Optimal dual frames and frame completions for majorization

P. G. Massey, M. A. Ruiz and D. Stojanoff
Depto. de Matemática, FCE-UNLP, La Plata, Argentina and IAM-CONICET

Abstract

In this paper we consider two problems of frame theory. On the one hand, given a fixed frame $\mathcal{F}$ we describe explicitly the spectral and geometric structure of optimal frames $\mathcal{G}$ that are in duality with $\mathcal{F}$ and such that the Frobenius norms of their analysis operators is bounded from below by a fixed constant, where optimality is measured with respect to submajorization. On the other hand, given a set of vectors $\mathcal{F}$ we describe the spectral and geometrical structure of optimal completions of $\mathcal{F}$ by a finite family of vectors with prescribed norms, under certain hypothesis. Again, optimality is measured with respect to majorization of the frames operators, which implies optimality with respect to a family of convex functionals that include the mean square error and the Bendetto-Fickus' potential. Our approach relies on the description of the spectral and geometrical structure of matrices that minimize submajorization on sets that are naturally associated with the problems above.

Contents

1 Introduction

Finite frame theory is a well established research field that has attracted the attention of many researchers (see [8, 14, 17] for a general reference to frame theory). On the one hand, finite frames provide with redundant linear encoding-decoding schemes that are useful when dealing with transmission of signals through noisy channels . Indeed, the redundancy of frames allow for reconstruction of signal, even when some frame coefficients are lost. Moreover, frames have also shown to be robust under erasures of the frame coefficients when a blind reconstruction strategy is considered (see [4, 5, 18]). On the other hand, there are several problems in frame theory that have deep relations with problems in other areas of mathematics (such as matrix analysis, operator theory and operator algebras) which constitute an strong motivation for research. For example, we can mention the relation between the Feichtinger conjecture in frame theory and some major open problems in operator algebra theory such as the Kadison-Singer problem (see [10, 11]). Other examples of this phenomenon are the design problem in frame theory, the so-called Paulsen problem in frame theory and frame completion problems ([1, 7, 9, 12, 21]) which are known to be equivalent to different aspects of the Schur-Horn theorem. Recently, matrix analysis has served as a tool to show some structural properties of minimizers of the Benedetto-Fickus's frame potential ([2, 13]) and other convex functionals in the finite setting ([22, 23]).

In this paper we explore some new connections of problems that arise naturally in frame theory with some results in matrix theory related with the notion of (sub)majorization between vectors and self-adjoint matrices. More explicitly, we consider the following two problems in frame theory (see Section 2 for the notation and terminology).

Given a fixed frame $\mathcal{F}$ for a finite dimensional Hilbert space $\mathcal{H} \cong \mathbb{F}^d$ (where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$) let $\mathcal{D}(\mathcal{F})$ denote the set of all frames $\mathcal{G}$ that are in duality with $\mathcal{F}$. It is well known that the canonical dual of $\mathcal{F}$, denoted $\mathcal{F}^{\#}$, has some optimality properties among the elements in $\mathcal{D}(\mathcal{F})$. Nevertheless, although optimal in some senses, $\mathcal{F}^{\#}$ may be ill-conditioned and therefore may not the best choice. In order to search for alternative duals for $\mathcal{F}$ we restrict attention the set $\mathcal{D}_t(\mathcal{F})$ which contains the frames $\mathcal{G}$ that are in duality with $\mathcal{F}$ and such that the Frobenius norm of their frame operators is bounded from below by a constant t. Hence we search within this set for optimal duals: but a problem arises, as to what measure of optimality are to be consider. As examples we can consider minimizing the frame potential of $\mathcal{G}$ or minimizing the condition number of the frame operator of $\mathcal{G}$. A way out of this problem is to consider optimality with respect to submajorization (which is a subtle measure of the spread of the eigenvalues of the frame operators), which implies optimality with respect to both the frame potential an the minimal eigenvalue. Therefore, in this paper we shall explicitly describe the spectral and geometrical structure of minimizers of submajorization in $\mathcal{D}_t(\mathcal{F})$ (see Theorems 6.2 and 6.3).

On the other hand, given a finite sequence of vectors $\mathcal{F}_0$ and a finite sequence of positive numbers $\mathbf{b}$ we are interested in computing optimal frame completions of $\mathcal{F}_0$, denoted by $\mathcal{F}$, obtained by adding vectors with norms prescribed by the entries of $\mathbf{b}$. This problems has been posed recently in [15], in the particular case where optimality is measured with respect to the mean square error of the completed frame $\mathcal{F}$ (see also [21] for some related completion problems). It is worth pointing out that the mean square error is not the only possible (desirable) measure for optimality: for example we can consider minimizing the Benedetto-Fickus's frame potential of the completed frame $\mathcal{F}$. This raises the question of whether optimal completions with respect to these different measures of optimality coincide. As before, a way out of such problems is to consider majorization as a measure of optimality. Therefore, in this paper we shall compute the spectral and geometrical structure of minimizers of majorization in the set of frame completions of $\mathcal{F}_0$ with norms prescribed by the entries of μ, under certain hypothesis (see Theorem 7.6).

Both problems above are related with the minimizers of (sub)majorization in certain sets $\mathcal{S}$ of positive semidefinite matrices that arise naturally. Although majorization is not a total order, we show that the sets $\mathcal{S}$ that we consider have minimal elements with respect to majorization, a fact that is of independent interest (see Theorems 4.7, 4.15 and 5.10). Notably, the existence of such minimizers is essentially obtained with tools and insights coming from frame theory.

The paper is organized as follows. In Section 2 we establish the notation and terminolgy used throughout the paper, and we state some basic facts from matrix theory and frame theory. In Section 3 we fix a frame $\mathcal{F}$ and characterize the set of eigenvalue lists corresponding to frame operators of frames $\mathcal{G}$ that are in duality with $\mathcal{F}$. This description allows us to establish a link with Fan-Pall's theorem on principal sub-matrices of self-adjoint matrices. In Section 4 we present an abstract version of the previous frame problem, in the general setting of positive semi-definite matrices and compute the spectral structure of minimizers of submajorization in this generality. In Section 5 we further describe the geometric structure of the minimizers of sub-majorization in the general setting. The techniques used in Sections 4 and 5 are those of matrix theory. Then, with the results of minimizers of the previous sections we consider the problems of minimal duals and minimal frame completions in Sections 6 and 7 respectively. With respect to the problem of minimal duals with respect to submajorization, we completely describe the spectral and geometrical structure of optimal duals. With respect to the problem of optimal completions with respect to majorization, we obtain a complete description in several cases, that include the case of uniform norms for the added vectors.

2 Preliminaries

In this section we describe the basic notions that we shall consider throughout the paper. We first establish the notation and then describe the interlacing inequalities and submajorization that are two notions from the theory of matrix analysis. Finally, we recall the basic facts from frame theory that are related with our main results.

2.1 General notations.

Given $m \in \mathbb{N}$ we denote by $\mathbb{I}_m = \{1, \dots, m\} \subseteq \mathbb{N}$ and $\mathbb{1} = \mathbb{1}_m \in \mathbb{R}^m$ denotes the vector with all its entries equal to 1. For a vector $x \in \mathbb{R}^m$ we denote by $x^\downarrow$ the rearrangement of x in a decreasing order, and $\mathbb{R}^{m\,\downarrow} = \{x \in \mathbb{R}^m : x = x^\downarrow\}$ the set of ordered vectors.

Given $\mathcal{H} \cong \mathbb{C}^d$ and $\mathcal{K} \cong \mathbb{C}^n$, we denote by $L(\mathcal{H}\,, \mathcal{K})$ the space of linear operators $T : \mathcal{H} \to \mathcal{K}$. Given an operator $T \in L(\mathcal{H}\,, \mathcal{K})$, $R(T) \subseteq \mathcal{K}$ denotes the image of T, $\ker T \subseteq \mathcal{H}$ the null space of T and $T^* \in L(\mathcal{K}\,, \mathcal{H})$ the adjoint of T. If $d \leq n$ we say that $U \in L(\mathcal{H}\,, \mathcal{K})$ is an isometry if $U^*U = I_{\mathcal{H}}$. In this case, U^* is called a coisometry. If $\mathcal{K} = \mathcal{H}$ we denote by $L(\mathcal{H}) = L(\mathcal{H}\,, \mathcal{H})$, by $\mathcal{G}l(\mathcal{H})$ the group of all invertible operators in $L(\mathcal{H})$, by $L(\mathcal{H})^+$ the cone of positive operators and by $\mathcal{G}l(\mathcal{H})^+ = \mathcal{G}l(\mathcal{H}) \cap L(\mathcal{H})^+$. If $T \in L(\mathcal{H})$, we denote by $\sigma(T)$ the spectrum of T, by $\operatorname{rk} T = \dim R(T)$ the rank of T, and by $\operatorname{tr} T$ the trace of T. By fixing an orthonormal basis (onb) of the Hilbert spaces involved, we shall identify operators with matrices, using the following notations:

By $\mathcal{M}_{n,d}(\mathbb{C}) \cong L(\mathbb{C}^d\,, \mathbb{C}^n)$ we denote the space of complex $n \times d$ matrices. If $n = d$ we write $\mathcal{M}_n(\mathbb{C}) = \mathcal{M}_{n,n}(\mathbb{C})$. $\mathcal{H}(n)$ is the $\mathbb{R}$-subspace of selfadjoint matrices, $\mathcal{G}l(n)$ the group of all invertible elements of $\mathcal{M}_n(\mathbb{C})$, $\mathcal{U}(n)$ the group of unitary matrices, $\mathcal{M}_n(\mathbb{C})^+$ the set of positive semidefinite matrices, and $\mathcal{G}l(n)^+ = \mathcal{M}_n(\mathbb{C})^+ \cap \mathcal{G}l(n)$. If $d \leq n$, we denote by $\mathcal{I}(d\,, n) \subseteq \mathcal{M}_{n\,,\,d}(\mathbb{C})$ the set of isometries, i.e. those $U \in \mathcal{M}_{n\,,\,d}(\mathbb{C})$ such that $U^*U = I_d$.

