3*: For $x, y \in \mathcal{H}$,

$$\langle ATx, y\rangle = \langle ATx, Ty + (I - T)y\rangle = \langle ATx, Ty\rangle = \langle T^*ATx, y\rangle$$

because $R(I - T) \subseteq R(AT)^\perp$. Therefore, $AT = T^*AT = T^*A$ and $T$ is $A$-selfadjoint. Then $T$ is an $A$-projection. Also, by Lemma 4.7, $E = I - T$ is an $A$-projection so that $AE = AE^2 = E^*AE \in L(\mathcal{H})^+$. Therefore, $A = A(T + E) = T^*AT + E^*AE \geq T^*AT$.

*3 → 1*: Since $T^*AT \leq A$, by Douglas' theorem, the equation $A^{1/2}X = T^*A^{1/2}$ admits a solution. Let $D$ be the reduced solution of $A^{1/2}X = T^*A^{1/2}$, i.e. $D$ satisfies $A^{1/2}D = T^*A^{1/2}$ and $R(D) \subseteq \overline{R(A)}$. Then, observe that

$$A^{1/2}D^2 = (T^*)^2A^{1/2} = T^*A^{1/2},$$

because $T$ is an $A$-idempotent. Also, $R(D^2) \subseteq R(D) \subseteq \overline{R(A)}$. Therefore $D^2$ is also a reduced solution of $A^{1/2}X = T^*A^{1/2}$, so that $D^2 = D$ by the uniqueness of the reduced solution. On the other hand, by Douglas' theorem, $\|D\|^2 = \inf\{\lambda : T^*AT \leq \lambda A\} \leq 1$, because $T^*AT \leq A$. Since $D^2 = D$ and $\|D\| \leq 1$, then automatically it holds that $D^* = D$, so that $T^*A = A^{1/2}DA^{1/2}$ is selfadjoint, i.e. $T^*A = AT$. □

**Corollary 4.10.** *Let $T \in L(\mathcal{H})$ be an $A$-idempotent. The following statements are equivalent:*

1. *$T$ is an $A$-projection,*
2. *$\|T\|_A = 1$,*
3. *$\langle Tx, x\rangle_A \geq 0, \quad \forall x \in \mathcal{H}$, i.e. $T$ is $A$-positive.*

*Proof.* *1.* → *2.*: Since $A$ is an $A$-projection, by Proposition 4.9, $T^*AT \leq A$. Then, for $x \in \mathcal{H}$,

$$\|Tx\|_A^2 = \langle ATx, Tx\rangle = \langle T^*ATx, x\rangle \leq \langle Ax, x\rangle = \|x\|_A^2,$$

so that $\|T\|_A \leq 1$. Also,

$$\|T(Tx)\|_A = \|A^{1/2}T^2x\| = \|A^{1/2}Tx\| = \|Tx\|_A,$$

because $T$ is $A$-idempotent. Therefore $\|T\|_A = 1$.

*2.* → *3.*: Consider $T \in L(\mathcal{H})$ such that $AT = AT^2$ and $\|T\|_A = 1$. Observe that

$$\langle T^*ATx, x\rangle = \langle ATx, Tx\rangle = \|Tx\|_A^2 \leq \|x\|_A^2 = \langle Ax, x\rangle,$$

so that $T^*AT \leq A$, and by Proposition 4.9, $T$ is an $A$-projection. By Proposition 4.4, it follows that $AT = T^*AT \in L(\mathcal{H})^+$.

*3.* → *1.*: Since $AT \in L(\mathcal{H})^+$, then $AT = T^*A$. Also $T$ is $A$-idempotent so that, by Proposition 4.4, $T$ is an $A$-projection. □

In the following paragraphs we study conditions for the existence of $A$-projections into $\mathcal{S}$ and we characterize the set of these projections.

Define

$$\Pi(A, \mathcal{S}) = \{T \in L(\mathcal{H}) : T \text{ is an } A\text{-projection into } \mathcal{S}\},$$

and

$$\Pi(A) = \{T \in L(\mathcal{H}) : T \text{ is an } A\text{-projection}\}.$$

By Proposition 4.5, it follows that

$$\Pi(A, \mathcal{S}) = \{T \in L(\mathcal{H}) : R(T) \subseteq \mathcal{S},\ AT = T^*A,\ ATP_{\mathcal{S}} = AP_{\mathcal{S}}\} \tag{3}$$

and

$$\Pi(A) = \{T \in L(\mathcal{H}) : AT = T^*A,\ AT^2 = AT\}.$$

In particular if $A = I$, then $\Pi(I, \mathcal{S}) = \{P_{\mathcal{S}}\}$ and $\Pi(I) = \mathcal{P}$.

The next result gives a characterization of $A$-projections into $\mathcal{S}$ in terms of the matrix decomposition induced by $P_{\mathcal{S}}$. Recall that $A = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}$ is the matrix representation of $A$, as in (1).

**Proposition 4.11.** $\Pi(A, \mathcal{S}) = \{T \in L(\mathcal{H}) : T = \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix},\ ax = a,\ ay = b\}$.

