Paths of inner-related functions

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Abstract

We characterize the connected components of the subset CN^* of H^∞ formed by the products bh, where b is Carleson-Newman Blaschke product and $h \in H^\infty$ is an invertible function. We use this result to show that, except for finite Blaschke products, no inner function in the little Bloch space is in the closure of one of these components. Our main result says that every inner function can be connected with an element of CN^* within the set of products uh, where u is inner and h is invertible. We also study some of these issues in the context of Douglas algebras.

1 Introduction

Let H^∞ be the Banach algebra of bounded analytic functions in the unit disk $\mathbb{D}$ with the norm $\|f\|_\infty = \sup_{z\in\mathbb{D}} |f(z)|$. A function in H^∞ is called inner if it has radial limits of modulus one at almost every point of the unit circle $\partial\mathbb{D}$. A Blaschke product is an inner function of the form

$$b(z) = \lambda z^m \prod_{n=1}^{\infty} \frac{\bar{z}_n}{|z_n|} \frac{z_n - z}{1 - \bar{z}_n z},$$

where m is a nonnegative integer, $\lambda \in \partial\mathbb{D}$ and $\{z_n\}$ is a sequence of points in $\mathbb{D} \setminus \{0\}$ satisfying the Blaschke condition $\sum_n (1 - |z_n|) < \infty$. If $\lambda = 1$, we say that b is normalized. Given a Blaschke product b, we denote by $Z(b) = \{z_n\}$ the sequence of its zeros repeated according to their multiplicity. A classical result by O. Frostman tells us that for any inner function u, there exists an exceptional set $E = E(u) \subset \mathbb{D}$ of logarithmic capacity zero such that the Mobius shift $(u - \alpha)/(1 - \bar{\alpha}u)$ is a Blaschke product for any $\alpha \in \mathbb{D} \setminus E$ (see [4] or [5, pp. 79]). Hence, any inner function can be uniformly approximated by Blaschke products.

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The set of invertible functions in H^∞ will be denoted by $(H^\infty)^{-1}$ and it consists of those functions $h \in H^\infty$ satisfying $\inf |h(z)| > 0$, where the infimum is taken over all points $z \in \mathbb{D}$. Let $\mathfrak{I}^*$ be the open set in H^∞ of functions of the form $f = uh$, where u is an inner function and $h \in (H^\infty)^{-1}$. Equivalently, $\mathfrak{I}^*$ consists of those functions in H^∞ whose non-tangential limits on the unit circle are bounded below away from zero. A result by Laroco asserts that $\mathfrak{I}^*$ is dense in H^∞ (see [14]).

A sequence of points $\{z_n\}$ of the unit disk is called an interpolating sequence if for any bounded sequence of complex values $\{w_n\}$, there exists a function $f \in H^\infty$ with $f(z_n) = w_n$, $n = 1, 2, \ldots$ A celebrated result by Carleson ([2] or [5, pp. 287]) asserts that $\{z_n\}$ is an interpolating sequence if and only if

$$\inf_n (1 - |z_n|^2)|b'(z_n)| > 0,$$

where b is the Blaschke product with zeros $\{z_n\}$. Although interpolating Blaschke products comprise a small subset of all inner functions, they play a central role in the theory of the algebra H^∞. See for instance the last three chapters of [5]. Marshall proved that finite linear combinations of Blaschke products are dense in H^∞ (see [15]). Later, this result was sharpened in [6] by showing that one can use interpolating Blaschke products. However, the following problem remains open.

Problem 1.1. *For any inner function u and $\varepsilon > 0$, does there exist an interpolating Blaschke product b such that $\|u - b\|_\infty < \varepsilon$?*

This question was posed in [13] and [5, pp. 420], and it is one of the most important open problems in the area. The following weaker version of Problem 1.1 is also open.

Problem 1.2. *For any inner function u and $\varepsilon > 0$, does there exist an interpolating Blaschke product b and an invertible function $h \in H^\infty$ such that $\|u - bh\|_\infty < \varepsilon$?*

This is really a question of approximation in BMO. Recall that a function $f \in L^1(\partial\mathbb{D})$ is in the space BMO if

$$\|f\|_* = \sup \frac{1}{|I|} \int_I |f - f_I| < \infty,$$

where the supremum is taken over all arcs $I \subset \partial\mathbb{D}$ of the unit circle and $f_I = |I|^{-1} \int_I f$ is the mean of f over the arc I. A classical result by Fefferman and Stein says that a function $f \in L^1(\partial\mathbb{D})$ is in BMO if and only if f can be written as $f = r + \tilde{s}$, where $r, s \in L^\infty(\partial\mathbb{D})$. Here $\tilde{s}$ means the harmonic conjugate of s. Moreover, $\|f\|_*$ is comparable to $\|f\|_{BMO} = \inf\{\|r\|_\infty + \|s\|_\infty\}$, where the infimum is taken over all possible decompositions $f = r + \tilde{s} + c$, where c is a constant. It is easy to see that Problem 1.2 has a positive answer if and only if for any inner function u and any $\varepsilon > 0$, there exists an interpolating Blaschke product b such that a suitable branch $\operatorname{Arg}(u/b)$ of the argument of the function $u(\xi)/b(\xi)$, $\xi \in \partial\mathbb{D}$, satisfies

$$\|\operatorname{Arg}(u/b)\|_{BMO} \le \varepsilon.$$

In other words, Problem 1.2 is the BMO-version of Problem 1.1.

In connection with the theory of Toeplitz operators on Hardy spaces and the existence of unconditional basis of reproducing kernels in model spaces, Nikolskii proposed the following weak version of Problem 1.2, which is also still open [23, pp. 210] (see also [24, pp. 91–93]).

Problem 1.3. *For any inner function u, is there an interpolating Blaschke product b such that*

$$\mathrm{dist}(u\bar{b}, H^\infty) < 1 \quad \textit{and} \quad \mathrm{dist}(b\bar{u}, H^\infty) < 1?$$

An equivalent formulation is to ask whether there is an interpolating Blaschke product b and $h \in (H^\infty)^{-1}$ such that $\|u - bh\|_\infty < 1$ (see [23, pp. 220]).

A Blaschke product b is called a Carleson-Newman Blaschke product if it can be decomposed as a finite product of interpolating Blaschke products, or equivalently, if

$$\mu_b = \sum_{b(z)=0} (1 - |z|)\delta_z$$

is a Carleson measure. Here δ_z denotes the Dirac measure at the point z. Recall that a complex-valued measure μ in the unit disk is called a Carleson measure if there exists a constant $C > 0$ such that

$$\int_{\mathbb{D}} |f(z)| d|\mu|(z) \leq C\|f\|_1$$

for any function f in the Hardy space H^1. The infimum of such constants C is denoted by $\|\mu\|_c$. It is well known that any Carleson-Newmann Blaschke product can be approximated uniformly by interpolating Blaschke products (see [16]). Thus, one can interchange interpolating and Carleson-Newman Blaschke products in the questions above, as well as in the rest of the paper. Let CN^* be the open set in H^∞ of functions of the form $f = bh$ where b is a Carleson-Newman Blaschke product and $h \in (H^\infty)^{-1}$. Equivalently, CN^* consists of those functions $f \in H^\infty$ for which there exist numbers $0 < r < 1$ and $c > 0$, such that for any $z \in \mathbb{D}$ one has $\sup\{|f(w)| : |w - z| < r(1 - |z|)\} > c$.

The main purpose of this paper is to study the connected components of $\mathfrak{I}^*$ and CN^*. This shall lead us to answer natural analogues of Problems 1.2 and 1.3, as well as to consider a number of related questions. The components of the set of inner functions has been considered by Herrero in [10] and by Nestoridis in [19] and [20].

A continuous version of Problem 1.2 can be stated as follows: given an inner function u, does there exist a path in CN^* (except for the final point) which ends at u? Or more basically, is any inner function in the closure of a connected component of CN^*? We prove that the answer to both questions is negative. Recall that an analytic function f on the unit disk is in the little Bloch space if

$$\lim_{|z|\to 1} (1 - |z|^2)\, |f'(z)| = 0.$$

Blaschke products with finitely many zeros are in the little Bloch space but it also contains many other inner functions. We prove in Theorem 2.10 that any inner function in the little Bloch space which is not a finite Blaschke product does not belong to the closure of a single component of CN^*. The proof uses a description of the connected components of CN^* given in Theorem 2.7. Roughly speaking, the component of a function $f \in CN^*$ is described in terms of the zeros of f and the Fefferman-Stein decomposition of the Cauchy integral of a path measure associated to those zeros.

In contrast to the situation described above for functions in the little Bloch space, any component of $\mathfrak{I}^*$ contains an element of CN^*. This is stated in the main result of the paper, Theorem 3.7. It can be understood as a positive answer to a weaker version of Problem 1.3. Actually, if $\|u - bh\|_\infty < 1$, the segment $\gamma(t) = u + t(bh - u)$, where $t \in [0, 1]$, is contained in $\mathfrak{I}^*$ and joins $\gamma(0) = u$ with $\gamma(1) = bh$. Our proof shows that there exists a universal constant N such that for any inner function u there is a polygonal $\gamma\colon [0, 1] \to \mathfrak{I}^*$ formed by at most N segments so that $\gamma(0) = u$ and $\gamma(1) \in CN^*$. The proof is constructive and it is the deepest part of the paper. It uses a Carleson contour decomposition and a discretization of harmonic measures in its interior. This provides a path in $L^\infty(\partial\mathbb{D})$ which can be lifted to $\mathfrak{I}^*$.

We will abbreviate Carleson-Newman Blaschke product by CNBP. The paper is organized as follows. Section 2 contains the description of the components of CN^* and the result on inner functions in the little Bloch space. Section 3 is devoted to the main result of the paper. Finally, in Section 4 we relate the previous results to Douglas algebras, present an example that illustrates the fact that two arbitrary functions in $\mathfrak{I}^*$ or CN^* can be multiplied by a CNBP into a single component, and mention some open problems.

2 The connected components of CN^*

Let μ be a finite complex measure in the complex plane. Let $C_\varepsilon(\mu)$ be its truncated Cauchy integral defined in the unit circle as

$$\mathcal{C}_\varepsilon(\mu)(e^{i\theta}) = \int_{|z-e^{i\theta}|>\varepsilon} \frac{d\mu(z)}{e^{i\theta} - z}.$$

It was shown by Mattila and Melnikov that the Cauchy integral defined as $\mathcal{C}(\mu)(e^{i\theta}) = \lim_{\varepsilon\to 0} C_\varepsilon(\mu)(e^{i\theta})$ exists at almost every point $e^{i\theta} \in \partial\mathbb{D}$ (see [17]). This was a consequence of the following weak-L^1 estimate: there is a universal constant C such that for any $\lambda > 0$, $|\{e^{i\theta} \in \partial\mathbb{D} : \mathcal{C}^*(\mu)(e^{i\theta}) > \lambda\}| < C\lambda^{-1}\|\mu\|$. Here $\mathcal{C}^*(\mu)$ denotes the maximal Cauchy transform defined as

$$\mathcal{C}^*(\mu)(e^{i\theta}) = \sup_{\varepsilon>0} \left| \int_{|z-e^{i\theta}|>\varepsilon} \frac{d\mu(z)}{e^{i\theta} - z} \right|.$$

We start with a well-known result on Cauchy's integrals of Carleson measures which will be used later. For $0 < p < \infty$ let H^p be the Hardy space of analytic functions f in the unit disk for which

$$\|f\|_p^p = \sup_{r<1} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty$$

and H_0^p be the subspace of those $f \in H^p$ with $f(0) = 0$.

