

# EQUIVALENCE BETWEEN REPRESENTATIONS FOR SAMPLABLE STOCHASTIC PROCESSES AND ITS RELATIONSHIP WITH RIESZ BASIS.

JUAN MIGUEL MEDINA AND BRUNO CERNUSCHI FRÍAS

ABSTRACT. In this paper we study the question of the representation of random variables by means of a countable Riesz basis. We study different representations for processes which are linearly determined by a countable Riesz basis. This concerns the representation of continuous time processes by means of discrete samples.

## 1. INTRODUCTION

Recall the definition of a Hilbert space representation given by Parzen [12] of a finite variance stochastic process. In this work we will study different equivalences between several representations for samplable processes. In this context, a samplable process, will mean a continuous time, or spatial process, which can be completely linearly determined by a series expansion, using a set of countable samples or measurements of the original process. This is related to the problem of reconstructing a signal from its samples. One of the most known results, related to this problem, is the Shannon-Whittaker-Kotelnikov (SWK) sampling theorem, which also has its stochastic version for wide sense stationary (wss) random processes :

**Theorem 1.1.** [13] *Let  $\mathcal{X} = \{X_t\}_{t \in \mathbb{R}}$  be a w.s.s. random process defined over a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , such that its spectral measure is concentrated in a finite interval  $(-B, B)$ , then*

$$X_t = \sum_{n \in \mathbb{Z}} \frac{\sin(Bt - \pi n)}{Bt - \pi n} X_{\frac{\pi n}{B}}.$$

Where the convergence is in the  $L^2(\Omega, \mathcal{F}, \mathbb{P})$ -norm.

This result admits some generalizations for related processes. Lloyd [9] gave a necessary and sufficient conditions, in terms of the spectral measure, for a w.s.s. process to be completely linearly determined by its samples. This result can be extended for some non stationary processes [8]. However, the condition, for a process, of being linearly determined by its samples is weaker then the condition of the samples forming a basis. The study of conditions for a w.s.s. process to have a basis or minimal system goes back to Kolmogorov [13] [14]. However, all these references, as in the case of the SWK theorem, deal with equidistant samples,

---

1991 *Mathematics Subject Classification.* 2010 MSC. Primary 47N30, 47N70, 60G12, 62M10, 94A11; Secondary: 62B10, 94A12.

*Key words and phrases.* Second Order Stochastic Processes, Riesz Basis, Sampling and Reconstruction of Signals.

and are mostly stated for w.s.s. processes. The stochastic version of the SWK theorem, under additional conditions, gives an orthogonal set or a Riesz basis of samples which spans the Hilbert space spanned by the whole process [11]. The representation of signals using Riesz basis has many practical applications [2], in particular, this gives a robust representation of the process under additive noise. A classical generalization of the (deterministic) SWK theorem was given by Kramer [7]. This result allows to treat the case of non uniform samples. In [4] a converse of this result is given, stated as conditions on the interpolating functions. Here, by means of the reproducing kernel Hilbert space [15] associated to the process, we will give an analogous to Kramer's result, and its converse, for random processes, which are the (stochastic) integral transform of an appropriate kernel function. W.s.s. processes are particular cases of this. We shall see that this is also a necessary condition for a Riesz basis of samples. Under additional conditions, we prove that this representation is obtained in a very similar manner to a Karhunen-Loève (KL) expansion, although, the classic KL result may not be applied in this general case.

## 2. GENERAL CONDITIONS FOR THE EQUIVALENCE BETWEEN REPRESENTATIONS

Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space, if  $X$  is an integrable random variable, we denote  $\mathbb{E}(X) = \int_{\Omega} X d\mathbb{P}$ . Let  $T$  be a set of indexes, which in our case of interest, is considered uncountable. In this work we will assume that  $\mathcal{X} = \{X_t\}_{t \in \mathfrak{T}}$  is a finite variance, zero mean random, real valued stochastic process with correlation function  $R(t, t') = \mathbb{E}(X_t X_{t'})$ . This positive definite function defines a reproducing kernel Hilbert space (RKHS) [12] [15], which we will denote  $H(R)$ . We are interested in some Riesz basis for  $H(\mathcal{X}) = \overline{\text{span}} \mathcal{X}$ , the closed linear span of  $\mathcal{X}$  in  $L^2(\Omega, \mathcal{F}, \mathbb{P})$ , and some of their properties.

**Definition 1.** A Riesz basis for  $H$ , a Hilbert space, is a family of vectors  $\{v_n\}_n \subset H$ , such that  $v_n = Ue_n$ , where  $\{e_n\}_n$  is an orthonormal basis of  $H$ , and  $U : H \rightarrow H$  is a bounded bijective operator.

A very useful characterization of Riesz basis is the following well known theorem:

**Theorem 2.1.** [2] *Let  $H$  be a Hilbert space,  $\{v_n\}_n$  is a Riesz basis for  $H \iff \{v_n\}_n$  is complete in  $H$ , and there exists constants  $0 < A < B < \infty$ , such that*

$$A \sum_k |c_k|^2 \leq \left\| \sum_k c_k v_k \right\|_H^2 \leq B \sum_k |c_k|^2.$$

**2.1. Condition for the existence of a countable Riesz basis of  $H(\mathcal{X})$ , an stochastic Kramer like theorem and its converse.** The Kramer sampling theorem [7] gives a method for obtaining orthogonal sampling formulas, for functions-signals which are in the range of an appropriate integral operator. The SWK theorem for band limited functions is a particular case of this. In the random case, we can see briefly that something similar happens if we consider processes which are the integral transform of a suitable random measure. Here,  $\mathfrak{T}$  denotes a set of indexes, which we assume to be, in general, uncountable, as this is the case of interest in sampling problems.