*Proof.* By equation (3), $T \in \Pi(A, \mathcal{S})$ if and only if $R(T) \subseteq \mathcal{S}$, $T^*A = AT$ and $ATP_{\mathcal{S}} = AP_{\mathcal{S}}$. Observe that $R(T) \subseteq \mathcal{S}$ if and only if the matrix representation of $T$ induced by $P_{\mathcal{S}}$ is $T = \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}$. In this case, $AT = T^*A$ if and only if $ax = x^*a,\ ay = x^*b$ and $b^*y = y^*b$. Also, $ATP_{\mathcal{S}} = AP_{\mathcal{S}}$ is equivalent to $ax = a$ and $b^*x = b^*$. Then $T \in \Pi(A, \mathcal{S})$ if and only if

$$ax = x^*a,\ ay = x^*b,\ b^*y = y^*b,\ ax = a \ \text{ and } \ b^*x = b^*. \tag{4}$$

It is not difficult to see that (4) is equivalent to $ax = a$ and $ay = b$. □

**Corollary 4.12.** *If the pair $(A, \mathcal{S})$ is compatible, then*

$$\mathcal{P}(A, \mathcal{S}) \subseteq \Pi(A, \mathcal{S}).$$

*Proof.* It follows from Corollary 3.2 and Proposition 4.11. □

Applying item *3* of Proposition 4.5, we obtain the following

**Corollary 4.13.** $\Pi(A, \mathcal{S}) = \{T \in L(\mathcal{H}) : R(T) \subseteq \mathcal{S}$ *and $T$ is a solution of the equation* $P_{\mathcal{S}}AX = P_{\mathcal{S}}A\}$.

The next result shows the relationship between the compatibility of the pair $(A, \mathcal{S})$ and the existence of $A$-projections into $\mathcal{S}$.

**Proposition 4.14.** *The pair $(A, \mathcal{S})$ is compatible if and only if there exists an $A$-projection into $\mathcal{S}$.*

*Proof.* By Proposition 4.11, the set $\Pi(A, \mathcal{S})$ is not empty if and only if the equation $ay = b$ admits a solution (observe that $ax = a$ always admits a solution). By Douglas' theorem this is equivalent to the condition $R(b) \subseteq R(a)$, or equivalently by Proposition 3.1, the pair $(A, \mathcal{S})$ is compatible. □

**Remark 4.15.** By the above proposition, it holds that $\Pi(A, \mathcal{S}) \neq \emptyset$ if and only if $\mathcal{P}(A, \mathcal{S}) \neq \emptyset$.

Recall that $\mathcal{N} = \mathcal{S} \cap A(\mathcal{S})^{\perp} = \mathcal{S} \cap N(A)$.

**Proposition 4.16.** *Let $T \in L(\mathcal{H})$ with $R(T) \subseteq \mathcal{S}$. Then $T$ is an $A$-projection into $\mathcal{S}$ if and only if $(A, \mathcal{S})$ is compatible and $P_{\mathcal{S}\ominus\mathcal{N}}T = P_{A,\mathcal{S}\ominus\mathcal{N}}$.*

*Proof.* Suppose $T$ is an $A$-projection into $\mathcal{S}$. Let $Q = P_{\mathcal{S}\ominus\mathcal{N}}T$. Then $R(Q) \subseteq \mathcal{S}$ and $AT = AQ$. Since $T$ is an $A$-projection, then $AQ = AT = T^*A = Q^*A$, so $Q$ is $A$-selfadjoint. Moreover $AQP_{\mathcal{S}} = ATP_{\mathcal{S}} = AP_{\mathcal{S}}$, then $Q$ is an $A$-projection into $\mathcal{S}$. Therefore $AQ^2 = AQ$ so that $R(Q^2 - Q) \subseteq N(A) \cap (\mathcal{S} \ominus \mathcal{N}) = \mathcal{N} \cap \mathcal{N}^{\perp} = \{0\}$, or equivalently $Q^2 = Q$. Moreover, from $AQP_{\mathcal{S}} = AP_{\mathcal{S}}$ and $AQ = Q^*A$ it follows that $AQP_{\mathcal{S}\ominus\mathcal{N}} = Q^*AP_{\mathcal{S}\ominus\mathcal{N}} = Q^*AP_{\mathcal{S}} = AQP_{\mathcal{S}} = AP_{\mathcal{S}} = AP_{\mathcal{S}\ominus\mathcal{N}}$. Therefore $A(QP_{\mathcal{S}\ominus\mathcal{N}} - P_{\mathcal{S}\ominus\mathcal{N}}) = 0$. Also $R(QP_{\mathcal{S}\ominus\mathcal{N}} - P_{\mathcal{S}\ominus\mathcal{N}}) \subseteq \mathcal{S} \ominus \mathcal{N}$. Hence $R(QP_{\mathcal{S}\ominus\mathcal{N}} - P_{\mathcal{S}\ominus\mathcal{N}}) \subseteq N(A) \cap (\mathcal{S} \ominus \mathcal{N}) = \{0\}$, so that $QP_{\mathcal{S}\ominus\mathcal{N}} = P_{\mathcal{S}\ominus\mathcal{N}}$ and then $\mathcal{S} \ominus \mathcal{N} \subseteq R(Q)$. Therefore $R(Q) = \mathcal{S}\ominus\mathcal{N}$. Since $AQ = Q^*A$, $Q^2 = Q$ and $R(Q) = \mathcal{S}\ominus\mathcal{N}$, it follows that $Q = P_{A,\mathcal{S}\ominus\mathcal{N}}$. Then $(A, \mathcal{S} \ominus \mathcal{N})$ is compatible, so that $(A, \mathcal{S})$ is compatible (see Section 3). Conversely, if $(A, \mathcal{S})$ is compatible and $P_{\mathcal{S}\ominus\mathcal{N}}T = P_{A,\mathcal{S}\ominus\mathcal{N}}$, then $T = P_{A,\mathcal{S}\ominus\mathcal{N}} + P_{\mathcal{N}}T$, so that $AT = T^*A$ and $ATP_{\mathcal{S}} = AP_{A,\mathcal{S}\ominus\mathcal{N}}P_{\mathcal{S}} = AP_{\mathcal{S}\ominus\mathcal{N}} = AP_{\mathcal{S}}$. Then $T$ is an $A$-projection into $\mathcal{S}$.