If $W \subseteq \mathcal{H}$ is a subspace we denote by $P_W \in L(\mathcal{H})^+$ the orthogonal projection onto W, i.e. $R(P_W) = W$ and $\ker\, P_W = W^\perp$. For vectors on $\mathbb{C}^n$ we shall use the euclidean norm. On the other hand, for matrices $T \in \mathcal{M}_n(\mathbb{C})$ we shall use both

1. The spectral norm $\|T\| = \|T\|_{sp} = \max\limits_{\|x\|=1} \|Tx\|$.
2. The Frobenius norm $\|T\|_2 = (\operatorname{tr}\, T^*T)^{1/2} = \big(\sum\limits_{i,j \in \mathbb{I}_n} |T_{ij}|^2 \big)^{1/2}$. This norm is induced by the inner product $\langle A,\ B\rangle = \operatorname{tr}\, B^*A$, for $A, B \in \mathcal{M}_n(\mathbb{C})$.

2.2 Interlacing inequalities and submajorization

Next we briefly describe two well known notions of matrix analysis that will be used throughout the paper.

Interlacing inequalities. Let $A \in \mathcal{H}(n)$ with $\lambda(A) \in \mathbb{R}^{n\,\downarrow}$ and let $P = P^2 = P^* \in \mathcal{M}_d(\mathbb{C})^+$ be a projection with $\operatorname{rk} P = k$. The interlacing inequalities (see [3]) relate the eigenvalues of A with the eigenvalues of $PAP \in \mathcal{H}(n)$ as follows:

$$\lambda_{n-k+i}(A) \le \lambda_i(PAP) \le \lambda_i(A) \quad \text{for every} \quad i \in \mathbb{I}_k \ . \tag{1}$$

On the other hand, if we have the equalities

$$\lambda_i(PAP) = \lambda_i(A) \quad \text{for every} \quad i \in \mathbb{I}_k \quad \text{then} \quad PA = AP \ , \tag{2}$$

and that $R(P)$ has an ONB $\{h_i\}_{i\in\mathbb{I}_k}$ such that $A\,h_i = \lambda_i\,h_i$ for every $i \in \mathbb{I}_k$. Indeed, if $Q = I - P$, then $\operatorname{tr} QAQ = \sum\limits_{i=k+1}^{n} \lambda_i(A)$. The interlacing inequalities applied to QAQ imply that

$$\lambda_{k+j}(A) \le \lambda_j(QAQ) \quad \text{for} \quad j \in \mathbb{I}_{n-k} \implies \lambda_j(QAQ) = \lambda_{k+j}(A) \quad \text{for} \quad j \in \mathbb{I}_{n-k} \, .$$

Taking Frobenius norms, we get that

$$\|A\|_2^2 = \sum_{i=1}^{n} \lambda_i(A)^2 = \|PAP\|_2^2 + \|QAQ\|_2^2 \implies PAQ = QAP = 0 \ ,$$

so that $A = PAP + QAQ$. The Ky-Fan inequalities (see [3]) assure that

$$\sum_{i=1}^{k} \lambda_i(A) = \max \Big\{ \operatorname{tr}\, PAP : P \in \mathcal{M}_d(\mathbb{C})^+ \ , \quad P = P^2 = P^* \quad \text{and} \quad \operatorname{rk} P = k \Big\} \ . \tag{3}$$

As before, given an orthogonal projection P with $\operatorname{rk} P = k$ such that

$$\operatorname{tr}\, PAP = \sum_{i=1}^{k} \lambda_i(A) \overset{(1)}{\implies} \lambda_i(PAP) = \lambda_i(A) \quad \text{for} \quad i \in \mathbb{I}_k \overset{(2)}{\implies} PA = AP \ , \tag{4}$$

and $R(P)$ has an ONB of eigenvectors for A associated to $\lambda_1(A)\,, \, \dots \,, \lambda_k(A)$. If we further assume that $\lambda_k(A) > \lambda_{k+1}(A)$ then in both cases (2) and (4) the projection P is unique, since the eigenvectors associated to the first k eigenvalues of A generate a unique subspace of $\mathbb{C}^n$.

2.1 (Submajorization). Given $x, y \in \mathbb{R}^n$ we say that x is *submajorized* by y, and write $x \prec_w y$, (see [3]) if

$$\sum_{i=1}^{k} x_i^{\downarrow} \le \sum_{i=1}^{k} y_i^{\downarrow} \quad \text{for every} \quad k \in \mathbb{I}_n \, .$$

If also $\operatorname{tr} x = \sum\limits_{i=1}^{n} x_i = \sum\limits_{i=1}^{n} y_i = \operatorname{tr} y$, then we say that x is *majorized* by y, and write $x \prec y$. On the other hand we write $x \leqslant y$ if $x_i \le y_i$ for every $i \in \mathbb{I}_n$. It is a standard exercise to show that $x \leqslant y \implies x^{\downarrow} \leqslant y^{\downarrow} \implies x \prec_w y$. Majorization is usually considered because of its relation with tracial inequalities for convex functions.

Indeed, given $x, y \in \mathbb{R}^n$ and $f : I \to \mathbb{R}$ a convex fuction defined on an interval $I \subseteq \mathbb{R}$ such that $x, y \in I^n$, then (see for example [3]):

1. If one assumes that $x \prec y$, then

$$\operatorname{tr} f(x) \overset{\text{def}}{=} \sum_{i=1}^{n} f(x_i) \le \sum_{i=1}^{n} f(y_i) = \operatorname{tr} f(y) \ .$$

2. If just $x \prec_w y$, but the map f is also increasing, then still $\operatorname{tr} f(x) \le \operatorname{tr} f(y)$. $\triangle$

2.3 Basic framework of finite frames and their dual frames

In what follows we consider (n,d)-frames. See [2], [4], [5], [6], [18] and [22] for detailed expositions of this notion.

Let $d, n \in \mathbb{N}$, with $d \leq n$. Fix a Hilbert space $\mathcal{H} \cong \mathbb{C}^d$. A family $\mathcal{F} = \{f_i\}_{i\in \mathbb{I}_n} \in \mathcal{H}^n$ is an (n,d)-frame for $\mathcal{H}$ if there exist constants $A, B > 0$ such that

$$A\|x\|^2 \leq \sum_{i=1}^{n} |\langle x\,,\,f_i\rangle|^2 \leq B\|x\|^2 \quad \text{for every} \quad x \in \mathcal{H}\ . \tag{5}$$

The **frame bounds**, denoted by $A_\mathcal{F}, B_\mathcal{F}$ are the optimal constants in (5). Tight frames are those which have $A_\mathcal{F} = B_\mathcal{F}$. Since $\dim \mathcal{H} < \infty$, a system $\mathcal{F} = \{f_i\}_{i\in \mathbb{I}_n}$ is an (n,d)-frame if and only if $\operatorname{span}\{f_i : i \in \mathbb{I}_n\} = \mathcal{H}$. We shall denote by $\mathbf{F} = \mathbf{F}(n,d)$ the set of all (n,d)-frames for $\mathcal{H}$.

Given $\mathcal{F} = \{f_i\}_{i\in \mathbb{I}_n} \in \mathcal{H}^n$, the operator $T_\mathcal{F} \in L(\mathcal{H}\,,\,\mathbb{C}^n)$ given by

$$T_\mathcal{F}\,x = \big(\,\langle x\,,\,f_i\rangle\,\big)_{i\in\mathbb{I}_n}\ , \quad \text{for every} \quad x \in \mathcal{H} \tag{6}$$

is the **analysis** operator of $\mathcal{F}$. Its adjoint $T^*_\mathcal{F}$ is called the **synthesis** operator:

$$T^*_\mathcal{F} \in L(\mathbb{C}^n\,,\,\mathcal{H}) \quad \text{given by} \quad T^*_\mathcal{F}\,v = \sum_{i\in\,\mathbb{I}_m} v_i\,f_i \quad \text{for every} \quad v = (v_1\,,\,\dots\,,\,v_n) \in \mathbb{C}^n\ .$$

Finally, we define the **frame operator** of $\mathcal{F}$ as $S_\mathcal{F} = T^*_\mathcal{F}\,T_\mathcal{F} \in L(\mathcal{H})^+$. Notice that, if $\mathcal{F} \in \mathbf{F}(n,d)$, then $\langle S_\mathcal{F}\,x\,,\,x\rangle = \sum_{i\in\mathbb{I}_n} \big|\langle x\,,\,f_i\rangle\big|^2$ for every $x \in \mathcal{H}$. Then $S_\mathcal{F} \in \mathcal{G}l(\mathcal{H})^+$ and

$$A_\mathcal{F}\,\|x\|^2 \,\leq\, \langle S_\mathcal{F}\,x\,,\,x\rangle \,\leq\, B_\mathcal{F}\,\|x\|^2 \quad \text{for every} \quad x \in \mathcal{H}\ . \tag{7}$$

In particular, $A_\mathcal{F} = \lambda_{\min}(S_\mathcal{F}) = \|S_\mathcal{F}^{-1}\|^{-1}$ and $\lambda_{\max}(S_\mathcal{F}) = \|S_\mathcal{F}\| = B_\mathcal{F}$. Moreover, $\mathcal{F}$ is tight if and only if $S_\mathcal{F} = \frac{\tau}{d}\,I_\mathcal{H}$, where $\tau = \operatorname{tr} S_\mathcal{F} = \sum_{i\in\mathbb{I}_n} \|f_i\|^2$. The frame operator plays an important role in the reconstruction of a vector x using its frame coefficients $\{\langle x\,,\,f_i\rangle\}_{i\in\mathbb{I}_n}$. This leads to the definiton of the canonical dual frame associated to $\mathcal{F}$:

Definition 2.2. For every $\mathcal{F} = \{f_i\}_{i\in \mathbb{I}_n} \in \mathbf{F}(n,d)$, the **canonical dual** frame associated to $\mathcal{F}$ is the sequence $\mathcal{F}^\# \in \mathbf{F}$ defined by

$$\mathcal{F}^\# \overset{\text{def}}{=} S_\mathcal{F}^{-1}\cdot\mathcal{F} = \{S_\mathcal{F}^{-1}\,f_i\}_{i\in\,\mathbb{I}_m} \in \mathbf{F}(n,d)\ .$$

Therefore, we obtain the reconstruction formulas

$$x = \sum_{i\in\mathbb{I}_n} \langle x\,,\,f_i\rangle\,S_\mathcal{F}^{-1}\,f_i = \sum_{i\in\mathbb{I}_n} \langle S_\mathcal{F}^{-1}\,x\,,\,f_i\rangle\,f_i \quad \text{for every} \quad x \in \mathcal{H}\ . \tag{8}$$

Observe that the canonical dual $\mathcal{F}^\#$ satisfies that given $x \in \mathcal{H}$, then

$$T_{\mathcal{F}^\#}\,x = \big(\,\langle x\,,\,S_\mathcal{F}^{-1}\,f_i\rangle\,\big)_{i\in\mathbb{I}_n} = \big(\,\langle S_\mathcal{F}^{-1}\,x\,,\,f_i\rangle\,\big)_{i\in\mathbb{I}_n} \quad \text{for} \quad x \in \mathcal{H} \implies T_{\mathcal{F}^\#} = T_\mathcal{F}\,S_\mathcal{F}^{-1}\ . \tag{9}$$

Hence $T^*_{\mathcal{F}^\#}\,T_\mathcal{F} = I_\mathcal{H}$ and $S_{\mathcal{F}^\#} = S_\mathcal{F}^{-1}\,T^*_\mathcal{F}\,T_\mathcal{F}\,S_\mathcal{F}^{-1} = S_\mathcal{F}^{-1}$. △

Next we recall the more general notion of (alternate) dual frames:

Definition 2.3. Let $\mathcal{F} = \{f_i\}_{i\in \mathbb{I}_n} \in \mathbf{F}(n,d)$.

1. We say that $\mathcal{G} = \{g_i\}_{i\in \mathbb{I}_n} \in \mathbf{F}(n,d)$ is a **dual** frame for $\mathcal{F}$ if $T^*_\mathcal{G}\,T_\mathcal{F} = I_\mathcal{H}$, or equivalently if $x = \sum_{i\in\mathbb{I}_n} \langle x\,,\,f_i\rangle\,g_i$ for every $x \in \mathcal{H}$.

2. We denote by $\mathcal{D}(\mathcal{F}) \stackrel{\text{def}}{=} \{\mathcal{G} \in \mathbf{F}(n,d) : T_\mathcal{G}^* T_\mathcal{F} = I_\mathcal{H}\}$, the set of all dual frames for $\mathcal{F}$. Observe that $\mathcal{D}(\mathcal{F}) \neq \varnothing$ since $\mathcal{F}^\# \in \mathcal{D}(\mathcal{F})$. △

Remark 2.4. Let $\mathcal{F} = \{f_i\}_{i\in\mathbb{I}_n} \in \mathcal{H}^n$. Then $\mathcal{F} \in \mathbf{F} \iff T_\mathcal{F}^*$ is surjective. In this case, a sequence $\mathcal{G} \in \mathcal{D}(\mathcal{F})$ if and only if its synthesis operator $T_\mathcal{G}^*$ is a pseudo-inverse of $T_\mathcal{F}$. Indeed, $\mathcal{G} \in \mathcal{D}(\mathcal{F}) \iff T_\mathcal{G}^* T_\mathcal{F} = I_\mathcal{H}$. Observe that the map $\mathbf{F} \ni \mathcal{G} \mapsto T_\mathcal{G}^*$ is one to one.

Moreover, the synthesis operator $T_{\mathcal{F}^\#}^*$ of the canonical dual $\mathcal{F}^\#$ corresponds to the Moore-Penrose pseudo-inverse of $T_\mathcal{F}$. Indeed, notice that $T_\mathcal{F} T_{\mathcal{F}^\#}^* = T_\mathcal{F} S_\mathcal{F}^{-1} T_\mathcal{F}^* \in L(\mathbb{C}^n)^+$, so that it is an orthogonal projection. From this point of view, the canonical dual $\mathcal{F}^\#$ has some optimal properties that come from the theory of pseudo-inverses. On the other hand the map $\mathcal{H}^n \ni \mathcal{G} \mapsto T_\mathcal{G}^* \in L(\mathcal{K}, \mathcal{H})$ is $\mathbb{R}$-linear. Then, for every $\mathcal{F} \in \mathbf{F}$, the set $\mathcal{D}(\mathcal{F})$ of dual frames is convex in $\mathcal{H}^n$ because the set of pseudoinverses of $T_\mathcal{F}$ is convex in $L(\mathcal{K}, \mathcal{H})$. △

Definition 2.5. Let $\mathcal{F} \in \mathbf{F}(n,d)$. We denote by

$$\mathcal{SD}(\mathcal{F}) = \{S_\mathcal{G} : \mathcal{G} \in \mathcal{D}(\mathcal{F})\}$$

the set of frame operators of all dual frames for $\mathcal{F}$. △

Proposition 2.6. *Let $\mathcal{F} \in \mathbf{F}(n,d)$. Then*

$$\mathcal{SD}(\mathcal{F}) = \{S \in \mathcal{G}l(d)^+ : X = S - S_\mathcal{F}^{-1} \geq 0 \ \text{ and } \ \operatorname{rk} X \leq n-d\} \ . \tag{10}$$

In particular, if $n \geq 2\,d$, then $\mathcal{SD}(\mathcal{F}) = \{S \in \mathcal{G}l(d)^+ : S \geq S_\mathcal{F}^{-1}\}$ which is a **convex** *set.*

Proof. Given $\mathcal{G} \in \mathbf{F}(n,d)$, then $\mathcal{G} \in \mathcal{D}(\mathcal{F}) \iff Z = T_\mathcal{G} - T_{\mathcal{F}^\#} \in L(\mathcal{H}, \mathbb{C}^n)$ satisfies $Z^* T_\mathcal{F} = 0$. In this case, by Eq. (9), we know that $T_{\mathcal{F}^\#} = T_\mathcal{F} S_\mathcal{F}^{-1} \implies Z^* T_{\mathcal{F}^\#} = 0$, and

$$S_\mathcal{G} = (T_{\mathcal{F}^\#} + Z)^* (T_{\mathcal{F}^\#} + Z) = S_{\mathcal{F}^\#} + X = S_\mathcal{F}^{-1} + X \ , \qquad \text{where} \qquad X = Z^* Z \in \mathcal{M}_d(\mathbb{C})^+ \ .$$

Moreover, $S_\mathcal{F} = T_\mathcal{F}^* T_\mathcal{F} \in \mathcal{G}l(d)^+ \implies \operatorname{rk} T_\mathcal{F} = d$, and the equation $T_\mathcal{F}^* Z = 0$ implies that

$$R(Z) \subseteq \ker T_\mathcal{F}^* = R(T_\mathcal{F})^\perp \implies \operatorname{rk} X = \operatorname{rk}(Z^* Z) = \operatorname{rk} Z \leq n-d \ .$$

Since any $X \in \mathcal{M}_d(\mathbb{C})^+$ with $\operatorname{rk} X \leq n-d$ can be represented as $X = Z^* Z$ for some $Z \in L(\mathcal{H}, R(T_\mathcal{F})^\perp)$, we have proved Eq. (10). The case $n \geq 2\,d$ is now apparent. □

3 Spectral picture of $\mathcal{D}(\mathcal{F})$

Recall that $\mathbb{R}_+^{d\,\downarrow}$ is the set of vectors $\mu \in \mathbb{R}_+^d$ with non negative and non-increasing entries (i.e. $\mu \in \mathbb{R}_+^d$ with $\mu^\downarrow = \mu$). If all the entries are positive (i.e., if $\mu_d > 0$), we write $\mu \in \mathbb{R}_{>0}^{d\,\downarrow}$. Given $S \in \mathcal{M}_d(\mathbb{C})^+$, we write $\lambda(S) \in \mathbb{R}_+^{d\,\downarrow}$ the decreasing vector of eigenvalues of S, counting multiplicities. We denote by $S^\dagger$ the Moore-Penrose pseudo-inverse of S. We shall also use the following notations:

1. Given $x \in \mathbb{C}^d$ then $D(x) \in \mathcal{M}_d(\mathbb{C})$ denotes the diagonal matrix with main diagonal x.

2. If $d \leq n$ and $y \in \mathbb{C}^d$, we write $(y\,,\,0_{n-d}) \in \mathbb{C}^n$, where 0_{n-d} is the zero vector of $\mathbb{C}^{n-d}$. In this case, we denote by $D_n(y) = D\big((y\,,\,0_{n-d})\big) \in \mathcal{M}_n(\mathbb{C})$.

Definition 3.1. Let $\mathcal{F} \in \mathbf{F}(n,d)$. We denote by

$$\Lambda(\mathcal{D}(\mathcal{F})) \stackrel{\text{def}}{=} \{\lambda(S_\mathcal{G}) : \ \mathcal{G} \in \mathcal{D}(\mathcal{F})\} \subseteq \mathbb{R}_{>0}^{d\,\downarrow} \ , \tag{11}$$

that is, the spectral picture of the set of frame operators of all dual frames for $\mathcal{F}$. △

The following result gives a characterization of $\Lambda(\mathcal{D}(\mathcal{F})\,)$.