**Theorem 2.2.** *Let  $M$  be a random orthogonal measure over a measurable space  $(U, \mathcal{A})$ , and let  $\mathcal{X} = \{X_t\}_{t \in \mathfrak{T}} \subset L^2(\Omega, \mathcal{F}, \mathbb{P})$  be an stochastic process defined by*

$X_t = \int_U k(t, s) dM(s)$ , where  $k(t, s) = \sum_{n \in \mathbb{Z}} S_n(t) f_n(s)$ ;  $\{f_n\}_n$  is a Riesz basis of  $L^2(U, \mathcal{A}, \mu)$ , with the control measure  $\mu(\cdot) = \mathbb{E}|M(\cdot)|^2$ ,  $(S_n(t))_n \in l^2$  and  $\{k(t, \cdot)\}_{t \in \mathfrak{T}}$  is complete in  $L^2(\mu)$ . Then:

- i)  $S_n(t)$  belongs to the RKHS generated by  $R(t, t') = \mathbb{E}(X_t X_{t'})$  and if  $(\alpha_n)_n \in l^2$  is such that  $\sum_{n \in \mathbb{Z}} \alpha_n S_n(t) = 0$ ,  $\forall t$ , then  $\alpha_n = 0$ ,  $\forall n$ .
- ii) There exists  $\{Z_n\}_n$ , a Riesz basis of  $H(\mathcal{X})$  such that  $X_t = \sum_{n \in \mathbb{Z}} S_n(t) Z_n$ .

*Proof.* If  $\{f_n\}_n$  is a Riesz basis, then there exists a biorthogonal basis  $\{f'_n\}_n$ . Recalling the isometry between  $H(\mathcal{X})$  and  $L^2(\mu)$  define  $Z'_n = \int_U f'_n dM$  and  $Z_n = \int_U f_n dM$ .

Thus  $\{Z_n\}_n$  is a Riesz basis in  $H(\mathcal{X})$ , with a dual basis given by  $\{Z'_n\}_n$ . From this:  $S_n(t) = \mathbb{E}(X_t Z'_n) = \langle k(t, \cdot), f'_n \rangle_{L^2(\mu)}$ . We have:

$$\mathbb{E} \left| X_t - \sum_{|n| \leq N} S_n(t) Z_n \right|^2 = \left\| k(t, \cdot) - \sum_{|n| \leq N} S_n(t) f_n \right\|_{L^2(\mu)}^2 \xrightarrow{N \rightarrow \infty} 0.$$

Finally, suppose that  $(\alpha_n)_n \in l^2$  is such that  $\sum_{n \in \mathbb{Z}} \alpha_n S_n(t) = 0$  for all  $t$ . This is equivalent to  $\mathbb{E} \left( \left( \sum_{n \in \mathbb{Z}} \alpha_n Z'_n \right) X_t \right) = 0$  for all  $t$ , then  $\sum_{n \in \mathbb{Z}} \alpha_n Z'_n = 0$  and thus  $\alpha_n = 0$  for all  $n$ , since  $Z'_n$  is a Riesz basis.  $\square$

Note that in theorem 2.2,  $\{k(t, \cdot)\}_{t \in \mathfrak{T}}$  and  $L^2(\mu)$  are both *representations* of  $\mathcal{X}$ , in the sense given by Parzen in [12]:

**Definition 2.** A Hilbert space  $H$  is a representation of a random process  $\mathcal{X} = \{X_t\}_{t \in \mathfrak{T}}$  if  $H$  is congruent to  $H(\mathcal{X})$ .

**Definition 3.** A family of vectors  $\{v_t\}_{t \in \mathfrak{T}}$  in a Hilbert space  $H$ , is a representation of a random process  $\mathcal{X} = \{X_t\}_{t \in \mathfrak{T}}$  if for every  $s, t \in \mathfrak{T}$ :  $\langle v_t, v_s \rangle_H = \mathbb{E}(X_t X_s)$ .

As in [4] it is possible to give a converse of this result (theorem 2.3). Giving appropriate conditions on the sampling functions, it is possible to obtain a Riesz basis of the whole space  $H(\mathcal{X})$ . In particular, the random process is linearly determined by its samples. In contrast to Garcia's result [4], the hypothesis on the signal, in this random case, of being the image of an integral transform can be dropped. So, in principle, one may conjecture that there exists a larger class of processes with this property. However, is rather easy to see (theorem 2.5), that if there exists a Riesz basis of  $H(\mathcal{X})$  then the process is the integral transform of an appropriate random measure.

**Theorem 2.3.** Let  $\mathfrak{T}$  be a set of indexes, generally non countable, and let  $\mathcal{X} = \{X_t\}_{t \in \mathfrak{T}} \subset L^2(\Omega, \mathcal{F}, \mathbb{P})$  be a stochastic process, such that  $\mathbb{E}(X_t) = 0$ , for all  $t \in \mathfrak{T}$ . Let  $H(\mathcal{X})$  be the closed subspace spanned by  $\mathcal{X}$ . The following assertions are equivalent:

- i) There exists a subset of  $T$  valued functions  $\{S_n\}_{n \in \mathbb{Z}} \in H(R)$  and a subset of random variables  $\{Z_n\}_{n \in \mathbb{Z}} \subset H(\mathcal{X})$  such that:
- a.1)  $(S_n(t))_n \in l^2$ , for all  $t \in \mathfrak{T}$ ,  $((S_n, f)_{H(R)})_n \in l^2$  for all  $f \in H(R)$ , and if

$(a_n)_n \in l^2$  verifies  $\sum_{n \in \mathbb{Z}} a_n S_n(t) = 0$  for all  $t \in \mathfrak{T}$ , then  $a_n = 0$ , for all  $n \in \mathbb{Z}$ .

a.2)  $(\mathbb{E}(YZ_n))_n \in l^2$  for all  $Y \in H(\mathcal{X})$  and  $X_t = \sum_{n \in \mathbb{Z}} S_n(t) Z_n$ , where convergence is in the  $L^2$  norm.

ii) There exists  $\{Z_n\}_{n \in \mathbb{Z}}$ , a Riesz basis of  $H(\mathcal{X})$ .

iii) There exists  $\{S_n\}_{n \in \mathbb{Z}}$  a Riesz basis of  $H(R)$ .

