Unbounded symmetrizable idempotents

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Abstract

The relationship between closed unbounded idempotents and dense decompositions of a Hilbert space is explored by extending the notion of compatibility between closed subspaces and positive bounded operators.

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1 Introduction

Bounded linear projections on $\mathcal{H}$ are naturally identified with different sum decompositions $\mathcal{H} = \mathcal{S} \dot{+} \mathcal{T}$, where $\mathcal{S}$ and $\mathcal{T}$ are closed subspaces of $\mathcal{H}$ with $\mathcal{S} \cap \mathcal{T} = \{0\}$. In this case, the norm of the projection $P_{\mathcal{S}//\mathcal{T}}$ with range $\mathcal{S}$ and nullspace $\mathcal{T}$ is $1/\sin(\theta)$, where θ is the Dixmier angle between $\mathcal{S}$ and $\mathcal{T}$ (see definition in Section 2). However, the projection $P_{\mathcal{S}//\mathcal{T}}$ is well defined on $\mathcal{S} \dot{+} \mathcal{T}$ even if $\mathcal{S} \dot{+} \mathcal{T}$ is only a proper dense subspace of $\mathcal{H}$ with zero angle between $\mathcal{S}$ and $\mathcal{T}$. In this case, $P_{\mathcal{S}//\mathcal{T}}$ is an unbounded projection which is closed in the sense that its graph is a closed subspace of $\mathcal{H} \times \mathcal{H}$ (or, equivalently, its range and nullspace are closed subspaces of $\mathcal{H}$, see [37]). Closed unbounded projections appear in several different contexts. We only mention two of them. In the study of Krein spaces, a symmetric projection with respect to the fundamental symmetry J is bounded (for the subjacent Hilbert space structure) if and only if its range is regular (Langer [34], Ando [3]). Thus, a non degenerated closed subspace $\mathcal{S}$ which is not regular induces an unbounded J-symmetric closed projection with range $\mathcal{S}$ and nullspace $\mathcal{S}^{[\perp]}$ (Gheondea [24], Maestripieri and Martínez-Pería [36]), where $\mathcal{S}^{[\perp]}$ denotes the J-orthogonal complement of $\mathcal{S}$. In a different setting, the Moore-Penrose pseudoinverse of PQ, where P, Q are orthogonal projections in a Hilbert space, is a closed projection which is bounded if and only if the range of PQ is closed (Greville [23], Corach and Maestripieri [14]).

Our motivation for the study of closed unbounded projections is the following. Let $\mathcal{H}$ be a Hilbert space, $\mathcal{S}$ a closed subspace of $\mathcal{H}$ and A a positive (semidefinite bounded) operator acting on $\mathcal{H}$. It is said that $\mathcal{S}$ and A are *compatible* if there exists some (bounded linear) projection with range $\mathcal{S}$ which is Hermitian with respect to the sesquilinear form defined by A. In many recent papers this notion has proved useful for studying problems on splines [16], frames [6], selfadjoint projections in Krein spaces [36], approximation [11], Schur complements [17], sampling [5] and so on. The condition has been implicitly used by Sard in 1950 (see [43] and [11]), and, more recently, it has been studied, for a selfadjoint operator A, by Hassi and Nordström [27]. The compatibility condition depends on an angle between $\mathcal{S}^{\perp}$ and the closure of $A\mathcal{S}$. More precisely, $\mathcal{S}$ and A are compatible if and only if the Friedrichs angle (see the definition in Section 2) between $\mathcal{S}^{\perp}$ and $\overline{A\mathcal{S}}$ is not zero; or, equivalently if $\mathcal{S} + (A\mathcal{S})^{\perp} = \mathcal{H}$. It turns out that, among non compatible pairs,

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there is a class which has a "quasi-compatibility" property. A pair $\mathcal{S}$, A is called *quasi-compatible* if there exists an unbounded closed projection with range $\mathcal{S}$ and nullspace contained in $(A\mathcal{S})^\perp$; or equivalently, if the subspace $\mathcal{S}+(A\mathcal{S})^\perp$ is dense in $\mathcal{H}$. The main goal of this paper is the study of this general notion and its applications. Section 2 contains some preliminaries and a quite complete description of all pairs of closed subspaces with some properties which appear in the different compatibility notions. Many of the results of Section 2 are well-known. However, we present some new results which are relevant for the subsequent sections. There is also a description of closed unbounded projections which is needed in the sequel. Section 3 is devoted to describe unbounded projections which are A-symmetric for a given positive bounded operator A. We show that the injectivity or invertibility of $P_1 + P_2$ and $P_1 - P_2$ or properties of $\|P_1 + P_2 - I\|$ and $\|P_1P_2\|$ (for a convenient choice, in each case, of the orthogonal projections P_1, P_2) give equivalent conditions for the compatibility and quasi-compatibility of a given pair $\mathcal{S}$, A as before. In addition, we describe the set $\mathcal{P}(A,\mathcal{S})$ of all A-symmetric projections with range $\mathcal{S}$. We also prove the existence of a distinguished $P_{A,\mathcal{S}} \in \mathcal{P}(A,\mathcal{S})$ with several nice properties. Section 4 shows how to calculate, more o less explicitely, the projection $P_{A,\mathcal{S}}$. Finally, Section 5 contains applications of quasi-compatibility to abstract interpolation, splines and least squares problems.

2 On sums and differences of orthogonal projections

In this paper, $\mathcal{H}$ is an infinite dimensional separable Hilbert space, $L(\mathcal{H})$ denotes the algebra of all bounded linear operators from $\mathcal{H}$ to $\mathcal{H}$ and $L(\mathcal{H})^+$ the cone of bounded linear positive operators of $L(\mathcal{H})$. Given a densely defined linear operator $T : \mathcal{D}(T) \subseteq \mathcal{H} \longrightarrow \mathcal{H}$, $R(T)$ and $N(T)$ denote the range and the nullspace of T, respectively. Throughout, $\mathcal{S}$ and $\mathcal{T}$ denote two closed subspaces of $\mathcal{H}$. By $\mathcal{S} \dot{+} \mathcal{T}$ we denote the direct sum between them and by $\mathcal{S} \oplus \mathcal{T}$ the orthogonal sum. Furthermore, $\mathcal{S} \ominus \mathcal{T} = \mathcal{S} \cap (\mathcal{S} \cap \mathcal{T})^\perp$. If $\mathcal{S} \cap \mathcal{T} = \{0\}$, $P_{\mathcal{S}//\mathcal{T}}$ denotes the (not necessarily bounded) projection (or idempotent) onto $\mathcal{S}$ with nullspace $\mathcal{T}$ and $P_\mathcal{S} = P_{\mathcal{S}//\mathcal{S}^\perp}$ is the orthogonal projection onto $\mathcal{S}$.

The *angle of Friedrichs* between the subspaces $\mathcal{S}$ and $\mathcal{T}$ is the angle $\theta(\mathcal{S},\mathcal{T})$ in $[0, \frac{\pi}{2}]$ whose cosine is defined by

$$c(\mathcal{S},\mathcal{T}) = \sup\{|\langle \xi, \eta\rangle| : \xi \in \mathcal{S} \ominus \mathcal{T}, \eta \in \mathcal{T} \ominus \mathcal{S}; \|\xi\|, \|\eta\| \leq 1\},$$

and the *angle of Dixmier* between the subspaces $\mathcal{S}$ and $\mathcal{T}$ is the angle $\theta_0(\mathcal{S},\mathcal{T})$ in $[0, \frac{\pi}{2}]$ whose cosine is defined by

$$c_0(\mathcal{S},\mathcal{T}) = \sup\{|\langle \xi, \eta\rangle| : \xi \in \mathcal{S}, \eta \in \mathcal{T}; \|\xi\|, \|\eta\| \leq 1\}.$$

Observe that $c_0(\mathcal{S},\mathcal{T})$ gives a sharp bound for the Cauchy-Schwarz inequality, in the sense that $|\langle \xi, \eta\rangle| \leq c_0(\mathcal{S},\mathcal{T})\|\xi\|\|\eta\|$ for every $\xi \in \mathcal{S}$, $\eta \in \mathcal{T}$. For many results on these notions of angles we refer the reader to the survey of Deutsch [19] and his book [20]. Here we collect some facts of [19] and [20] that we shall use along these notes:

Proposition 2.1. *The following assertions hold:*

1. $0 \leq c(\mathcal{S},\mathcal{T}) \leq c_0(\mathcal{S},\mathcal{T}) \leq 1$;
2. *If* $\mathcal{S} \cap \mathcal{T} = \{0\}$ *then* $c(\mathcal{S},\mathcal{T}) = c_0(\mathcal{S},\mathcal{T})$;
3. $c_0(\mathcal{S},\mathcal{T}) = \|P_\mathcal{S}P_\mathcal{T}\|$;
4. $c(\mathcal{S},\mathcal{T}) = \|P_\mathcal{S}P_\mathcal{T} - P_{\mathcal{S}\cap\mathcal{T}}\|$;
5. $c(\mathcal{S},\mathcal{T}) = c(\mathcal{S}^\perp,\mathcal{T}^\perp)$.

We start studying the operators $P_{\mathcal{S}} + P_{\mathcal{T}}$ and $P_{\mathcal{S}} - P_{\mathcal{T}}$. Observe that the first one is positive and the second one is selfadjoint. These operators verify the following properties:

Proposition 2.2. *The following assertions hold:*

1. $R((P_{\mathcal{S}} + P_{\mathcal{T}})^{1/2}) = \mathcal{S} + \mathcal{T}$;
2. $N(P_{\mathcal{S}} + P_{\mathcal{T}}) = \mathcal{S}^{\perp} \cap \mathcal{T}^{\perp}$;
3. $N(P_{\mathcal{S}} - P_{\mathcal{T}}) = \mathcal{S} \cap \mathcal{T} \oplus \mathcal{S}^{\perp} \cap \mathcal{T}^{\perp}$;
4. $(P_{\mathcal{S}} - P_{\mathcal{T}})^2 + (P_{\mathcal{S}} + P_{\mathcal{T}} - I)^2 = I$;
5. $\|P_{\mathcal{S}} - P_{\mathcal{T}}\| = \max\{\|P_{\mathcal{S}}(I - P_{\mathcal{T}})\|, \|P_{\mathcal{T}}(I - P_{\mathcal{S}})\|\}$;
6. $\|P_{\mathcal{S}} + P_{\mathcal{T}} - I\| \leq 1$.

The first identity follows applying [[22], Theorem 2.2]. $N(P_{\mathcal{S}} + P_{\mathcal{T}}) = \mathcal{S}^{\perp} \cap \mathcal{T}^{\perp}$ is evident. A proof for item *3* can be found in [[33], Lemma 2.2]. Observe that if $\mathcal{S} \cap \mathcal{T} = \{0\}$ then it holds $N(P_{\mathcal{S}}+P_{\mathcal{T}}) = N(P_{\mathcal{S}}-P_{\mathcal{T}}) = \mathcal{S}^{\perp}\cap\mathcal{T}^{\perp}$. On the other hand, identity $(P_{\mathcal{S}}-P_{\mathcal{T}})^2+(P_{\mathcal{S}}+P_{\mathcal{T}}-I)^2 = I$, which follows by computation, is due to Kato ([29], [30]) and it is equivalent to

$$\|(P_{\mathcal{S}} - P_{\mathcal{T}})\xi\|^2 + \|(P_{\mathcal{S}} + P_{\mathcal{T}} - I)\xi\|^2 = \|\xi\|^2 \quad \text{for every} \quad \xi \in \mathcal{H}. \tag{1}$$

Item *5* is the Krein-Krasnoselskii-Milman equality and its proof can be found in [1], [28] and [32].

Observe that $P_{\mathcal{S}} + P_{\mathcal{T}} - I = P_{\mathcal{S}} - P_{\mathcal{T}^{\perp}}$. Observe also that, for orthogonal projections P_1 and P_2 there are three alternatives for the norm of $P_1 - P_2$: *a*) $\|P_1 - P_2\| < 1$; *b*) $\|P_1 - P_2\| = 1$ but the norm is not attained; and *c*) there exists $\xi \in \mathcal{H}$ with $\|\xi\| = 1$ and $\|(P_1 - P_2)\xi\| = 1$. We shall prove in Section 3 that these three alternatives (for a convenient choice of P_1 and P_2) describe the different notions of compatibility. The next theorem collects several sets of equivalent conditions which are central for the different types of compatibility. Most of these results are well-known, and we briefly indicate their proofs; for the new results we include complete proofs. We start with a simple lemma.