□

The following result shows that $\Pi(A, \mathcal{S})$ is an affine manifold.

**Proposition 4.17.** *If the pair $(A, \mathcal{S})$ is compatible, then*

$$\Pi(A, \mathcal{S}) = P_{A,\mathcal{S}} + L(\mathcal{H}, \mathcal{N}).$$

*Proof.* Let $T \in \Pi(A, \mathcal{S})$, then by Proposition 4.16, it follows that $T = P_{A,\mathcal{S}\ominus\mathcal{N}} + P_{\mathcal{N}}T = P_{A,\mathcal{S}} + P_{\mathcal{N}}(T - I) \in P_{A,\mathcal{S}} + L(\mathcal{H}, \mathcal{N})$, see Section 3. Conversely, if $T = P_{A,\mathcal{S}} + W$ with $W \in L(\mathcal{H}, \mathcal{N})$, then $P_{\mathcal{S}\ominus\mathcal{N}}T = P_{A,\mathcal{S}\ominus\mathcal{N}}$. By Proposition 4.16, it holds that $T$ is an $A$-projection into $\mathcal{S}$. □

**Remark 4.18.** Given $T \in \Pi(A, \mathcal{S})$, observe that $AT = AP_{A,\mathcal{S}}$ since $\mathcal{N} \subseteq N(A)$. Hence $A(R(T)) = R(AT) = A(\mathcal{S})$.

A natural question is whether $\mathcal{P}(A, \mathcal{S})$ equals $\Pi(A, \mathcal{S})$. We prove now that this happens if and only if $\mathcal{P}(A, \mathcal{S})$ and/or $\Pi(A, \mathcal{S})$ has cardinal 1.

**Theorem 4.19.** *Suppose that the pair $(A, \mathcal{S})$ is compatible. Then the following statements are equivalent:*

1. $\mathcal{P}(A, \mathcal{S}) = \Pi(A, \mathcal{S})$,
2. $\mathcal{N} = \{0\}$.
3. $card(\Pi(A, \mathcal{S})) = 1$,
4. $card(\mathcal{P}(A, \mathcal{S})) = 1$.

*Proof.* *1→2*: Suppose $\mathcal{N} \neq \{0\}$ and consider $T = P_{A,\mathcal{S}} + P_{\mathcal{N}}$. Then, by the previous theorem $T \in \Pi(A, \mathcal{S})$. But it is not difficult to see that $T^2 \neq T$, so that $T \notin \mathcal{P}(A, \mathcal{S})$.

*2→3*: If $\mathcal{N} = \{0\}$, by the previous theorem $\Pi(A, \mathcal{S}) = \{P_{A,\mathcal{S}}\}$.

*3→4*: It follows by Corollary 4.12 and Remark 4.15.

*4→1*: If $card(\mathcal{P}(A, \mathcal{S})) = 1$, by [8, Theorem 3.5] it holds that $\mathcal{N} = \{0\}$ and $\mathcal{P}(A, \mathcal{S}) = \{P_{A,\mathcal{S}}\}$. Hence, by the previous result, $\Pi(A, \mathcal{S}) = \{P_{A,\mathcal{S}}\} = \mathcal{P}(A, \mathcal{S})$. $\square$

**Corollary 4.20.** *If $A$ is invertible, then $\Pi(A, \mathcal{S}) = \mathcal{P}(A, \mathcal{S}) = \{P_{A,\mathcal{S}}\}$.*

**Remark 4.21.** Under the hypothesis of the above corollary, $P_{A,\mathcal{S}}$ can be compute as

$$P_{A,\mathcal{S}} = A^{-1/2} P_{\overline{A^{1/2}(\mathcal{S})}} A^{1/2},$$

see [29, Section 3] or, more generally [11, Proposition 3.3].

Some minimality properties of $P_{A,\mathcal{S}}$ respect to $\mathcal{P}(A, \mathcal{S})$ are proved in [8, Theorem 3.5] and [9, Theorem 3.2]. The next result extends these properties to the set $\Pi(A, \mathcal{S})$.