Theorem 3.2. *Let* $\mathcal{F} = \{f_i\}_{i\in\mathbb{I}_n} \in \mathbf{F}(n,d)$ *and* $\mu \in \mathbb{R}_{>0}^{d\,\downarrow}$. *Then the following conditions are equivalent:*

1. *The vector* $\mu \in \Lambda(\mathcal{D}(\mathcal{F})\,)$.
2. $\mu = \lambda(S_{\mathcal{F}}^{-1} + X)$ *for some* $X \in \mathcal{M}_d(\mathbb{C})^+$ *with* $\operatorname{rk} X \le n-d$.
3. *There exists an orthogonal projection* $P \in \mathcal{M}_n(\mathbb{C})$ *such that* $\operatorname{rk} P = d$ *and*
$$\lambda\,(P\,D_n(\mu)\,P) = \big(\,\lambda(S_{\mathcal{F}}^{-1})\,,\,0_{n-d}\,\big) = \lambda(G_{\mathcal{F}}^{\dagger})\ , \tag{12}$$
where $G_{\mathcal{F}} = T_{\mathcal{F}}\,T_{\mathcal{F}}^* \in \mathcal{M}_n(\mathbb{C})^+$ *is the Gram matrix of* $\mathcal{F}$.

Proof. The equivalence $1 \Leftrightarrow 2$ follows from Proposition 2.6.

$1 \Rightarrow 3$. Let $\mathcal{G} \in \mathcal{D}(\mathcal{F})$ with $\lambda(S_{\mathcal{G}}) = \mu$. Then $T_{\mathcal{G}}^*\,T_{\mathcal{F}} = I$ and
$$G_{\mathcal{F}}\,G_{\mathcal{G}}\,G_{\mathcal{F}} = T_{\mathcal{F}}\,(T_{\mathcal{F}}^*\,T_{\mathcal{G}})\,(T_{\mathcal{G}}^*\,T_{\mathcal{F}})\,T_{\mathcal{F}}^* = T_{\mathcal{F}}\,T_{\mathcal{F}}^* = G_{\mathcal{F}} \implies Q\,G_{\mathcal{G}}\,Q = G_{\mathcal{F}}^{\dagger}\ , \tag{13}$$
where $Q = G_{\mathcal{F}}\,G_{\mathcal{F}}^{\dagger} = P_{R(T_{\mathcal{F}})}$. Note that $\operatorname{rk} Q = \operatorname{rk} T_{\mathcal{F}} = d$, since $\mathcal{F}$ is a frame. Also
$$\lambda(G_{\mathcal{G}}) = \lambda(T_{\mathcal{G}}\,T_{\mathcal{G}}^*) = (\lambda(T_{\mathcal{G}}^*\,T_{\mathcal{G}}), 0_{n-d}) = (\lambda(S_{\mathcal{G}}), 0_{n-d}) = (\mu\,,\,0_{n-d})\ .$$
Then there exists $U \in \mathcal{U}(n)$ such that
$$U^*\,D(\mu\,,\,0_{n-d})\,U = U^*\,D_n(\mu)\,U = U^*\ D_n\big(\,\lambda(S_{\mathcal{G}})\,\big)\ U = T_{\mathcal{G}}\,T_{\mathcal{G}}^* = G_{\mathcal{G}}\ . \tag{14}$$
Let $P = U\,Q\,U^*$. Note that $\operatorname{rk} P = \operatorname{rk} Q = d$. Using (13) and (14) we get the item 3 :
$$\lambda\,(P\,D_n(\mu)\,P) = \lambda(U\,Q\,U^*\,D_n(\mu)\,U\,Q\,U^*) \overset{(14)}{=} \lambda(Q\,G_{\mathcal{G}}\,Q) \overset{(13)}{=} \lambda(G_{\mathcal{F}}^{\dagger}) = (\lambda(S_{\mathcal{F}}^{-1})\,,\,0_{n-d})\ .$$
$3 \Rightarrow 1$. Assume that there exists the projection $P \in \mathcal{M}_n(\mathbb{C})^+$ of item 3. Observe that there always exists $\mathcal{U} \in \mathbf{F}(n,d)$ such that $\lambda(S_{\mathcal{U}}) = \lambda(T_{\mathcal{U}}^*\,T_{\mathcal{U}}) = \mu$. Then
$$\lambda(G_{\mathcal{U}}) = \lambda(T_{\mathcal{U}}\,T_{\mathcal{U}}^*) = (\mu\,,\,0_{n-d}) \in \mathbb{R}_+^{n\,\downarrow}\ .$$
Let $V \in \mathcal{U}(n)$ such that $V^*\,G_{\mathcal{U}}\,V = D_n(\mu)$. Denote by $Q = VPV^*$. Then we get that
$$\lambda(Q\,G_{\mathcal{U}}\,Q) = \lambda(P\,V^*\,G_{\mathcal{U}}\,V\,P) = \lambda\,(P\,D_n(\mu)\,P) \overset{(12)}{=} (\lambda(S_{\mathcal{F}}^{-1}), 0_{n-d}) = \lambda(G_{\mathcal{F}}^{\dagger})\ . \tag{15}$$
Thus, there exists $W \in \mathcal{U}(n)$ such that $W^*\,(Q\,G_{\mathcal{U}}\,Q)\,W = G_{\mathcal{F}}^{\dagger}$. Observe that
$$\operatorname{rk} Q = d \quad \text{and} \quad W^*(R(Q)\,) \supseteq R(G_{\mathcal{F}}^{\dagger}) = R(G_{\mathcal{F}}) = R(T_{\mathcal{F}}) \implies W^*QW = P_{R(T_{\mathcal{F}})}\ .$$
Moreover, $G_{\mathcal{F}}\,G_{\mathcal{F}}^{\dagger} = G_{\mathcal{F}}^{\dagger}\,G_{\mathcal{F}} = P_{R(G_{\mathcal{F}})} = P_{R(T_{\mathcal{F}})} = W^*QW$. Hence
$$\begin{aligned} G_{\mathcal{F}} &= G_{\mathcal{F}}\,G_{\mathcal{F}}^{\dagger}\,G_{\mathcal{F}} = G_{\mathcal{F}}\,(W^*\,Q\,G_{\mathcal{U}}\,Q\,W)\,G_{\mathcal{F}} \\ &= G_{\mathcal{F}}\,P_{R(G_{\mathcal{F}})}\,(W^*\,G_{\mathcal{U}}\,W)\,P_{R(G_{\mathcal{F}})}\,G_{\mathcal{F}} = G_{\mathcal{F}}\,(W^*\,G_{\mathcal{U}}\,W)\,G_{\mathcal{F}}\ . \end{aligned}$$
We can rewrite this fact as $T_{\mathcal{F}}\big(\,T_{\mathcal{F}}^*\,W^*T_{\mathcal{U}}\,T_{\mathcal{U}}^*\,W\,T_{\mathcal{F}}\,\big)\,T_{\mathcal{F}}^* = T_{\mathcal{F}}\,T_{\mathcal{F}}^*$. Since $T_{\mathcal{F}}^*$ is surjective,
$$(T_{\mathcal{F}}^*\,W^*T_{\mathcal{U}})\,(T_{\mathcal{U}}^*\,W\,T_{\mathcal{F}}) = I_{\mathcal{H}} \implies M \overset{\text{def}}{=} T_{\mathcal{U}}^*\,W\,T_{\mathcal{F}} \in \mathcal{U}(d)\ . \tag{16}$$
Finally, take $\mathcal{G} = \{g_i\}_{i\in\mathbb{I}_n} \overset{\text{def}}{=} \{M^*\,T_{\mathcal{U}}^*\,W\,e_i\}_{i\in\mathbb{I}_m} \in L(n,d)$, where $\{e_i\}_{i\in\mathbb{I}_n}$ is the canonical basis of $\mathbb{C}^n$. Observe that, given $v = (v_i)_{i\in\mathbb{I}_n} \in \mathbb{C}^n$, we have that
$$T_{\mathcal{G}}^*\,v = \sum_{i\in\mathbb{I}_n} v_i\,g_i = M^*\,T_{\mathcal{U}}^*\,W \sum_{i\in\mathbb{I}_n} v_i\,e_i = M^*\,T_{\mathcal{U}}^*\,W\,v \implies T_{\mathcal{G}}^* = M^*\,T_{\mathcal{U}}^*\,W$$
Then $T_{\mathcal{G}}^*\,T_{\mathcal{F}} = M^*\,T_{\mathcal{U}}^*\,W\,T_{\mathcal{F}} = M^*\,M = I_{\mathcal{H}}$. Also $S_{\mathcal{G}} = T_{\mathcal{G}}^*\,T_{\mathcal{G}} = M^*\,S_{\mathcal{U}}\,M \in \mathcal{G}l(d)^+$. This last fact assures that $\mathcal{G} \in \mathbf{F}(n,d)$ and $\lambda(S_{\mathcal{G}}) = \lambda(S_{\mathcal{U}}) = \mu$. $\square$

Remark 3.3. Let $\mathcal{F} \in \mathbf{F}(n,d)$ and $\mu \in \mathbb{R}^d_{>0}{}^{\downarrow}$ as in Theorem 3.2. It turns out that condition (12) can be characterized in terms of interlacing inequalities.