Lemma 2.3. $\mathcal{S} \cap \mathcal{T} \neq \{0\}$ *if and only if there exists* $0 \neq \xi \in \mathcal{H}$ *such that* $\|P_{\mathcal{S}}P_{\mathcal{T}}\xi\| = \|\xi\|$.

Proof. It is clear that if $\mathcal{S}\cap\mathcal{T} \neq \{0\}$ then there exists $\xi \neq 0$ such that $\|P_{\mathcal{S}}P_{\mathcal{T}}\xi\| = \|\xi\|$. Conversely, if there exists $\xi \neq 0$ such that $\|P_{\mathcal{S}}P_{\mathcal{T}}\xi\| = \|\xi\|$ then $\|\xi\|^2 \geq \|P_{\mathcal{T}}\xi\|^2 = \|P_{\mathcal{S}}P_{\mathcal{T}}\xi\|^2 + \|P_{\mathcal{S}^{\perp}}P_{\mathcal{T}}\xi\|^2 = \|\xi\|^2 + \|P_{\mathcal{S}^{\perp}}P_{\mathcal{T}}\xi\|^2 \geq \|\xi\|^2$. So, $\|P_{\mathcal{S}^{\perp}}P_{\mathcal{T}}\xi\| = 0$ and, in consequence, $P_{\mathcal{S}}P_{\mathcal{T}}\xi = P_{\mathcal{T}}\xi$. Also, $\|\xi\|^2 = \|P_{\mathcal{T}}\xi\|^2+\|P_{\mathcal{T}^{\perp}}\xi\|^2 = \|\xi\|^2+\|P_{\mathcal{T}^{\perp}}\xi\|^2$. Then $\|P_{\mathcal{T}^{\perp}}\xi\|^2 = 0$. Therefore $\xi \in \mathcal{T}$ and $\xi = P_{\mathcal{T}}\xi = P_{\mathcal{S}}P_{\mathcal{T}}\xi$, so that $\xi \in \mathcal{S}$. Then $\xi \in \mathcal{S}\cap\mathcal{T}$. □

Theorem 2.4.

1. $\mathcal{S} + \mathcal{T}$ *is closed* $\Leftrightarrow c(\mathcal{S},\mathcal{T}) < 1 \Leftrightarrow P_{\mathcal{S}} + P_{\mathcal{T}}$ *has closed range* $\Leftrightarrow P_{\mathcal{S}} - P_{\mathcal{T}}$ *has closed range.*
2. $\overline{\mathcal{S} + \mathcal{T}} = \mathcal{H} \Leftrightarrow \mathcal{S}^{\perp}\cap\mathcal{T}^{\perp} = \{0\} \Leftrightarrow \overline{P_{\mathcal{S}^{\perp}}(\mathcal{T})} = \mathcal{S}^{\perp} \Leftrightarrow \overline{P_{\mathcal{T}^{\perp}}(\mathcal{S})} = \mathcal{T}^{\perp} \Leftrightarrow P_{\mathcal{S}} + P_{\mathcal{T}}$ *is an injective operator.*
3. $\mathcal{S} + \mathcal{T} = \mathcal{H} \Leftrightarrow P_{\mathcal{S}} + P_{\mathcal{T}}$ *is invertible* $\Leftrightarrow N(P_{\mathcal{S}} - P_{\mathcal{T}}) = \mathcal{S} \cap \mathcal{T}$ *and* $R(P_{\mathcal{S}} - P_{\mathcal{T}})$ *is closed* $\Leftrightarrow c_0(\mathcal{S}^{\perp}, \mathcal{T}^{\perp}) < 1 \Leftrightarrow \|P_{\mathcal{S}^{\perp}}P_{\mathcal{T}^{\perp}}\| < 1$.
4. $\overline{\mathcal{S} + \mathcal{T}}$ *is a proper subspace* $\Leftrightarrow \mathcal{S}^{\perp}\cap\mathcal{T}^{\perp} \neq \{0\} \Leftrightarrow$ *there exists* $\xi \neq 0$ *such that* $\|P_{\mathcal{S}^{\perp}}P_{\mathcal{T}^{\perp}}\xi\| = \|\xi\|$ *(and then* $\|P_{\mathcal{S}^{\perp}}P_{\mathcal{T}^{\perp}}\| = 1$*).*

5. *$\mathcal{S}+\mathcal{T}$ is a proper dense subspace $\Leftrightarrow$ $P_\mathcal{S}+P_\mathcal{T}$ is injective non invertible $\Leftrightarrow$ $N(P_\mathcal{S}-P_\mathcal{T}) = \mathcal{S}\cap\mathcal{T}$ and $R(P_\mathcal{S}-P_\mathcal{T})$ is non closed $\Leftrightarrow$ $\|P_{\mathcal{S}^\perp}P_{\mathcal{T}^\perp}\| = 1$ but for every $\xi\neq 0$ it holds that $\|P_{\mathcal{S}^\perp}P_{\mathcal{T}^\perp}\xi\| < \|\xi\|$.*

6. *$\mathcal{S}+\mathcal{T}$ is a proper closed subspace $\Leftrightarrow$ $c(\mathcal{S},\mathcal{T}) < 1$ and $\mathcal{S}^\perp\cap\mathcal{T}^\perp \neq \{0\}$.*

If, in addition, $\mathcal{S}\cap\mathcal{T} = \{0\}$ then

7. *$\overline{\mathcal{S}\dot{+}\mathcal{T}} = \mathcal{H}$ $\Leftrightarrow$ there exists a densely defined closed idempotent $P_{\mathcal{S}//\mathcal{T}}$ $\Leftrightarrow$ $P_\mathcal{S}-P_\mathcal{T}$ is an injective operator.*

8. *$\mathcal{S}\dot{+}\mathcal{T} = \mathcal{H}$ $\Leftrightarrow$ there exists a bounded linear idempotent $P_{\mathcal{S}//\mathcal{T}}$ $\Leftrightarrow$ $\|P_\mathcal{S}+P_\mathcal{T}-I\| < 1$.*

9. *$\overline{\mathcal{S}\dot{+}\mathcal{T}}$ is a proper subspace $\Leftrightarrow$ $P_\mathcal{S}+P_\mathcal{T}$ is not injective $\Leftrightarrow$ there exists $\xi\neq 0$ such that $\|(P_\mathcal{S}+P_\mathcal{T}-I)\xi\| = \|\xi\|$.*

10. *$\mathcal{S}\dot{+}\mathcal{T}$ is a proper dense subspace $\Leftrightarrow$ $\|P_\mathcal{S}+P_\mathcal{T}-I\| = 1$ but for every $\xi\neq 0$ it holds that $\|(P_\mathcal{S}+P_\mathcal{T}-I)\xi\| < \|\xi\|$ $\Leftrightarrow$ $\|P_\mathcal{S}P_\mathcal{T}\| = 1$ and $\mathcal{S}^\perp\cap\mathcal{T}^\perp = \{0\}$ $\Leftrightarrow$ $\|P_{\mathcal{S}^\perp}P_{\mathcal{T}^\perp}\| = 1$ and $\mathcal{S}^\perp\cap\mathcal{T}^\perp = \{0\}$.*

11. *$\mathcal{S}\dot{+}\mathcal{T}$ is a proper closed subspace $\Leftrightarrow$ $\|P_\mathcal{S}P_\mathcal{T}\| < 1$ and $\|P_{\mathcal{S}^\perp}P_{\mathcal{T}^\perp}\| = 1$ $\Leftrightarrow$ $\|P_\mathcal{S}P_\mathcal{T}\| < 1$ and $\|P_\mathcal{S}+P_\mathcal{T}-I\| = 1$.*

Proof.

1. The first equivalence is proven in [[19], Theorem 13]. The rest of the assertions are proven in [[22], Theorem 2.2] and [[33], Lemma 2.4].

2. It follows from [[19], Lemma 11 and Theorem 13], [[35], Lemma 2.1] and item *2* of Proposition 2.2.

3. The first three equivalences follow by combining items *1* and *2*. The last equivalence follows from Proposition 2.1.

4. The first equivalence follows applying item *2*. The second one is Lemma 2.3.

Item *5* follows from items *1*, *2* and *4*. Item *6* follows from [[19], Lemma 11 and Theorem 12].

7. The first equivalence is proven in [[37], Lemma 3.5]. On the other hand, since $\mathcal{S}\cap\mathcal{T} = \{0\}$ then $N(P_\mathcal{S}-P_\mathcal{T}) = \mathcal{S}^\perp\cap\mathcal{T}^\perp$. Now $\overline{\mathcal{S}\dot{+}\mathcal{T}} = \mathcal{H}$ if and only if $\{0\} = \mathcal{S}^\perp\cap\mathcal{T}^\perp = N(P_\mathcal{S}-P_\mathcal{T})$ if and only if $P_\mathcal{S}-P_\mathcal{T}$ is injective.

Item *8* is proven in [[9], Theorem 1].

Item *9* follows from item *2* of this proposition, items *2* and *3* of Proposition 2.2 and identity (1).

10. Suppose $\mathcal{S}\dot{+}\mathcal{T}$ dense and non closed. Since $\mathcal{S}\dot{+}\mathcal{T}$ is direct and dense, it holds that $\mathcal{S}\cap\mathcal{T} = \{0\}$ and $\mathcal{S}^\perp\cap\mathcal{T}^\perp = \{0\}$. Then, by item *3* of Proposition 2.2, $N(P_\mathcal{S}-P_\mathcal{T}) = \{0\}$. Therefore, by (1), $\|(P_\mathcal{S}+P_\mathcal{T}-I)\xi\| < \|\xi\|$ for every $\xi\neq 0$ and since $\mathcal{S}\dot{+}\mathcal{T}$ is not closed, by item *8*, $\|P_\mathcal{S}+P_\mathcal{T}-I\| = 1$. Conversely, if $\|(P_\mathcal{S}+P_\mathcal{T}-I)\xi\| < \|\xi\|$ for every $\xi\neq 0$, by (1), the operator $P_\mathcal{S}-P_\mathcal{T}$ is injective. So, by item *3* of Proposition 2.2, $\mathcal{S}\dot{+}\mathcal{T}$ is dense. But, since $\|P_\mathcal{S}+P_\mathcal{T}-I\| = 1$ then $\mathcal{S}\dot{+}\mathcal{T}$ is dense non closed and the first equivalence follows. On the other hand, since $\mathcal{S}\cap\mathcal{T} = \{0\}$ then $\mathcal{S}\dot{+}\mathcal{T}$ is dense non closed if and only if $c_0(\mathcal{S},\mathcal{T}) = 1$ and $\mathcal{S}^\perp\cap\mathcal{T}^\perp = \{0\}$ or, equivalently, $\|P_\mathcal{S}P_\mathcal{T}\| = 1$ and $\mathcal{S}^\perp\cap\mathcal{T}^\perp = \{0\}$. The proof that $\mathcal{S}\dot{+}\mathcal{T}$ is dense non closed if and only if $\|P_{\mathcal{S}^\perp}P_{\mathcal{T}^\perp}\| = 1$ and $\mathcal{S}^\perp\cap\mathcal{T}^\perp = \{0\}$ is similar.

Item *11* follows from the Krein-Krasnoselskii-Milman equality. □

Remark 2.5. In a recent paper, Ando [4] found several nice formulae for the (may be unbounded) projection with nullspace $\mathcal{T}$ and range $\mathcal{S}$ provided that $\mathcal{S} \dot{+} \mathcal{T}$ is dense in $\mathcal{H}$. Ando proves that if E is a densely defined projection with closed range $\mathcal{S}$ and closed nullspace $\mathcal{T}$ then E is bounded if and only if $P_{\mathcal{S}} + P_{\mathcal{T}}$ is invertible. In such case it holds $E = P_{\mathcal{S}}(P_{\mathcal{S}} + P_{\mathcal{T}})^{-1}$. Moreover, E is not bounded if and only if $P_{\mathcal{S}} + P_{\mathcal{T}}$ is not invertible and, in this case, $E = P_{\mathcal{S}}(P_{\mathcal{S}} + P_{\mathcal{T}})^{-1/2}(P_{\mathcal{S}} + P_{\mathcal{T}})^{-1/2}$. Compare these formulae with that one of Greville [14] $E = (P_{\mathcal{T}^{\perp}} P_{\mathcal{S}})^{\dagger}$.