Lemma 2.1. *Let μ be a complex-valued Carleson measure on $\mathbb{D}$ and for $0 < r < 1$, let μ_r be its restriction to the disk $r\mathbb{D}$. Then*

1. *$\mathcal{C}(\mu_r)$ converges in L^2-norm to $C(\mu)$ as $r \to 1$.*
2. *$\overline{\mathcal{C}(\mu)} \in H_0^2$ and $\|\mathcal{C}(\mu)\|_2 \le \|\mu\|_c^{1/2} \, |\mu|(\mathbb{D})^{1/2}$,*
3. *$\overline{\mathcal{C}(\mu)} \in BMO$ and $\|\mathcal{C}(\mu)\|_{BMO} \le C\|\mu\|_c$, where C is an absolute constant.*

Proof. Given two functions $f, g \in L^2(\partial\mathbb{D})$, let $\langle f, g\rangle$ denote their scalar product in $L^2(\partial\mathbb{D})$. It is obvious that $h_r(e^{i\theta}) = \overline{e^{i\theta}\mathcal{C}(\mu_r)(e^{i\theta})}$ can be extended to a function in H^∞, and for $f \in H^2$ we have

$$\langle f, h_r\rangle = \int_0^{2\pi} f(e^{i\theta}) \int_{\mathbb{D}} \frac{1}{1 - ze^{-i\theta}} \, d\mu_r(z) \, \frac{d\theta}{2\pi} = \int_{\mathbb{D}} f(z) \, d\mu_r(z).$$

By the Cauchy-Schwarz inequality

$$|\langle f, h_r\rangle| \le \left(\int_{\mathbb{D}} |f(z)|^2 \, d|\mu_r|(z) \right)^{1/2} |\mu_r|(\mathbb{D})^{1/2} \le \|\mu_r\|_c^{1/2} \, \|f\|_2 \, |\mu_r|(\mathbb{D})^{1/2}.$$

Since $\|\mu_r\|_c \le \|\mu\|_c$, we get $\|h_r\|_2 \le \|\mu\|_c^{1/2} \, |\mu_r|(\mathbb{D})^{1/2}$. Similarly,

$$\|h_r - h_s\|_2 \le \|\mu\|_c^{1/2} \, |\mu_r - \mu_s|(\mathbb{D})^{1/2},$$

from which h_r converges in $L^2(\partial\mathbb{D})$ to a function h when $r \to 1$, with $\|h\|_2 \le \|\mu\|_c^{1/2} \, |\mu|(\mathbb{D})^{1/2}$. Observe that $|\mathcal{C}_{1-r}(\mu)(e^{i\theta}) - \mathcal{C}(\mu_r)(e^{i\theta})| \le \mathcal{C}^*(\mu - \mu_r)(e^{i\theta})$. Then the weak-$L^1$ estimate tells us that $\mathcal{C}(\mu_r)$ converges in measure to $\mathcal{C}(\mu)$ as r tends to 1. Thus $h(e^{i\theta}) = \overline{e^{i\theta}\mathcal{C}(\mu)(e^{i\theta})}$ at almost every point $e^{i\theta} \in \partial\mathbb{D}$ and (1) and (2) follow.

Let $f \in H^\infty$ and write $f(e^{i\theta}) - f(0) = e^{i\theta} g(e^{i\theta})$, with $g \in H^\infty$. Then

$$|\langle f, \overline{\mathcal{C}(\mu)}\rangle| = |\langle f - f(0), \overline{\mathcal{C}(\mu)}\rangle| = \left| \int_{\mathbb{D}} \int_0^{2\pi} \frac{g(e^{i\theta})}{1 - ze^{-i\theta}} \, \frac{d\theta}{2\pi} \, d\mu(z) \right| = \left| \int_{\mathbb{D}} g(z) \, d\mu(z) \right|,$$

which is bounded by $\|\mu\|_c \|f - f(0)\|_1 \le 2\|\mu\|_c \|f\|_1$. Since H^∞ is dense in H^1, $\overline{\mathcal{C}(\mu)}$ induces a bounded linear functional on H^1 with norm bounded by $2\|\mu\|_c$. This means that $\overline{\mathcal{C}(\mu)} \in BMO$ with $\|\overline{\mathcal{C}(\mu)}\|_{BMO} \le C\|\mu\|_c$. □

Remark 2.2. It is clear that the truncated measure μ_r in Lemma 2.1 can be replaced by $\chi_{E_s}\mu$, where E_s, with $0 < s < 1$, is any continuum of increasing compact sets in $\mathbb{D}$ such that for any $0 < r < 1$ there is s with $r\mathbb{D} \subset E_s$.

Let $\varphi_z(w) = (z - w)/(1 - \overline{w}z)$ be the involution on $\mathbb{D}$ that interchanges 0 and z. The pseudohyperbolic and hyperbolic distance between z and w in $\mathbb{D}$ are respectively defined by $\rho(z, w) = |\varphi_z(w)|$ and

$$\beta(z, w) = \log \frac{1 + \rho(z, w)}{1 - \rho(z, w)}.$$

Let b, b^* be two CNBP and let $\{z_k\}$, $\{z_k^*\}$ be their zero sequences. Assume that there exists a constant $M > 0$ such that $\beta(z_k, z_k^*) \le M$ for all $k \ge 1$. If σ_k is a path measure from z_k to z_k^* with bounded hyperbolic diameter. Consider the measures $S_N = \sum_{k=1}^N \sigma_k$ and $\sigma = \sum_{k=1}^\infty \sigma_k$. Is is clear that the Carleson norm of both measures is bounded by a constant depending only on M and $\| \sum_{k=1}^\infty (1 - |z_k|)\delta_{z_k} \|_c$. Thus, the previous lemma and remark say that $\mathcal{C}(S_N) \to \mathcal{C}(\sigma)$ in L^2-norm. The next result tells us that on the unit circle the function $2\mathrm{Im}\, \mathcal{C}(\sigma)$ is an argument of the quotient b/b^*.

Lemma 2.3. *Let b (respectively b^*) be a normalized CNBP with zero sequence $\{z_k\}$ (respectively $\{z_k^*\}$) such that $\sup_k \beta(z_k, z_k^*) < \infty$. Then*

$$\exp\left[\, i2\, \mathrm{Im}\, \mathcal{C}(\sigma)(e^{i\theta})\right] = e^{i\gamma} b(e^{i\theta}) \overline{b^*(e^{i\theta})}$$

at almost every point $e^{i\theta} \in \partial\mathbb{D}$, where $e^{i\gamma} = \prod_{k\ge1} \frac{z_k}{z_k^*} \frac{|z_k^*|}{|z_k|}$, and we are interpreting here that $z/|z| = |z|/z = -1$ when $z = 0$.

Proof. We can assume that $z_k \neq 0 \neq z_k^*$ for all $k \ge 1$. Let $\alpha_z(w) = (\overline{z}/|z|)(z - w)/(1 - \overline{z}w)$. A straightforward calculation shows that

$$\frac{\alpha_{z_k}(e^{i\theta})}{\alpha_{z_k^*}(e^{i\theta})} = \frac{z_k^*}{z_k} \left| \frac{z_k}{z_k^*} \right| \left(\frac{1 - z_k e^{-i\theta}}{1 - z_k^* e^{-i\theta}} \right)^2 \left| \frac{1 - z_k^* e^{-i\theta}}{1 - z_k e^{-i\theta}} \right|^2. \tag{2.1}$$

Observe that

$$\int_{z_k}^{z_k^*} \frac{2}{e^{i\theta} - z}\, dz = 2\left[\mathrm{Log}\,(1 - z_k e^{-i\theta}) - \mathrm{Log}\,(1 - z_k^* e^{-i\theta})\right],$$

where the imaginary part of $\mathrm{Log}\,(1 - w)$ varies between $-\pi/2$ and $\pi/2$ for $w \in \mathbb{D}$. Hence,

$$2\mathrm{Im}\, \mathcal{C}(S_N)(e^{i\theta}) = \mathrm{Im} \sum_{k=1}^N \int_{z_k}^{z_k^*} \frac{2}{e^{i\theta} - z}\, dz = 2\sum_{k=1}^N \left[\mathrm{Arg}\,(1 - z_k e^{-i\theta}) - \mathrm{Arg}\,(1 - z_k^* e^{-i\theta})\right],$$

where $\operatorname{Arg}(1-w)$ denotes the principal branch of the argument, which for $w \in \mathbb{D}$ takes values between $-\pi/2$ and $\pi/2$. According to (2.1) one deduces

$$\exp\left[i2\operatorname{Im}\mathcal{C}(S_N)(e^{i\theta})\right] = \left(\prod_{k=1}^{N} \frac{z_k}{z_k^*}\left|\frac{z_k^*}{z_k}\right|\right) b_N(e^{i\theta})\overline{b_N^*(e^{i\theta})}. \tag{2.2}$$

Here b_N (respectively b_N^*) denotes the Blaschke product formed with the first N zeros of b (respectively b^*). Since $\|\mathcal{C}(S_N)-\mathcal{C}(\sigma)\|_2 \to 0$, there is a subsequence N_j such that the convergence holds pointwise almost everywhere on $\partial\mathbb{D}$, and since the same holds for the partial Blaschke products b_N and b_N^*, we can assume that the subsequence N_j achieves the three convergences at once. Furthermore, since $\sum |z_k^* - z_k| < \infty$, we have that $\prod_{k=1}^{N} \frac{z_k}{z_k^*}\left|\frac{z_k^*}{z_k}\right|$ converges to a certain point $e^{i\gamma}$ of the unit circle. Therefore, the Corollary follows by taking limits in both members of (2.2). □

Given a Carleson-Newman Blaschke product we denote by μ_b the Carleson measure given by $\mu_b = \sum(1-|z|)\delta_z$ where the sum is taken over all zeros $z \in \mathbb{D}$ of b counting multiplicities. Given a function $v \in L^1(\partial\mathbb{D})$, let $\tilde{v}$ be its harmonic conjugate normalized so that $\tilde{v}(0) = 0$.