More explicitly, given $\mu \in \mathbb{R}^d_{>0}{}^{\downarrow}$, by the Fan-Pall inequalities (see [20]), the existence of a projection P satisfying (12) for μ is equivalent to the following inequalities:

1. $\mu \geqslant \lambda(S_{\mathcal{F}}^{-1})$, i.e. $\mu_i \geq \lambda_i(S_{\mathcal{F}}^{-1})$ for every $i \in \mathbb{I}_d$.
2. If $n < 2\,d$ and we denote $m = 2\,d - n \in \mathbb{N}$, then μ also satisfies

$$\mu_{d-m+i} \leq \lambda_i(S_{\mathcal{F}}^{-1}) \quad \text{for every} \quad i \in \mathbb{I}_m \ ,$$

   where the last inequalities compare the first m entries of $\lambda(S_{\mathcal{F}}^{-1})$ with the last m of μ.

These facts together with Theorem 3.2 give a complete description of the spectral picture of the frame operators $S_{\mathcal{G}}$ for every $\mathcal{G} \in \mathcal{D}(\mathcal{F})$, which we write as follows. △

Recall that, if $x,\, y \in \mathbb{R}^d$, we write $x \leqslant y$ if $x_i \leq y_i$ for every $i \in \mathbb{I}_d$.

Corollary 3.4. *Let $\mathcal{F} = \{f_i\}_{i\in \mathbb{I}_n} \in \mathbf{F}(n,d)$ and fix $\mu \in \mathbb{R}^d_{>0}{}^{\downarrow}$. Then, the set $\Lambda(\mathcal{D}(\mathcal{F})\,)$ can be characterized as follows:*

1. *If $n \geq 2\,d$, we have that*
$$\mu \in \Lambda(\mathcal{D}(\mathcal{F})\,) \iff \mu \geqslant \lambda(S_{\mathcal{F}}^{-1}) \ . \tag{17}$$
2. *If $2\,d > n$ and $m = 2\,d - n$, then*
$$\mu \in \Lambda(\mathcal{D}(\mathcal{F})\,) \iff \mu \geqslant \lambda(S_{\mathcal{F}}^{-1}) \quad \textit{and} \quad \mu_{d-m+i} \leq \lambda_i(S_{\mathcal{F}}^{-1}) \quad \textit{for} \quad i \in \mathbb{I}_m \ . \tag{18}$$

Proof. It is a direct consequence of Theorem 3.2 and the Fan-Pall inequalities described in Remark 3.3. □

Corollary 3.5. *Let $\mathcal{F} \in \mathbf{F}(n,d)$. Then $\Lambda(\mathcal{D}(\mathcal{F})\,)$ is a convex set.*

Proof. It is clear that the inequalities given in Eqs. (17) and (18) are preserved by convex combinations. Observe that also the set $\mathbb{R}^d_{>0}{}^{\downarrow}$ is convex. □

Remark 3.6. Let $\mathcal{F} = \{f_i\}_{i\in \mathbb{I}_n}$, $\mathcal{G} = \{g_i\}_{i\in \mathbb{I}_n} \in \mathbf{F}(n,d)$. The reader should note that the fact that $\lambda(S_{\mathcal{G}}) \in \Lambda(\mathcal{D}(\mathcal{F}))$ does not imply that $\mathcal{G} \in \mathcal{D}(\mathcal{F})$. Indeed, it is fairly easy to produce examples of this phenomenon. Therefore, the spectral picture of $\Lambda(\mathcal{D}(\mathcal{F}))$ does not determine the set $\mathcal{D}(\mathcal{F})$. This last assertion is a consequence of the fact that $\mathcal{SD}(\mathcal{F})$ is not saturated by unitary equivalence. Nevertheless, $\Lambda(\mathcal{D}(\mathcal{F}))$ allow to compute *global* minimizers of any continuous function of the eigenvalues of duals of $\mathcal{F}$.

4 Minimizers for submajorization 1: Vectors

The spectral pictures studied in the previous section motivates the definition of following sets.

Definition 4.1. Let $\lambda \in \mathbb{R}^d_{+}{}^{\downarrow}$ and take an integer $m < d$. We consider the set

$$\Lambda(\lambda\,,\,m) = \begin{cases} \{\mu \in \mathbb{R}^d_{+}{}^{\downarrow} :\ \mu \geqslant \lambda\,\} & \text{if } \ m \leq 0 \\ \\ \{\mu \in \Lambda(\lambda\,,\,0) : \mu_{d-m+i} \leq \lambda_i \quad \text{for every} \quad i \in \mathbb{I}_m\} & \text{if } \ m \geq 1 \ . \end{cases} \tag{19}$$

Denote by $t_0 = \operatorname{tr}\lambda$. For every and $t \geq t_0$, we also consider the set

$$\Lambda_t(\lambda\,,\,m) = \{\mu \in \Lambda(\lambda\,,\,m) :\ \operatorname{tr}\mu \geq t\} \ . \qquad \triangle$$

Now Corollary 3.4 can be rewritten as

Corollary 4.2. *Let $\mathcal{F} \in \mathbf{F}(n,d)$, $m = 2\,d - n$ and $\lambda = \lambda(S_{\mathcal{F}}^{-1})$. Then $\Lambda(\mathcal{D}(\mathcal{F})) = \Lambda(\lambda\,,\,m)$.* $\square$

Observe that if $m = d$ then $\Lambda(\lambda\,,\,m) = \{\lambda\}$. This condition corresponds with the case $d = n$, where the frame becomes a basis and has a unique dual frame. Since this case has no interest, we have assumed and we shall assume that $d < n$ and therefore $m < d$.

In this section we show that the sets $\Lambda_t(\lambda\,,\,m)$ have minimal elements with respect to submajorization and we describe explicitly these elements. We shall apply these results in the context optimal dual frames and optimal frame completions in Sections 6 and 7. We begin with the following notion of irregularity.

4.1 Irregularity

Definition 4.3. Let $\lambda \in \mathbb{R}_+^{d\,\downarrow}$ and $t \in \mathbb{R}$ such that $\operatorname{tr}\lambda \le t < d\,\lambda_1$. Consider the set

$$A_\lambda(t) \stackrel{\text{def}}{=} \Big\{\, r \in \mathbb{I}_{d-1} :\ p_\lambda(r\,,\,t) \stackrel{\text{def}}{=} \frac{t - \sum_{j=1}^{r} \lambda_j}{d-r} \ge \lambda_{r+1} \,\Big\}\ .$$

Observe that $t \ge \operatorname{tr}\lambda \implies t - \sum\limits_{j=1}^{d-1} \lambda_j \ge \lambda_d$, so that $d-1 \in A_\lambda(t) \neq \varnothing$. The t-irregularity of the ordered vector λ, denoted $r_\lambda(t)$, is defined by

$$r_\lambda(t) \stackrel{\text{def}}{=} \min\, A_\lambda(t) = \min\{r \in \mathbb{I}_{d-1} : p_\lambda(r\,,\,t) \ge \lambda_{r+1}\}\ . \tag{20}$$

If $t \ge d\,\lambda_1$, we set $r_\lambda(t) \stackrel{\text{def}}{=} 0$ and $p_\lambda(0\,,\,t) = t/d$. $\triangle$

For example, if $t = \operatorname{tr}\,\lambda$, then for every $r \in \mathbb{I}_{d-1}$ we have that

$$p_\lambda(r\,,\,t) = \frac{t - \sum_{j=1}^{r} \lambda_j}{d-r} = \frac{\sum_{j=r+1}^{d} \lambda_j}{d-r} \ge \lambda_{r+1} \iff \lambda_{r+1} = \lambda_d\ .$$

Therefore in this case it is easy to see that, if we denote by $t_0 = \operatorname{tr}\,\lambda$, then

- If $\lambda = c\,\mathbb{1}_d$ for some $c \in \mathbb{R}_{>0}$, then $r_\lambda(t_0) = 0$.
- If $\lambda_1 > \lambda_d$, then

$$r_\lambda(t_0) + 1 = \min\{i \in \mathbb{I}_d : \lambda_i = \lambda_d\} \quad \text{and} \quad r_\lambda(t_0) = \max\{r \in \mathbb{I}_{d-1} : \lambda_r > \lambda_d\}\ . \tag{21}$$

Definition 4.4. Let $\lambda \in \mathbb{R}_+^{d\,\downarrow}$ and $t_0 = \operatorname{tr}\lambda$. We define the functions

$$r_\lambda : [t_0\,,\,+\infty) \to \{0, \dots, d-1\} \quad \text{given by} \quad r_\lambda(s) \stackrel{(20)}{=} \text{the } s\text{-irregularity of } \lambda \tag{22}$$

$$c_\lambda : [t_0\,,\,+\infty) \to \mathbb{R}_{\ge 0} \quad \text{given by} \quad c_\lambda(s) = p_\lambda(r_\lambda(s)\,,\,s) = \frac{s - \sum_{i=1}^{r_\lambda(s)} \lambda_i}{d - r_\lambda(s)}\ , \tag{23}$$

for every $s \in [t_0\,,\,+\infty)$, where we set $\sum\limits_{i=1}^{0} \lambda_i = 0$. $\triangle$

In the following Lemma we state several properties of these maps, which we shall use below. The proofs are technical but elementary, so that we only sketch the essential arguments.