2.1 Closed unbounded projections

Let T be a densely defined operator from $\mathcal{H}$ to $\mathcal{H}$. It is said that T is *closed* if its graph is a closed subspace of $\mathcal{H} \times \mathcal{H}$ and T is called *closable* if it admits a closed extension (i.e. if there exists a closed linear operator T_1 such that $T \subset T_1$ or, which is the same, $\mathcal{D}(T) \subset \mathcal{D}(T_1)$ and $T\xi = T_1\xi$ for every $\xi \in \mathcal{D}(T)$). Every closable operator T has a smallest closed extension called the *closure* which is denoted $\overline{T}$. On the other hand, it is well known that a densely defined operator Q is an idempotent if $R(Q) \subseteq \mathcal{D}(Q)$ and $Q^2\xi = Q\xi$ for all $\xi \in \mathcal{D}(Q)$. In such case, $\mathcal{D}(Q) = R(Q) \dot{+} N(Q)$. It holds that Q is closed if and only if $R(Q)$ and $N(Q)$ are closed subspaces. If the idempotent Q is closable then $\overline{Q}$ and Q^* are closed idempotents and it holds that $\overline{Q} = Q_{\overline{R(Q)}//\overline{N(Q)}}$, $R(Q^*) = N(Q)^{\perp}$ and $N(Q^*) = R(Q)^{\perp}$. Furthermore, an idempotent Q is bounded if and only if Q is closed and $\mathcal{D}(Q) = \mathcal{H}$. A classical reference for unbounded projections is Ota's paper [37]. We also refer the reader to the papers by Popovych [38], Samoilenko and Turowska [42], Booß et al. [8] and Ando [4]. We finish this section by giving a characterization of closed densely defined idempotents in terms of matrix representations. The following result allows to get a matrix representation of unbounded operators (see [4] and [14] for the proof).

Lemma 2.6. *If $Q = Q_{\mathcal{S}//\mathcal{T}}$ is a closed idempotent then $\mathcal{D}(Q) = \mathcal{S} \dot{+} \mathcal{T} = \mathcal{S} \oplus P_{\mathcal{S}^{\perp}}(\mathcal{T})$.*

A related matrix form to the next one can be found in the paper by Ando [4].

Proposition 2.7. *Consider the decomposition $\mathcal{H} = \mathcal{S} \oplus \mathcal{S}^{\perp}$. If*

$$Q = \begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix}, \tag{2}$$

where 1 denotes the identity operator of $\mathcal{S}$ and $x : \mathcal{D}(x) \subseteq \mathcal{S}^{\perp} \longrightarrow \mathcal{S}$ is a densely defined linear operator, then the following assertions hold:

1. *$Q : \mathcal{S} \oplus \mathcal{D}(x) \longrightarrow \mathcal{S}$ is a densely defined idempotent with $R(Q) = \mathcal{S}$.*
2. *The idempotent Q is closed if and only if $\Gamma(x)$ (the graph of x) is closed.*
3. *If the idempotent Q is closed then $Q^* = \begin{pmatrix} 1|_{\mathcal{D}(x^*)} & 0 \\ x^* & 0 \end{pmatrix}$ on $\mathcal{D}(x^*) \oplus \mathcal{S}^{\perp}$.*

Proof.

1. It is easy to check that $\mathcal{D}(Q) = \mathcal{S} \oplus \mathcal{D}(x)$ is dense in $\mathcal{H}$ and $R(Q) = \mathcal{S}$. Then Q^2 is well defined because $R(Q) \subseteq \mathcal{D}(Q)$ and $Q^2 = Q$. So the assertion follows.

2. This equivalence is an immediate consequence of the fact that $N(Q)$ and $\Gamma(x)$ are isometrically isomorph. In fact, $N(Q) = \{-x\omega + \omega : \omega \in \mathcal{D}(x)\}$ and $T : N(Q) \longrightarrow \Gamma(x)$ defined by $T(-x\omega + \omega) = (\omega, x\omega)$ is an isometric isomorphism.

3. Since Q is a densely defined closed idempotent we know that Q^* is a closed densely defined projection too. In addition, as $\mathcal{D}(Q) = \mathcal{S} \oplus \mathcal{D}(x)$ then $\mathcal{D}(Q^*) = \mathcal{D}(x^*) \oplus \mathcal{S}^{\perp}$. In fact, if $\xi = \xi_1 + \xi_2 \in$

$\mathcal{D}(Q^*)$, where $\xi_1 \in \mathcal{S}$ and $\xi_2 \in \mathcal{S}^\perp$ then there exists $\rho = \rho_1 + \rho_2$, where $\rho_1 \in \mathcal{S}$ and $\rho_2 \in \mathcal{S}^\perp$ such that $\langle Q\omega, \xi \rangle = \langle \omega, \rho \rangle$ for every $\omega = \omega_1 + \omega_2 \in \mathcal{D}(Q)$, where $\omega_1 \in \mathcal{S}$ and $\omega_2 \in \mathcal{D}(x)$. In particular, for every $\omega = \omega_2$ we get $\langle x\omega_2, \xi_1 \rangle = \langle Q\omega_2, \xi_1 + \xi_2 \rangle = \langle \omega_2, \rho_1 + \rho_2 \rangle = \langle \omega_2, \rho_2 \rangle$. So, $\xi_1 \in \mathcal{D}(x^*)$ and therefore $\mathcal{D}(Q) \subseteq \mathcal{D}(x^*) \oplus \mathcal{S}^\perp$. Conversely, if $\xi = \xi_1 + \xi_2 \in \mathcal{D}(x^*) \oplus \mathcal{S}^\perp$, where $\xi_1 \in \mathcal{D}(x^*)$ and $\xi_2 \in \mathcal{S}^\perp$ then for every $\omega = \omega_1 + \omega_2 \in \mathcal{D}(Q)$, where $\omega_1 \in \mathcal{S}$ and $\omega_2 \in \mathcal{D}(x)$ there exists $\rho = \xi_1 + x^*\xi_1$ such that $\langle Q\omega, \xi \rangle = \langle \omega, \rho \rangle$. In fact, $\langle Q\omega, \xi \rangle = \langle \omega_1 + x\omega_2, \xi_1 \rangle = \langle \omega_1, \xi_1 \rangle + \langle x\omega_2, \xi_1 \rangle = \langle \omega_1, \xi_1 \rangle + \langle \omega_2, x^*\xi_1 \rangle = \langle \omega_1 + \omega_2, \xi_1 \rangle + \langle \omega_1 + \omega_2, x^*\xi_1 \rangle = \langle \omega, \xi_1 + x^*\xi_1 \rangle$. Therefore, $\mathcal{D}(x^*) \oplus \mathcal{S}^\perp \subseteq \mathcal{D}(Q^*)$. Moreover, for every $\xi = \xi_1 + \xi_2 \in \mathcal{D}(Q^*) = \mathcal{D}(x^*) \oplus \mathcal{S}^\perp$ it holds $Q^*\xi = \xi_1 + x^*\xi_1$. In consequence, $Q^* = \begin{pmatrix} 1|_{\mathcal{D}(x^*)} & 0 \\ x^* & 0 \end{pmatrix}$ as asserted. □

Remark 2.8. It follows from Proposition 2.7 that $\mathcal{D}(Q) = \mathcal{S} \oplus \mathcal{D}(x)$ and, by [[4], Theorem 2.7], $\mathcal{D}(x) = P_{\mathcal{S}^\perp}(N(Q))$.

3 A-symmetric projections and quasi-compatibility

In this section we study unbounded projections which are symmetric for the semi-inner product defined by a bounded positive operator A by $\langle \xi, \eta \rangle_A = \langle A\xi, \eta \rangle$ for all $\xi, \eta \in \mathcal{H}$. Also, we characterize the existence of such projections with a prescribed range $\mathcal{S}$. Finally, we describe the set of symmetrizable projections for A with fixed range and domain. In the sequel we deal with closed projections.

3.1 A-symmetric projections

Remember that a densely defined operator T is *symmetric* if $T \subset T^*$. Furthermore, T is *selfadjoint* if $T = T^*$; this means that T is symmetric and $\mathcal{D}(T) = \mathcal{D}(T^*)$.

Definition 3.1. *Let $A \in L(\mathcal{H})^+$ and Q a closed densely defined projection; we say that Q is A-symmetric if AQ is symmetric and it is A-selfadjoint if AQ is selfadjoint.*

Since A is bounded then $\mathcal{D}(AQ) = \mathcal{D}(Q)$ and $(AQ)^* = Q^*A$. Therefore Q is A-symmetric if and only if $AQ\xi = Q^*A\xi$ for every $\xi \in \mathcal{D}(Q)$ and Q is A-selfadjoint if and only if $AQ = Q^*A$.

From now on A denotes an operator in $L(\mathcal{H})^+$ and $\mathcal{S}$ a closed subspace of $\mathcal{H}$. In the next result we give conditions which guarantee that a densely defined projection is A-symmetric. For it, we consider the following matrix representation of A under the decomposition $\mathcal{H} = \mathcal{S} \oplus \mathcal{S}^\perp$:

$$A = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}, \tag{3}$$

where $a = P_\mathcal{S} A P_\mathcal{S}|_\mathcal{S}$, $b = P_\mathcal{S} A P_{\mathcal{S}^\perp}|_{\mathcal{S}^\perp}$ and $c = P_{\mathcal{S}^\perp} A P_{\mathcal{S}^\perp}|_{\mathcal{S}^\perp}$. Then $a \in L(\mathcal{S})^+$, $b \in L(\mathcal{S}^\perp, \mathcal{S})$ and $c \in L(\mathcal{S}^\perp)^+$. Equivalence $1 \leftrightarrow 3$ of the following proposition is due to Krein [32].

Proposition 3.2. *Let A with matrix representation (3) and Q a densely defined closed idempotent with $R(Q) = \mathcal{S}$ and matrix representation (2). The following assertions are equivalent:*

1. *Q is A-symmetric;*
2. *$ax \subset b$;*
3. *$N(Q) \subseteq (A\mathcal{S})^\perp$;*

4. $\mathcal{D}(Q)=\mathcal{D}(Q^*AQ)$ *and* $Q^*AQ\le A|_{\mathcal{D}(Q)}$.

Proof.

1$\leftrightarrow$ *2*. If Q is A-symmetric then $AQ=Q^*A|_{\mathcal{D}(Q)}$. By the matrix representation of A, Q and Q^* we get that $\begin{pmatrix} a & ax\\ b^* & b^*x\end{pmatrix}=\begin{pmatrix} a & b|_{\mathcal{D}(x)}\\ x^*a & x^*b|_{\mathcal{D}(x)}\end{pmatrix}$. So, $ax=b|_{\mathcal{D}(x)}$. Conversely, suppose that $ax\subset b$. Then $b^*\subset(ax)^*=x^*a$. So that, $b^*=x^*a\in L(\mathcal{S},\mathcal{S}^\perp)$ and $R(a)\subseteq\mathcal{D}(x^*)$. By the matrix representation of Q we get

$$AQ=\begin{pmatrix} a & ax\\ b^* & b^*x\end{pmatrix}=\begin{pmatrix} a & b|_{\mathcal{D}(x)}\\ b^* & b^*x\end{pmatrix}\text{ and}$$

$$Q^*A|_{\mathcal{D}(Q)}=Q^*A|_{\mathcal{S}\oplus\mathcal{D}(x)}=\begin{pmatrix} 1|_{\mathcal{D}(x^*)} & 0\\ x^* & 0\end{pmatrix}\begin{pmatrix} a & b|_{\mathcal{D}(x)}\\ b^* & c|_{\mathcal{D}(x)}\end{pmatrix}=\begin{pmatrix} a & b|_{\mathcal{D}(x)}\\ x^*a & x^*b|_{\mathcal{D}(x)}\end{pmatrix}.$$

Therefore, since $x^*a=b^*$, $x^*b|_{\mathcal{D}(x)}=x^*ax=b^*x$ then $AQ=Q^*A|_{\mathcal{D}(Q)}$. So, the assertion follows.

2$\leftrightarrow$ *3*. First observe that by the matrix representation of Q it holds $N(Q)=\{-x\xi+\xi:\xi\in\mathcal{D}(x)\}$. If $ax=b|_{\mathcal{D}(x)}$, $\xi\in\mathcal{D}(x)$ and $\eta\in\mathcal{S}$ then $\langle -x\xi+\xi,A\eta\rangle=\langle -x\xi+\xi,a\eta+b^*\eta\rangle=\langle -x\xi,a\eta\rangle+\langle\xi,b^*\eta\rangle=\langle -ax\xi,\eta\rangle+\langle b\xi,\eta\rangle=0$. Then $N(Q)\subseteq(A\mathcal{S})^\perp$. Conversely, if $N(Q)\subseteq(A\mathcal{S})^\perp$ then given $\xi\in\mathcal{D}(x)$, for every $\eta\in\mathcal{S}$ it holds $0=\langle -x\xi+\xi,A\eta\rangle=\langle -x\xi+\xi,a\eta+b^*\eta\rangle=\langle -ax\xi,\eta\rangle+\langle b\xi,\eta\rangle$. So that $\langle -ax\xi+b\xi,\eta\rangle=0$ for every $\eta\in\mathcal{S}$ and then $ax=b|_{\mathcal{D}(x)}$.