**Proposition 4.22.** *Suppose that the pair $(A, \mathcal{S})$ is compatible. Then*

1. $\|P_{A,\mathcal{S}}\| = \min\{\|T\| : T \in \Pi(A, \mathcal{S})\}$.
2. $\|(I - P_{A,\mathcal{S}})x\| \leq \|(I - T)x\|$, *for all $x \in \mathcal{H}$ and every $T \in \Pi(A, \mathcal{S})$.*

*Proof. 1.* Consider $T \in \Pi(A, \mathcal{S})$. Then, by Proposition 4.16, $T = P_{A,\mathcal{S}\ominus\mathcal{N}} + W$ for some $W \in L(\mathcal{H}, \mathcal{N})$. Then

$$\|Tx\|^2 = \|P_{A,\mathcal{S}\ominus\mathcal{N}}x\|^2 + \|Wx\|^2 \geq \|P_{A,\mathcal{S}\ominus\mathcal{N}}x\|^2, \qquad \text{for all } x \in \mathcal{H}.$$

Therefore, $\|T\| \geq \|P_{A,\mathcal{S}\ominus\mathcal{N}}\|$. Finally, observe that

$$\begin{aligned} \|P_{A,\mathcal{S}}\|^2 &= \|P_{A,\mathcal{S}}(P_{A,\mathcal{S}})^*\| = \|P_{\mathcal{N}} + P_{A,\mathcal{S}\ominus\mathcal{N}}(P_{A,\mathcal{S}\ominus\mathcal{N}})^*\| \\ &= \max\{\|P_{\mathcal{N}}\|, \|P_{A,\mathcal{S}\ominus\mathcal{N}}\|\} = \|P_{A,\mathcal{S}\ominus\mathcal{N}}\|^2, \end{aligned}$$

because $P_{A,\mathcal{S}} = P_{A,\mathcal{S}\ominus\mathcal{N}} + P_{\mathcal{N}}$, $P_{A,\mathcal{S}\ominus\mathcal{N}}P_{\mathcal{N}} = 0 = P_{\mathcal{N}}(P_{A,\mathcal{S}\ominus\mathcal{N}})^*$ and $P_{\mathcal{N}}P_{A,\mathcal{S}\ominus\mathcal{N}} = 0 = (P_{A,\mathcal{S}\ominus\mathcal{N}})^*P_{\mathcal{N}}$.

*2.* Consider $T \in \Pi(A, \mathcal{S})$. By Proposition 4.17, $T = P_{A,\mathcal{S}} + W$ for some $W \in L(\mathcal{H}, \mathcal{N})$. Observe that

$$\|(I - T)x\|^2 = \|(I - P_{A,\mathcal{S}})x\|^2 + \|Wx\|^2 \leq \|(I - P_{A,\mathcal{S}})x\|^2, \quad \text{for all } x \in \mathcal{H},$$

because $R(I - P_{A,\mathcal{S}}) = N(P_{A,\mathcal{S}}) = A(\mathcal{S})^\perp \ominus \mathcal{N} \subseteq \mathcal{N}^\perp$ □

As an application, we characterize the abstract splines in terms of weighted projections. The theory of abstract splines is due to Atteia [3]. The reader is referred to [10] for some relationships between the notion of compatibility and abstract splines in Hilbert spaces.

Given $C \in L(\mathcal{H})$, $\mathcal{S}$ a closed subspace of $\mathcal{H}$ and $x \in \mathcal{H}$, an *abstract spline* or a $(C, \mathcal{S})$-*spline interpolant* to $x$ is any element of the set

$$sp(C, \mathcal{S}, x) = \{y \in x + \mathcal{S} : \|Cy\| = \min_{s \in \mathcal{S}} \|C(x + s)\|\}.$$

If $A = C^*C \in L(\mathcal{H})^+$, observe that $\|y\|_A = \|Cy\|$, for $y \in \mathcal{H}$. Then

$$sp(C, \mathcal{S}, x) = \{y \in x + \mathcal{S} : \|y\|_A = d_A(x, \mathcal{S})\|\}.$$

where $d_A(x, \mathcal{S}) = \inf_{s \in \mathcal{S}} \|x + s\|_A$.

The next proposition contains some results on splines, the proofs can be found in [10].

**Proposition 4.23.** *Given $C \in L(\mathcal{H})$, consider $A = C^*C$. Then*

1. $sp(C, \mathcal{S}, x) = (x + \mathcal{S}) \cap A(\mathcal{S})^\perp$, *for $x \in \mathcal{H}$.*
2. *$sp(C, \mathcal{S}, x)$ is not empty for every $x \in \mathcal{H}$ if and only if the pair $(A, \mathcal{S})$ is compatible.*

3. *If* $(A, \mathcal{S})$ *is compatible and* $x \in \mathcal{H} \setminus \mathcal{S}$*, then* $sp(C, \mathcal{S}, x) = \{(I-Q)x : Q \in \mathcal{P}(A, \mathcal{S})\}$.

**Proposition 4.24.** *Consider* $C \in L(\mathcal{H})$ *and suppose that* $(A, \mathcal{S})$ *is compatible, where* $A = C^*C$*. For every nonzero* $x \in \mathcal{H}$*, it holds that*

$$sp(C, \mathcal{S}, x) = \{(I-T)x : T \in \Pi(A, \mathcal{S})\}.$$

*Proof.* Consider $y \in sp(C, \mathcal{S}, x)$. By item *1.* of Proposition 4.23 there exists $s \in \mathcal{S}$ such that $x = s + y$ and $y = x - s \in A(\mathcal{S})^\perp$. We are looking for $T \in \Pi(A, \mathcal{S})$ such that $(I-T)x = y$, or equivalently $Tx = s$. Note that $y_1 = (I - P_{A,\mathcal{S}})x \in N(P_{A,\mathcal{S}}) \subseteq A(\mathcal{S})^\perp$. Then

$$y_1 - y = (I - P_{A,\mathcal{S}})x - (x - s) = s - P_{A,\mathcal{S}}x \in \mathcal{S} \cap A(\mathcal{S})^\perp = \mathcal{N}.$$

Therefore, since $x \neq 0$, we can consider $W \in L(\mathcal{H}, \mathcal{N})$ sucht that $Wx = s - P_{A,\mathcal{S}}x$. By Proposition 4.17 it follows that $T = P_{A,\mathcal{S}} + W \in \Pi(A, \mathcal{S})$; moreover $Tx = s$.