Lemma 4.5. *Let $\lambda \in \mathbb{R}_+^{d\,\downarrow}$ and $t_0 = \operatorname{tr}\lambda$.*

1. *The function r_λ is non-increasing and right-continuous, with $\lambda_{r_\lambda(t_0)+1} = \lambda_d$.*

2. *The image of r_λ is the set $\mathcal{B} = \{k \in \mathbb{I}_{d-1} : \lambda_k > \lambda_{k+1}\} \cup \{0\}$.*

3. *The map c_λ is piece-wise linear, strictly increasing and continuous.*

4. *It satisfies that $c_\lambda(t_0) = \lambda_d$ and $c_\lambda(t) = t/d$ for $t \geq d\,\lambda_1$.*

5. *For every $t \in [t_0\,,\, d\,\lambda_1)$, if $r = r_\lambda(t)$ then $\lambda_{r+1} \leq c_\lambda(t) < \lambda_r$.*

6. *For any $k \in \mathcal{B}$ let $s_k = \sum\limits_{i=1}^{k} \lambda_i + (d-k\,)\,\lambda_{k+1}$. Then $r_\lambda(s_k) = k$ and $c_\lambda(s_k) = \lambda_{k+1}$. Moreover, the set A of discontinuity points of r_λ satisfies that*

$$A = \{t \in (t_0\,,\,+\infty) :\; c_\lambda(t) = \lambda_{r_\lambda(t)+1}\} = c_\lambda^{-1}\{\lambda_i :\; \lambda_i \neq \lambda_d\} = \{s_k : k \in \mathcal{B}\}\ .$$

7. *Given $t \in [t_0\,,\,+\infty)$, such that $c_\lambda(t) = \lambda_m$ (even if $m \notin \mathcal{B}$), then*
   - $t \in A \iff \lambda_m \neq \lambda_d$.
   - $r_\lambda(t) = 0 \iff c_\lambda(t) = \lambda_1 \iff t = d\,\lambda_1$.
   - *If $\lambda_m \neq \lambda_1$, then $r_\lambda(t) = \max\{j \in \mathbb{I}_d : \lambda_j > \lambda_m\}$ and*

$$t = \sum_{i=1}^{m} \lambda_i + (d-m)\,\lambda_m = \sum_{i=1}^{r_\lambda(t)} \lambda_i + (d - r_\lambda(t)\,)\,\lambda_m\ . \tag{24}$$

Proof. Given $t \in [t_0\,,\,d\,\lambda_1)$ and $1 \leq r \leq d-1$, then $r = r_\lambda(t)$ if and only if

$$c_\lambda(t) = p_\lambda(r\,,\,t) \geq \lambda_{r+1} \qquad \text{and} \qquad p_\lambda(r-1\,,\,t) < \lambda_r\ . \tag{25}$$

On the other hand the map $t \mapsto p_\lambda(r,t)$ is linear, continuous and increasing for any r fixed. From these facts one easily deduces the right continuity of the map r_λ , and that the map c_λ is continuous at the points where r_λ is. We can also deduce that if $c_\lambda(t) \neq \lambda_{r_\lambda(t)+1}$ then r_λ is continuous (i.e. constant) near the point t. Observe that, if $r = r_\lambda(t)$, then

$$\lambda_r \overset{(25)}{>} p_\lambda(r-1\,,\,t) = \frac{(d-r)p_\lambda(r\,,\,t) + \lambda_r}{d-r+1} \implies \lambda_r > p_\lambda(r\,,\,t) \geq \lambda_{r+1} \implies r \in B\ . \tag{26}$$

Using that $r_\lambda(t) = 0$ for $t \geq d\,\lambda_1$, that $c_\lambda(t_0) = \lambda_d$, and the right continuity of the map r_λ , we have that the set $A = \{t \in (t_0\,,\,+\infty) :\; c_\lambda(t) = \lambda_{r_\lambda(t)+1}\} = c_\lambda^{-1}\{\lambda_i :\; \lambda_i \neq \lambda_d\}$.

Hence, in order to check the continuity of c_λ we have to verify the continuity of c_λ from the left at the points $t > t_0$ for which $c_\lambda(t) = \lambda_{r_\lambda(t)+1}$. Note that, if $r = r_\lambda(t)$, then $r \in B$ and

$$c_\lambda(t) = p_\lambda(r\,,\,t) = \frac{t - \sum_{j=1}^{r} \lambda_j}{d-r} = \lambda_{r+1} \implies t = \sum_{j=1}^{r} \lambda_j + (d-r)\lambda_{r+1}\ . \tag{27}$$

Assume that $c_\lambda(t) = \lambda_{r_\lambda(t)+1} > \lambda_d$. Then $\hat{r} = \max\{j \in \mathbb{I}_{d-1} :\; \lambda_j = \lambda_{r+1}\}$ is the first element of B after r. Note that $\lambda_{\hat{r}+1} < \lambda_{\hat{r}} = \lambda_{r+1}$. We shall see that if $s < t$ near t, then $r_\lambda(s) = \hat{r}$. Indeed, as in Eq. (27),

$$p_\lambda(\hat{r}\,,\,t+x) = \frac{(d-r)\lambda_{r+1} - \sum_{j=r+1}^{\hat{r}} \lambda_j + x}{d-\hat{r}} = \lambda_{r+1} + \frac{x}{d-\hat{r}} > \lambda_{\hat{r}+1} \quad \text{and}$$

$$p_\lambda(\hat{r}-1\,,\,t+x) = \frac{(d-r)\lambda_{r+1} - \sum_{j=r+1}^{\hat{r}-1} \lambda_j + x}{d-\hat{r}+1} = \lambda_{r+1} + \frac{x}{d-\hat{r}+1} < \lambda_{r+1} = \lambda_{\hat{r}}\ .$$

for $x \in (-\varepsilon, 0]$ if $\varepsilon > 0$ sufficiently small. By Eq. (25) we deduce that $r_\lambda(t+x) = \hat{r} \neq r_\lambda(t)$ for such an x, so that $t \in A$ (r_λ is discontinuous at t). On the other hand,

$$c_\lambda(t+x) = p_\lambda(\hat{r}, t+x) = \lambda_{r_\lambda(t)+1} + \frac{x}{d-\hat{r}} \implies \lim_{x\to 0^-} c_\lambda(t+x) = \lambda_{r_\lambda(t)+1} = c_\lambda(t) \ .$$

This last fact implies that c_λ is continuous and, since r_λ is right-continuous, that c_λ is a piece-wise linear and strictly increasing function. With the previous remarks, the proof of all other statements of the lemma becomes now straightforward. $\square$

4.2 Minimizers for submajorization in $\Lambda_t(\lambda\,,\,m)$ for $m \leq 0$.

The following Lemma is a standard fact in majorization theory, and it was already stated in [24]. We include a short proof of it for the sake of completeness.

Lemma 4.6. Let $\alpha\,,\,\gamma \in \mathbb{R}^n$, $\beta \in \mathbb{R}^m$ and $b \in \mathbb{R}$ such that $b \leq \min_{k\in\mathbb{I}_n} \gamma_k$. Then, if

$$\operatorname{tr}(\gamma\,,\,b\,\mathbb{1}_m) \leq \operatorname{tr}(\alpha\,,\,\beta) \quad \text{ and } \quad \gamma \prec_w \alpha \implies (\gamma\,,\,b\,\mathbb{1}_m) \prec_w (\alpha\,,\,\beta) \ .$$

Observe that we are not assuming that $(\alpha\,,\,\beta) = (\alpha\,,\,\beta)^\downarrow$.

Proof. Let $h = \operatorname{tr}\beta$ and $\rho = \frac{h}{m}\,\mathbb{1}_m$. Then it is easy to see that

$$\sum_{i\in\mathbb{I}_k} (\gamma^\downarrow\,,\,b\,\mathbb{1}_m)_i \leq \sum_{i\in\mathbb{I}_k} (\alpha^\downarrow\,,\,\rho)_i \leq \sum_{i\in\mathbb{I}_k} (\alpha^\downarrow\,,\,\beta^\downarrow)_i \quad \text{for every} \quad k \in \mathbb{I}_{n+m} \ .$$

Since $(\gamma^\downarrow\,,\,b\,\mathbb{1}_m) = (\gamma\,,\,b\,\mathbb{1}_m)^\downarrow$, we can conclude that $(\gamma\,,\,b\,\mathbb{1}_m) \prec_w (\alpha\,,\,\beta)$. $\square$

In the following statement we shall use the maps r_λ and c_λ defined in 4.4.

Theorem 4.7. *Fix $m \leq 0$. Let $\lambda \in \mathbb{R}_+^{d\,\downarrow}$, $t_0 = \operatorname{tr}\lambda$ and $t \in [t_0\,,\,+\infty)$. Consider the vector*

$$\nu = \nu_\lambda(t) \stackrel{def}{=} \big(\lambda_1\,,\,\ldots\,,\,\lambda_{r_\lambda(t)}\,,\,c_\lambda(t)\,,\,\ldots\,,\,c_\lambda(t)\big) \quad \textit{if} \quad r_\lambda(t) > 0 \ , \tag{28}$$

or $\nu = \frac{t}{d}\,\mathbb{1}_d = c_t(\lambda)\,\mathbb{1}_d \in \Lambda_t(\lambda\,,\,m)$ if $r_\lambda(t) = 0$. Then ν satisfies that

$$\nu \in \Lambda_t(\lambda\,,\,m) \ , \quad \operatorname{tr}\nu = t \quad \textit{and} \quad \nu \prec_w \mu \quad \textit{for every} \quad \mu \in \Lambda_t(\lambda\,,\,m) \ . \tag{29}$$

Proof. Given $t \in [t_0\,,\,+\infty)$, we denote by $r = r_\lambda(t)$. If $r = 0$ then,

$$t \geq d\,\lambda_1 \quad \text{ and } \quad \lambda = \lambda^\downarrow \implies c_\lambda(t) = \frac{t}{d} \geq \lambda_1 \implies \nu = c\,\mathbb{1}_d \in \Lambda_t(\lambda\,,\,m) \ .$$

It is clear that such a vector must satisfy that $\nu \prec_w \mu$ for every $\mu \in \Lambda_t(\lambda\,,\,m)$.

Suppose now that $r \geq 1$, so that $t < d\,\lambda_1$. Recall from Lemma 4.5 that in this case we have that $\lambda_{r+1} \leq c_\lambda(t) < \lambda_r$. Hence $\nu \geqslant \lambda$ and $\nu = \nu^\downarrow$. It is clear from Eq. (23) that $\operatorname{tr}(\nu) = t$. From these facts we can conclude that $\nu \in \Lambda_t(\lambda\,,\,m)$ as claimed.