1$\leftrightarrow$ *4*. If Q is A-symmetric then $AQ=Q^*A|_{\mathcal{D}(Q)}$. It is clear that $\mathcal{D}(Q^*AQ)\subseteq\mathcal{D}(Q)$. Now, if $\xi\in\mathcal{D}(Q)$ then $AQ\xi=AQ^2\xi=Q^*AQ\xi$. So that, $\xi\in\mathcal{D}(Q^*AQ)$. On the other hand, since $AQ=Q^*A|_{\mathcal{D}(Q)}$ and $A(\mathcal{D}(Q))\subseteq\mathcal{D}(Q^*)=\mathcal{D}(I-Q^*)$ then $A(I-Q)=(I-Q^*)A|_{\mathcal{D}(Q)}$ and $A(I-Q)=(I-Q^*)A(I-Q)$. Then, as Q^*AQ and $(I-Q^*)A(I-Q)$ are positive we get that $A|_{\mathcal{D}(Q)}=Q^*AQ+(1-Q)^*A(1-Q)\ge Q^*AQ$. Conversely, since $\mathcal{D}(Q)=\mathcal{D}(Q^*AQ)$ then $R(A^{1/2}Q)\subseteq\mathcal{D}((A^{1/2}Q)^*)$ and so $Q^*AQ=(A^{1/2}Q)^*A^{1/2}Q$. Now, following the proof of Theorem 2 of Douglas [21],define $C:A^{1/2}(\mathcal{D}(Q))\longrightarrow\mathcal{H}$ as $C(A^{1/2}\xi)=A^{1/2}Q\xi$ for every $\xi\in\mathcal{D}(Q)$. We claim that C is well defined. Indeed if $A^{1/2}\xi=A^{1/2}\eta$ for some $\xi,\eta\in\mathcal{D}(Q)$ then $\xi-\eta\in N(A)\cap\mathcal{D}(Q)$. Now, by the hypothesis we obtain that $\left\langle A^{1/2}Q(\xi-\eta),A^{1/2}Q(\xi-\eta)\right\rangle=\langle Q^*AQ(\xi-\eta),\xi-\eta\rangle\le\langle A(\xi-\eta),\xi-\eta\rangle=0$. So, $\|A^{1/2}Q(\xi-\eta)\|=0$ and then $A^{1/2}Q\xi=A^{1/2}Q\eta$ which proves that C is well defined. On the other hand, $\|CA^{1/2}\xi\|^2=\|A^{1/2}Q\xi\|^2=\langle Q^*AQ\xi,\xi\rangle\le\langle A\xi,\xi\rangle=\|A^{1/2}\xi\|^2$. Then, C is bounded on $A^{1/2}(\mathcal{D}(Q))$ (dense in $R(A^{1/2})$) so that we can extend C continuously to $\overline{R(A)}$ and if it is defined as 0 in $N(A)$ it holds that $C\in L(\mathcal{H})$, $\|C\|\le 1$ and $CA^{1/2}|_{\mathcal{D}(Q)}=A^{1/2}Q$. In addition, if $\xi\in\mathcal{D}(Q)$ then $C^2(A^{1/2}\xi)=C(A^{1/2}Q\xi)=A^{1/2}Q\xi=C(A^{1/2}\xi)$. So $C^2=C$ on $A^{1/2}(\mathcal{D}(Q))$ and therefore, since $A^{1/2}(\mathcal{D}(Q))$ is dense in $\overline{R(A^{1/2})}$ and $N(A)\subseteq N(C)$, $C^2=C$. Hence, since C is an idempotent of $L(\mathcal{H})$ and $\|C\|\le1$ then $C=C^*$. Moreover, it can be checked that $C=P_{\overline{A^{1/2}(R(Q))}}$. In consequence, $A^{1/2}P_{\overline{A^{1/2}(R(Q))}}A^{1/2}|_{\mathcal{D}(Q)}=AQ$ is symmetric and so Q is A-symmetric. □

It is well known that every bounded projection Q is A-selfadjoint with respect to some $A\in L(\mathcal{H})^+$: take $A=Q^*Q+(I-Q^*)(I-Q)$. The next corollary extends this result to closed unbounded projections.

Corollary 3.3. *Let Q be a densely defined closed projection. Then there exists $A\in L(\mathcal{H})^+$ such that Q is A-symmetric.*

Proof. Let $\mathcal{S}=R(Q)$ and $\mathcal{T}=N(Q)$. Then $\overline{\mathcal{S}\dotplus\mathcal{T}}=\mathcal{H}$ and, by Theorem 2.4, this is equivalent to $\overline{P_{\mathcal{T}^\perp}(\mathcal{S})}=\mathcal{T}^\perp$. Now, take $A=P_{\mathcal{S}^\perp}+P_{\mathcal{T}^\perp}\in L(\mathcal{H})^+$. Then $\overline{A\mathcal{S}}=\mathcal{T}^\perp$ and so, by Proposition 3.2, Q is A-symmetric. □

The semi-inner product $\langle ., . \rangle_A$ defines a semi-norm $\|.\|_A : \mathcal{H} \longrightarrow \mathbb{R}^+$ by means of $\|\xi\|_A = \langle \xi, \xi \rangle_A^{1/2} = \|A^{1/2}\xi\|$. Then, it defines a seminorm on certain class of unbounded operators. More precisely, a densely defined operator T is said A-*bounded* if for every $\xi \in \mathcal{D}(T)$ there exists a constant $c > 0$ such that $\|T\xi\|_A \leq c\|\xi\|_A$. In such case, $\|T\|_A = \sup\{\|T\xi\|_A : \ \xi \in \mathcal{D}(T) \text{ and } \|\xi\|_A = 1\}$ is a seminorm on the set of A-bounded operators.

Proposition 3.4. *Let Q be an A-symmetric projection and $\mathcal{M} = \overline{R(A^{1/2}Q)}$. Then*

1. *$A^{1/2}Q$ admits an unique bounded extension to $\mathcal{H}$. In fact, $A^{1/2}Q = P_{\mathcal{M}}A^{1/2}$ in $\mathcal{D}(Q)$. Therefore, $AQ = A^{1/2}P_{\mathcal{M}}A^{1/2}$ in $\mathcal{D}(Q)$.*
2. *Q is A-bounded and $\|Q\|_A = 0$ or $\|Q\|_A = 1$.*
3. *$(AQ)^*$ is bounded and selfadjoint and it holds $(AQ)^* = \overline{AQ} = A^{1/2}P_{\mathcal{M}}A^{1/2}$. If, in addition, AQ is closed then Q is bounded and A-selfadjoint.*

Proof.
1. It follows from the proof of equivalence *1↔4* of Proposition 3.2.
2. Let Q be an A-symmetric projection. If $\xi \in \mathcal{D}(Q)$ then, by Proposition 3.2, $\|Q\xi\|_A^2 = \langle AQ\xi, Q\xi \rangle = \langle Q^*AQ\xi, \xi \rangle \leq \langle A\xi, \xi \rangle = \|\xi\|_A^2$. So, $\|Q\|_A \leq 1$. Now, if $R(Q) \subseteq N(A)$ then it is clear that $\|Q\|_A = 0$. On the contrary if $R(Q) \nsubseteq N(A)$ then there exists $0 \neq \eta \in R(Q) \setminus N(A)$ and it holds $\|Q\eta\|_A = \|\eta\|_A$. Therefore, $\|Q\|_A = 1$ and the assertion follows.
3. It follows from item *1* that AQ admits (a unique) bounded extension $S = A^{1/2}P_{\mathcal{M}}A^{1/2}$. Observe that $S = S^*$. Also, by the general fact that if $T_1 \subseteq T_2$ implies that $T_2^* \subseteq T_1^*$, it follows that $AQ \subseteq S = S^* \subseteq (AQ)^*$. Therefore $(AQ)^* = S$ because $S \in L(\mathcal{H})$, so that $(AQ)^*$ is bounded and selfadjoint. Moreover, by [[40], Theorem VIII.1], it holds that $S = (AQ)^{**} = \overline{AQ}$. Observe that, if AQ is closed then $AQ = \overline{AQ} = (AQ)^{**} = (AQ)^*$. Then Q is A-selfadjoint and AQ is bounded. Thus, $\mathcal{H} = \mathcal{D}(AQ) = \mathcal{D}(Q)$ and therefore $Q \in L(\mathcal{H})$. □

Corollary 3.5. *Let $A \in L(\mathcal{H})^+$ injective, Q an A-symmetric projection and $\mathcal{M} = \overline{R(A^{1/2}Q)}$. Then $Q = A^{-1/2}P_{\mathcal{M}}A^{1/2}$ on $\mathcal{D}(Q)$.*

Proof. It follows from item *1* of Proposition 3.4. □

Corollary 3.6. *If Q is an A-selfadjoint projection then Q is bounded.*

Proof. If Q is A-seladjoint then AQ is closed. Then apply item *3* of Proposition 3.4. □

3.2 Quasi-compatibility

Given $A \in L(\mathcal{H})^+$ and a closed subspace $\mathcal{S} \subseteq \mathcal{H}$, the theory of compatibility studies the existence of A-selfadjoint bounded projections with range $\mathcal{S}$. More precisely, the pair $(A, \mathcal{S})$ is called *compatible* if there exists a projection $Q \in L(\mathcal{H})$ with range $\mathcal{S}$ such that Q is A-selfadjoint (i.e. $AQ = Q^*A$). This fact is equivalent to $\mathcal{S} + (A\mathcal{S})^\perp = \mathcal{H}$. See the references [17, 16, 12, 13] for several characterizations, examples and applications of this concept. In the next result we characterize the compatibility of a pair $(A, \mathcal{S})$ in terms of unbounded projections.

Theorem 3.7. *If there exists an A-selfadjoint (densely defined closed) projection onto $\mathcal{S}$ then $(A, \mathcal{S})$ is compatible.*

Proof. The proof follows applying Corollary 3.6. □

Observe that the converse of Theorem 3.7 is immediate. Taking into account the above theorem, in what follows we study a weaker notion than compatibility, namely, the existence of an A-symmetric projection onto $\mathcal{S}$ for given $A \in L(\mathcal{H})^+$ and $\mathcal{S} \subseteq \mathcal{H}$ a closed subspace.

Definition 3.8. *Let $A \in L(\mathcal{H})^+$ and $\mathcal{S} \subseteq \mathcal{H}$ a closed subspace of $\mathcal{H}$. We say that the pair $(A, \mathcal{S})$ is quasi-compatible if there exists an A-symmetric projection Q with $R(Q) = \mathcal{S}$.*

Proposition 3.9. *The pair $(A, \mathcal{S})$ is quasi-compatible if and only if $\mathcal{S} + (A\mathcal{S})^\perp$ is dense in $\mathcal{H}$.*

Proof. If $(A, \mathcal{S})$ is quasi-compatible then there exists an A-symmetric projection with range $\mathcal{S}$ and, by Proposition 3.2, $\mathcal{D}(Q) = \mathcal{S} \dotplus N(Q) \subseteq \mathcal{S} + (A\mathcal{S})^\perp$. Since Q is densely defined then $\mathcal{S} + (A\mathcal{S})^\perp$ is dense in $\mathcal{H}$. Conversely, let $\mathcal{N} = \mathcal{S} \cap (A\mathcal{S})^\perp$. Note that $\mathcal{S} + (A\mathcal{S})^\perp = \mathcal{S} \dotplus (A\mathcal{S})^\perp \ominus \mathcal{N}$ and define $Q = P_{\mathcal{S}//(A\mathcal{S})^\perp \ominus \mathcal{N}}$. Then Q is a closed densely defined projection and, by Proposition 3.2, Q is A-symmetric. So that $(A, \mathcal{S})$ is quasi-compatible. $\square$

In the sequel, given $A \in L(\mathcal{H})^+$ and a closed subspace $\mathcal{S}$, we denote

$$\mathcal{S}_A := \mathcal{S} + (A\mathcal{S})^\perp \quad \text{and} \quad \mathcal{N} := \mathcal{S} \cap (A\mathcal{S})^\perp.$$

Note that $\mathcal{N} = \mathcal{S} \cap N(A)$.