*Proof.* ((ii)  $\implies$  (i)) If  $\{Z_n\}_n$  is a Riesz basis of  $H(\mathcal{X})$ , then there exists a biorthogonal basis  $\{Z'_n\}_{n \in \mathbb{Z}}$ , such that  $S_n(t) = \mathbb{E}(Z'_n X_t)$  and  $X_t = \sum_{n \in \mathbb{Z}} S_n(t) Z_n$ . These  $S_n(t)$

are unique, since  $Z_n$  is a Riesz basis. Is immediate that  $(\mathbb{E}(Z_n Y))_n \in l^2$ , for all  $Y \in H(\mathcal{X})$  and  $S_n \in H(R)$  from the definition. On the other hand, recalling the theory of reproducing kernels, for every  $f \in H(R)$ , there exists  $Y \in H(\mathcal{X})$  such that  $f(t) = \mathbb{E}(X_t Y)$ . This can be written as  $f(t) = \tilde{J}(Y)(t)$ , where

$$(2.1) \quad \begin{aligned} \tilde{J} : H(\mathcal{X}) &\longrightarrow H(R), \\ Y &\longmapsto \tilde{J}(Y) = \mathbb{E}(X_t Y) \end{aligned}$$

and where the reproducing kernel Hilbert space  $H(R) = \text{Ran}(\tilde{J})$ , is equipped with the norm [15]:  $\|v\|_{H(R)} = \inf \left\{ \|w\|_{H(\mathcal{X})} : \tilde{J}(w) = v \right\}$ . In this way, an isometry is defined and thus  $\langle S_n, f \rangle_{H(R)} = \mathbb{E}(Z'_n Y)$ , but  $(\mathbb{E}(Z'_n Y))_n \in l^2$  because  $\{Z'_n\}_n$  is also a Riesz basis. In particular, for all  $t \in \mathfrak{T}$ :  $\sum_{n \in \mathbb{Z}} |S_n(t)|^2 < \infty$ . Finally, let  $(\alpha_n)_n \in l^2$  be such that  $\sum_{n \in \mathbb{Z}} \alpha_n S_n(t) = 0$ ,  $\forall t \in \mathfrak{T}$ . Hence:

$$\sum_{n \in \mathbb{Z}} \alpha_n S_n(t) = \sum_{n \in \mathbb{Z}} \alpha_n \mathbb{E}(Z'_n X_t) = \mathbb{E} \left( X_t \left( \sum_{n \in \mathbb{Z}} \alpha_n Z'_n \right) \right) = 0 \quad \forall t \in T.$$

Then  $\sum_{n \in \mathbb{Z}} \alpha_n Z'_n \in H(\mathcal{X})^\perp = \{0\}$ , thus  $\sum_{n \in \mathbb{Z}} \alpha_n Z'_n = 0$  but  $\{Z'_n\}_n$  is a basis, then  $\alpha_n = 0$ , for all  $n$ .

((i)  $\implies$  (ii)) We shall see that  $\{Z_n\}_n$  of (i) is, indeed, a Riesz basis.

(Step I) Let  $T_k$  and  $T$  be defined as:

$$\begin{aligned} T_k : H(\mathcal{X}) &\longrightarrow l^2 & \text{and} & & T : H(\mathcal{X}) &\longrightarrow l^2, \\ Y &\longmapsto (\mathbb{E}(Z'_n Y)) \mathbf{1}_{[-k, k]}(n) & & & Y &\longmapsto (\mathbb{E}(Z'_n Y))_n \end{aligned}$$

where the  $Z'_n$ 's are such that  $\mathbb{E}(Z'_n X_t) = S_n(t)$ , thus,

$$\|T_k(Y) - T(Y)\|_{l^2}^2 = \sum_{|n| > k} |\mathbb{E}(Z'_n Y)|^2 \xrightarrow[k \rightarrow \infty]{} 0,$$

and then pointwise convergence follows from this. On the other hand

$$\|T_k(Y)\|_{l^2}^2 \leq \left( \sum_{|n| < k} \mathbb{E}|Z'_n|^2 \right) \mathbb{E}|Y|^2,$$

then, by the Banach-Steinhaus theorem,  $T$  is a bounded operator, so there exists  $B$  such that :

$$\sum_{n \in \mathbb{Z}} |\mathbb{E}(Z'_n Y)|^2 \leq B \mathbb{E}|Y|^2.$$

Then by Lemma 3.1.6. of [2], we have that

$$(2.2) \quad \left\| \sum_{n \in \mathbb{Z}} a_n Z'_n \right\|_{H(\mathcal{X})}^2 \leq B \sum_{n \in \mathbb{Z}} |a_n|^2.$$

For all  $(a_n)_n \in l^2$ .

(Step II) Let  $H(S)$  be the reproducing kernel Hilbert space induced by the linear operator

$$J : l^2 \longrightarrow H(S), \\ (\alpha_k)_k \longmapsto J(\alpha) = \sum_k \alpha_k S_k(t)$$

note that  $J$  is well defined, and  $H(S)$  is the range of  $J$ :  $Ran(J)$  equipped with the norm  $\|v\|_{H(S)} = \inf \{ \|w\|_{l^2} : J(w) = v \}$ .

Let us see that  $H(S) = H(R)$ , in the sense of set inclusions. Let  $v \in H(S)$ , then  $v(t) = \sum_{n \in \mathbb{Z}} \alpha_n S_n(t)$ , taking in account that  $S_n(t) = \mathbb{E}(Z'_n X_t) = \tilde{J}(Z'_n)$  and that

$\sum_{n \in \mathbb{Z}} \alpha_n Z'_n$  converges by (Step I) eq. 2.2, then by eq. 2.1  $v(t) = \tilde{J} \left( \sum_{n \in \mathbb{Z}} \alpha_n Z'_n \right) (t)$ ,

and thus  $v \in Ran(\tilde{J}) = H(R)$ . On the other hand if  $v \in H(R)$ , then  $v = \tilde{J}(Y)$ , for some  $Y \in H(\mathcal{X})$ , but if  $X_t = \sum_{n \in \mathbb{Z}} Z_n S_n(t)$  and recalling again eq. 2.1:  $v(t) =$

$\sum_{n \in \mathbb{Z}} \mathbb{E}(Z_n Y) S_n(t) = J(\beta)(t)$ , with  $\beta_n = \mathbb{E}(Z_n Y)$ ,  $\beta \in l^2$ , then  $v \in Ran(J) = H(S)$ .