Corollary 2.4. *Let b be a CNBP with zeros $\{z_k\}$ and $\varepsilon > 0$. Then there is $\alpha = \alpha(\varepsilon, \|\mu_b\|_c) > 0$ such that for any sequence $\{z_k^*\}$ with $\sup_k \beta(z_k, z_k^*) \le \alpha$ and its corresponding Blaschke product b^*, there is $h \in (H^\infty)^{-1}$ such that*

$$\|b - b^*h\|_\infty < \varepsilon \quad \textit{and} \quad 2Im\mathcal{C}(\sigma) - \widetilde{\log|h|} \in L^\infty(\partial\mathbb{D}).$$

Proof. We can assume that b and b^* are normalized. Write $\sigma = \sum \sigma_k$, where σ_k is the path measure in the segment from z_k to z_k^*. Then $\|\sigma\|_c \le C(\alpha)\|\mu_b\|_c$, with $C(\alpha)\to 0$ as $\alpha \to 0$. Therefore, given $\eta > 0$, we can choose $\alpha = \alpha(\eta)$ small enough so that $\|\sigma\|_c < \eta$. By Lemma 2.1 one has that $\|C(\sigma)\|_{BMO} \le C\eta$, and the Fefferman-Stein decomposition of BMO gives $2\operatorname{Im} C(\sigma) = u + \tilde{v}$, where $\|u\|_\infty + \|v\|_\infty \le C'\eta$. Here $C' > 0$ is a fixed constant. Pick $h = e^{-i\gamma + v + i\tilde{v}}$, where γ is the constant appearing in Lemma 2.3. By Lemma 2.3, at almost every point of the unit circle one has

$$b\overline{b}^* h^{-1} = e^{i2\operatorname{Im}\mathcal{C}(\sigma)} e^{-v - i\tilde{v}} = e^{-v+iu},$$

and consequently

$$\|b - b^*h\|_\infty \le \|h\|_\infty \|b\overline{b}^* h^{-1} - 1\|_\infty \le e^{\|v\|_\infty}\|e^{-v+iu} - 1\|_\infty \to 0$$

as $\eta \to 0$. □

It is worth mentioning that two interpolating Blaschke products could be uniformly close but still have zero sets which are hyperbolically far away one from the other. For instance consider the singular inner function $s(z) = \exp((z+1)/(z-1))$, $z \in \mathbb{D}$. For $\alpha \in \mathbb{D}$ let s_α be its Mobius shift given by $s_\alpha = (\alpha - s)/(1 - \overline{\alpha} s)$. It is easy to see that for any $\alpha \in \mathbb{D} \setminus \{0\}$, the function s_α is an interpolating Blaschke product. It is clear that s_α and s_β are uniformly close if $|\alpha| + |\beta|$ is small, but in this case, the hyperbolic distance between its zero sets is bounded below by $\log(\log|\alpha|/\log|\beta|)$ when $|\alpha| < |\beta|$. Taking $\alpha = \beta^2$ we see that the distance between their respective zeros is bounded below by $\log 2$.

Lemma 2.5. *For $0 \leq t \leq 1$, let u_t be an inner function and $h_t \in (H^\infty)^{-1}$. Assume that $|u_t h_t|$ varies continuously in $L^\infty(\mathbb{D})$. Then both $|u_t|$ and $|h_t|$ vary continuously in $L^\infty(\mathbb{D})$.*

Proof. Since $|h_t(e^{i\theta})| = |h_t(e^{i\theta}) u_t(e^{i\theta})|$ varies continuously in $L^\infty(\partial\mathbb{D})$ and

$$|h_t(z)| = \exp\left(\int_0^{2\pi} \frac{1 - |z|^2}{|1 - \overline{z} e^{i\theta}|^2} \log|h_t(e^{i\theta})| \frac{d\theta}{2\pi}\right),$$

then $|h_t(z)|$ varies continuously on $L^\infty(\mathbb{D})$. Thus, $|u_t(z)| = |h_t(z)|^{-1} |h_t(z) u_t(z)|$ varies continuously on $L^\infty(\mathbb{D})$. □

We remark that the above lemma is false without taking modulus, that is, the continuity of $t \mapsto u_t h_t$ in H^∞ does not imply the continuity of $t \mapsto u_t$. An example will be given in Section 4 as a consequence of Proposition 4.5.

Lemma 2.6. *For $0 \leq t \leq 1$, let b_t be a CNBP and $h_t \in (H^\infty)^{-1}$. Assume that $|b_t h_t|$ varies continuously in $L^\infty(\mathbb{D})$. Then there is a reordering $\{z_k^1\}$ of the zeros of b_1 such that $\sup_k \beta(z_k^0, z_k^1) < \infty$, where $\{z_k^0\}$ are the zeros of b_0.*

Proof. Let $\{z_k(t) : k = 1, 2, \ldots\}$ be the zeros of b_t. By compactness it is enough to prove that given $0 < \varepsilon < 1$, for any $t_0 \in [0, 1]$ there is an open neighborhood V of t_0 in $[0, 1]$ depending on t_0, such that whenever $t', t'' \in V$, there is a reordering of the zeros of $b_{t''}$ such that $\beta(z_k(t'), z_k(t'')) \leq \varepsilon$, for any $k \geq 1$.

Fix $t_0 \in [0, 1]$, since b_{t_0} is a CNBP, its zero sequence $Z(b_{t_0})$ can be split into $n(t_0)$ sequences $Z(b_{t_0}) = S_1 \cup \ldots \cup S_{n(t_0)}$ such that $\beta(z, w) \geq 1$ for $z, w \in S_k$ with $z \neq w$. Consider the family of open hyperbolic disks $\Delta_j = \{z \in \mathbb{D} : \beta(z, z_j) < \frac{\varepsilon}{4n(t_0)}\}$ for $z_j \in Z(b_{t_0})$. We claim that any connected component of $\bigcup \Delta_j$ contains no more than $n(t_0)$ points of $Z(b_{t_0})$. In fact, suppose that $\mathcal{O} = \Delta_{j_1} \cup \ldots \cup \Delta_{j_m}$ is a maximal connected set such that $z_{j_1}, \ldots, z_{j_m}$ belong to different sequences S_k (not necessarily a component of $\bigcup \Delta_j$). Then

$$\operatorname{diam}_\beta \mathcal{O} \leq m \frac{2\varepsilon}{4n(t_0)} \leq n(t_0) \frac{2\varepsilon}{4n(t_0)} = \frac{\varepsilon}{2} \leq \frac{1}{2}. \tag{2.3}$$

Now, if $z_i \in Z(b_{t_0})$ belongs to the same sequence S_k as some of the points z_{j_l}, say z_{j_1}, then for every $z \in \mathcal{O}$,

$$\beta(z_i, z) \geq \beta(z_i, z_{j_1}) - \beta(z_{j_1}, z) \geq 1 - \frac{1}{2}.$$

So, $\beta(z_i, \mathcal{O}) \geq 1/2$ and consequently $\beta(\Delta_i, \mathcal{O}) \geq \frac{1}{2} - \frac{\varepsilon}{4n(t_0)} \geq \frac{1}{4}$. Thus, Δ_i cannot meet $\mathcal{O}$, which implies that $\mathcal{O}$ is indeed one of the connected components of $\bigcup \Delta_j$, and that the hyperbolic distance between two of these components is $\geq 1/4$.

Since b_{t_0} is a CNBP, there is some $\eta > 0$ such that $|b_{t_0}| \geq \eta$ in $\mathbb{D} \setminus \bigcup \Delta_j$. Let $V_{t_0} \subset [0,1]$ be a relatively open neighborhood of t_0 such that $||b_t(z)| - |b_{t_0}(z)|| < \frac{\eta}{2}$ for all $z \in \mathbb{D}$ and $t \in V_{t_0}$. Then

$$\{|b_t| < \eta/2\} \subset \{|b_{t_0}| < \eta\} \subset \bigcup \Delta_j \quad \text{for all} \quad t \in V_{t_0}.$$

Together with (2.3), this implies that every (simply) connected component Ω of the set $\{|b_{t_0}| < \eta\}$ has hyperbolic diameter bounded by $\varepsilon/2$.

The lemma will follow if we show that b_{t_0} and b_t have the same number of zeros in Ω for $t \in V_{t_0}$. By a conformal mapping between Ω and $\mathbb{D}$ and the first paragraph of the proof, it is enough to show that if B_t are finite Blaschke products such that $|B_t|$ varies continuously for $0 \leq t \leq 1$, then they have the same degree. By compactness, the degrees are bounded and there is some $0 < r < 1$ such that $Z(B_t) \subset r\mathbb{D}$ for all t. Furthermore, composing at the right side by some automorphism of $\mathbb{D}$ we can also assume that $B_t(0) \neq 0$ for all t. If we write α_j, with $1 \leq j \leq n(t)$, for the zeros of B_t counting multiplicities, Jensen's formula gives

$$r^{n(t)} = |B_t(0)| \prod_{j=1}^{n(t)} \frac{r}{|\alpha_j|} = \exp\left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |B_t(re^{i\theta})| \, d\theta \right\}.$$

Since the right member of the equality is a continuous function of t, so is $r^{n(t)}$, which means that $n(t)$ is constant. □

We are ready now to prove our characterization of the components of CN^*.

Theorem 2.7. *Let b, b^* be CNBP and $h \in (H^\infty)^{-1}$. Then b and b^*h can be joined by a path contained in CN^* if and only if the following two conditions hold*

1. *there is a reordering $\{z_k^*\}$ of the zeros of b^* such that $\sup_k \beta(z_k, z_k^*) < \infty$, where $\{z_k\}$ are the zeros of b.*
2. *if $\sigma = \sum \sigma_k$, where σ_k is the path measure on the segment from z_k to z_k^*, then*

$$2 \operatorname{Im} \mathcal{C}(\sigma) - \widetilde{\log |h|} \in L^\infty(\partial\mathbb{D}).$$

Proof. Let us first discuss the sufficiency of the conditions (1) and (2). Suppose that $\{z_k\}$ and $\{z_k^*\}$ satisfy (1). Write $M = \sup_k \beta(z_k, z_k^*)$ and $z_k(t) = z_k + t(z_k^* - z_k)$ for $0 \leq t \leq 1$. If b_t is the Blaschke product with zeros $\{z_k(t) : k = 1, 2, \ldots\}$ then $\|\mu_{b_t}\|_c \leq C_1 = C_1(\|\mu_b\|_c, M)$ for all t. Hence, there is a constant $\varepsilon > 0$ independent of t such that

$$|b_t(z)| \geq \varepsilon \quad \text{if} \quad \beta(z, Z(b_t)) \geq 1. \tag{2.4}$$

Let $\alpha = \alpha(\varepsilon/2, C_1) < 1$ be the quantifier of Corollary 2.4 and choose points $0 = t_0 < \cdots < t_n = 1$ in the interval $[0, 1]$ such that $\beta(z_k(t), z_k(t')) < \alpha$ for all $k \geq 1$, whenever t and t' belong to the same interval $[t_j, t_{j+1}]$, $j = 0, 1, \ldots, n-1$. By Corollary 2.4, for each $0 \leq j < n$ there is $g_{j+1} \in (H^\infty)^{-1}$ such that

$$\|b_{t_j} - b_{t_{j+1}} g_{j+1}\| < \varepsilon/2 \quad \text{and} \quad 2\mathrm{Im}\, \mathcal{C}(\sigma_{t_j,t_{j+1}}) - \widetilde{\log |g_{j+1}|} \in L^\infty(\partial\mathbb{D}). \tag{2.5}$$

Here $\sigma_{t_j,t_{j+1}}$ is the sum of the path measures from $z_k(t_j)$ to $z_k(t_{j+1})$. Since $|b_{t_j}(z)| \geq \varepsilon$ when $\beta(z, Z(b_{t_j})) \geq 1$, for any $0 \leq s \leq 1$, the zeros of the function $b_{t_j} + s(b_{t_{j+1}} g_{j+1} - b_{t_j})$ are contained in $\Omega_j = \{z \in \mathbb{D} : \beta(z, Z(b_{t_j})) \leq 1\}$. Moreover, by Rouche's Theorem on each connected component of Ω_j it has as many zeros as b_{t_j}. Hence,

$$\{b_{t_j} + s(b_{t_{j+1}} g_{j+1} - b_{t_j}) : 0 \leq s \leq 1\}$$

is a segment contained in CN^* which joins b_{t_j} and $b_{t_{j+1}} g_{j+1}$. Thus, b and $b_{t_n} g_1 \ldots g_n$ can be joined by a polygonal contained in CN^*. Write $g = \prod_1^n g_j \in (H^\infty)^{-1}$ and observe that

$$2\mathrm{Im}\, \mathcal{C}(\sigma_{t_0,t_n}) - \widetilde{\log |g|} = \sum_{j=0}^{n-1} (2\mathrm{Im}\, \mathcal{C}(\sigma_{t_j,t_{j+1}}) - \widetilde{\log |g_{j+1}|}) \in L^\infty(\partial\mathbb{D}).$$

and that $\mathcal{C}(\sigma) = \mathcal{C}(\sigma_{t_0,t_n})$ on the unit circle . So far we have proved that if b and b^* satisfy (1) there is $g \in (H^\infty)^{-1}$ that satisfies (2) and such that b and $b^* g$ can be joined by a path contained in CN^*. If $h \in (H^\infty)^{-1}$ is any function that satisfies (2) then

$$\widetilde{\log \frac{|h|}{|g|}} = \widetilde{\log |h|} - \widetilde{\log |g|} \in L^\infty(\partial\mathbb{D}),$$

which implies that $h = g e^f$ for $f \in H^\infty$. This means that h and g are in the same connected component in $(H^\infty)^{-1}$. This proves the sufficiency.