Lemma 3.10. *The following assertions hold:*

1. $\mathcal{S}_A = \mathcal{S} \dotplus (A\mathcal{S})^\perp \ominus \mathcal{N} = \mathcal{S} \oplus P_{\mathcal{S}^\perp}((A\mathcal{S})^\perp) = (A\mathcal{S})^\perp \oplus P_{\overline{A\mathcal{S}}}(\mathcal{S})$.
2. $P_{\overline{A\mathcal{S}}}(\mathcal{S}) = \mathcal{S}_A \cap \overline{A\mathcal{S}}$.

Proof.
1. The first equality is an elementary result of linear algebra. In order to see that $\mathcal{S}_A = \mathcal{S} \oplus R(P_{\mathcal{S}^\perp} P_{(A\mathcal{S})^\perp})$, take $\xi = \eta + \omega \in \mathcal{S}_A$, where $\eta \in \mathcal{S}$ and $\omega \in (A\mathcal{S})^\perp$. Since $\omega = P_{\mathcal{S}}\omega + P_{\mathcal{S}^\perp}\omega$ then $\xi = \eta + P_{\mathcal{S}}\omega + P_{\mathcal{S}^\perp}\omega \in \mathcal{S} \oplus R(P_{\mathcal{S}^\perp} P_{(A\mathcal{S})^\perp})$. Conversely, let $\xi = \eta + \omega \in \mathcal{S} \oplus R(P_{\mathcal{S}^\perp} P_{(A\mathcal{S})^\perp})$, where $\eta \in \mathcal{S}$ and $\omega \in R(P_{\mathcal{S}^\perp} P_{(A\mathcal{S})^\perp})$. Then, $\omega = P_{\mathcal{S}^\perp}\mu = (I - P_{\mathcal{S}})\mu$ for some $\mu \in (A\mathcal{S})^\perp$ and so $\xi = \eta - P_{\mathcal{S}}\mu + \mu \in \mathcal{S} + (A\mathcal{S})^\perp = \mathcal{S}_A$. The proof of the third equality is similar.
2. By item *1* $P_{\overline{A\mathcal{S}}}(\mathcal{S}) \subseteq \mathcal{S}_A$. So, $P_{\overline{A\mathcal{S}}}(\mathcal{S}) \subseteq \mathcal{S}_A \cap \overline{A\mathcal{S}}$. Conversely, let $\xi = \eta + \mu \in \mathcal{S}_A \cap \overline{A\mathcal{S}}$ where $\eta \in (A\mathcal{S})^\perp$ and $\mu \in P_{\overline{A\mathcal{S}}}(\mathcal{S})$. Then $\xi - \mu = \eta \in (A\mathcal{S})^\perp \cap \overline{A\mathcal{S}} = \{0\}$. So $\xi = \mu \in P_{\overline{A\mathcal{S}}}(\mathcal{S})$. $\square$

The next elementary lemma will be useful to provide some examples of quasi-compatible pairs.

Lemma 3.11. *It holds $\mathcal{S}^\perp \cap \overline{A\mathcal{S}} \cap R(A) = \{0\}$.*

Proof. Let $\xi \in \mathcal{S}^\perp \cap \overline{A\mathcal{S}} \cap R(A)$. Then $\xi = A\eta$ for some $\eta \in \mathcal{H}$ and there exists a sequence $\{\mu_n\}_{n\in\mathbb{N}} \subseteq \mathcal{S}$ such that $A\mu_n \xrightarrow[n\to\infty]{} \xi = A\eta$. Furthermore, $0 = \langle \mu_n, \xi \rangle = \langle \mu_n, A\eta \rangle = \langle A\mu_n, \eta \rangle \xrightarrow[n\to\infty]{} \langle A\eta, \eta \rangle$. Hence, $\|A^{1/2}\eta\| = 0$. Therefore, $\eta \in N(A)$ and so $\xi = A\eta = 0$. $\square$

Corollary 3.12. *If $A \in L(\mathcal{H})^+$ has closed range and $\mathcal{S}$ is a closed subspace then $\mathcal{S}^\perp \cap \overline{A\mathcal{S}} = \{0\}$. Therefore, for every $A \in L(\mathcal{H})^+$ with closed range the pair $(A, \mathcal{S})$ is quasi-compatible. Furthermore, $(A, \mathcal{S})$ is compatible if and only if $c(\mathcal{S}, (A\mathcal{S})^\perp) < 1$.*

Proof. Since $\overline{R(A)} = R(A)$ then $\overline{A\mathcal{S}} \subseteq R(A)$. Therefore, $\mathcal{S}^\perp \cap \overline{A\mathcal{S}} = \mathcal{S}^\perp \cap \overline{A\mathcal{S}} \cap R(A) = \{0\}$ and so $(A, \mathcal{S})$ is quasi-compatible. For the last assertion, note that since $\overline{\mathcal{S} + (A\mathcal{S})^\perp} = \mathcal{H}$ then $(A, \mathcal{S})$ is compatible if and only if $\mathcal{S} + (A\mathcal{S})^\perp$ is closed if and only if $c(\mathcal{S}, (A\mathcal{S})^\perp) < 1$. $\square$

Example 3.13. *Let P_1 and P_2 be orthogonal projections. Then:*

1. $(P_1, R(P_2))$ *is compatible if and only if* $R(P_1P_2)$ *is closed (see [[17], Theorem 7.1]).*

2. $(P_1, R(P_2))$ *is quasi-compatible non compatible if and only if* $R(P_1P_2)$ *is non closed. In fact, since* P_1 *has closed range then* $(P_1, R(P_2))$ *is quasi-compatible. Then the assertion follows by item 1.*

Corollary 3.12 shows that there are many strictly quasi-compatible pairs. In the following example we exhibit a pair $(A, \mathcal{S})$ which is not quasi-compatible.

Example 3.14. *Given* $A \in L(\mathcal{H})^+$ *with non closed range, choose* $\xi \in \overline{R(A)} \setminus R(A)$ *and define a closed subspace* $\mathcal{S}$ *by* $\mathcal{S}^\perp = span\{\xi\}$. *Observe that* $(A\mathcal{S})^\perp = A^{-1}(\mathcal{S}^\perp) = A^{-1}(span\{\xi\}) = A^{-1}(span\{\xi\} \cap R(A)) = A^{-1}(\{0\}) = N(A)$. *Then* $\overline{A\mathcal{S}} = \overline{R(A)}$ *and by density* $\mathcal{S}^\perp \cap \overline{A\mathcal{S}} = \mathcal{S}^\perp \cap \overline{R(A)} = \mathcal{S}^\perp \neq \{0\}$. *Therefore, by Theorem 3.16, the pair* $(A, \mathcal{S})$ *is not quasi-compatible.*

The next result characterizes the quasi-compatibility in terms of a bounded operator. This proposition is related with a result of Crimmins (see Radjavi and Williams [39]) which proves that a bounded linear operator T can be factorized as the product of two orthogonal projections if and only if $T^2 = TT^*T$. The proof of the following proposition follows from [[14], Theorem 6.2].

Proposition 3.15. *The pair* $(A, \mathcal{S})$ *is quasi-compatible if and only if there exists* $T \in L(\mathcal{H})$ *such that* $TT^*T = T^2$, $\overline{R(T)} = \overline{A\mathcal{S}}$ *and* $N(T) = (\mathcal{S} \ominus \mathcal{N})^\perp$.

Items *3* and *4* of the following result are a particular case of [[35], Lemma 2.1]. Here we include them as a manifestation of quasi-compatibility.

Theorem 3.16. *The following assertions are equivalent:*

1. *The pair* $(A, \mathcal{S})$ *is quasi-compatible;*

2. *There exists a closed densely defined projection* Q *with* $R(Q) = \mathcal{S}$ *and* $N(Q) \subseteq (A\mathcal{S})^\perp$;

3. $\mathcal{S}^\perp \cap \overline{A\mathcal{S}} = \{0\}$;

4. $\overline{P_{\overline{A\mathcal{S}}}(\mathcal{S})} = \overline{A\mathcal{S}}$;

5. $\overline{\mathcal{S}_A \cap \overline{A\mathcal{S}}} = \overline{A\mathcal{S}}$;

6. $I - P_{(\mathcal{S}\ominus\mathcal{N})^\perp} P_{\overline{A\mathcal{S}}}$ *is injective.*

Proof.

1 ↔ *2*. Follows from Proposition 3.2.

1 ↔ *3*. $(A, \mathcal{S})$ is quasi-compatible if and only if $\mathcal{H} = \overline{\mathcal{S} + (A\mathcal{S})^\perp}$ if and only if $\{0\} = \mathcal{S}^\perp \cap \overline{A\mathcal{S}}$.

1 ↔ *4*. Using Proposition 3.9 and Lema 3.10, $(A, \mathcal{S})$ is quasi-compatible if and only if $\overline{\mathcal{S}_A} = \mathcal{H}$ if and only if $\overline{P_{\overline{A\mathcal{S}}}(\mathcal{S})} = \overline{A\mathcal{S}}$.

1 ↔ *5*. By Proposition 3.9 and Lemma 3.10, $(A, \mathcal{S})$ is quasi-compatible if and only if $\overline{P_{\overline{A\mathcal{S}}}(\mathcal{S})} = \overline{A\mathcal{S}}$ if and only if $\overline{\mathcal{S}_A \cap \overline{A\mathcal{S}}} = \overline{A\mathcal{S}}$.

1 ↔ *6*. Observe that $N(I - P_{(\mathcal{S}\ominus\mathcal{N})^\perp} P_{\overline{A\mathcal{S}}}) = (\mathcal{S} \ominus \mathcal{N})^\perp \cap \overline{A\mathcal{S}}$. In fact, it is clear that $(\mathcal{S} \ominus \mathcal{N})^\perp \cap \overline{A\mathcal{S}} \subseteq N(I - P_{(\mathcal{S}\ominus\mathcal{N})^\perp} P_{\overline{A\mathcal{S}}})$. Conversely, if $\xi \in N(I - P_{(\mathcal{S}\ominus\mathcal{N})^\perp} P_{\overline{A\mathcal{S}}})$ then $\xi = P_{(\mathcal{S}\ominus\mathcal{N})^\perp} P_{\overline{A\mathcal{S}}}\xi$. Therefore $\xi \in (\mathcal{S} \ominus \mathcal{N})^\perp$ and $P_{(\mathcal{S}\ominus\mathcal{N})^\perp}\xi = P_{(\mathcal{S}\ominus\mathcal{N})^\perp} P_{\overline{A\mathcal{S}}}\xi$, or, equivalently, $P_{(\mathcal{S}\ominus\mathcal{N})^\perp} P_{(A\mathcal{S})^\perp}\xi = 0$. So, $P_{(A\mathcal{S})^\perp}\xi \in (A\mathcal{S})^\perp \cap (\mathcal{S} \ominus \mathcal{N}) = \{0\}$ and then $\xi \in \overline{A\mathcal{S}}$ because $\xi = P_{\overline{A\mathcal{S}}}\xi + P_{(A\mathcal{S})^\perp}\xi$. Then $\xi \in (\mathcal{S} \ominus \mathcal{N})^\perp \cap \overline{A\mathcal{S}}$. Hence $N(I - P_{(\mathcal{S}\ominus\mathcal{N})^\perp} P_{\overline{A\mathcal{S}}}) \subseteq (\mathcal{S} \ominus \mathcal{N})^\perp \cap \overline{A\mathcal{S}}$. Now we get that $(A, \mathcal{S})$ is quasi-compatible if and only if $\mathcal{H} = \overline{\mathcal{S} \ominus \mathcal{N} \dot{+} (A\mathcal{S})^\perp}$ if and only if $(\mathcal{S} \ominus \mathcal{N})^\perp \cap \overline{A\mathcal{S}} = \{0\}$ if and only if $I - P_{(\mathcal{S}\ominus\mathcal{N})^\perp} P_{\overline{A\mathcal{S}}}$ is injective. □

Observe that item *4* of the above proposition means that for every $\eta \in \overline{A\mathcal{S}}$ there exists a sequence $\{P_{\overline{A\mathcal{S}}}\xi_n\}_{n\in\mathbb{N}}$, $\xi_n \in \mathcal{S}$, such that $\lim_{n\to\infty} P_{\overline{A\mathcal{S}}}\xi_n = \eta$. If, in addition, the sequence $\{P_{\overline{A\mathcal{S}}}\xi_n\}_{n\in\mathbb{N}}$ is bounded then it is equivalent to the compatibility of the pair $(A, \mathcal{S})$. In fact, by the hypothesis it holds that $\overline{P_{\overline{A\mathcal{S}}}\mathcal{S}} = \overline{A\mathcal{S}}$. Now, as $\{P_{\overline{A\mathcal{S}}}\xi_n\}_{n\in\mathbb{N}}$ is bounded then, by [[35], Proposition 2.2] it holds that $\overline{A\mathcal{S}} \subseteq P_{\overline{A\mathcal{S}}}\mathcal{S}$ and so, $P_{\overline{A\mathcal{S}}}\mathcal{S} = \overline{A\mathcal{S}}$. This last assertion means that $\mathcal{S} + (A\mathcal{S})^\perp = \mathcal{H}$, or, which is the same, the pair $(A, \mathcal{S})$ is compatible. The converse is immediate. In order to get other equivalent conditions to the compatibility of $(A, \mathcal{S})$, all conditions of Proposition 3.16 can be adapted as follows:

2) there exists a bounded projection Q with $R(Q) = \mathcal{S}$ and $N(Q) \subseteq (A\mathcal{S})^\perp$;
3) $\mathcal{S}^\perp \cap \overline{A\mathcal{S}} = \{0\}$ and $\mathcal{S} + (A\mathcal{S})^\perp$ is closed;
4) $P_{\overline{A\mathcal{S}}}(\mathcal{S}) = \overline{A\mathcal{S}}$;
5) $\mathcal{S}_A \cap \overline{A\mathcal{S}} = \overline{A\mathcal{S}}$;
6) $I - P_{(\mathcal{S}\ominus\mathcal{N})^\perp} P_{\overline{A\mathcal{S}}}$ is invertible.