(Step III) Now, let us see that their norms are equivalent. In fact, consider the inclusion map  $H(R) \xrightarrow{i} H(S)$  and  $v_n = i(v_n) \xrightarrow{n \rightarrow \infty} w$  in the  $H(S)$ -norm and such that  $v_n \xrightarrow{n \rightarrow \infty} 0$  in the  $H(R)$ -norm. But as  $H(R)$  and  $H(S)$  are both RKHS, then convergence in norm implies pointwise convergence for each  $t \in \mathfrak{T}$ , so we have that  $v_n(t) = i(v_n)(t) \xrightarrow{n \rightarrow \infty} w(t) = 0$  for all  $t \in \mathfrak{T}$ , thus  $w = 0$ , and then by the closed graph theorem  $i$  is continuous, but  $i$  is also a bijective map, then there exist constants  $0 < A \leq B < \infty$  such that:

$$A \|v\|_{H(R)} \leq \|i(v)\|_{H(S)} \leq B \|v\|_{H(R)},$$

but we have seen that  $v = J(\beta)$ , with  $\beta_n = \mathbb{E}(Z_n Y)$ , and since this coefficients are unique (condition a.1)  $\|v\|_{H(S)} = \sum_{n \in \mathbb{Z}} |\mathbb{E}(Z_n Y)|^2$ . On the other hand,  $v = \tilde{J}(Y)$ , thus

$$A \|Y\|_{H(\mathcal{X})} \leq \left( \sum_{n \in \mathbb{Z}} |\mathbb{E}(Z_n Y)|^2 \right)^{\frac{1}{2}} \leq B \|Y\|_{H(\mathcal{X})},$$

and then  $\{Z_n\}_n$  is a frame.

(Step IV) Now, we shall see that  $\{Z_n\}_n$  is a Riesz basis of  $H(\mathcal{X})$ . Indeed  $\{Z_n, Z'_n\}_n$  is a biorthogonal system, with  $Z'_n$  as in (Step I). Since  $X_t = \sum_{n \in \mathbb{Z}} Z_n S_n(t)$  and

then  $S_m(t) = \sum_{n \in \mathbb{Z}} \mathbb{E}(Z'_m Z_n) S_n(t)$ , thus  $\mathbb{E}(Z'_m Z_n) = \delta_{nm}$ . Finally, let us see that

$\text{span}\{Z_k\}_k = H(\mathcal{X})$ . For this, take  $Y \in H(\mathcal{X})$  such that  $\mathbb{E}(Z_k Y) = 0 \forall k \in \mathbb{Z}$ , but this implies  $\mathbb{E}(Y X_t) = 0$  for all  $t \in \mathfrak{T}$ , and then  $Y = 0$  a.s.

((ii)  $\iff$  (iii)) is immediate.

□

The fact that every samplable process admits a representation as an stochastic integral, in this case, is a consequence of the fact that  $H(\mathcal{X})$  is separable and that all separable Hilbert spaces are isometrically isomorphic. We will give a complete proof of this, in order to make the development of the work self contained. First, we need the following theorem:

**Theorem 2.4.** [5] *Let the covariance function of a random process  $\mathcal{X} = \{X_t\}_{t \in \mathfrak{T}}$  admit the following representation:  $R(t, t') = \int_U k(t, \cdot) \overline{k(t', \cdot)} d\mu$ , where  $\mu$  is a positive measure over  $(U, \mathcal{A})$ , and  $\{k(t, \cdot)\}_{t \in \mathfrak{T}} \subset L^2(\mu)$  is complete. Then  $\mathcal{X} = \{X_t\}_{t \in \mathfrak{T}}$  admits the following representation:  $X_t = \int_U k(t, \cdot) dM$  a.s., where  $\{M(A), A \in \mathcal{A}\} \subset H(\mathcal{X})$  is an orthogonal random measure, such that  $\mathbb{E}|M(\cdot)|^2 = \mu(\cdot)$ .*

With this result, we can prove:

**Theorem 2.5.** *If there exists a Riesz basis,  $\{Z_n\}_n$  of  $H(\mathcal{X})$ , given  $(U, \mathcal{A}, \mu)$  a measure space, such that  $L^2(\mu)$  is separable, then:*

- i) *There exists an orthogonal random measure  $M$  over  $(U, \mathcal{A})$ , with control measure  $\mu$ , i.e.  $\mu(\cdot) = \mathbb{E}|M(\cdot)|^2$ .*
- ii) *There exists a Riesz basis  $\{f_n\}_n$  of  $L^2(\mu)$ , such that:  $X_t = \int_U k(t, s) dM(s)$ , with  $k(t, s) = \sum_{n \in \mathbb{Z}} S_n(t) f_n(s)$  and  $\{S_n\}_n$  as in a.1) of theorem 2.3, and  $\{k(t, \cdot)\}_{t \in \mathfrak{T}}$  a complete system.*

*Proof.* If  $\{Z_n\}_{n \in \mathbb{Z}}$  is a Riesz basis of  $H(\mathcal{X})$ , then  $H(\mathcal{X})$  is separable as it is  $L^2(\mu)$ , so there exists an isometric isomorphism  $J : H(\mathcal{X}) \rightarrow L^2(\mu)$ , and taking  $f_n = J(Z_n)$ , then  $\{f_n\}_{n \in \mathbb{Z}}$  is a Riesz basis of  $L^2(\mu)$ . So, we take,

$$k(t, \cdot) = \sum_{n \in \mathbb{Z}} \mathbb{E}(X_t Z'_n) f_n = \sum_{n \in \mathbb{Z}} S_n(t) f_n.$$