The necessity of (1) follows from Lemma 2.6. Let us now prove that (2) is also necessary. Let $\gamma : [0, 1] \to CN^*$ be a path joining $\gamma(0) = b$ and $\gamma(1) = b^* h$. Thus $\gamma(t) = b_t h_t$, where b_t is a CNBP and $h_t \in (H^\infty)^{-1}$. By compactness, there exists a constant $K \geq 1$ such that $K^{-1} \leq |h_t(z)| \leq K$ for any $z \in \mathbb{D}$ and any $0 \leq t \leq 1$. Let $0 < \varepsilon < K^{-2}$ be a number satisfying (2.4) and choose points $0 = t_0 < t_1 < \ldots < t_n = 1$ that simultaneously satisfy (2.5) and

$$\|b_t h_t - b_s h_s\|_\infty < \frac{\varepsilon}{2K}$$

for any $t, s \in [t_{j-1}, t_j]$ and $j = 1, 2, \ldots, n$. The existence of such points follows from Corollary 2.4 and the first paragraph in the proof of Lemma 2.6. Hence, on $\partial\mathbb{D}$ we have

$$\|b_{t_j} \overline{b}_{t_{j+1}} - g_{j+1}\|_\infty < \varepsilon/2 \quad \text{and} \quad \|b_{t_j} \overline{b}_{t_{j+1}} - h_{t_j}^{-1} h_{t_{j+1}}\|_\infty < \varepsilon/2,$$

which leads to

$$\|g_{j+1}h_{t_j}h_{t_{j+1}}^{-1} - 1\|_\infty \leq K^2\|g_{j+1} - h_{t_j}^{-1}h_{t_{j+1}}\|_\infty < K^2\varepsilon < 1.$$

Consequently, $g_{j+1}h_{t_j}h_{t_{j+1}}^{-1} = e^f$ with $f \in H^\infty$, which means that

$$\widetilde{\log|g_{j+1}|} - (\widetilde{\log|h_{t_{j+1}}|} - \widetilde{\log|h_{t_j}|}) \in L^\infty(\partial\mathbb{D}).$$

Summing from $j = 0$ to $n-1$ we get

$$\widetilde{\log|g|} - \widetilde{\log|h|} = \sum_{j=0}^{n-1} \widetilde{\log|g_{j+1}|} - (\widetilde{\log|h_{t_n}|} - \widetilde{\log|h_{t_0}|}) \in L^\infty(\partial\mathbb{D}).$$

Since g satisfies (2), so does h. □

It is important to notice that the invertible function h of the above theorem is associated to the particular reordering of the zeros of b^*. Indeed, there could exist two different reorderings of $\{z_k^*\}$ satisfying condition (1) of the theorem that lead to respective functions $h_1, h_2 \in (H^\infty)^{-1}$ with

$$\widetilde{\log|h_1|} - \widetilde{\log|h_2|} \notin L^\infty(\partial\mathbb{D}).$$

An example of this phenomenon is given in Section 4, where in addition $b = b^*$. We also remark that instead of taking segments in the above proof, we can use an equicontinuous family (with respect to the hyperbolic metric) of curves joining z_k with z_k^* with bounded hyperbolic length. In the proof of the theorem we have also showed the following

Corollary 2.8. *Let b and b^* be two CNBP. Then there exists a function $h \in (H^\infty)^{-1}$ such that b and b^*h can be joined by a continuous path contained in CN^* if and only if there is a reordering $\{z_k^*\}$ of the zeros of b^* such that $\sup_k \beta(z_k, z_k^*) < \infty$, where $\{z_k\}$ are the zeros of b.*

We end this section applying Theorem 2.7 to the little Bloch space. The following lemma is well known. It follows immediately from a couple of results given by Guillory, Izuchi and Sarason: Theorem 1 of [8] and the first theorem in Section 3 of [9].

Lemma 2.9. *Let u be an inner function and let b be a CNBP with zeros $Z(b)$. The following two conditions are equivalent*

(1) $\lim_{|w|\to 1} \sup\{|u(w)| : \beta(w, Z(b)) < \alpha\} = 0$ *for any* $\alpha > 0$,

(2) $\lim_{|w|\to 1} |u(w)|(1 - |b(w)|) = 0$.

In particular, if these conditions hold, $\sup\{||u(z)| - |b(z)|| : z \in \mathbb{D}\} = 1$.

Theorem 2.10. *Let u be an inner function in the little Bloch space that is not a finite Blaschke product and let Ω be a connected component of CN^*. Then $|u|$ cannot be approximated uniformly in $\mathbb{D}$ by functions $|f|$, with $f \in \Omega$. In particular, u does not belong to the closure of any component of CN^*.*

Proof. We argue by contradiction. Assume there exists a sequence b_n of CNBP and a sequence of functions $h_n \in (H^\infty)^{-1}$ such that $b_n h_n \in \Omega$ and $\sup\{||u(z)| - |b_n(z)h_n(z)|| : z \in \mathbb{D}\} \to 0$. Then $|h_n|$ tend to 1 uniformly on $\partial\mathbb{D}$, and since h_n are invertible, it follows that $|h_n| \to 1$ uniformly on $\mathbb{D}$. Hence,

$$\sup\{||u(z)| - |b_n(z)|| : z \in \mathbb{D}\} \to 0.$$

Therefore, there is n_0 such that $||u(z)| - |b_{n_0}(z)|| < 1/2$ for all $z \in \mathbb{D}$. Consequently, Lemma 2.9 says that there are constants $m > 0$, $\eta > 0$ and a subsequence $\{z_k\}$ of zeros of b_{n_0} such that $\sup\{|u(z)| : \beta(z, z_k) \leq m\} > \eta$ for all $k \geq 1$. Since u is in the little Bloch space, $|u(z_k)| \geq \eta/2$ for all k sufficiently large. Now fix n such that $||u(z)| - |b_n(z)|| < \eta/4$ for all $z \in \mathbb{D}$. In particular, $|u(w)| < \eta/4$ for any zero w of b_n. Since u is in the little Bloch space,

$$\beta(z_k, Z(b_n)) \geq \beta\big(\{z : |u(z)| \geq \eta/2 \text{ and } |z| \geq |z_k|\}, \{z : |u(z)| \leq \eta/4\}\big) \to \infty$$

when $k \to \infty$. By Theorem 2.7 there is no $h \in (H^\infty)^{-1}$ such that $b_n h$ and b_{n_0} connect in CN^*, which is a contradiction. □

3 On the components of $\mathfrak{I}^*$

We say that a Blaschke product b is floating if there is a sequence $0 < r_n < 1$, tending to 1, such that $\inf_\theta |b(r_n e^{i\theta})| \to 1$ as $n \to \infty$.

Lemma 3.1. *Every Blaschke product b can be factorized as $b = b_1 b_2$, where b_1 and b_2 are floating Blaschke products.*

Proof. Let $Z(b)$ be the zeros of b counting multiplicities. Given any $0 < r < 1$ and $\beta < 1$, there are constants r_0, r_1, with $r < r_0 < r_1 < 1$, such that if B_0 is the Blaschke product whose zeros are the zeros of b that lie in $\{|z| \leq r\} \cup \{|z| \geq r_1\}$, then $\inf_\theta |B_0(r_0 e^{i\theta})| > \beta$. Thus, if $0 < \beta_k < 1$ is a sequence tending to 1 and $0 < r_1 < 1$ is given, we can inductively construct a sequence $r_k < r_{k+1} \to 1$, such that if B_k is the Blaschke product whose zeros are those of b that lie in $\{|z| \leq r_{k-1}\} \cup \{|z| \geq r_{k+1}\}$, then $\inf_\theta |B_k(r_k e^{i\theta})| > \beta_k$ for all $k > 1$. Define b_1 and b_2 as the Blaschke products whose zeros are respectively

$$Z(b_1) = \{z \in Z(b) : \; |z| \leq r_1 \text{ or } r_{4k+3} \leq |z| \leq r_{4k+5}, \text{ for } k \geq 0\},$$
$$Z(b_2) = \{z \in Z(b) : \; r_{4k+1} < |z| < r_{4k+3}, \text{ for } k \geq 0\}.$$

Then $|b_1(z)| > \beta_{4k+2}$ if $|z| = r_{4k+2}$ and $|b_2(z)| > \beta_{4k}$ if $|z| = r_{4k}$ for all $k \geq 1$. It is also clear that $b = b_1 b_2$. □

We say that an open set $G \subset \mathbb{D}$ is non-tangentially dense if for almost every $e^{i\theta} \in \partial\mathbb{D}$, G contains truncated cones

$$\Lambda_\alpha^r(e^{i\theta}) = \{z \in \mathbb{D} : |z - e^{i\theta}| < \alpha(1 - |z|), \ |z| > r\}, \quad r < 1 < \alpha$$

of arbitrarily large opening α. Since an inner function u has non-tangential limits of modulus 1 at almost every point of $\partial\mathbb{D}$, the set $\{z \in \mathbb{D} : |u(z)| > \delta\}$ is non-tangentially dense for any $0 < \delta < 1$. Next we state several technical results that will be used in the proof of our main theorem.

Proposition 3.2. *Let u_0 be a floating Blaschke product and u_1 be an inner function. Assume that there exist a function $h \in (H^\infty)^{-1}$ with $\|h\|_\infty \leq 1$, an open set $\Omega \subset \mathbb{D}$ and a constant $0 < \delta < 1$ such that*

(1) *For $i = 0, 1$, one has $\Omega \subset \{|u_i| < \delta\}$*

(2) *Arclength $\lambda_{\partial\Omega}$ on $\partial\Omega$ is a Carleson measure.*

(3) *There exists an analytic branch of the logarithm of $u_1 h/u_0$ in an open set of the unit disk containing $\mathbb{D} \setminus \Omega$, which we denote by $\log(u_1 h/u_0)$, whose non-tangential limits*

$$\lim_{z \in \mathbb{D}\setminus\Omega,\, z \to e^{i\theta}} \log(u_1 h/u_0)(z)$$

exist at almost every point $e^{i\theta} \in \partial\mathbb{D}$ and define a function in $L^\infty(\partial\mathbb{D})$

(4) $10\,\delta\|\lambda_{\partial\Omega}\|_c \leq \inf_{\partial\mathbb{D}} |h|$.