Conversely, let $T \in \Pi(A, \mathcal{S})$. Then $(I-T)x \in (x+\mathcal{S}) \cap R(I-T)$. But, by Proposition 4.9 and Remark 4.18, $R(I-T) \subseteq R(AT)^\perp = A(\mathcal{S})^\perp$. Therefore

$$(I-T)x \in (x + \mathcal{S}) \cap A(\mathcal{S})^\perp = sp(C, \mathcal{S}, x),$$

by the above proposition. □

Observe that, by item *1* of Proposition 4.23, $sp(C, \mathcal{S}, 0) = \mathcal{N}$.

# 5 Weighted inverses

Throughout this section, $A \in L(\mathcal{H})^+$ and $B \in L(\mathcal{H})$ is a closed range operator.

**Definition 5.1.** Given $y \in \mathcal{H}$, $x_0 \in \mathcal{H}$ is an *A-least squares solution* or an *A-LSS* of $Bx = y$ if

$$\|y - Bx_0\|_A \leq \|y - Bx\|_A, \quad x \in \mathcal{H}. \tag{5}$$

**Remark 5.2.** Given $y \in \mathcal{H}$, $x_0$ satisfies (5) if and only if $\|A^{1/2}(y - Bx_0)\| \leq \|A^{1/2}(y - Bx)\| = \|A^{1/2}(y - Bx_0) + A^{1/2}B(x_0 - x)\|$ for all $x \in \mathcal{H}$, or equivalently $\langle A^{1/2}y - A^{1/2}Bx_0, A^{1/2}Bz\rangle = 0$ for all $z \in \mathcal{H}$ (recall that given $a, b \in \mathcal{H}$, it holds that $\|a\| \leq \|a + tb\|$ for all $t \in \mathbb{C}$ if and only if $\langle a, b\rangle = 0$). Then $x_0$ is an $A$-LSS of $Bx = y$ if and only if $x_0$ is a solution of

$$B^*ABx = B^*Ay. \tag{6}$$

Equation (6) is the *normal equation* associated to (5).

The next two results generalize [7, Proposition 4.4] and [7, Lemma 4.6].

**Proposition 5.3.** *Suppose $(A, R(B))$ is compatible and consider $y \in \mathcal{H}$, $y \neq 0$. Then $u \in \mathcal{H}$ is an $A$-LSS of $Bx = y$ if and only if there exists $T \in \Pi(A, R(B))$ such that $Bu = Ty$.*

*Proof.* Observe that $u \in \mathcal{H}$ is an $A$-LSS of $Bx = y$ if and only if $\|Bu - y\|_A = \inf_{\sigma \in R(B)} \|\sigma + y\|_A$, or $y - Bu \in sp(A^{1/2}, R(B), y)$; or equivalently, by Proposition 4.24, $Bu = Ty$ for some $T \in \Pi(A, R(B))$. □

**Corollary 5.4.** *Let $(A, N(B))$ be compatible and consider $x_0 \in N(B)^\perp$, $x_0 \neq 0$ and $u \in x_0 + N(B)$. Then $\|u\|_A \leq \|x\|_A$ for all $x \in x_0 + N(B)$ if and only if there exists $T \in \Pi(A, N(B))$ such that $u = (I - T)x_0$.*

*Proof.* Since $x_0 \in N(B)^\perp$ and $u \in x_0 + N(B)$, then $u = x_0 + P_{N(B)}u$. Consider $T \in \Pi(A, N(B))$ such that $u = (I - T)x_0$, so that $P_{N(B)}u = -Tx_0$. Since $x_0 \neq 0$, by the previous proposition it holds that $u$ is an $A$-LSS of $P_{N(B)}x = -x_0$, then $\|P_{N(B)}u + x_0\|_A \leq \|P_{N(B)}x + x_0\|_A$ for all $x \in \mathcal{H}$, or equivalently $\|u\|_A \leq \|x\|_A$ for all $x \in x_0 + N(B)$. The converse follows by [7, Lemma 4.6]. □

The following concept was introduced by Rao and Mitra for finite dimensional spaces, [24].

**Definition 5.5.** An operator $G \in L(\mathcal{H})$ is called an *$A$-inverse* of $B$ if for each $y \in \mathcal{H}$, $Gy$ is an $A$-LSS of $Bx = y$, i.e.

$$\|y - BGy\|_A \leq \|y - Bx\|_A, \quad x \in \mathcal{H}.$$

**Remark 5.6.** If $G$ is an $A$-inverse of $B$ then $R(G)$ is not necessarily closed. In fact, if $A$ has infinite dimensional nullspace consider $G_1 \in L(N(A))$ such that $R(G_1)$ is not closed. It is easy to see that $G = G_1 P_{N(A)} + P_{N(A)^\perp}$ is an $A$-inverse of $I$.

The following result gives a necessary and sufficient condition for an operator $B$ with closed range to admit an $A$-inverse.