The coefficients are the same, unique,  $S_n(t)$ 's of the previous result, so a.1) of theorem 2.3 holds. On the other hand  $X_t = \sum_{n \in \mathbb{Z}} \mathbb{E}(X_t Z'_n) Z_n$ , thus  $J(X_t) = k(t, \cdot)$

and then

$$\mathbb{E}(X_t X_{t'}) = \int_U J(X_t) J(X_{t'}) d\mu = \int_U k(t, \cdot) \overline{k(t', \cdot)} d\mu,$$

and  $\{k(t, \cdot)\}_{t \in \mathfrak{T}}$  is complete, indeed, take  $h \in L^2(\mu)$  such that  $\langle k(t, \cdot), h \rangle_{L^2(\mu)} = 0$ , for all  $t \in \mathfrak{T}$ , and since  $\langle f_n, h \rangle_{L^2(\mu)} = \mathbb{E}(Z_n, J^{-1}(h))$  and from the bi orthogonality of  $\{Z_n\}_{n \in \mathbb{Z}}$  and  $\{Z'_n\}_{n \in \mathbb{Z}}$ , we have:

$$0 = \sum_{n \in \mathbb{Z}} \mathbb{E}(X_t Z'_n) \langle f_n, h \rangle_{L^2(\mu)} = \sum_{n \in \mathbb{Z}} \mathbb{E}(X_t Z'_n) \mathbb{E}(Z_n J^{-1}(h)) = \mathbb{E}(X_t J^{-1}(h)).$$

As this holds, for every  $t \in T$ , then  $J^{-1}(h) = 0$  a.s. and thus  $h = 0$  a.e.- $[\mu]$ . Finally, from the representation theorem 2.4, it follows that there exists a random measure  $M$ , such that  $X_t = \int_U k(t, \cdot) dM$ , and  $\mathbb{E}|M(\cdot)|^2 = \mu(\cdot)$ .

□

*Remark:* Alternatively, one may construct the random measure  $M$ , and the Riesz basis  $\{f_n\}_{n \in \mathbb{Z}}$  in the following way: as  $H(\mathcal{X})$  and  $L^2(\mu)$  are both separable, take

any pair of orthonormal basis  $\{Y_n\}_{n \in \mathbb{Z}}$  and  $\{g_n\}_{n \in \mathbb{Z}}$  of  $H(\mathcal{X})$  and  $L^2(\mu)$  respectively, and define over the algebra of  $\mathcal{A}$ -measurable subsets with finite  $\mu$  measure:

$$M(A) = \sum_{n \in \mathbb{Z}} \langle \mathbf{1}_A, g_n \rangle_{L^2(\mu)} Y_n, \text{ and } f_n = \sum_{m \in \mathbb{Z}} \mathbb{E}(Z_n Y_m) Y_m.$$

Then one can verify that if  $A \cap B = \emptyset$  then  $M(A \cup B) = M(A) + M(B)$ , and

$$\mathbb{E}|M(A)|^2 = \sum_{n \in \mathbb{Z}} |\langle g_n, \mathbf{1}_A \rangle_{L^2(\mu)}|^2 = \int_U |\mathbf{1}_A|^2 d\mu = \mu(A).$$

From this, the measure  $M$  extends as usual. Now, given  $M$ , the stochastic integral  $\int_U f dM$ , for  $f \in L^2(\mu)$  is constructed in the standard way, defining an isometry.

**2.2. Application: A sampling theorem.** The following result, shows how the previous results may be applied to the problem of characterizing process which are linearly determined by its samples, and which also form a Riesz basis

**Corollary 2.1.** *Let  $\{X_t\}_{t \in \mathfrak{T}}$  be a zero mean, finite variance stochastic process. The following are equivalent:*

i) *There exists  $S_n \in H(R)$ ,  $Z_n \in H(\mathcal{X})$  and  $\{t_n\}_n \subset \mathfrak{T}$  such that:*

(b.1.)  $\sum_{n \in \mathbb{Z}} |S_n(t)|^2 < \infty, \forall t \in \mathfrak{T}$ ,  $S_n(t_r) = \delta_{nr}$  and  $(\langle S_n, f \rangle_{H(R)})_n \in l^2$  for all  $f \in H(R)$ .

(b.2.)  $(\mathbb{E}(Y X_{t_n}))_n \in l^2 \forall Y \in H(\mathcal{X})$  and  $X_t = \sum_{n \in \mathbb{Z}} X_{t_n} S_n(t)$ .

ii) *There exist  $\{t_n\}_{n \in \mathbb{Z}}$  such that  $\{X_{t_n}\}_n$  is a Riesz basis of  $H(\mathcal{X})$ .*

iii) *Given  $(U, \mathcal{A}, \mu)$  a measure space, such that  $L^2(\mu)$  is separable, there exists  $M$  an orthogonal random measure over  $(U, \mathcal{A})$ ,  $\{t_n\}_{n \in \mathbb{Z}}$  and  $k(t, \cdot) \in L^2(\mu)$  such that  $\{k(t_n, \cdot)\}_{n \in \mathbb{Z}}$  is a Riesz basis of  $L^2(\mu)$ , with  $\mathbb{E}|M(\cdot)|^2 = \mu(\cdot)$  and  $X_t = \int_U k(t, \cdot) dM$ .*