Now let $\mu \in \Lambda_t(\lambda\,,\,m)$ and notice that, since $\mu \geqslant \lambda$, we get that

$$\sum_{i=1}^k \mu_i \geq \sum_{i=1}^k \lambda_i = \sum_{i=1}^k \nu_i \quad \text{for every} \quad 1 \leq k \leq r_\lambda(t) \ .$$

Now we can apply Lemma 4.6 (with $n = r_\lambda(t)$ and $b = c_\lambda(t)$) and deduce that $\nu \prec_w \mu$. $\square$

4.3 Minimizers for submajorization in $\Lambda_t(\lambda\,,\,m)$. The general case.

Recall that $\Lambda_t(\lambda\,,\,m) = \big\{\mu \in \mathbb{R}_+^{d\,\downarrow} : \ \mu \succcurlyeq \lambda\,,\ \operatorname{tr}\mu \geq t \text{ and } \mu_{d-m+i} \leq \lambda_i \text{ for every } i \in \mathbb{I}_m\big\}$, for each $m \in \mathbb{I}_{d-1}$. In what follows we shall compute a minimal element in $\Lambda_t(\lambda\,,\,m)$ with respect to submajorization in terms of the maps r_λ and c_λ defined in 4.4.

Definition 4.8. Let $\lambda \in \mathbb{R}_+^{d\,\downarrow}$, and $m \in \mathbb{I}_{d-1}$. We denote by $s^* = s^*(\lambda\,,\,m) \overset{\text{def}}{=} c_\lambda^{-1}(\lambda_m)$, the unique $s \in [t_0\,,\,+\infty)$ such that $c_\lambda(s) = \lambda_m$. Observe that

$$s^* = d\,\lambda_1 \quad \text{if } \ \lambda_m = \lambda_1\ , \qquad \text{and} \qquad s^* = \sum_{i=1}^{m} \lambda_i + (d-m)\,\lambda_m \quad \text{if} \quad \lambda_m \neq \lambda_1\ , \tag{30}$$

by Lemma 4.5. △

Proposition 4.9. *Let $\lambda \in \mathbb{R}_+^{d\,\downarrow}$, $t_0 = \operatorname{tr}\lambda$, $m \in \mathbb{I}_d$. If $t \in [t_0\,,\,s^*(\lambda\,,\,m)]$, then the vector $\nu = \big(\lambda_1\,,\,\dots\,,\,\lambda_{r_\lambda(t)}\,,\,c_\lambda(t)\,,\,\dots\,,\,c_\lambda(t)\big)$ of Eq. (28) satisfies that $\nu \in \Lambda_t(\lambda\,,\,m)$. Hence*

$$\operatorname{tr}\nu = t\ , \quad \nu_d = c_\lambda(t) \quad \textit{and} \quad \nu \prec_w \mu \quad \textit{for every} \quad \mu \in \Lambda_t(\lambda\,,\,m)\ .$$

Proof. We already know by Theorem 4.7 that $\nu \in \Lambda_t(\lambda)$ and $\operatorname{tr}\nu = t$. Using the inequality $c_\lambda(t) \leq c_\lambda(s^*) = \lambda_m$, the verification of the fact that $\nu \in \Lambda_t(\lambda\,,\,m)$ is direct. By Theorem 4.7, we conclude that $\nu \prec_w \mu$ for every $\mu \in \Lambda_t(\lambda\,,\,m) \subseteq \Lambda_t(\lambda)$. □

Remark 4.10. Let $\lambda \in \mathbb{R}_+^{d\,\downarrow}$, $t_0 = \operatorname{tr}\lambda$, $m \in \mathbb{I}_d$ and $t \in (s^*(\lambda\,,\,m)\,,\,+\infty)$. Then, arguing by induction on m we can show that there exists a unique vector $\nu \in \Lambda_t(\lambda\,,\,m)$ such that

$$\operatorname{tr}(\nu) = t\ \ , \ \ \nu_d = \lambda_m \quad \text{and} \quad \nu \prec_w \mu \quad \text{for every} \quad \mu \in \Lambda_t(\lambda\,,\,m)\ . \tag{31}$$

Indeed, if $m = 1$, let $c = \frac{t-\lambda_1}{d-1}$ and define $\nu = (c\,\mathbb{1}_{d-1}\,,\,\lambda_1) \in \mathbb{R}_{\geq 0}^d$. Then it can be checked that (31) holds in this case.

If we now assume that $m > 1$ and that our claim holds for all smaller values of m we define the new parameters

$$\lambda' = (\lambda_1, \dots, \lambda_{d-1}) \in \mathbb{R}_+^{d-1\,\downarrow}\ , \quad t' = t - \lambda_m \quad \text{and} \quad m' = m-1 \in \mathbb{I}_{d-1}\ . \tag{32}$$

By the inductive hypothesis and Proposition 4.9, there exists $\nu' \in \Lambda_{t'}(\lambda'\,,\,m')$ such that $\operatorname{tr}\nu' = t'$ and $\nu' \prec_w \mu'$ for every $\mu' \in \Lambda_{t'}(\lambda')$. Let

$$\nu = (\nu'_1\,,\,\dots\,,\,\nu'_{d-1}\,,\,\lambda_m) \in \mathbb{R}_{\geq 0}^d\ .$$

Using that $\nu' \in \Lambda_{t'}(\lambda'\,,\,m')$ and $\nu_d = \lambda_m$, it is clear that $\nu = \nu^{\downarrow} \succcurlyeq \lambda$, and that $\nu_{d-m+i} \leq \lambda_i$ for every $1 \leq i \leq m$. Since $\operatorname{tr}\nu = \operatorname{tr}\nu' + \lambda_m = t$, we conclude that $\nu \in \Lambda_t(\lambda\,,\,m)$. Now, if $\mu \in \Lambda_t(\lambda\,,\,m)$ it is straightforward to check that $\mu' = (\mu_1, \dots, \mu_{d-1}) \in \Lambda_{t'}(\lambda'\,,\,m')$. Therefore $\nu' \prec_w \mu'$ and $\operatorname{tr}\mu \geq \operatorname{tr}\nu \implies \nu \prec_w \mu$, which shows that (31) holds in this case. △

Although the inductive process described in Remark 4.10 together with Proposition 4.9 show the existence of $\nu \in \Lambda_t(\lambda\,,\,m)$ satisfying (31), this approach is not constructive. Nevertheless, one can follow the rule of construction in each step to obtain a concrete description of ν. In what follows we introduce some new notions that will allow us to compute ν explicitly.

Definition 4.11. Let $\lambda \in \mathbb{R}_+^{d\,\downarrow}$, $t_0 = \operatorname{tr} \lambda$. Take an integer $m < d$. If $m > 0$ recall that $s^* = s^*(\lambda\,,\,m) = c_\lambda^{-1}(\lambda_m)$. For $t \in [t_0\,,\,+\infty)$ let

$$c_{\lambda\,,\,m}(t) \stackrel{\text{def}}{=} \begin{cases} c_\lambda(t) & \text{if } \ t \le s^* \\ \\ \lambda_m + \frac{t-s^*}{d-m} & \text{if } \ t > s^* \end{cases} \qquad \text{and}$$

$$r_{\lambda\,,\,m}(t) \stackrel{\text{def}}{=} \begin{cases} \quad r_\lambda(t) & \text{if } \ t \le s^* \\ \\ \min\{r \in \mathbb{I}_{d-1} \cup \{0\} : c_{\lambda\,,\,m}(t) \ge \lambda_{r+1}\} & \text{if } \ t > s^* \end{cases} .$$

If $m \le 0$ we just define $c_{\lambda\,,\,m}(t) = c_\lambda(t)$ and $r_{\lambda\,,\,m}(t) = r_\lambda(t)$ for every $t \in [t_0\,,\,+\infty)$. △

4.12. Observe that the map $c_{\lambda\,,\,m}(\cdot)$ is continuous and strictly increasing. Indeed, by Eq. (30) we know that $s^* = \sum\limits_{i=1}^m \lambda_i + (d-m)\,\lambda_m$. Hence $c_{\lambda\,,\,m}(t) = \frac{t-\sum_{j=1}^m \lambda_j}{d-m}$ for every $t > s^*$. By Definition 4.4 we can also deduce that $c_{\lambda\,,\,m}(t) \ge c_\lambda(t)$ for every t.

Let us abbreviate by $r = r_{\lambda\,,\,m}(t)$ for any fixed $t > s^*$. Then,

$$r < m \quad \text{ and } \quad \lambda_r \ge c_{\lambda\,,\,m}(t) = \lambda_m + \frac{t-s^*}{d-m} \ge \lambda_{r+1} \ . \tag{33}$$

In particular this shows that the map $r_{\lambda\,,\,m}(t)$ is also non-increasing and right-continuous for $t > s^*$, where the discontinuity points are:

$$\{t > s^* : \ t = (d-m)\lambda_k + \sum_{i=1}^m \lambda_i \quad \text{ for } \quad k \in \mathcal{B},\, k < m\} \ ,$$

where the index set $\mathcal{B}$ is that defined on Lemma 4.5. Finally we write

$$s^{**} = c_{\lambda\,,\,m}^{-1}(\lambda_1) = (d-m)\,\lambda_1 + \sum_{j=1}^m \lambda_j \ge s^* \quad (\text{with equality } \iff \lambda_1 = \lambda_m) \ . \tag{34}$$

Definition 4.13. Let $\lambda \in \mathbb{R}_+^{d\,\downarrow}$, $t_0 = \operatorname{tr} \lambda$ and $m \in \mathbb{Z}$ such that $m < d$. Fix $t \in [t_0\,,\,+\infty)$ and denote by $r = r_{\lambda\,,\,m}(t)$. Consider the vector $\nu_{\lambda\,,\,m}(t) \in \mathbb{R}_+^{d\,\downarrow}$ given by the following rule:

- If $m \le 0$ then $\nu_{\lambda\,,\,m}(t) = \nu_\lambda(t) \stackrel{(28)}{=} \big(\lambda_1\,,\,\dots\,,\,\lambda_r\,,\,c_{\lambda\,,\,m}(t)\,\mathbb{1}_{d-r}\big)$.