Then u_0 and $u_1 h$ can be joined by a path contained in $\mathfrak{I}^*$.

Proof. First observe that for any $t \in [0,1]$, the function $g_t = u_0 \exp(t \log(u_1 h/u_0))$ is a bounded analytic function on a neighborhood of $\mathbb{D} \setminus \Omega$. Moreover,

$$\left| u_0 e^{t \log\left(\frac{u_1 h}{u_0}\right)} \right| = |u_0|^{1-t}\, |u_1|^t\, |h|^t.$$

Observe that $|h| \leq |g_t| \leq 1$ on the unit circle and $|g_t| \leq \delta$ on $\partial\Omega$. Fix $0 \leq t \leq 1$. By duality (see [5, IV, Thm. 1.3]), one has

$$\begin{aligned} \operatorname{dist}_{L^\infty(\partial\mathbb{D})}(g_t, H^\infty) &= \sup_{F \in H_0^1,\, \|F\|_1 \leq 1} \left| \int_0^{2\pi} g_t(e^{i\theta}) F(e^{i\theta}) \frac{d\theta}{2\pi} \right| \\ &= \sup_{F \in H^1,\, \|F\|_1 \leq 1} \left| \int_{\partial\mathbb{D}} g_t(z) F(z) \frac{dz}{2\pi} \right| \end{aligned}$$

Fix $F \in H^1$. Cauchy's Theorem and a limit argument shows that

$$\int_{\partial\mathbb{D}} g_t(z)F(z)\frac{dz}{2\pi} = \int_{\partial\Omega} g_t(z)F(z)\frac{dz}{2\pi} \tag{3.6}$$

Indeed, since u_0 is a floating Blaschke product, there are $r_j \to 1$ such that $\inf_{|z|=r_j} |u_0(z)| \to 1$. By condition (1) the circles $|z| = r_j$ do not meet Ω if j is sufficiently large. Let Ω_k, $k \geq 1$, be the connected components of Ω. By Cauchy Theorem,

$$\int_{\partial(r_j\mathbb{D})} gF\frac{dz}{2\pi} = \sum_{\Omega_k \subset r_j\mathbb{D}} \int_{\partial\Omega_k} gF\frac{dz}{2\pi}.$$

By condition (2),

$$\lim_{j\to\infty} \int_{\partial(r_j\mathbb{D})} gF\frac{dz}{2\pi} = \int_{\partial\Omega} gF\frac{dz}{2\pi}.$$

Now,

$$\int_{\partial(r_j\mathbb{D})} g(z)F(z)\frac{dz}{2\pi} = \int_{\partial\mathbb{D}} g(r_jw)F(r_jw)r_j\,\frac{dw}{2\pi} \to \int_{\partial\mathbb{D}} g(w)F(w)\,\frac{dw}{2\pi}$$

by the dominated convergence theorem, observing that $|g(r_jw)F(r_jw)|$ is bounded by the non-tangential maximal function of F at w. This proves (3.6). Hence,

$$\left|\int_{\partial\mathbb{D}} g_t(z)F(z)\frac{dz}{2\pi}\right| \leq \delta\|\lambda_{\partial\Omega}\|_c\|F\|_1.$$

Consequently, (4) says to $\mathrm{dist}_{L^\infty(\partial\mathbb{D})}(g_t, H^\infty) < \inf_{\partial\mathbb{D}} |h|/10$. Therefore, there is $f_t \in H^\infty$ such that

$$\|g_t - f_t\|_{L^\infty(\partial\mathbb{D})} \leq \frac{1}{5}\inf_{\partial\mathbb{D}} |h|, \tag{3.7}$$

implying that at almost every point of $\partial\mathbb{D}$ one has

$$|f_t| \geq |h|^t - \frac{|h|}{5} \geq |h| - \frac{|h|}{5} = \frac{4}{5}|h| \tag{3.8}$$

In particular, $f_t \in \mathfrak{I}^*$ for every $t \in [0,1]$.

Since $\log(u_1h/u_0) \in L^\infty(\partial\mathbb{D})$, the mapping from $[0,1]$ to $L^\infty(\partial\mathbb{D})$ given by $t \mapsto e^{t\log(u_1h/u_0)}$ is continuous, and consequently there is a finite partition of $[0,1]$, $0 = t_0 < t_1 < \cdots < t_n = 1$, such that

$$\|u_0e^{t_j\log\left(\frac{u_1h}{u_0}\right)} - u_0e^{t_{j+1}\log\left(\frac{u_1h}{u_0}\right)}\|_{L^\infty(\partial\mathbb{D})} < \frac{1}{5}\inf_{\partial\mathbb{D}} |h|,$$

which together with (3.7) implies that the three quantities

$$\|u_0 - f_{t_0}\|_{L^\infty(\partial\mathbb{D})}, \quad \|f_{t_j} - f_{t_{j+1}}\|_{L^\infty(\partial\mathbb{D})}, \quad \|u_1h - f_{t_n}\|_{L^\infty(\partial\mathbb{D})}$$

are bounded above by $\frac{3}{5}\inf_{\partial\mathbb{D}}|h|$ for $0 \le j < n$. So, for any function in the segment joining f_{t_j} with $f_{t_{j+1}}$, that is for any $0 \le s \le 1$, (3.8) says that

$$|f_{t_j} + s(f_{t_{j+1}} - f_{t_j})| \ge |f_{t_j}| - |f_{t_{j+1}} - f_{t_j}| \ge \frac{4}{5}\inf_{\partial\mathbb{D}}|h| - \frac{3}{5}\inf_{\partial\mathbb{D}}|h| = \frac{1}{5}\inf_{\partial\mathbb{D}}|h|,$$

and the same holds for the segments joining u_0 with f_{t_0}, and f_{t_n} with $u_1 h$. Hence, all these segments are contained in $\mathfrak{I}^*$ and their union is a path in $\mathfrak{I}^*$ between u_0 and $u_1 h$. □

We shall also use the following version of Proposition 3.2.

Proposition 3.3. *Let u_0 be a floating Blaschke product and u_1 be an inner function. Suppose that $\Omega \subset \mathbb{D}$ is an open set and $0 < \delta < 1$ is a constant satisfying properties* (1) *and* (2) *of Proposition 3.2. Instead of* (3) *and* (4) *assume that*

(3') *There exists an analytic branch of the logarithm of the function u_1/u_0 in an open set of the unit disk containing $\mathbb{D}\setminus\Omega$, which we denote by $\log(u_1/u_0)$, whose non-tangential limits*

$$\lim_{z\in\mathbb{D}\setminus\Omega, z\to e^{i\theta}} \log(u_1/u_0)(z)$$

exist at almost every point $e^{i\theta} \in \partial\mathbb{D}$ and define a function in BMO($\partial\mathbb{D}$).

(4') $10\,\delta\|\lambda_{\partial\Omega}\|_c \le e^{-2\|\log(u_1/u_0)\|_{BMO}}$.

Then there is $h \in (H^\infty)^{-1}$ with $\|h\|_\infty \le 1$ such that (3) *and* (4) *of Proposition 3.2 hold.*

Proof. Write $\gamma = \|\log(\frac{u_1}{u_0})\|_{BMO}$. By hypothesis, on the unit circle one can decompose

$$\log(\frac{u_1}{u_0}) = \text{Im}\,\log(\frac{u_1}{u_0}) = r + \tilde{s}, \quad \text{with} \quad \|r\|_\infty + \|s\|_\infty \le \gamma.$$

Taking $h = e^{-(s+\gamma+i\tilde{s})}$ we have $e^{-2\gamma} \le |h| \le 1$. Thus (4') implies (4). Define $\log(u_1 h/u_0) = \log(u_1/u_0) - (s + \gamma + i\tilde{s})$. Therefore, at almost every point of the unit circle one has

$$\log(\frac{u_1 h}{u_0}) = -(s+\gamma) + ir$$

and (3) of Proposition 3.2 holds. Observe also that $\|\log(\frac{u_1 h}{u_0})\|_{L^\infty(\partial\mathbb{D})} \le 3\gamma$. □

In certain cases, at almost every point of the unit circle the logarithm of the quotient of two Blaschke products can be written as a Cauchy integral of a Carleson measure.

Lemma 3.4. *Let u, b be Blaschke products. Let Ω be an open set of the unit disk containing all the zeros of u and b such that $\mathbb{D}\setminus\overline{\Omega}$ is non-tangentially dense. Let $\nu = \nu_u - \nu_b$, where ν_u (respectively ν_b) is the sum of the harmonic measures $\omega(z, -, \Omega)$ on $\partial\Omega$ from the zeros z of u (respectively b). Suppose that the boundary Γ_j of each connected component Ω_j of Ω is a Jordan rectifiable curve with $0 \notin \Gamma_j$ satisfying*

(1) *Both functions u and b have the same finite number of zeros in each Ω_j*

(2) *Arclength on the union of Γ_j is a Carleson measure*

(3) *There is a constant $C_0 > 0$ such that for any arc $\gamma \subset \Gamma_j$, $|\nu(\gamma)| < C_0$.*

For each $j \geq 1$ fix a point $\xi_j \in \Gamma_j$. Then, there is a constant C_1 such that for any $z \in \mathbb{D} \setminus \overline{\Omega}$,

$$\log \frac{u}{b}(z) = C_1 - \sum_j \int_{\Gamma_j} \nu\big(\gamma(\xi_j, \xi)\big) \frac{d\xi}{\xi - z} - \sum_j \int_{\Gamma_j} \nu\big(\gamma(\xi_j, \xi)\big) \frac{d\overline{\xi}}{(1 - \overline{\xi} z)\overline{\xi}}$$

where $\gamma(\xi_j, \xi)$ denotes the arc contained in Γ_j which goes from ξ_j to ξ in the counterclockwise direction, defines a logarithm of u/b in $\mathbb{D} \setminus \overline{\Omega}$. Moreover, there exists a constant C_2 such that

$$\log \frac{u}{b}(z) = C_2 + 2i \operatorname{Im} \mathcal{C} \left[\sum_j \nu\big(\gamma(\xi_j, \xi)\big)\, d\xi|_{\Gamma_j} \right](z)$$

for almost every $z \in \partial\mathbb{D}$.