### 2.3. Examples:

2.3.1. *The SKW theorem with a Riesz basis.* Let  $\mathcal{X} = \{X_t\}_{t \in \mathbb{R}}$ , be defined by:

$$X_t = \int_{\mathbb{R}} e^{it\lambda} dM(\lambda),$$

where  $M(\cdot)$  is an orthogonal random measure, such that  $\mu(A) = \mathbb{E}|M(A)|^2 = \int_A \phi(\lambda) d\lambda$ <sup>1</sup>. For some non negative  $\phi \in L^1(\mathbb{R})$ , is a standard result that such  $M$ , exists [5]. Moreover we can take  $\phi$ , such that  $A \leq \phi \leq B$ , a.e. on  $[-\pi, \pi]$ , for some  $A, B > 0$ , and  $\phi = 0$  a.e. on  $[-\pi, \pi]^c$ . The resulting process is w.s.s. and it is easy to verify that  $\{\frac{e^{i\pi n \lambda}}{\sqrt{2\pi}}\}_{n \in \mathbb{Z}}$  is a Riesz basis of  $L^2(\mathbb{R}, d\mu)$  and that,  $\{X_n\}_{n \in \mathbb{Z}}$  is also a Riesz basis of  $H(\mathcal{X})$ . Moreover, the dual basis is given by  $\{\frac{e^{i\pi n \lambda} (\phi(\lambda))^{-1}}{\sqrt{2\pi}}\}_{n \in \mathbb{Z}}$ , and then  $S_n(t) = \mathbb{E}(X_t Z'_n) = \frac{\sin(\pi(t-n))}{\pi(t-n)}$ , and obviously,  $k(t, \lambda) = e^{it\lambda} \mathbf{1}_{[-\pi, \pi]}(\lambda)$ .

<sup>1</sup>In this case, the condition on  $\mu$  of being absolutely continuous with respect to the Lebesgue measure, is necessary, as it was proved in [11].

**2.3.2. Bessel-Hankel Transforms.** Consider  $\mathcal{X} = \{X_t\}_{t \in \mathbb{R}_{\geq 0}}$ , a process defined in the following way: take  $\{W_t\}_{t \in [0,1]}$  a Wiener process, and the orthogonal basis  $\{\sqrt{x}J_r(x\lambda_n)\}_{n \in \mathbb{N}} \subset L^2[0,1]$ , where  $\lambda_n$  is the  $n$ -th positive zero of the Bessel function  $J_r$ ,  $r > -1$ . Thus, if

$$X_t = \int_{[0,1]} \sqrt{xt}J_r(xt)dW(x), \quad \text{then} \quad X_t = \sum_{n \in \mathbb{N}} X_{\lambda_n} \frac{2\sqrt{t\lambda_n}J_r(t)}{J'_r(\lambda_n)(t^2 - \lambda_n^2)},$$

where convergence is in the m.s. sense, this follows from the  $L^2[0,1]$  convergent, classic formula  $\sqrt{xt}J_r(xt) = \sum_{n \in \mathbb{N}} \sqrt{x}J_r(x\lambda_n) \frac{2\sqrt{t\lambda_n}J_r(t)}{J'_r(\lambda_n)(t^2 - \lambda_n^2)}$ .

### 3. KARHUNEN-LOÈVE TYPE EXPANSIONS

In the previous section, we have seen that the random variables  $Z_n$  and the functions  $S_n$  are related. Indeed if  $\{Z'_n\}_n$  is the associated dual basis we know that  $S_n(t) = \mathbb{E}(Z'_n X_t)$ . We would like to give some additional condition under which the  $Z_n$ 's are obtained in a similar fashion to “random Fourier coefficients” as in the original K-L expansion. This point of view may be of more practical use in some applications. However, in this case, we shall confine to the case when  $\mathfrak{T}$  is an open subset of  $\mathbb{R}^d$ . First we begin with an auxiliary result:

**Proposition 3.1.** *Let  $\mathfrak{T} \subset \mathbb{R}^d$  be an open subset, let  $\{X_t\}_{t \in \mathfrak{T}}$  be a m.s. continuous random process and let  $\nu$  be a finite Borel Measure, such that  $\nu$  is equivalent to the Lebesgue measure ( $\nu \equiv \lambda$ ) and  $\mathbb{E}|X_t|^2 = R(t, t) \in L^1(\nu)$ . If there exists  $\{Z_n\}_{n \in \mathbb{Z}}$  a Riesz basis of  $H(\mathcal{X})$  and if  $\{S_n\}_{n \in \mathbb{Z}}$  is as in the previous result, then  $\{S_n\}_n$  is a Bessel sequence (with respect to  $\|\cdot\|_{L^2(\nu)}$ ). And  $T: l^2 \rightarrow L^2(\nu)$ , defines a linear,*

$$(c_n)_{n \in \mathbb{Z}} \mapsto \sum_{n \in \mathbb{Z}} c_n S_n$$

*bounded and injective operator.*

*Remark:* Note that under this conditions there exists an stochastically equivalent measurable version of  $\{X_t\}_{t \in \mathfrak{T}}$  [5], i.e. a version which is  $\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{F}$ -measurable, as a function of  $(t, \omega) \in \mathfrak{T} \times \Omega$ . Indeed, we shall work with that measurable version.

*Proof.*

$$\left\| \sum_{k=1}^N c_k S_k \right\|_{L^2(\nu)}^2 = \int_{\mathfrak{T}} \left| \sum_{k=1}^N c_k S_k(t) \right|^2 d\nu(t) = \int_{\mathfrak{T}} \left| \sum_{k=1}^N c_k \mathbb{E}(Z'_k X_t) \right|^2 d\nu(t).$$

This integral can be bounded, using the Cauchy-Schwartz inequality and the Bessel condition on the  $Z'_n$ s:

$$\begin{aligned} & \int_{\mathfrak{T}} \left| \mathbb{E} \left( X_t \left( \sum_{k=1}^N c_k Z'_k \right) \right) \right|^2 d\nu(t) \\ & \leq \int_{\mathfrak{T}} R(t, t) \mathbb{E} \left| \sum_{k=1}^N c_k Z'_k \right|^2 d\nu(t) \leq B \left( \int_{\mathfrak{T}} R(t, t) d\nu(t) \right) \sum_{k=1}^N |c_k|^2. \end{aligned}$$

Now, if  $K$  is a compact subset,  $(c_n)_n \in l^2$ , and since  $X_t$  is m.s. continuous then  $R(t, t)$  is continuous and bounded, In a similar manner as in the previous bound,



by the Cauchy-Schwartz inequality, we have:

$$\left| \sum_{k=M}^N c_k S_k(t) \right| \leq \sup_{t \in K} (R(t, t))^{\frac{1}{2}} \left( \sum_{k=N}^M |c_k|^2 \right)^{\frac{1}{2}} \xrightarrow{N, M \rightarrow \infty} 0.$$