Denote

$$T_{A,\mathcal{S}} := P_{\mathcal{S}\ominus\mathcal{N}} + P_{(A\mathcal{S})^\perp} \quad \text{and} \quad R_{A,\mathcal{S}} := P_{\mathcal{S}\ominus\mathcal{N}} - P_{(A\mathcal{S})^\perp}.$$

The next theorem offers a more precise description of the type of compatibility (or non compatibility) of a pair $(A, \mathcal{S})$ in terms of $\|T_{A,\mathcal{S}} - I\|$ and $\|P_{\mathcal{S}^\perp} P_{\overline{A\mathcal{S}}}\|$:

Theorem 3.17.

1. *$(A, \mathcal{S})$ is quasi-compatible $\Leftrightarrow$ $T_{A,\mathcal{S}}$ is injective $\Leftrightarrow$ $R_{A,\mathcal{S}}$ is injective.*
2. *$(A, \mathcal{S})$ is compatible $\Leftrightarrow$ $T_{A,\mathcal{S}}$ is invertible $\Leftrightarrow$ $R_{A,\mathcal{S}}$ is invertible $\Leftrightarrow$ $\|T_{A,\mathcal{S}} - I\| < 1 \Leftrightarrow \|P_{\mathcal{S}^\perp} P_{\overline{A\mathcal{S}}}\| < 1$.*
3. *$(A, \mathcal{S})$ is not quasi-compatible $\Leftrightarrow$ there exists $\xi \neq 0$ such that $\|(T_{A,\mathcal{S}} - I)\xi\| = \|\xi\|$ $\Leftrightarrow$ there exists $\xi \neq 0$ such that $\|P_{\mathcal{S}^\perp} P_{\overline{A\mathcal{S}}}\xi\| = \|\xi\|$.*
4. *$(A, \mathcal{S})$ is quasi-compatible non compatible $\Leftrightarrow$ $\|T_{A,\mathcal{S}} - I\| = 1$ and for every $\xi \neq 0$, $\|(T_{A,\mathcal{S}} - I)\xi\| < \|\xi\|$ $\Leftrightarrow$ $\|P_{\mathcal{S}^\perp} P_{\overline{A\mathcal{S}}}\| = 1$ and for every $\xi \neq 0$, $\|P_{\mathcal{S}^\perp} P_{\overline{A\mathcal{S}}}\xi\| < \|\xi\|$.*

The last result of this subsection relates properties of $\mathcal{S} + \mathcal{T}$ with the existence of some type of compatibility. Item *3* has appeared in [[13], Theorem 3.10].

Proposition 3.18. *The following assertions hold:*

1. *$\overline{\mathcal{S} + \mathcal{T}} = \mathcal{H}$ if and only in there exists $A \in L(\mathcal{H})^+$ such that $\overline{A\mathcal{S}} = \mathcal{T}^\perp$ and the pair $(A, \mathcal{S})$ is quasi-compatible.*
2. *$\mathcal{S} + \mathcal{T}$ is dense non closed in $\mathcal{H}$ if and only if there exists $A \in L(\mathcal{H})^+$ with non closed range such that $\overline{A\mathcal{S}} = \mathcal{T}^\perp$ and the pair $(A, \mathcal{S})$ is quasi-compatible but non compatible.*
3. *$\mathcal{S} + \mathcal{T} = \mathcal{H}$ if and only if there exists $A \in L(\mathcal{H})^+$ with closed range such that $\overline{A\mathcal{S}} = \mathcal{T}^\perp$ and $(A, \mathcal{S})$ is compatible.*

Proof.

1. Suppose $\overline{\mathcal{S} + \mathcal{T}} = \mathcal{H}$ and take $A = P_{\mathcal{S}^\perp} + P_{\mathcal{T}^\perp}$. It is clear that $A \in L(\mathcal{H})^+$. Furthermore, $\overline{A\mathcal{S}} = \overline{P_{\mathcal{T}^\perp}(\mathcal{S})} = \mathcal{T}^\perp$, where the last equality holds by Theorem 2.4. Therefore $(A, \mathcal{S})$ is quasi-compatible because $\overline{\mathcal{S} + (A\mathcal{S})^\perp} = \overline{\mathcal{S} + \mathcal{T}} = \mathcal{H}$. The converse is immediate.

2. Suppose that $\mathcal{S} + \mathcal{T}$ is a dense non closed subspace and take $A = P_{\mathcal{S}^\perp} + P_{\mathcal{T}^\perp} \in L(\mathcal{H})^+$. By Theorem 2.4 the operator A has non closed range. As in the above item, $\overline{A\mathcal{S}} = \mathcal{T}^\perp$. Now, since

$\overline{\mathcal{S}+(A\mathcal{S})^\perp}=\overline{\mathcal{S}+\mathcal{T}}$ then $(A,\mathcal{S})$ is quasi-compatible but it is non compatible because $\mathcal{S}+(A\mathcal{S})^\perp$ is non closed. The converse is immediate.

3. The proof of item *3* is similar. □

Remark 3.19. If $\overline{\mathcal{S}+\mathcal{T}}=\mathcal{H}$ then $A=P_{\mathcal{T}^\perp}\in L(\mathcal{H})^+$ satisfies that $\overline{A\mathcal{S}}=\mathcal{T}^\perp$ and $(A,\mathcal{S})$ is quasi-compatible. However, the quasi-compatibility is straightforward because A has closed range.

3.3 The set $\mathcal{P}(A,\mathcal{S})$

If Q is an A-symmetric projection onto $\mathcal{S}$ then, by Proposition 3.2, $\mathcal{D}(Q)\subseteq\mathcal{S}_A$ and this inclusion may be strict as shows the following example: consider $A\in L(\mathcal{H})^+$ such that $dim(N(A))=\infty$ and $\mathcal{S}\subseteq N(A)$ a closed subspace of $\mathcal{H}$ such that $dim(\mathcal{S})=dim(\mathcal{S}^\perp)=\infty$. Observe that any closed projection Q such that $R(Q)\subseteq N(A)$ is trivially A-symmetric because $AQ=0$ and, since $(A(R(Q)))^\perp=\mathcal{H}$, it follows that $\mathcal{S}_A=\mathcal{H}$. In this particular case it is possible to construct a closed subspace $\mathcal{M}$ such that $\mathcal{S}\dot{+}\mathcal{M}$ is dense in $\mathcal{H}$ and $c(\mathcal{S},\mathcal{M})=1$ (see [[26], pages 28-29]) so that $\mathcal{S}\dot{+}\mathcal{M}$ is not closed. The projection $Q=P_{\mathcal{S}//\mathcal{M}}$ is an unbounded closed A-symmetric projection onto $\mathcal{S}$ such that $\mathcal{D}(Q)=\mathcal{S}\dot{+}\mathcal{M}\subsetneq\mathcal{H}=\mathcal{S}_A$.

If the pair $(A,\mathcal{S})$ is quasi-compatible then $\mathcal{S}_A$ is dense in $\mathcal{H}$ and $\mathcal{S}_A=\mathcal{S}\oplus(A\mathcal{S})^\perp\ominus\mathcal{N}=\mathcal{S}\ominus\mathcal{N}\oplus(A\mathcal{S})^\perp$. Consider the A-symmetric projections:

$$P_{A,\mathcal{S}}=P_{\mathcal{S}//(A\mathcal{S})^\perp\ominus\mathcal{N}}\quad \textit{and}\quad P_{A,\mathcal{S}\ominus\mathcal{N}}=P_{\mathcal{S}\ominus\mathcal{N}//(A\mathcal{S})^\perp},$$

and denote by $\mathcal{P}(A,\mathcal{S})$ the set of all A-symmetric idempotents with domain $\mathcal{S}_A$ and range $\mathcal{S}$.

The notations $P_{A,\mathcal{S}}$, $P_{A,\mathcal{S}\ominus\mathcal{N}}$ and $\mathcal{P}(A,\mathcal{S})$ have been used in [17, 16] in the context of bounded projections. Observe that if $(A,\mathcal{S})$ is quasi-compatible then $\mathcal{P}(A,\mathcal{S})$ is not empty because $P_{A,\mathcal{S}}\in\mathcal{P}(A,\mathcal{S})$. Moreover, if $\mathcal{S}\cap N(A)=\{0\}$ then $\mathcal{P}(A,\mathcal{S})=\{P_{A,\mathcal{S}}\}$. The next theorem is the "unbounded version" of [[17], Theorem 3.5].

Theorem 3.20. *If the pair $(A,\mathcal{S})$ is quasi-compatible then*

$$\mathcal{P}(A,\mathcal{S})=P_{A,\mathcal{S}}+\{W\in L(\mathcal{H}):R(W)\subseteq\mathcal{N}\ \textit{and}\ \mathcal{S}\subseteq N(W)\}.$$

Proof. Let $Q=P_{A,\mathcal{S}}+W$, where $W\in L(\mathcal{H})$ is such that $R(W)\subseteq\mathcal{N}$ and $\mathcal{S}\subseteq N(W)$. Observe that $P_{A,\mathcal{S}}W=W$, $WP_{A,\mathcal{S}}=0$ and $W^2=0$. Then Q is an idempotent with domain $\mathcal{S}_A$. Furthermore $R(Q)=\mathcal{S}$. In fact, it is clear that $R(Q)\subseteq\mathcal{S}$ and if $\xi\in\mathcal{S}$ then $Q\xi=(P_{A,\mathcal{S}}+W)\xi=P_{A,\mathcal{S}}\xi=\xi$. So, $\mathcal{S}\subseteq R(Q)$. Also, $N(Q)\subseteq(A\mathcal{S})^\perp$. Indeed, let $\xi\in N(Q)$. Then $P_{A,\mathcal{S}}\xi=-W\xi$ and so $\xi=(I-P_{A,\mathcal{S}})\xi+P_{A,\mathcal{S}}\xi\in(A\mathcal{S})^\perp$. In order to see that $Q\in\mathcal{P}(A,\mathcal{S})$ it only remains to prove that Q is a closed operator but this is a consequence of the fact that Q is the sum of a closed operator and a bounded operator. Conversely, let $Q\in\mathcal{P}(A,\mathcal{S})$ and define $W=Q-P_{A,\mathcal{S}}$. It clear that $\mathcal{D}(W)=\mathcal{S}_A$ and $\mathcal{S}\subseteq N(W)$. Now, $R(W)\subseteq\mathcal{S}$ and since $W=(I-P_{A,\mathcal{S}})-(I-Q)$ then $R(W)\subseteq(A\mathcal{S})^\perp$. So, $R(W)\subseteq\mathcal{N}$. On the other hand, since $W|_{(A\mathcal{S})^\perp\ominus\mathcal{N}}=Q|_{(A\mathcal{S})^\perp\ominus\mathcal{N}}$ and Q is closed then $W|_{(A\mathcal{S})^\perp\ominus\mathcal{N}}$ is closed and then bounded. Now, as $W|_{\mathcal{S}}=0$ then W is bounded on $\mathcal{S}_A$ which is a dense subspace of $\mathcal{H}$. Therefore, W has a unique bounded linear extension $\tilde{W}$ to $\mathcal{H}$ and it satisfies that $Q=P_{A,\mathcal{S}}+\tilde{W}$, $\mathcal{S}\subseteq N(\tilde{W})$ and $R(\tilde{W})\subseteq\mathcal{N}$. □

Proposition 3.21. *If the pair $(A,\mathcal{S})$ is quasi-compatible then*

1. $P_{A,\mathcal{S}}=P_{A,\mathcal{S}\ominus\mathcal{N}}+P_{\mathcal{N}}$.
2. *If $\xi\in\mathcal{S}_A$ then $\|(I-P_{A,\mathcal{S}})\xi\|=\min\{\|(I-Q)\xi\|:Q\in\mathcal{P}(A,\mathcal{S})\}$. Moreover, $(I-P_{A,\mathcal{S}})\xi$ is the unique vector with minimal norm.*

Proof. We omit the proof because it is similar to that one of the bounded case (see [[17], Theorem 3.6]). □

4 Formulas for $P_{A,\mathcal{S}}$

For the sake of simplicity, along this section, we will consider $A \in L(\mathcal{H})^+$ injective. In what follows, we describe several formulae for the element $P_{A,\mathcal{S}}$ for a compatible pair $(A,\mathcal{S})$. We begin with a matrix representation of $P_{A,\mathcal{S}}$.