Proof. Let $\varphi_\xi(z) = (\xi - z)/(1 - \overline{\xi} z)$. For any $z \in \mathbb{D} \setminus \Omega$, the function

$$\frac{\varphi'_\xi}{\varphi_\xi}(z) = \frac{|\xi|^2 - 1}{(1 - \overline{\xi} z)(\xi - z)} = \frac{\overline{\xi}}{(1 - \overline{\xi} z)} + \frac{1}{(z - \xi)}$$

is harmonic with respect to ξ in the interior of $\Gamma = \cup \Gamma_j$. Then

$$\frac{u'(z)}{u(z)} = \sum_{u(\xi_n) = 0} \frac{\varphi'_{\xi_n}(z)}{\varphi_{\xi_n}(z)} = \int_\Gamma \frac{\varphi'_\xi(z)}{\varphi_\xi(z)}\, d\nu_u(\xi),$$

and the same holds for b. So, for any $z \in \mathbb{D} \setminus \overline{\Omega}$,

$$\frac{u'(z)}{u(z)} - \frac{b'(z)}{b(z)} = \int_\Gamma \frac{1}{(z - \xi)} + \frac{\overline{\xi}}{(1 - \overline{\xi} z)}\, d\nu(\xi) \tag{3.9}$$

On the other hand, for $j = 1, 2, \ldots$ and $z \in \mathbb{D} \setminus \overline{\Omega}$,

$$\begin{aligned} &\frac{d}{dz} \int_{\Gamma_j} \left[\int_{\gamma(\xi_j, \xi)} \frac{dv}{v - z} + \int_{\gamma(\xi_j, \xi)} \frac{1}{(1 - \overline{v} z)} \frac{d\overline{v}}{\overline{v}} \right] d\nu(\xi) \\ &= \int_{\Gamma_j} \left[\int_{\gamma(\xi_j, \xi)} \frac{dv}{(v - z)^2} + \int_{\gamma(\xi_j, \xi)} \frac{d\overline{v}}{(1 - \overline{v} z)^2} \right] d\nu(\xi) \\ &= \int_{\Gamma_j} \left[\frac{1}{(z - \xi)} + \frac{\overline{\xi}}{(1 - \overline{\xi} z)} \right] d\nu(\xi), \end{aligned} \tag{3.10}$$

because $\int_{\Gamma_j} d\nu = 0$. From (3.9) and (3.10), we deduce that there exists a constant C_1 such that on $\mathbb{D} \setminus \overline{\Omega}$ the function

$$\log \frac{u}{b}(z) = C_1 + \sum_j \int_{\Gamma_j} \left[\int_{\gamma(\xi_j,\xi)} \frac{dv}{v-z} + \int_{\gamma(\xi_j,\xi)} \frac{1}{(1-\overline{v}z)} \frac{d\overline{v}}{\overline{v}} \right] d\nu(\xi)$$

is a logarithmic branch of u/b. Using Fubini, one gets

$$\log \frac{u}{b}(z) = C_1 - \sum_j \int_{\Gamma_j} \nu\big(\gamma(\xi_j,v)\big) \frac{dv}{v-z} - \sum_j \int_{\Gamma_j} \nu\big(\gamma(\xi_j,v)\big) \frac{d\overline{v}}{(1-\overline{v}z)\overline{v}} \,,$$

given that $\chi_{\gamma(\xi_j,v)}(v) = \chi_{\gamma(v,\xi_j)}(v)$ and $\nu(\Gamma_j) = 0$. This gives the first statement. To prove the second identity, consider the functions

$$f(z) = \sum_j \int_{\Gamma_j} \nu\big(\gamma(\xi_j,v)\big) \frac{\overline{z}dv}{1-\overline{z}v},$$

$$g(z) = \ C_1 - \sum_j \int_{\Gamma_j} \nu\big(\gamma(\xi_j,v)\big) \frac{d\overline{v}}{(1-\overline{v}z)\overline{v}},$$

which according to Lemma 2.1, $\overline{f} \in H^2_0$ and $g \in H^2$. Observe that $\log(u/b) = f + g$ on $\partial\mathbb{D}$. Since $\log|u/b| = 0 = \operatorname{Re}(f+g)$, the real part of the function $g + \overline{f} \in H^2$ vanishes. Hence, $g = -\overline{f} + ic$, where $c \in \mathbb{R}$ is a constant, meaning that at almost every point of the unit circle,

$$\log \frac{u}{b} = f - \overline{f} + ic = 2i \operatorname{Im} f + ic.$$

□

Given a Blaschke product u, we will construct an interpolating Blaschke product b and a Carleson contour $\Gamma = \partial\Omega$ verifying Lemma 3.4. The system of rectifiable Jordan curves Γ_j appearing in Lemma 3.4 is presented in the following result which is part of the proof of [22, Lemma 3.2]. An explicit proof can be found in [11, Lemma 2]. This is a variation of the classical corona construction given by Carleson in [3].

Lemma 3.5. *Let $u \in H^\infty$ with $\|u\|_\infty = 1$. Let $0 < \delta < 1$ be a fixed constant. Then there exist a constant $\varepsilon = \varepsilon(\delta) > 0$ and a system $\Gamma = \cup\Gamma_j$ of disjoint rectifiable Jordan curves Γ_j such that*

(a) $|u(z)| \le \delta$ *when* $\beta(z, \operatorname{int}\Gamma) \le 1$

(b) $\sup\{|u(w)| : \beta(w,z) \le 15\} > \varepsilon$ *when* $z \notin \operatorname{int}\Gamma$

(c) *The arclength on Γ is a Carleson measure λ_Γ with $\|\lambda_\Gamma\|_c \le C$, where C is a universal constant independent of u and δ.*

Lemma 3.6. *Let u be a Blaschke product and Γ be a Jordan curve contained in $\mathbb{D}$. Let* $\operatorname{int}\Gamma$ *denote the interior of Γ, and consider the sum of harmonic measures*

$$\nu_\Gamma = \sum_{z\in \operatorname{int}\Gamma,\, u(z)=0} \omega(z, -, \operatorname{int}\Gamma).$$

If $L \subset \Gamma$ then

$$\operatorname{diam}_\rho L \geq (\inf_L |u|)^{1/\nu_\Gamma(L)},$$

where $\operatorname{diam}_\rho L = \sup\{\rho(z,w) : z, w \in L\}$.

Proof. By harmonicity

$$\omega(z, L, \operatorname{int}\Gamma)\, \log(\operatorname{diam}_\rho L)^{-1} \leq \log |\varphi_w(z)|^{-1}$$

for $z \in \operatorname{int}\Gamma$ and $w \in L$. Summing on $z \in Z(u)$ we obtain

$$\nu_\Gamma(L)\, \log(\operatorname{diam}_\rho L)^{-1} \leq \log (\inf_L |u|)^{-1}.$$

□

We are ready now to prove the main result of the paper.

Theorem 3.7. *Let u be an inner function. Then there exists a path $\gamma : [0,1] \to \mathfrak{I}^*$ such that $\gamma(0) = u$ and $\gamma(1) = bh$, where b is a CNBP and $h \in (H^\infty)^{-1}$.*

Proof. Using a Mobius transformation we can assume that u is a Blaschke product. By Lemma 3.1 we can assume that u is a floating Blaschke product. Let $0 < \delta < 1$ be a small constant to be chosen later. Consider the contour Γ given by Lemma 3.5 and decompose u as $u = u_1 u_2$ into two Blaschke products u_1, u_2, where u_1 is formed with the zeros z of u that lie inside the interior of Γ such that $\beta(z, \Gamma) > 1$. For each zero z of u_2, part (b) of Lemma 3.5 provides a point $w \in \mathbb{D}$ such that $\beta(z, w) \leq 16$ and $|u_2(w)| \geq |u(w)| > \varepsilon(\delta)$. This implies that u_2 is a Carleson-Newman Blaschke product. For each component Γ_k of Γ consider the measure

$$d\nu_{u_1}(\xi) = \sum_{k\geq 1} \sum_{u_1(z)=0} \omega(z, \xi, \operatorname{Int}\Gamma_k), \quad \xi \in \Gamma,$$

where $\omega(z, \xi, \Omega)$ denotes the harmonic measure from a point $z \in \Omega$ in the domain $\Omega \subset \mathbb{D}$. Hence, the total mass $\nu_{u_1}(\Gamma_k)$ is the number of zeros of u_1 in the interior of Γ_k, which is finite by (a) of Lemma 3.5, given that u is floating. Split each Γ_k into closed arcs that are pairwise disjoint except for the extremes $\{\Gamma_{k,i} : 1 \leq i \leq \nu_{u_1}(\Gamma_k)\}$, with $\nu_{u_1}(\Gamma_{k,i}) = 1$ for all i, and locate a point $w_{k,i}$ in $\Gamma_{k,i}$. Let b_1 be the Blaschke product with zeros $\{w_{k,i} : 1 \leq i \leq \nu_{u_1}(\Gamma_k),\ k \geq 1\}$. Part (c) of Lemma 3.5 and Lemma 3.6 show that b_1 is a CNBP.

The theorem will follow if we show that the functions u and b_1u_2 satisfy the four conditions of Proposition 3.3 when δ is sufficiently small. Applying Lemma 3.4 to u_1 and b_1, we see that at almost every point of $\partial\mathbb{D}$,

$$\log(u_1/b_1) = C_2 + 2i\mathrm{Im}\,\mathcal{C}(\sum_j \nu(\gamma(\xi_j,\xi))d\xi|_{\Gamma_j}),$$

where $\nu = \nu_{u_1} - \nu_{b_1}$. By (c) of Lemma 3.5 and Lemma 2.1, $\log(u_1/b_1)$ belongs to $\mathrm{BMO}(\partial\mathbb{D})$, where $\|\log(u_1/b_1)\|_{BMO}$ is bounded by an absolute constant (independent of u and δ). Since $u/u_2b_1 = u_1/b_1$, only (1) of Proposition 3.2 remains to be proved. This will follow if we show that there is a constant $c(\delta)$ such that

$$\sup_\Gamma |u_2b_1| \le c(\delta) \to 0 \quad \text{when } \delta \to 0. \tag{3.11}$$

Fix $z \in \mathbb{D}$ with $\beta(z, \mathrm{Int}\,\Gamma) \ge 1$ and observe that

$$\log\frac{1}{|u_1(z)|} = \int_\Gamma \log\frac{1}{|\varphi_w(z)|}\, d\nu_{u_1}(w). \tag{3.12}$$

Split the integral over Γ as integrals over $\Gamma_{k,i}$ and consider the families of short and long arcs defined by

$$\mathcal{S} = \{\Gamma_{k,i} : \mathrm{diam}_\beta(\Gamma_{k,i}) \le 1/4\} \quad \text{and} \quad \mathcal{L} = \{\Gamma_{k,i} : \mathrm{diam}_\beta(\Gamma_{k,i}) > 1/4\}.$$

Fix $\Gamma_{k,i} \in \mathcal{S}$. Since $\beta(z, \mathrm{Int}\,\Gamma) \ge 1$, for $w, w_{k,i} \in \Gamma_{k,i}$,

$$\log\frac{1}{|\varphi_w(z)|} < C_1(1 - |\varphi_w(z)|^2) < C_2(1 - |\varphi_{w_{k,i}}(z)|^2) < 2C_2 \log\frac{1}{|\varphi_{w_{k,i}}(z)|}, \tag{3.13}$$

where C_1 and C_2 are universal constants. Hence

$$\int_{\Gamma_{k,i}} \log\frac{1}{|\varphi_w(z)|}\, d\nu_{u_1}(w) < 2C_2 \log\frac{1}{|\varphi_{w_{k,i}}(z)|} \tag{3.14}$$

Now for each $\Gamma_{k,i} \in \mathcal{L}$ let $\alpha_{k,i} = \alpha_{k,i}(z) \in \Gamma_{k,i}$ such that

$$\log\frac{1}{|\varphi_{\alpha_{k,i}}(z)|} = \sup\left\{\log\frac{1}{|\varphi_w(z)|} : w \in \Gamma_{k,i}\right\}.$$