Then  $\sum_{k \in \mathbb{Z}} c_k S_k$  converges uniformly over compact sets. On the other hand,  $|S_n(t) - S_n(s)|^2 \leq \mathbb{E}|Z'_n|^2 \mathbb{E}|X_t - X_s|^2$ , so the  $S_n$  are continuous over  $K$ , and the same holds for  $\sum_{k \in \mathbb{Z}} c_k S_k$  from the uniform convergence. Now, if  $(c_n)_n$  is such that

$\left\| \sum_k c_k S_k \right\|_{L^2(\nu)} = 0$ , thus  $\sum_{|k| \leq N} c_k S_k \xrightarrow[N \rightarrow \infty]{\nu} 0$ , and then we have  $N_1 < \dots < N_{k-1} < N_k \nearrow \infty$ , a subsequence, such that  $\sum_{|k| \leq N_r} c_k S_k \xrightarrow[r \rightarrow \infty]{} 0$  a.e.  $[\nu]$ , and since  $\nu \equiv \lambda$ , then  $\sum_{k \in \mathbb{Z}} c_k S_k(t) = 0$ , for all  $t$ . But the condition a.1) from theorem 2.2, implies  $c_n = 0$ , for all  $n$ . Thus,

$$T : l^2 \longrightarrow L^2(\nu), \\ (c_n)_n \longmapsto \sum_{n \in \mathbb{Z}} c_n S_n$$

is a well defined, linear and bounded operator, which is injective.  $\square$

Now, if  $(\mathfrak{T}, \mathfrak{F}, \nu)$  is a measure space, given the process  $X_t$ , and if  $\{S_n\}_n$  is a basic sequence, we can find the random variables  $Z_n$ , calculating an integral over  $\mathfrak{T}$ :

**Theorem 3.1.** *Let  $\mathcal{X} = \{X_t\}_{t \in \mathfrak{T}}$  be measurable (i.e.  $X(t, \omega)$  is  $\mathfrak{F} \otimes \mathcal{F}$ -measurable), and let  $\nu$  be a  $\sigma$ -finite measure, such that  $R(t, t) \in L^1(\nu)$ . If  $\{S_n\}_{n \in \mathbb{Z}}$  is a basis of  $\mathcal{A} = \overline{\text{span}}\{S_n\}_n$ , then:*

$$(3.1) \quad Tf(\omega) = \int_{\mathfrak{T}} X(t, \omega) f(t) d\nu(t),$$

*defines a bounded linear operator from  $L^2(\nu)$  to  $H(\mathcal{X})$ , and  $T(S'_n) = Z_n$ , where  $\{S'_n\}_n$  is the dual basis of  $\{S_n\}_n$  with respect to the  $L^2(\nu)$ -norm.*

$$(3.2) \quad T'f(t) = \int_{\mathfrak{T}} R(t, t') f(t') d\nu(t'),$$

*defines a bounded linear operator from  $L^2(\nu)$  to  $H(R)$ , and  $T'(S'_n) = S_n^*$ , where  $\{S_n^*\}_n$  is the dual basis of  $\{S_n\}_n$  with respect to the  $H(R)$ -norm.*

*Proof.* i) Let us prove first that  $T : L^2(\nu) \longrightarrow L^2(\mathbb{P})$ , is well defined and bounded. Indeed, if  $X(t, \omega)$  is  $\mathfrak{F} \otimes \mathcal{F}$ -measurable, and as  $R(t, t) \in L^1(\nu)$  then  $X(t, \omega) \in L^2(\nu)$  for almost all  $\omega$   $[\mathbb{P}]$ . Thus, as  $f \in L^2(\nu)$ , the integral 3.1 is well defined. On the other hand, by Minkowski's and Cauchy-Schwartz's inequalities, respectively:

$$\mathbb{E}|Tf|^2 = \mathbb{E} \left| \int_{\mathfrak{T}} X(t, \cdot) f(t) d\nu(t) \right|^2$$

$$(3.3) \quad \leq \left( \int_{\mathfrak{T}} (\mathbb{E}|X_t|^2)^{\frac{1}{2}} |f(t)| d\nu \right)^2 \leq \int_{\mathfrak{T}} R(t, t) d\nu(t) \|f\|_{L^2(\nu)}^2.$$

Now, let us prove that  $\text{Ran}(T) \subset H(\mathcal{X})$ , indeed, for each  $N \in \mathbb{N}$ , define  $T_N(f) = \sum_{|n| \leq N} Z_n \langle f, S_n \rangle_{L^2(\nu)}$ . Is easy to verify that this  $T_N$ 's are bounded linear operators from  $L^2(\nu)$  to  $H(\mathcal{X})$ , and  $T_N \xrightarrow{N \rightarrow \infty} T$  strongly, since in a similar way to eq. 3.3, we can get:

$$\mathbb{E}|T_N f - T f|^2 \leq \left( \int_{\mathfrak{T}} \mathbb{E} \left| X_t - \sum_{|n| \leq N} Z_n S_n(t) \right|^2 d\nu(t) \right) \|f\|_{L^2(\nu)}^2.$$

If  $\{Z_n\}_n$  is Riesz basis, there exist  $0 < A < B < \infty$  basis constants, so that:

$$\begin{aligned} \mathbb{E} \left| X_t - \sum_{|n| \leq N} Z_n S_n(t) \right|^2 &= \mathbb{E} \left| \sum_{|n| > N} Z_n S_n(t) \right|^2 \\ &\leq A \sum_{n \in \mathbb{Z}} |S_n(t)|^2 \leq \frac{A}{B} \mathbb{E} \left| \sum_{n \in \mathbb{Z}} Z_n S_n(t) \right|^2 = \frac{A}{B} R(t, t) \in L^1(\nu). \end{aligned}$$