Proposition 4.1. *Let $A \in L(\mathcal{H})^+$ an injective operator such that $(A,\mathcal{S})$ is a quasi-compatible pair. Then*

$$P_{A,\mathcal{S}} = \begin{pmatrix} 1 & -P_{\mathcal{S}}(P_{\mathcal{S}^\perp}P_{(A\mathcal{S})^\perp}|_{(A\mathcal{S})^\perp})^{-1} \\ 0 & 0 \end{pmatrix} \quad on \quad \mathcal{S} \oplus R(P_{\mathcal{S}^\perp}P_{(A\mathcal{S})^\perp}).$$

Furthermore, if we consider the matrix representation (3) of A then $x_0 = -P_{\mathcal{S}}(P_{\mathcal{S}^\perp}P_{(A\mathcal{S})^\perp}|_{(A\mathcal{S})^\perp})^{-1}$ is the unique solution of the equation $ax = b|_{R(P_{\mathcal{S}^\perp}P_{(A\mathcal{S})^\perp})}$.

Proof. Since A is an injective operator then $P_{A,\mathcal{S}} = P_{\mathcal{S}//(A\mathcal{S})^\perp}$ and $P_{\mathcal{S}^\perp}P_{(A\mathcal{S})^\perp}|_{(A\mathcal{S})^\perp} : (A\mathcal{S})^\perp \longrightarrow R(P_{\mathcal{S}^\perp}P_{(A\mathcal{S})^\perp})$ is invertible. Therefore the matrix representation of $P_{A,\mathcal{S}}$ follows from [[4], Theorem 2.6]. The last part of this result follows from Proposition 3.2.

□

In order to present different formulas to the element $P_{A,\mathcal{S}}$ recall that if $T : \mathcal{D}(T) \subseteq \mathcal{H} \longrightarrow \mathcal{H}$ is a closed densely defined operator, the *Moore-Penrose inverse of* T is the unique linear operator $T^\dagger$ with domain $\mathcal{D}(T^\dagger) = R(T) \oplus R(T)^\perp$ which satisfies the following properties:

1. $TT^\dagger = P_{\overline{R(T)}}|_{\mathcal{D}(T^\dagger)}$;
2. $T^\dagger T = P_{N(T)^\perp}|_{\mathcal{D}(T)}$;
3. $N(T^\dagger) = R(T)^\perp$.

It holds that the operator $T^\dagger$ is bounded if and only if $R(T)$ is closed (see [7] for the proof of this assertion).

Proposition 4.2. *Let $A \in (\mathcal{H})^+$ an injective operator such that $(A,\mathcal{S})$ is a quasi-compatible pair. If $P = P_{\mathcal{S}}$ then*

1. $P_{A,\mathcal{S}} = A^{-1/2}P_{\overline{A^{1/2}\mathcal{S}}}A^{1/2}|_{\mathcal{S}_A}$;

2. $P_{A,\mathcal{S}} = (P_{\overline{A\mathcal{S}}}P)^\dagger$;

3. $P_{A,\mathcal{S}} = P(P + P_{(A\mathcal{S})^\perp})^{-1/2}(P + P_{(A\mathcal{S})^\perp})^{-1/2}$.

Proof. Remember that if $(A,\mathcal{S})$ is quasi-compatible then the subspace $\mathcal{S}_A = \mathcal{S} \dot{+} (A\mathcal{S})^\perp$ is dense in $\mathcal{H}$.

1. Since $(A,\mathcal{S})$ is quasi-compatible and A is injective then the formula follows from Corollary 3.5.

2. Since A is injective then $P_{A,\mathcal{S}} = P_{\mathcal{S}//(A\mathcal{S})^\perp}$. On the other hand, as $P_{A,\mathcal{S}}$ is a closed densely defined operator then $R(P_{A,\mathcal{S}})$ is closed and $P_{A,\mathcal{S}}^\dagger \in L(\mathcal{H})$. Now, $P_{A,\mathcal{S}}^\dagger = P_{A,\mathcal{S}}^\dagger P_{A,\mathcal{S}} P_{A,\mathcal{S}}^\dagger = (P_{A,\mathcal{S}}^\dagger P_{A,\mathcal{S}})(P_{A,\mathcal{S}} P_{A,\mathcal{S}}^\dagger) = P_{\overline{A\mathcal{S}}}|_{\mathcal{S}_A} P_{\mathcal{S}} = P_{\overline{A\mathcal{S}}}P_{\mathcal{S}}$. Then $P_{A,\mathcal{S}} = (P_{\overline{A\mathcal{S}}}P_{\mathcal{S}})^\dagger$ as we claimed.

3. This formula is due to Ando [[4], Theorem 2.2]. □

Remark 4.3. The formulas for $P_{A,\mathcal{S}}$ given in the above proposition are still valid if $A \in L(\mathcal{H})^+$ is not injective but $\mathcal{S} \cap N(A) = \{0\}$. In this case, in the formula of item *1* the operator $A^{-1/2}$ must be replaced by $(A^{1/2})^\dagger$.

We finish this section with a characterizaction of compatibility in terms of a closable idempotent. Before that we present the following lemma.

Lemma 4.4. *Let $E_0 : R(A^{1/2}) \longrightarrow \mathcal{H}$ defined by $E_0 = A^{1/2} P_{\overline{A^{1/2}\mathcal{S}}} A^{-1/2}$. Then, E_0 is an idempotent with $R(E_0) = A^{1/2}(\overline{A^{1/2}\mathcal{S}})$ and $N(E_0) = \mathcal{S}^{\perp} \cap R(A^{1/2})$.*

Proof. It is easy to check that $E_0^2 = E_0$. Furthermore, $R(E_0) = A^{1/2} P_{\overline{A^{1/2}\mathcal{S}}} A^{-1/2}(R(A^{1/2})) = R(A^{1/2} P_{\overline{A^{1/2}\mathcal{S}}}) = A^{1/2}(\overline{A^{1/2}\mathcal{S}})$. Now, observe that $A^{1/2}((A^{1/2}\mathcal{S})^{\perp}) = A^{1/2}(A^{-1/2}\mathcal{S}^{\perp}) = \mathcal{S}^{\perp} \cap R(A^{1/2})$. If $\xi \in N(E_0) \subseteq R(A^{1/2})$ then there exists a unique $\eta \in \mathcal{H}$ such that $\xi = A^{1/2}\eta$ and it holds that $P_{\overline{A^{1/2}\mathcal{S}}}\eta = 0$. So that, $\xi \in A^{1/2}((A^{1/2}\mathcal{S})^{\perp}) = \mathcal{S}^{\perp} \cap R(A^{1/2})$. Conversely, if $\xi \in \mathcal{S}^{\perp} \cap R(A^{1/2}) = A^{1/2}((A^{1/2}\mathcal{S})^{\perp})$ then there exists a unique $\eta \in (A^{1/2}\mathcal{S})^{\perp}$ such that $\xi = A^{1/2}\eta$. Then $E_0\xi = A^{1/2} P_{\overline{A^{1/2}\mathcal{S}}}\eta = 0$. Therefore, $N(E_0) = \mathcal{S}^{\perp} \cap R(A^{1/2})$. □

Theorem 4.5. *The pair $(A, \mathcal{S})$ is compatible if and only if E_0 is closable, $\overline{E_0}$ is bounded and $\mathcal{S}^{\perp} = \overline{\mathcal{S}^{\perp} \cap R(A^{1/2})}$.*

Proof. Since A is injective and $(A, \mathcal{S})$ is compatible then $P_{A,\mathcal{S}} = A^{-1/2} P_{\overline{A\mathcal{S}}} A^{1/2}$ is bounded. Observe that $P_{A,\mathcal{S}}^* \supseteq (P_{\overline{A^{1/2}\mathcal{S}}} A^{1/2})^*(A^{-1/2}) \supseteq A^{1/2} P_{\overline{A^{1/2}\mathcal{S}}} A^{-1/2} = E_0$. Then, E_0 is closable and bounded. Therefore, $\overline{E_0}$ is a closed densely defined projection and $\overline{E_0} = P_{\overline{A^{1/2}(\overline{A^{1/2}\mathcal{S}})}//\overline{\mathcal{S}^{\perp}\cap R(A^{1/2})}}$. Now, as $A^{1/2}(A^{1/2}\mathcal{S}) \subseteq A^{1/2}(\overline{A^{1/2}\mathcal{S}}) \subseteq \overline{A\mathcal{S}}$ then $\overline{A^{1/2}(\overline{A^{1/2}\mathcal{S}})} = \overline{A\mathcal{S}}$. In consequence, $\overline{A\mathcal{S}} \dotplus \overline{\mathcal{S}^{\perp} \cap R(A^{1/2})}$ is dense in $\mathcal{H}$. Let see that $\overline{A\mathcal{S}} \dotplus \overline{\mathcal{S}^{\perp} \cap R(A^{1/2})}$ is also closed. By the hypothesis we get that $\overline{A\mathcal{S}} \dotplus \mathcal{S}^{\perp} = \mathcal{H}$. Thus $c_0(\overline{A\mathcal{S}}, \overline{\mathcal{S}^{\perp} \cap R(A^{1/2})}) \leq c_o(\overline{A\mathcal{S}}, \mathcal{S}^{\perp}) < 1$. Hence $\overline{A\mathcal{S}} \dotplus \overline{\mathcal{S}^{\perp} \cap R(A^{1/2})} = \overline{A\mathcal{S}} \dotplus \mathcal{S}^{\perp} = \mathcal{H}$. So that $\overline{E_0} \in L(\mathcal{H})$ and $\mathcal{S}^{\perp} = \overline{\mathcal{S}^{\perp} \cap R(A^{1/2})}$. Conversely, if E_0 is closable and $\overline{E_0}$ is bounded then $\overline{E_0} \in L(\mathcal{H})$ and $\overline{E_0} = P_{\overline{A\mathcal{S}}//\mathcal{S}^{\perp}}$. Then $\overline{A\mathcal{S}} \dotplus \mathcal{S}^{\perp} = \mathcal{H}$, or which is the same, $\mathcal{S} \dotplus (A\mathcal{S})^{\perp} = \mathcal{H}$. So that, $(A, \mathcal{S})$ is compatible. □

5 Applications

5.1 An interpolation problem

Consider the following two decompositions of the Hilbert space $L^2(\mathbb{T})$:

$$L^2(\mathbb{T}) = H^2 \oplus \overline{H}^2 = L^2(K) \oplus L^2(\mathbb{T} \setminus K), \tag{4}$$

where K is a compact subset of $\mathbb{T}$ such that K and $\mathbb{T} \setminus K$ have positive measure and $L^2(K)$ denotes the subspace of functions of $L^2(\mathbb{T})$ which vanish almost everywhere on $\mathbb{T} \setminus K$. Furthermore, H^2 denotes the Hardy space which is the subspace of $L^2(\mathbb{T})$ whose Fourier coefficients of strictly negative indices vanish and its orthogonal complement $\overline{H}^2$ is the subspace of functions of $L^2(\mathbb{T})$ whose Fourier coefficient of non-negative indices vanish. This and other examples are the starting point of [35] for the study of interpolation and constrained approximation problems in Hilbert function spaces. See [35] and references therein for applications of (4) and other decompositions. The problems studied in [35] have the following general framework: $\mathcal{H} = \mathcal{S} \oplus \mathcal{S}^{\perp} = \mathcal{T} \oplus \mathcal{T}^{\perp}$ are two different decompositions of $\mathcal{H}$ such that $\mathcal{S}^{\perp} \cap \mathcal{T}^{\perp} = \{0\}$. Observe that, by Theorem 2.4, this condition means that $\mathcal{S} + \mathcal{T}$ is dense in $\mathcal{H}$ or, equivalently, $\overline{P_{\mathcal{T}^{\perp}}(\mathcal{S})} = \mathcal{T}^{\perp}$. Under this hypothesis, the following problem is one of those studied by Leblond and Partington in [[35], Section 4]:

(i) describe $P_{\mathcal{T}^{\perp}}(\mathcal{S})$;

(ii) for each $\eta \in P_{\mathcal{T}^\perp}(\mathcal{S})$, find a vector $\xi \in \mathcal{S}$ such that $\eta = P_{\mathcal{T}^\perp}\xi$.