Clearly,

$$\sum_{\Gamma_{k,i}\in\mathcal{L}} \int_{\Gamma_{k,i}} \log\frac{1}{|\varphi_w(z)|}\, d\nu_{u_1}(w) \le \sum_{\Gamma_{k,i}\in\mathcal{L}} \log\frac{1}{|\varphi_{\alpha_{k,i}}(z)|} \le C_1 \int_{\mathbb{D}} \frac{1-|z|^2}{|1-\bar{\xi}z|^2} dm(\xi),$$

where C_1 is the universal constant in (3.13) and $m = \sum(1 - |\alpha_{k,i}|^2)\delta_{\alpha_{k,i}}$. Next we will show that m is a Carleson measure whose Carleson norm is bounded independently of z. Let Q

be a Carleson square. If $\alpha_{k,i} \in Q$, since $\Gamma_{k,i}$ is long, then $1 - |\alpha_{k,i}|^2 < C$ length$(\Gamma_{k,i} \cap 2Q)$, where C is a universal constant. Thus $m(Q) < C$ length$(\Gamma \cap 2Q)$. Hence m is a Carleson measure whose Carleson norm is bounded by a fixed multiple of the Carleson norm of the arclength of Γ. Therefore, by [5, VI, Lemma 3.3],

$$\sum_{\Gamma_{k,i} \in \mathcal{L}} \int_{\Gamma_{k,i}} \log \frac{1}{|\varphi_w(z)|} \, d\nu_{u_1}(w) \leq K, \tag{3.15}$$

where K is another universal constant. Applying (3.14) and (3.15) in (3.12) we get

$$\log \frac{1}{|u_1(z) u_2(z)|} \leq \log \frac{1}{|u_2(z)|} + 2C_2 \sum_{\Gamma_{k,i} \in \mathcal{S}} \log \frac{1}{|\varphi_{w_{k,i}}(z)|} + K,$$

for $\beta(z, \operatorname{int} \Gamma) \geq 1$. Since (a) of Lemma 3.5 says that $|u(z)| \leq \delta$ when $\beta(z, \operatorname{int} \Gamma) = 1$, and we can assume that $2C_2 \geq 1$,

$$\log \frac{1}{\delta} \leq 2C_2 \left[\log \frac{1}{|u_2(z)|} + \log \frac{1}{|b_1(z)|} \right] + K.$$

Therefore

$$|u_2(z) b_1(z)| \leq (e^K \delta)^{1/2C_2} = c(\delta),$$

which together with the maximum modulus principle proves (3.11). Summing up, if δ is sufficiently small, we can apply Propositions 3.2 and 3.3 to deduce that there exists $h \in (H^\infty)^{-1}$ such that u and $u_2 b_1 h$ can be joined by a path contained in $\mathfrak{I}^*$. □

Remark 3.8. A careful examination of the proofs of Proposition 3.2 and Theorem 3.7 shows that there exists a universal constant N such that any inner function can be joined in $\mathfrak{I}^*$ to a function in CN^* by a polygonal formed by the union of at most N segments.

4 Applications and examples

4.1 The invertible group of a Douglas algebra

Given an inner function u, the Douglas algebra $H^\infty[\overline{u}]$ is the closed subalgebra of $L^\infty(\partial\mathbb{D})$ generated by H^∞ and $\overline{u}$. The maximal ideal space of $H^\infty[\overline{u}]$ is naturally identified with the subset of the maximal ideal space $M(H^\infty)$ of H^∞ given by

$$M_u = \{x \in M(H^\infty) : |u(x)| = 1\}.$$

Here we are looking at the functions of H^∞ as defined on the whole maximal space $M(H^\infty)$ (that is, we are identifying $f \in H^\infty$ with its Gelfand transform). Given two inner functions u_0 and u_1, it is well known that $H^\infty[\overline{u}_0] \subset H^\infty[\overline{u}_1]$ if and only if $M_{u_0} \supset M_{u_1}$ (see [5, IX]).

Lemma 4.1. *For $0 \le t \le 1$ let u_t be an inner function such that the mapping $t \to |u_t|$ is continuous from $[0,1]$ to $L^\infty(\mathbb{D})$. Then, given $\varepsilon > 0$ there is $\delta > 0$ such that*

$$|u_0(z)| > 1-\delta \ \Rightarrow \ |u_t(z)| > 1-\varepsilon \ \textit{ for all } \ 0 \le t \le 1.$$

In particular, $H^\infty[\overline{u}_0] = H^\infty[\overline{u}_t]$ for all $0 \le t \le 1$.

Proof. For any $t_0 \in [0,1]$ we have $|u_t(z)| - 1/8 \le |u_{t_0}(z)| \le |u_t(z)| + 1/8$ for all $z \in \mathbb{D}$ whenever $|t - t_0|$ is small enough. Therefore, for these values of t one has

$$\{z \in \mathbb{D} : |u_t(z)| > \frac{1}{2}\} \subset \{z \in \mathbb{D} : |u_{t_0}(z)| > \frac{3}{8}\} \subset \{z \in \mathbb{D} : |u_t(z)| > \frac{1}{4}\}.$$

The first inclusion implies that $u_{t_0}^{-1} \in H^\infty(\{|u_t| > 1/2\})$, and since $u_{t_0}^{-1}(e^{i\theta}) = \overline{u}_{t_0}(e^{i\theta})$ for almost every θ, [5, IX, Thm. 5.2] says that $\overline{u}_{t_0} \in H^\infty[\overline{u}_t]$. Analogously, the second inclusion shows that $\overline{u}_t \in H^\infty[\overline{u}_{t_0}]$. That is, $H^\infty[\overline{u}_t] = H^\infty[\overline{u}_{t_0}]$. Furthermore, let $\delta_n > 0$ be a sequence that tends to 0 and denote $I = [0,1]$. The set

$$\{(t,x) \in I \times M(H^\infty) : |u_t(x)| > 1-\varepsilon\}$$

is an open neighborhood of $I \times M_{u_0} = \bigcap_n \{(t,x) \in I \times M(H^\infty) : |u_0(x)| > 1-\delta_n\}$. So, by compactness there is some n such that

$$\{(t,x) : |u_0(x)| > 1-\delta_n\} \subset \{(t,x) : |u_t(x)| > 1-\varepsilon\}.$$

□

An immediate corollary of Lemma 4.1 is that if u_t $(0 \le t \le 1)$ are inner functions such that $|u_t|$ varies continuously in $\| \ \|_\infty$ and u_0 is a floating Blaschke product, then

$$\inf_\theta |u_0(r_n e^{i\theta})| \to 1 \ \Rightarrow \ \inf_\theta |u_t(r_n e^{i\theta})| \to 1 \ \text{ uniformly on } \ 0 \le t \le 1.$$

In particular, u_t is a floating Blaschke product for all t. Observe that this argument shows that in Propositios 3.2 and 3.3, the inner function u_1 is also a floating Blaschke product.

If A is a commutative Banach algebra with unit, and A^{-1} is the group of invertible elements, the connected component of the unit in A^{-1} is $\exp A = \{e^a : a \in A\}$. Therefore, two elements $a, b \in A^{-1}$ are in the same component if and only if $b \in a \exp A$.

Theorem 4.2. *Let u_0, u_1 be two inner functions such that $M_{u_0} = M_{u_1}$ and $h_1 \in (H^\infty)^{-1}$. The following conditions are equivalent*

(1) $h_1 u_1 \in u_0 \exp(H^\infty[\overline{u}_0])$.

(2) *for some open neighborhood U of M_{u_0} in $M(H^\infty)$ with $\inf_U |u_0 u_1| > 0$, there is a bounded analytic branch of $\log(h_1 u_1 / u_0)$ on $U \cap \mathbb{D}$.*

(3) *There exists a CNBP b (or $b \equiv 1$) with $M_b \supseteq M_{u_0}$ such that bu_0 and bh_1u_1 can be joined by a path contained in $\mathfrak{I}^*$.*

Proof. (1) $\Rightarrow$ (2). Let $f \in H^\infty[\overline{u}_0]$ such that $h_1u_1 = u_0e^f$. Since $f \in H^\infty[\overline{u}_0]$, given $\eta > 0$ there are $g \in H^\infty$ and $n \geq 0$ integer such that $\sup_{M_{u_0}} |\overline{u}_0^n g + f| < \eta$. The set $\{x \in M(H^\infty) : |u_0(x)| > 1/2\}$ is a neighborhood of M_{u_0} where the function

$$\Lambda_\eta := \left| \frac{u_1h_1}{u_0} \exp(\frac{g}{u_0^n}) - 1 \right|$$

is continuous. In addition, on M_{u_0}: $\Lambda_\eta = |e^{f+\overline{u}_0^n g} - 1| \leq e^\eta - 1$. Thus, by choosing $\eta > 0$ small enough so that $\sup_{M_{u_0}} \Lambda_\eta < 1/4$, we get that

$$U := \{x \in M(H^\infty) : |u_0(x)| > \frac{1}{2} \text{ and } |\Lambda_\eta(x)| < \frac{1}{2}\}$$

is an open neighborhood of M_{u_0} such that the function $(u_1h_1/u_0)\exp(g/u_0^n)$ has a bounded analytic logarithm on $U \cap \mathbb{D}$. Clearly, so does u_1h_1/u_0.
(2) $\Rightarrow$ (1). Let $q \in H^\infty(U \cap \mathbb{D})$ such that $h_1u_1/u_0 = e^q$ on $U \cap \mathbb{D}$, where U is an open neighborhood of M_{u_0}. Then $U \cap \mathbb{D}$ is non-tangentially dense, and the function q has a non-tangential limit at almost every point of $\partial\mathbb{D}$ that belongs to $H^\infty[\overline{u}_0]$ (see [5, IX, Thm. 5.1 and Thm. 5.2]).
(1) $\Rightarrow$ (3). Let $f \in H^\infty[\overline{u}_0]$ such that $h_1u_1 = u_0e^f$. Since the set

$$\left\{ g\frac{a}{b} :\ g \in (H^\infty)^{-1},\ a, b \in \text{CNBP} \cap H^\infty[\overline{u}_0]^{-1} \right\}$$

is dense in $H^\infty[\overline{u}_0]^{-1}$ (see [26, Thm. 3.3]), the homotopy e^{tf}, $0 \leq t \leq 1$, in $H^\infty[\overline{u}_0]^{-1}$ can be approximated by a polygonal $p(t)$ formed by segments joining finitely many functions of the form

$$p(0) = 1,\ g_0\frac{a_0}{b_0},\ g_1\frac{a_1}{b_1}, \ldots,\ g_n\frac{a_n}{b_n},\ e^f = p(1),$$

where $g_j \in (H^\infty)^{-1}$, and $a_j, b_j \in \text{CNBP} \cap H^\infty[\overline{u}_0]^{-1}$. Setting $b = \prod_{j=0}^n b_j$, we have that $bu_0p(t)$, $0 \leq t \leq 1$, implements a path in $\mathfrak{I}^*$ between bu_0 and $bu_0e^f = bh_1u_1$. Since each $b_j \in H^\infty[\overline{u}_0]^{-1}$, so is b, meaning that $M_b \supset M_{u_0}$.
(3) $\Rightarrow$ (1). Let $\gamma : [0,1] \to \mathfrak{I}^*$ be a path joining $\gamma(0) = bu_0$ with $\gamma(1) = bh_1u_1$, where b is as in (3), and denote by v_t the inner part of $\gamma(t)$. By Lemma 2.5, the map $t \mapsto |v_t|$ is continuous from $[0,1]$ into $L^\infty(\mathbb{D})$, and consequently Lemma 4.1 says that $\gamma(t) \in (H^\infty[\overline{v}_0])^{-1}$ for all $t \in [0,1]$. Hence, bu_0 and bh_1u_1 are in the same connected component of $(H^\infty[\overline{bu_0}])^{-1}$, meaning that

$$bh_1u_1 \in bu_0 \exp(H^\infty[\overline{bu_0}]) = bu_0 \exp(H^\infty[\overline{u}_0]),$$

where the last equality holds because the hypothesis $M_b \supset M_{u_0}$ implies that $H^\infty[\overline{bu_0}] = H^\infty[\overline{u}_0]$. So, multiplying the above formula by $\overline{b}$ we obtain (1). □

4.2 Nice Blaschke products

For a Blaschke product b and $r > 0$ write

$$\alpha_b(r) = \inf\{|b(z)| : \ \beta(z, Z(b)) > r\}.$$

This function increases with r, so $\alpha_b(\infty) := \sup_r \alpha_b(r) = \lim_{r\to\infty} \alpha_b(r) \in [0,1]$. It is well known that if b is a CNBP then $\alpha_b(\infty) > 0$. For a while it was mistakingly believed that the converse is also true. However, in [7] the authors exhibit a Blaschke product b constructed by Treil that satisfies $\alpha_b(\infty) > 0$ but it is not a CNBP. Moreover, a quick examination of the example shows that $\alpha_b(\infty) = 1$. Notice that $\alpha_b(\infty) = 1$ just means that $|b(z)| \to 1$ when $\beta(z, Z(b)) \to \infty$.