**Proposition 5.7.** *The operator $B$ admits an $A$-inverse if and only if $(A, R(B))$ is compatible.*

*Proof.* Let $G \in L(\mathcal{H})$ be an $A$-inverse of $B$ and consider $T = BG$. Then $R(T) \subseteq R(B)$ and $\|y - Ty\|_A = \|y - BGy\|_A \leq \|y - Bx\|_A$ for all $x \in \mathcal{H}$, so that $T$ is an $A$-projection into $R(B)$. Then $(A, R(B))$ is compatible by Proposition 4.14. Conversely, if $(A, R(B))$ is compatible, using again Proposition 4.14, let $T$ be an $A$-projection into $R(B)$. Since $R(T) \subseteq R(B)$, by Douglas' theorem there exists $G \in L(\mathcal{H})$ such that $T = BG$. Therefore,

$$\|y - BGy\|_A = \|y - Ty\|_A \leq \|y - Bx\|_A, \qquad \text{for } x \in \mathcal{H},$$

so that $G$ is an $A$-inverse of $B$. □

**Remark 5.8.** It follows from the above proof that, if $G$ is an $A$-inverse of $B$ then $T = BG$ is an $A$-projection into $R(B)$. Conversely, given $T$ an $A$-projection into $R(B)$, the solutions of $BX = T$ are $A$-inverses of $B$.

The next result gives necessary and sufficient conditions for an operator $G \in L(\mathcal{H})$ to be an $A$-inverse of $B$.

**Proposition 5.9.** *Given $G \in L(\mathcal{H})$ then $G$ is an $A$-inverse of $B$ if and only if $B^*ABG = B^*A$.*

*Proof.* Let $G \in L(\mathcal{H})$ be an $A$-inverse of $B$. By Remark 5.8, it holds that $T = BG$ is an $A$-projection into $R(B)$. Hence, by Proposition 4.5, it follows that $P_{R(B)}AT = P_{R(B)}A$ so that $B^*ABG = B^*A$. Conversely, consider $T = BG$, then $B^*AT = B^*A$, or equivalently, $P_{R(B)}AT = P_{R(B)}A$. Therefore, by Proposition 4.5, $T = BG$ is an $A$-projection into $R(B)$. Finally, by Remark 5.8, $G$ is an $A$-inverse of $B$. □

**Corollary 5.10.** *If $(A, R(B))$ is compatible, then the set of $A$-inverses of $B$ is*

$$(B^*AB)^\dagger B^*A + L(\mathcal{H}, N(B^*AB)).$$

## 5.1 Restricted weighted inverses

Throughout this paragraph, $\mathcal{M}$ is a closed subspace of $\mathcal{H}$ such that $B(\mathcal{M})$ is closed, or equivalently, since $B$ has closed range, $\mathcal{M} + N(B)$ is a closed subspace of $\mathcal{H}$.

**Definition 5.11.** An operator $G \in L(\mathcal{H})$ is called an *$A$-inverse of $B$ restricted to $\mathcal{M}$* if $R(G) \subseteq \mathcal{M}$ and for each $y \in \mathcal{H}$ it holds that

$$\|y - BGy\|_A \leq \|y - Bx\|_A, \qquad \forall x \in \mathcal{M}.$$

The concept of $A$-inverses restricted to $\mathcal{M}$ was introduced by Rao and Mitra [24] for finite dimensional spaces.

In what follows we show that the existence of an $A$-inverse of $B$ restricted to $\mathcal{M}$ is equivalent to the compatibility of the pair $(A, B(\mathcal{M}))$.

**Lemma 5.12.** *An operator $G \in L(\mathcal{H})$ is an $A$-inverse of $B$ restricted to $\mathcal{M}$ if and only if $R(G) \subseteq \mathcal{M}$ and $G$ is an $A$-inverse of $BP_{\mathcal{M}}$.*

*Proof.* Straightforward. □

**Remark 5.13.** By the previous lemma and Proposition 5.9 applied to $BP_{\mathcal{M}}$, it holds that $G$ is an $A$-inverse of $B$ restricted to $\mathcal{M}$ if and only if $R(G) \subseteq \mathcal{M}$ and $P_{\mathcal{M}}(B^*ABG - B^*A) = 0$.

**Proposition 5.14.** *Suppose $(A, B(\mathcal{M}))$ is compatible and consider $T \in \Pi(A, B(\mathcal{M}))$. Then the reduced solution of*

$$BP_{\mathcal{M}}X = T,$$

*is an $A$-inverse of $B$ restricted to $\mathcal{M}$.*

*Proof.* Let $G_0$ be the reduced solution of $BP_{\mathcal{M}}X = T$, then $R(G_0) \subseteq N(BP_{\mathcal{M}})^{\perp}$. By Remark 5.8, $G_0$ is an $A$-inverse of $BP_{\mathcal{M}}$. Since $N(BP_{\mathcal{M}}) = (\mathcal{M} \cap N(B)) \oplus \mathcal{M}^{\perp}$, then $R(G_0) \subseteq \mathcal{M}$. Therefore, by Lemma 5.12, $G_0$ is an $A$-inverse of $B$ restricted to $\mathcal{M}$. □

**Corollary 5.15.** *The operator $B$ admits an $A$-inverse restricted to $\mathcal{M}$ if and only if $(A, B(\mathcal{M}))$ is compatible.*

*Proof.* If $G$ is an $A$-inverse of $B$ restricted to $\mathcal{M}$, then by Remark 5.12, $G$ is an $A$-inverse of $BP_{\mathcal{M}}$, so that $(A, B(\mathcal{M}))$ is compatible (see Proposition 5.7). The converse follows by Proposition 5.14. □

## 5.2 $A_1A_2$-inverses and weak weighted inverses

Throughout this section, we consider $B \in L(\mathcal{H})$ a closed range operator and $A_1, A_2 \in L(\mathcal{H})^+$.