Observe that, by Proposition 3.18, the condition $\mathcal{S}^\perp \cap \mathcal{T}^\perp = \{0\}$ is equivalent to the existence of $A \in L(\mathcal{H})^+$ such that $\overline{A\mathcal{S}} = \mathcal{T}^\perp$ and $(A, \mathcal{S})$ is quasi-compatible. Moreover, we know that there are at least two possible choices for A, namely, $A = P_{\mathcal{S}^\perp} + P_{\mathcal{T}^\perp}$ and $A = P_{\mathcal{T}^\perp}$. In the next result we study the problems (i) and (ii) in the context of quasi-compatibility. This approach allows to get simple descriptions of (i) and (ii).

Proposition 5.1. *Let $(A, \mathcal{S})$ a quasi-compatible pair. If $\eta \in P_{\overline{A\mathcal{S}}}(\mathcal{S}) \setminus \mathcal{S}$ then*

$$\{\xi \in \mathcal{S} : \eta = P_{\overline{A\mathcal{S}}}\xi\} = \{Q\eta : Q \in \mathcal{P}(A, \mathcal{S})\}. \tag{5}$$

Moreover, if $\mathcal{N} = \{0\}$ then

$$\{\xi \in \mathcal{S} : \eta = P_{\overline{A\mathcal{S}}}\xi\} = \{P_{A,\mathcal{S}}\eta\}. \tag{6}$$

Proof. Recall that, by Lemma 3.10, $P_{\overline{A\mathcal{S}}}(\mathcal{S}) = \mathcal{S}_A \cap \overline{A\mathcal{S}}$. Given $\eta \in P_{\overline{A\mathcal{S}}}(\mathcal{S}) \setminus \mathcal{S}$ let $\xi \in \mathcal{S}$ such that $\eta = P_{\overline{A\mathcal{S}}}\xi$. Since $P_{\overline{A\mathcal{S}}}\eta = P_{\overline{A\mathcal{S}}}\xi$ it follows that $\eta - \xi \in (A\mathcal{S})^\perp$. Furthermore, as $\eta \notin \mathcal{S}$ then $\eta - \xi \notin \mathcal{S}$. Observe that $span\{\eta - \xi\} \dot{+} \mathcal{N} \subseteq (A\mathcal{S})^\perp$ is a closed subspace. Define $\mathcal{T} = (span\{\eta - \xi\} \dot{+} \mathcal{N})^\perp \cap (A\mathcal{S})^\perp$ and $\mathcal{W} = \mathcal{T} \dot{+} span\{\eta - \xi\} \subseteq (A\mathcal{S})^\perp$. Then $\mathcal{W}$ is a closed subspace and it holds that $\mathcal{S} \dot{+} \mathcal{W} = \mathcal{S}_A$. Therefore $Q = P_{\mathcal{S}//\mathcal{W}} \in \mathcal{P}(A, \mathcal{S})$ and we get $0 = Q(\eta - \xi) = Q\eta - \xi$. So, $\xi = Q\eta$ as claimed. Conversely, given $\eta \in P_{\overline{A\mathcal{S}}}(\mathcal{S}) \setminus \mathcal{S}$ take $Q \in \mathcal{P}(A, \mathcal{S})$ and define $\xi = Q\eta$. Then, $P_{\overline{A\mathcal{S}}}\xi = P_{\overline{A\mathcal{S}}}Q\eta = P_{\overline{A\mathcal{S}}}|_{\mathcal{S}_A}\eta = \eta$. So, identity (5) holds. To get (6) it is sufficient to note that if $\mathcal{N} = \{0\}$ then $\mathcal{P}(A, \mathcal{S}) = \{P_{A,\mathcal{S}}\}$. □

Remark 5.2. If $(A, \mathcal{S})$ is quasi-compatible and $\eta \in P_{\overline{A\mathcal{S}}}(\mathcal{S}) \cap \mathcal{S}$ then it is easy to check that the inclusion $\{Q\eta : Q \in \mathcal{P}(A, \mathcal{S})\} \subseteq \{\xi \in \mathcal{S} : \eta = P_{\overline{A\mathcal{S}}}\xi\}$ holds. However, the reverse inclusion does not hold in general. In fact, let $A \in L(\mathcal{H})^+$ such that $(A, \mathcal{S})$ is quasi-compatible and $\mathcal{N} = \mathcal{S} \cap N(A) \neq \{0\}$. Take $\xi = \rho + \eta \in \mathcal{S}$ with $0 \neq \rho \in \mathcal{N}$. Note that $\eta = P_{\overline{A\mathcal{S}}}\xi$. Now, observe that if there exists an idempotent Q with $R(Q) = \mathcal{S}$ such that $\xi = Q\eta$ then $\eta = \xi$. So, $\{\xi \in \mathcal{S} : \eta = P_{\overline{A\mathcal{S}}}\xi\} \not\subseteq \{Q\eta : Q \in \mathcal{P}(A, \mathcal{S})\}$.

5.2 Abstract splines

Given T be a bounded linear operator from $\mathcal{H}$ to a Hilbert space $\mathcal{K}$, consider $A = T^*T$ and $\xi \in \mathcal{H}$. The set of $(T, \mathcal{S})$-*spline interpolants* to ξ is

$$sp(T, \mathcal{S}, \xi) = \{\eta \in \xi + \mathcal{S} : \|T\eta\| = \min_{\sigma \in \mathcal{S}} \|T(\xi + \sigma)\|\}$$

(see [18] and [19] for a treatment of this subject). If $A = T^*T$ then this set can be rewritten as

$$sp(T, \mathcal{S}, \xi) = \{\eta \in \xi + \mathcal{S} : \|\eta\|_A = d_A(\xi, \mathcal{S})\},$$

where $d_A(\xi, \mathcal{S}) = \inf_{\sigma \in \mathcal{S}} \|\eta - \sigma\|_A$. Following the same steps as in the proof of [[16], Theorem 3.2, item 3] we obtain that if the pair $(A, \mathcal{S})$ is quasi-compatible then $sp(T, \mathcal{S}, \xi)$ is not empty for every ξ in a dense subset of $\mathcal{H}$, namely $\mathcal{S}_A$. Moreover, if $\xi \in \mathcal{S}_A \setminus \mathcal{S}$ then it holds

$$sp(T, \mathcal{S}, \xi) = \{(I - Q)\xi : Q \in \mathcal{P}(A, \mathcal{S})\}.$$

Furthermore, in this case $(I - P_{A,\mathcal{S}})\xi$ is the unique vector in $sp(T, \mathcal{S}, \xi)$ with minimal norm (Proposition 3.21).

5.3 A least squares problem

In this section we present a characterization of compatibility in term of the existence of least squares solution with a constraint of the equation $A^{1/2}\xi = \eta$.

Proposition 5.3. *The following assertions are equivalent:*

1. $(A, \mathcal{S})$ *is compatible;*
2. $\min_{\xi \in \mathcal{S}} \|A^{1/2}\xi - \eta\|$ *has solution for every* $\eta \in R(A^{1/2}) \oplus N(A^{1/2})$.

Proof. First, observe that to find $\min\{\|A^{1/2}\xi - \eta\| : \xi \in \mathcal{S}\}$ is equivalent to find $\min\{\|A^{1/2}P_{\mathcal{S}}\xi - \eta\| : \xi \in \mathcal{H}\}$. Furthermore, it holds

$$\min_{\xi \in \mathcal{S}} \|A^{1/2}\xi - \eta\| = \min_{\mu \in \mathcal{H}} \|A^{1/2}P_{\mathcal{S}}\mu - \eta\|. \tag{7}$$

If the pair $(A, \mathcal{S})$ is compatible then, by [[15], Proposition 2.14], it holds that $R(A^{1/2}) = A^{1/2}\mathcal{S} \oplus (A^{1/2}\mathcal{S})^{\perp} \cap R(A^{1/2})$. On the other hand, by (7) and [[25], Theorem 2.1.1] the equation $A^{1/2}\xi = \eta$ has least square solution in $\mathcal{S}$ for every $\eta \in R(A^{1/2}P_{\mathcal{S}}) \oplus R(A^{1/2}P_{\mathcal{S}})^{\perp} = A^{1/2}\mathcal{S} \oplus (A^{1/2}\mathcal{S})^{\perp}$. Observe that $(A^{1/2}\mathcal{S})^{\perp} = (A^{1/2}\mathcal{S})^{\perp} \cap \overline{R(A^{1/2})} \oplus N(A^{1/2})$. Then $R(A^{1/2}) \oplus N(A^{1/2}) \subseteq R(A^{1/2}P_{\mathcal{S}}) \oplus R(A^{1/2}P_{\mathcal{S}})^{\perp}$ and so, the equation $A^{1/2}\xi = \eta$ has a least squares solution for every in $\mathcal{S}$ for every $\xi \in R(A^{1/2}) \oplus N(A^{1/2})$. Conversely, if item *2* holds then, by (7) and [[25], Theorem 2.1.1], we get that $R(A^{1/2}) \oplus N(A^{1/2}) \subseteq A^{1/2}\mathcal{S} \oplus (A^{1/2}\mathcal{S})^{\perp} = A^{1/2}\mathcal{S} \oplus (A^{1/2}\mathcal{S})^{\perp} \cap \overline{R(A^{1/2})} \oplus N(A^{1/2})$. Let us see that $R(A^{1/2}) = A^{1/2}\mathcal{S} \oplus (A^{1/2}\mathcal{S})^{\perp} \cap R(A^{1/2})$. In fact, let $\xi = \mu + \rho + \theta \in R(A^{1/2})$; where $\mu \in A^{1/2}\mathcal{S}$, $\rho \in (A^{1/2}\mathcal{S})^{\perp} \cap \overline{R(A^{1/2})}$ and $\theta \in N(A^{1/2})$. Then $\theta = \xi - \mu - \rho \in \overline{R(A^{1/2})} \cap N(A^{1/2}) = \{0\}$. So, $\xi = \mu + \rho$. Furthermore, $\rho = \xi - \mu \in (A^{1/2}\mathcal{S})^{\perp} \cap R(A^{1/2})$. Therefore $R(A^{1/2}) \subseteq A^{1/2}\mathcal{S} \oplus (A^{1/2}\mathcal{S})^{\perp} \cap R(A^{1/2})$. The other inclusion is trivial. Then, by [[15], Proposition 2.14], we get that the pair $(A, \mathcal{S})$ is compatible. □

Remark 5.4. In a similar way it can be proven that if $(A, \mathcal{S})$ is quasi-compatible then $A^{1/2}\xi = \eta$ has a least squares solution in $\mathcal{S}$ for every η in a dense subset of $\mathcal{H}$, namely, for every $\eta \in A^{1/2}\mathcal{S} + (A^{1/2}\mathcal{S})^{\perp} \cap \overline{R(A^{1/2})} \oplus N(A^{1/2})$. Observe that $(A^{1/2}\mathcal{S})^{\perp} \cap \overline{R(A^{1/2})} = \overline{(A^{1/2})^{\perp} \cap R(A^{1/2})}$ and that, if $(A, \mathcal{S})$ is quasi-compatible, then $A^{1/2}\mathcal{S}_A = A^{1/2}\mathcal{S} \oplus (A^{1/2})^{\perp} \cap R(A^{1/2})$ is dense in $R(A^{1/2})$.

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