But  $\mathbb{E} \left| X_t - \sum_{|n| \leq N} Z_n S_n(t) \right|^2 \xrightarrow{N \rightarrow \infty} 0$ , so by Lebesgue's theorem:

$$\sup_{\|f\|_{L^2(\nu)}=1} \mathbb{E}|T_N f - T f|^2 \xrightarrow{N \rightarrow \infty} 0,$$

from this  $T f \in H(\mathcal{X})$ , since  $H(\mathcal{X})$  is closed. Finally, as  $T_N(S'_m) = Z_m$  if  $N \geq m$ , then  $T(S'_m) = Z_m$ .

ii) If  $\{X_t\}_t$  is such that  $X(t, \omega)$  is  $\mathfrak{F} \otimes \mathcal{F}$ -measurable, then  $\mathbb{E}(X_t X_{t'}) = R(t, t')$  is  $\mathfrak{F} \otimes \mathfrak{F}$ -measurable. On the other hand, by the Cauchy-Schwartz inequality:

$$\begin{aligned} |T' f(t)| &\leq \sqrt{R(t, t)} \int_{\mathfrak{T}} \sqrt{R(t', t')} |f(t')| d\nu(t') \\ &\leq \sqrt{R(t, t)} \left( \int_{\mathfrak{T}} R(t', t') d\nu(t') \right)^{\frac{1}{2}} \|f\|_{L^2(\nu)} < \infty. \end{aligned}$$

So  $T' f$  is well defined and, by Fubini's theorem:

$$T' f(t) = \int_{\mathfrak{T}} \mathbb{E}(X_t X_{t'}) f(t') d\nu(t') = \mathbb{E}(X_t T f) = J(T f)(t)$$

where  $T$ , is defined as in eq. 3.1, and  $\tilde{J}$  as in eq. 2.1. Thus  $T' = J \circ T$  defines a bounded operator. □

*Final Remarks.* Note that from the proof of theorem 3.1 we get that  $A \sum_n |\langle S_n, f \rangle|^2 \leq \mathbb{E}|Tf|^2$ , so if  $Tf = 0$  a.s. then  $\langle S_n, f \rangle = 0$  for all  $n$ , thus  $f \in \mathcal{A}^\perp$ . In a similar manner, if  $f \in \mathcal{A}^\perp$ , then  $Tf = 0$  and so  $\text{Ker}(T) = \mathcal{A}^\perp$ . In particular if  $\mathcal{A}$  is the whole space  $L^2(\nu)$ , then  $T$  is injective. A similar analysis holds for  $T'$ . In these last results  $\{S_n\}_n$  cannot be an unconditional (Riesz) basis with respect to the  $L^2(\nu)$ -norm, indeed, if this would be the case, as the  $\{Z_n\}_n$  are already a Riesz basis, we have that  $A \sum_n |S_n(t)|^2 \leq R(t, t) \leq B \sum_n |S_n(t)|^2$ , and integrating, we get  $\sum_n \|S_n\|_{L^2(\nu)}^2 < \infty$ , which contradicts the fact that  $\inf_n \|S_n\| > 0$ .

#### ACKNOWLEDGEMENTS

This work was partially financed by CONICET, Universidad de Buenos Aires, grant No. UBACyT 20020100100503, and grant PICT No. 209.

#### REFERENCES

- [1] Cambanis S. Masry E.: "Wavelet Approximation of deterministic and random signals: Convergence properties and rates". *IEEE Trans. Inf. Theory*, Vol. 40, pp.1013-1029, 1994.
- [2] Christensen O.: *Frames and Bases. Applied and Numerical Harmonic Analysis Series*, Birkhäuser, 2008.
- [3] Garcia A.G.: "Orthogonal Sampling Formulas: a unified approach", *SIAM Review*, 42(3), pp. 499-512, 2000.
- [4] Garcia, A. G.: "A converse of the Kramer Sampling Theorem". *Sampling Theory in Image and Signal Processing*, Vol 1 (1), 2002, pp. 53-61.
- [5] Gikhman, I.I. Skorokhod A.V., *The Theory of Stochastic Processes*, Springer (Classics in Mathematics), 2004 edition.
- [6] Jorgensen P.E., Song M. "Entropy Encoding, Hilbert Space, and Karhunen-Loève Transforms", *J. Math. Phys.*, 48, 103503 (2007).
- [7] Kramer, H.P.: "A Generalized Sampling Theorem". *J.Math. Phys.* 63, pp. 68-72, 1957.
- [8] Lee, A.J.: "Sampling theorems for non stationary random processes". *Trans. Amer. Math. Soc.* 242, pp.225 -241, 1978.
- [9] Lloyd S.P.: "A sampling theorem for stationary (wide sense) stochastic processes". *Trans. Amer. Math. Soc.* 92(1) pp.1-12, 1959.
- [10] Masry E., Steiglitz K., Liu B., "Bases in Hilbert Space Related to the Representation of Stationary Operators", *SIAM J. Appl. Math.*, Vol. 16, No. 3. (May, 1968), pp. 552-562.
- [11] Medina J.M. Cernuschi-Frías B.. "Wide Sense Stationary Random Processes Forming Frames, *IEEE Trans. Inf. Theory*, 57(7), 2011, pp. 4636-4646.
- [12] Parzen E. *Statistical Inference on time series by Hilbert space methods I*. Technical Report No. 23, 1959, Stanford University.
- [13] Rozanov, Y. A.: *Stationary Random Processes*. Holden-Day, 1967.
- [14] Rozanov Y.A.: "On stationary sequences forming a basis", *Soviet Math.-Doklady* 1, 91-93 (1960).
- [15] Saitoh S. *Integral Transforms, Reproducing Kernels and Their Applications*. Longman, 1997.

CORRESPONDENCE TO: JUAN M. MEDINA; ADDRESS: PASEO COLÓN 850 -C1063ACV- CAPITAL FEDERAL., DPTO. MATEMÁTICA, (U.B.A.) UNIVERSIDAD DE BUENOS AIRES -AND- INSTITUTO ARGENTINO DE MATEMÁTICA A.P. CALDERÓN- CONICET ARGENTINA  
E-mail address: jmedina@fi.uba.ar