**Definition 5.16.** An operator $G \in L(\mathcal{H})$ is called an $A_1A_2$-*inverse* of $B$ if $G$ is an $A_1$- inverse of $B$ and, for each $y \in \mathcal{H}$, $Gy$ has minimum $A_2$-seminorm among the $A_1$-LSS of $Bx = y$.

In [21, 24], $A_1A_2$-inverses are called minimum seminorm semileast squares inverses in the context of finite dimensional spaces.

The next two results are proved in [21], for finite dimensional Hilbert spaces. The proofs, which follow the same ideas, are included for the sake of completeness.

**Proposition 5.17.** *Consider $G \in L(\mathcal{H})$. Then $G$ is an $A_1A_2$-inverse of $B$ if and only if*

1. $B^*A_1BG = B^*A_1$,
2. $R(A_2G) \subseteq N(A_1B)^\perp$.

*Proof.* By Proposition 5.9, $G$ is an $A_1$-inverse of $B$ if and only if $B^*A_1BG = B^*A_1$. Let $G$ be an $A_1$-inverse of $B$. It is remains to prove that $Gy$ has minimum $A_2$-seminorm among the $A_1$-LSS of $Bx = y$ for each $y \in \mathcal{H}$ if and only if $R(A_2G) \subseteq \overline{R(B^*A_1^{1/2})}$. Observe that given $y \in \mathcal{H}$, by Remark 5.2, any $A_1$-LSS of $Bx = y$ can be written as

$$x_0 = \tilde{x} + P_{N(B^*A_1B)}z,$$

where $\tilde{x} = Gy$ is a solution of (6) (i.e $B^*A_1B\tilde{x} = B^*A_1y$) and $z \in \mathcal{H}$. Then $\|Gy\|_{A_2} \leq \|Gy + P_{N(B^*A_1B)}z\|_{A_2}$, for all $z \in \mathcal{H}$, if and only if $\|A_2^{1/2}Gy\| \leq \|A_2^{1/2}Gy + A_2^{1/2}P_{N(B^*A_1B)}z)\|$, for all $z \in \mathcal{H}$, or equivalently $\langle A_2Gy, P_{N(B^*A_1B)}z\rangle = 0$ for all $z \in \mathcal{H}$, or $P_{N(B^*A_1B)}A_2G = 0$. Therefore, $G$ is an $A_1A_2$-inverse of $B$ if and only if $B^*A_1BG = B^*A_1$ and $R(A_2G) \subseteq N(B^*A_1B)^\perp = N(A_1B)^\perp$. □

**Proposition 5.18.** *If $G$ is an $A_1A_2$-inverse of $B$, then*

1. $A_1BGB = A_1B, \quad A_1BG = (BG)^*A_1$,
2. $A_2GBG = A_2G, \quad A_2GB = (GB)^*A_2$.

*Proof.* If $G$ is an $A_1A_2$-inverse of $B$ then $G$ is an $A_1$-inverse of $B$. Therefore, by Proposition 5.9, $B^*A_1BG = B^*A_1$ then $A_1BG = (BG)^*A_1BG \geq 0$ so that $BG$ is $A_1$-selfadjoint. Also, $A_1B = (BG)^*A_1B = A_1BGB$ and item *1* holds. To prove item *2* observe that $R(I - GB) \subseteq N(B^*A_1B)$ because $B^*A_1BGB = B^*A_1B$. Therefore for each $y \in \mathcal{H}$, by Remark 5.2, it follows that

$$x = Gy + (I - GB)z, \quad z \in \mathcal{H}$$

is a solution of the normal equation (6) and then it is an $A_1$-LSS of $Bx = y$. Since $G$ is an $A_1A_2$-inverse of $B$, then $\|Gy\|_{A_2} \leq \|Gy + (I - GB)z\|_{A_2}$, for all $z \in \mathcal{H}$, or equivalently $\|A_2^{1/2}Gy\| \leq \|A_2^{1/2}Gy + A_2^{1/2}(I - GB)z)\|$, for all $z \in \mathcal{H}$, then $\langle A_2Gy, (I - GB)z\rangle = 0$ for all $z \in \mathcal{H}$, or $G^*A_2(I - GB) = 0$. Finally, in the same way as we did in item *1*, $G^*A_2 = G^*A_2GB$ implies item *2* (actually, both conditions are equivalent). □

**Corollary 5.19.** *Suppose the pairs* $(A_1, R(B))$ *and* $(A_2, N(A_1B))$ *are compatible. Then*

$$G = (I - T_2)B^\dagger T_1$$

*is an* $A_1A_2$*-inverse of* $B$ *for every* $T_1 \in \Pi(A_1, R(B))$ *and for every* $T_2 \in \Pi(A_2, N(A_1B))$.