The significance of this constant is that if $w \in \mathbb{D}$ satisfies $0 < |w| < \alpha_b(\infty)$ then $b_w = (w - b)/(1 - \overline{w}b)$ is a CNBP. Indeed, if $r > 0$ is such that $|w| < \alpha_b(r)$ then for every $z \in Z(b_w)$ there is some point $\xi \in Z(b)$ with $\beta(z, \xi) \le r$. So, $|b_w(\xi)| = |w|$, implying that b_w is a CNBP.

Let Γ_{H^∞} be the set of trivial points in $M(H^\infty)$, that is, the points of $M(H^\infty)$ whose Gleason part is a singleton. It is well known that an inner function is a CNBP if and only if it never vanishes on Γ_{H^∞}.

Corollary 4.3. *Let b be a CNBP. Then*

$$\alpha_b(\infty) = \inf\{|b(x)| : x \in \Gamma_{H^\infty}\}.$$

Proof. Let us denote the above infimum by γ. If $|w| < \alpha_b(\infty)$ then $b_w = (w - b)/(1 - \overline{w}b)$ is a Carleson-Newman Blaschke product, and consequently never vanishes on Γ_{H^∞}. So, $\gamma \ge \alpha_b(\infty)$.

If $\gamma > \alpha_b(\infty)$ there is a sequence $\{z_n\}$ such that $\beta(z_n, Z(b)) \to \infty$ and $b(z_n) \to \lambda$, with $\alpha_b(\infty) < |\lambda| < \gamma$. The last of these inequalities implies that $b_\lambda = (\lambda - b)/(1 - \overline{\lambda}b)$ is a CNBP, and consequently $\beta(z_n, Z(b_\lambda)) \to 0$ when $n \to \infty$. Hence, there is a subsequence of zeros of b_λ, say $\{w_k\}$, such that $\beta(w_k, Z(b)) \to \infty$. By Theorem 2.7, b and b_λ cannot be joined by a continuous path contained in CN^*, which means the path $b_{t\lambda}$, with $0 \le t \le 1$, cannot consist entirely of CNBP. In other words, there is t_0, $0 \le t_0 \le 1$ such that $b_{t_0\lambda}$ vanishes at some point of Γ_{H^∞}, a contradiction. $\square$

A description of the CNBPs b that satisfy $\alpha_b(\infty) = 1$ in terms of the distribution of their zeros can be found in [21]. The techniques are based on a previous result by Bishop [1], where he characterized the Blaschke products in the little Bloch space $\mathcal{B}_0$ in terms of their zeros. Not surprisingly, the distribution of the zeros in both cases are diametrically opposed. Similarly, we have seen in Theorem 2.10 the bad behaviour of the Blaschke products in $\mathcal{B}_0$ with respect to the components of CN^*, and next we show how nicely behaves a CNBP b with $\alpha_b(\infty) = 1$.

Corollary 4.4. *Let b be a CNBP and $h \in (H^\infty)^{-1}$. Then $\alpha_b(\infty) = 1$ if and only if $CN^*(hb) = \mathfrak{I}^*(hb)$, where $CN^*(hb)$ (respectively $\mathfrak{I}^*(hb)$) is the component of hb in CN^* (respectively $\mathfrak{I}^*$).*

Proof. First assume that $\alpha_b(\infty) = 1$. We prove the nontrivial inclusion. So, suppose that $u_t h_t \in \mathfrak{I}^*$ is a path, where u_t is inner, $h_t \in (H^\infty)^{-1}$ and $u_0 h_0 = bh$. By Corollary 4.3 and Lemma 4.1,

$$\Gamma_{H^\infty} \subset \{x \in M(H^\infty) : |b(x)| = 1\} = \{x \in M(H^\infty) : |u_t(x)| = 1\}$$

for all t. In particular, u_t never vanishes on Γ_{H^∞} and therefore it is a CNBP. Thus, the path is actually in CN^*. Now suppose that $\alpha_b(\infty) < 1$. Then by Corollary 4.3 there is some $w \in \mathbb{D}$ such that $b_w = (w - b)/(1 - \overline{w}b)$ vanishes at some point of Γ_{H^∞}. Thus, $hb_w \in \mathfrak{I}^*(hb) \setminus CN^*(hb)$. □

In [20] Nestoridis proved that if u is an inner function such that for every $0 < \varepsilon < 1$, the hyperbolic diameter of the components of $\{z \in \mathbb{D} : |u(z)| < \varepsilon\}$ is bounded by a constant depending on ε, then u and zu cannot be joined by a path of inner functions. Since it is clear that such u must be a CNBP satisfying $\alpha_u(\infty) = 1$, Corollaries 4.4 and 2.8 imply that there is no $h \in (H^\infty)^{-1}$ such that uh and zuh are in the same component of $\mathfrak{I}^*$.

4.3 Oddities

Proposition 4.5. *Let $f, g \in \mathfrak{I}^*$. Then there is a CNBP b such that bf and bg can be joined by a path contained in $\mathfrak{I}^*$. Moreover, if $f, g \in CN^*$, then b can be chosen such that bf and bg are joined by a path contained in CN^*.*

Proof. As will be explained later, only the second statement needs to be proved. So let $f, g \in CN^*$. It is known that Γ_{H^∞} is totally disconnected (see [25, Thm. 3.4]), and that the set of functions

$$\{hb_1/b_2 : h \in (H^\infty)^{-1} \text{ and } b_1, b_2 \text{ are CNBP}\}$$

is dense in $C(\Gamma_{H^\infty})$ (see the comments preceding Lemma 4.3 in [25]). Since Γ_{H^∞} is totally disconnected, $C(\Gamma_{H^\infty})^{-1}$ is connected (see [18, Thm. III.4]), and consequently there is a path in $C(\Gamma_{H^\infty})^{-1}$ joining f with g. We can assume that this path is a polygonal $\gamma : [0,1] \to C(\Gamma_{H^\infty})^{-1}$ joining finitely many functions:

$$f,\ h_0\frac{a_0}{b_0},\ h_1\frac{a_1}{b_1}, \ldots,\ h_n\frac{a_n}{b_n},\ g$$

where $h_j \in (H^\infty)^{-1}$ and a_j, b_j are CNBP. Consider $b = \prod_{j=0}^n b_j$, then $b\gamma(t)$ is a polygonal in CN^* that joins bf with bg. The proof of the first statement is analogous once Γ_{H^∞} is replaced by the Shilov boundary of H^∞. □

Let $h \in (H^\infty)^{-1}$ which is not in the connected component of the unity. By Proposition 4.5 there exists a CNBP b such that b and bh are in the same component of CN^*. Let $b_t h_t$, $0 \le t \le 1$, be the path joining b and bh in CN^*. Observe that $t \mapsto b_t$ cannot be continuous because $t \mapsto h_t$ is not.

In [12] Jones used interpolating Blaschke products to find a constructive method of obtaining the Fefferman-Stein decomposition of a BMO function. Our next result points in the same direction.

Corollary 4.6. *Let u be a real-valued function in $L^\infty(\partial\mathbb{D})$, and let $\tilde{u}$ be its harmonic conjugate. Then there exists a CNBP b with zeros $\{z_k\}$ and a permutation of these zeros $\{z_k^*\}$ with $\sup \beta(z_k, z_k^*) < \infty$ such that if σ is the measure associated to these zeros by the comments preceding Lemma 2.3, then*

$$\tilde{u} \in \operatorname{Im} \mathcal{C}(\sigma) + L^\infty(\partial\mathbb{D}).$$

Proof. Consider the function $h = e^{u+i\tilde{u}} \in (H^\infty)^{-1}$ and apply the Proposition 4.5 to h and 1. Then there is a CNBP b such that bh and b can be joined by a path contained in CN^*. Applying Theorem 2.7 the proof is completed. □

4.4 Open questions

This subsection is devoted to mention several questions that appear naturally in this context.

The first one is to find a description of the connected components of $\mathfrak{I}^*$ from which one could deduce our main result, Theorem 3.7. Let u and b be inner functions and $h \in (H^\infty)^{-1}$. According to Lemma 4.1 and Theorem 4.2 if u and bh are in the same component of $\mathfrak{I}^*$ then $M_u = M_b$ and there is a bounded branch of the logarithm of u/bh in a natural open subset of the unit disk. However, these conditions do not seem to be sufficient.

Let u be an inner function and let $\mathfrak{I}^*(u)$ be the connected component of $\mathfrak{I}^*$ containing u. Does there exist a function f in CN^* such that the segment $\{u + t(f-u) : 0 \le t \le 1\}$ is contained in $\mathfrak{I}^*(u)$? In other words, can we take $N = 1$ in Remark 3.8?

Our main result says that any inner function can be joined in $\mathfrak{I}^*$ to a function in CN^*. Can it be also joined in $\mathfrak{I}^*$ to a CNBP? Observe that a positive answer to Problem 1.1 would imply a positive answer to this question. Indeed, if $\|u - b\|_\infty < 1$, the segment $\{u + t(b-u) : 0 \le t \le 1\}$ is contained in $\mathfrak{I}^*$. Also, it is not difficult to show that there are plenty of components of $\mathfrak{I}^* \setminus (H^\infty)^{-1}$ which contain no inner function. An easy example is provided by a finite Blaschke product b and $h \in (H^\infty)^{-1} \setminus \exp H^\infty$. It follows immediately from Theorem 2.7 that there is no inner function in $\mathfrak{I}^*(bh)$.

Theorem 2.10 tells us that the boundary of a single connected component of CN^* cannot contain any inner function in the little Bloch space except for finite Blaschke products. It is natural to ask for a description of the functions which are in the boundary of a connected component of CN^*.

Given a component U of $\mathfrak{I}^*$, describe all the components of CN^* contained in U. Corollary 4.4 gives an answer in a particular case.

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