*Proof.* Consider $T_1 \in \Pi(A_1, R(B))$, $T_2 \in \Pi(A_2, N(A_1B))$ and $G = (I - T_2)B^\dagger T_1$. Then

$$A_1BGB = A_1BB^\dagger T_1B - A_1BT_2B^\dagger T_1B = A_1T_1B = A_1B,$$

because $R(T_1) \subseteq R(B)$, $R(T_2) \subseteq N(A_1B)$ and $A_1T_1P_{R(B)} = A_1P_{R(B)}$. Also, observe that

$$\begin{aligned}(BG)^*A_1 &= (BB^\dagger T_1 - BT_2B^\dagger T_1)^*A_1 = T_1^*A_1 - (A_1BT_2B^\dagger T_1)^* \\ &= A_1T_1 = A_1BG.\end{aligned}$$

Finally, by Proposition 4.9 and Remark 4.18, it holds that $R(A_2G) \subseteq A_2R(I - T_2) \subseteq A_2[R(A_2T_2)^\perp] = A_2[(A_2N(A_1B))^\perp] = A_2[A_2^{-1}(N(A_1B)^\perp)] \subseteq N(A_1B)^\perp$. Therefore, by Proposition 5.17, it follows that $G$ is an $A_1A_2$-inverse of $B$. □

**Proposition 5.20.** *The operator* $B$ *admits an* $A_1A_2$*-inverse if and only if the pairs* $(A_1, R(B))$ *and* $(A_2, N(A_1B))$ *are compatible.*

*Proof.* Suppose $B$ admits an $A_1A_2$-inverse. Then, by Proposition 5.7, the pair $(A_1R(B))$ is compatible. The pair $(A_2, N(A_1B))$ turns out to be compatible by [18, Proposition 3.9]. The converse follows by the previous result. □

**Definition 5.21.** An operator $G \in L(\mathcal{H})$ is called a *weak* $A_1A_2$*-inverse* of $B$ if satisfies

$$\left\{\begin{array}{ll} A_1BGB = A_1B, & A_1BG = (BG)^*A_1 \\ A_2GBG = A_2G, & A_2GB = (GB)^*A_2. \end{array}\right. \tag{7}$$

If $A_1 = A_2 = I$ and $G$ is a weak $A_1A_2$-inverse of $B$, then $G = B^\dagger$.

Observe that if $G \in L(\mathcal{H})$ is an $A_1A_2$-inverse of $B$ then, by Proposition 5.18, $G$ is a weak $A_1A_2$-inverse of $B$.

**Remark 5.22.** Observe that (7) is equivalent to $B^*A_1BG = B^*A_1$ and $G^*A_2GB = G^*A_2$.

**Lemma 5.23.** *Consider $G \in L(\mathcal{H})$. Then $G$ is a weak $A_1A_2$-inverse of $B$ if and only if $G$ is an $A_1$-inverse of $B$ and $B$ is an $A_2$-inverse of $G$.*

*Proof.* Apply Remark 5.22 and Proposition 5.9. □

In [7], the authors called *weighted generalized inverse* of $B$ to an operator $C \in L(\mathcal{H})$ such that

$$(8) \quad BCB = B, \quad CBC = C, \quad A_1BC = (BC)^*A_1, \quad A_2CB = (CB)^*A_2.$$

In [7, Theorem 3.1], it is proved that the pairs $(A_1, R(B))$ and $(A_2, N(B))$ are compatible if and only if $B$ admits a weighted generalized inverse. Observe that in this case, $C$ is a weak $A_1A_2$-inverse of $B$.

Also, it holds that $C$ is a weighted generalized inverse of $B$ if and only if $BC \in \mathcal{P}(A, R(B))$ and $I - CB \in \mathcal{P}(A, N(B))$. In order to generalize this, we now consider the solutions of the system

$$(9) \qquad \begin{cases} BG \in \Pi(A_1, R(B)) \\ I - GB \in \Pi(A_2, N(B)). \end{cases}$$

**Proposition 5.24.** *Consider $G \in L(\mathcal{H})$, then $G \in L(\mathcal{H})$ is a solution of (9) if and only if*

$$(10) \qquad BGB = B, \quad A_1BG = (BG)^*A_1, \quad A_2GB = (GB)^*A_2.$$

*Proof.* Note that $G$ is a solution of (9) if and only if $A_1BG = (BG)^*A_1$, $A_1BGP_{R(B)} = A_1P_{R(B)}, R(I-GB) \subseteq N(B), A_2(I-GB) = (I-GB)^*A_2$ and $A_2(I-GB)P_{N(B)} = A_2P_{N(B)}$. Equivalently, $A_1BG = (BG)^*A_1, A_1BGB = A_1B, B(I-GB) = 0, A_2GB = (GB)^*A_2$, or $BGB = B, A_1BG = (BG)^*A_1$ and $A_2GB = (GB)^*A_2$. □

**Corollary 5.25.** *The following statements are equivalent:*

1. *system (9) admits a solution,*
2. *the pairs $(A_1, R(B))$ and $(A_2, N(B))$ are compatible,*

*3. $B$ admits a weighted generalized inverse.*

*Proof.* *1* → *2*: It is straightforward. *2* → *3*: It follows by [7, Theorem 3.1]. *3* → *1*: The assertion follows by the previous proposition. □

By Propositon 5.24 and Corollary 5.25 it follows that (8) has a solution if and only if (10) has a solution.

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