If $m \ge 1$ we define

- $\nu_{\lambda\,,\,m}(t) = \big(\lambda_1\,,\,\dots\,,\,\lambda_r\,,\,c_{\lambda\,,\,m}(t)\,\mathbb{1}_{d-r}\big)$ for $t \le s^*$ (so that $r \ge m$ and $c_{\lambda\,,\,m}(t) \le \lambda_m$).
- $\nu_{\lambda\,,\,m}(t) = \Big(\lambda_1\,,\,\dots\,,\,\lambda_r\,,\,c_{\lambda\,,\,m}(t)\,\mathbb{1}_{d-m}\,,\,\lambda_{r+1}\,,\,\dots\,,\,\lambda_m\Big)$ for $t \in (s^*\,,\,s^{**})$, and
- $\nu_{\lambda\,,\,m}(t) = \big(c_{\lambda\,,\,m}(t)\,\mathbb{1}_{d-m}\,,\,\lambda_1\,,\,\dots\,,\,\lambda_m\big)$ for $t \ge s^{**}$.

If $\lambda_1 = \lambda_m$, the second case of the definition of $\nu_{\lambda\,,\,m}(t)$ disappears. △

In the following Lemma we state several properties of this map which are easy to see:

Lemma 4.14. Let $\lambda \in \mathbb{R}_+^{d\,\downarrow}$, and $m \in \mathbb{Z}$ such that $m < d$. The map $\nu_{\lambda\,,\,m}(\cdot)$ has the following properties:

1. It is continuous.

2. It is increasing in the sense that $t_1 < t_2 \implies \nu_{\lambda\,,\,m}(t_1) \leqslant \nu_{\lambda\,,\,m}(t_2)$.

3. More precisely, for any fixed $k \in \mathbb{I}_d$, the k-th entry $\nu^{(k)}_{\lambda\,,\,m}(t)$ of $\nu_{\lambda\,,\,m}(t)$ is given by

$$\nu^{(k)}_{\lambda\,,\,m}(t) = \begin{cases} \max\{\lambda_k\,,\, c_{\lambda\,,\,m}(t)\} & \text{if} \quad k \le d-m\,, \\ \\ \min\Big\{\ \max\{\lambda_k\,,\, c_{\lambda\,,\,m}(t)\}\,,\ \lambda_i\Big\} & \text{if} \quad k = d-m+i\,, \quad i \in \mathbb{I}_m\ . \end{cases}$$

4. The vector $\nu_{\lambda\,,\,m}(t) \in \Lambda_t(\lambda\,,\, m)$ and $\operatorname{tr} \nu_{\lambda\,,\,m}(t) = t$ for every $t \in [t_0\,,\, +\infty)$. □

We can now state the main result of this section.

Theorem 4.15. *Let $\lambda \in \mathbb{R}^{d\,\downarrow}_+$, $t_0 = \operatorname{tr}\lambda$ and $t \in [t_0\,,\,+\infty)$. Fix $m \in \mathbb{Z}$ such that $m < d$. Then the vector $\nu_{\lambda\,,\,m}(t)$ defined in 4.13 is the unique element of $\Lambda_t(\lambda\,,\, m)$ such that*

$$\nu_{\lambda\,,\,m}(t) \prec_w \mu \quad \textit{for every} \quad \mu \in \Lambda_t(\lambda\,,\, m)\ . \tag{35}$$

Proof. If $m \le 0$ the result follows from Theorem 4.7. Suppose now that $m \ge 1$. By Lemma 4.14, the vector $\nu_{\lambda\,,\,m}(t) \in \Lambda_t(\lambda\,,\, m)$ and $\operatorname{tr} \nu_{\lambda\,,\,m}(t) = t$ for $t \in [t_0\,,\,+\infty)$. In Proposition 4.9 we have shown that $\nu_{\lambda\,,\,m}(t)$ satisfies (35) for every $t \in [t_0\,,\, s^*(\lambda\,,\, m)\,]$. Hence we check the other two cases:

Case $t \in (s^*\,,\, s^{**})$: Fix $\mu \in \Lambda_t(\lambda\,,\, m)$ such that $\operatorname{tr}\mu = t$. Let us denote by $r = r_{\lambda\,,\,m}(t)$,

$$\alpha = (\mu_1\,,\,\dots\,,\,\mu_r)\ ,\quad \beta = (\mu_{r+1}\,,\,\dots\,,\,\mu_{r+d-m})\ ,\quad \gamma = (\mu_{r+d-m+1}\,,\,\dots\,,\,\mu_d)\ ,$$

$\rho = (\lambda_1\,,\,\dots\,,\,\lambda_r)$ and $\omega = (\lambda_{r+1}\,,\,\dots\,,\,\lambda_m)$. Then

$$\mu = (\alpha\,,\,\beta\,,\,\gamma) \qquad \text{and} \qquad \nu_{\lambda\,,\,m}(t) = (\rho\,,\, c_{\lambda\,,\,m}(t)\,\mathbb{1}_{d-m}\,,\,\omega)\ .$$

Since $\mu \in \Lambda_t(\lambda\,,\, m)$ and $\operatorname{tr}\ \nu_{\lambda\,,\,m}(t) = \operatorname{tr}\ \mu = t$, then

$$\rho \leqslant \alpha \quad , \quad \gamma \leqslant \omega \qquad \text{and} \qquad \operatorname{tr}(\alpha\,,\,\beta) \ge \operatorname{tr}(\rho\,,\, c_{\lambda\,,\,m}(t)\,\mathbb{1}_{d-m})\ .$$

Then we can apply Lemma 4.6 to deduce that $(\rho\,,\, c_{\lambda\,,\,m}(t)\,\mathbb{1}_{d-m}) \prec_w (\alpha\,,\,\beta)$. Using this fact jointly with $\gamma \leqslant \omega$ one easily deduces that $\nu^*(\lambda\,,\,t) \prec \mu$ (because $\operatorname{tr}\mu = \operatorname{tr}\nu_{\lambda\,,\,m}(t) = t$).

The case $t \ge s^{**}$ for vectors $\mu \in \Lambda_t(\lambda\,,\, m)$ such that $\operatorname{tr}\mu = t$ follows similarly.

If we have that $\mu \in \Lambda_t(\lambda\,,\, m)$ with $\operatorname{tr}\mu = a > t$, then

$$\mu \in \Lambda_a(\lambda\,,\, m) \implies \nu_{\lambda\,,\,m}(t) \leqslant \nu_{\lambda\,,\,m}(a) \prec \mu \implies \nu_{\lambda\,,\,m}(t) \prec_w \mu\ ,$$

where the first inequality follows from Lemma 4.14. □

5 Minimizers for submajorization 2: Matrices

Given $S_0 \in \mathcal{M}_d(\mathbb{C})^+$ with $\lambda(S_0) = \lambda$ and $m \in \mathbb{I}_d$, we introduce the sets $U_t(S_0, m)$ whose spectral pictures are the sets $\Lambda_t(\lambda, m)$ defined in the previous section (see Proposition 5.3). Hence by Theorem 4.15, the sets $U_t(S_0, m)$ have minimal elements with respect to submajorization. In this section we characterize the minimal $S \in U_t(S_0, m)$ in terms of the geometry of S_0.

Definition 5.1. Let $S_0 \in \mathcal{M}_d(\mathbb{C})^+$ with $\lambda(S_0) = \lambda \in \mathbb{R}^{d\,\downarrow}_+$, $t_0 = \operatorname{tr}\, S_0$, and $t \ge t_0$. For any integer $m < d$ we consider the following sets:

$$U(S_0\,,\, m) = \{S_0 + B :\ B \in \mathcal{M}_d(\mathbb{C})^+\,,\ \operatorname{rk} B \le d-m\ \} \qquad \text{and}$$

$$U_t(S_0, m) = \{S \in U(S_0, m) :\ \operatorname{tr}\, S \ge t\}\ . \tag{36}$$

Observe that if $m \le 0$ then $U(S_0\,,\, m) = \{S \in \mathcal{M}_d(\mathbb{C})^+ \,:\, S \ge S_0\}$. △

Remark 5.2. Let $\mathcal{F} = \{f_i\}_{i\in \mathbb{I}_n} \in \mathbf{F}(n,d)$. Then Proposition 2.6 shows that $\mathcal{SD}(\mathcal{F}) = U(S_{\mathcal{F}}^{-1}\,,\,m)$, where $d-m = n-d \implies m = 2d-n$. This fact, together with Corollary 3.4, gives the characterization of the spectral pictures of the sets $U(S_{\mathcal{F}}^{-1}\,,\,m)$. △

Proposition 5.3. *Let $S_0 \in \mathcal{M}_d(\mathbb{C})^+$. Denote by $\lambda = \lambda(S_0)$ and take an integer $m < d$. Then*

$$\Lambda(U(S_0\,,\,m)) = \Lambda(\lambda\,,\,m) \quad \textit{and} \quad \Lambda(U_t(S_0\,,\,m)) = \Lambda_t(\lambda\,,\,m)\ .$$

Proof. If $m \le 0$ then $U(S_0\,,\,m) = \{S \in \mathcal{M}_d(\mathbb{C})^+ : S \ge S_0\}$. Hence the first equality follows easily from Weyl theorem (see [3]). The second equality is a straightforward consequence of the first one.

If $m \ge 1$, let us assume first that $S_0 \in \mathcal{G}l(d)^+$ and let $n = 2d-m > d$. Take $\mathcal{F} = \{f_i\}_{i\in\mathbb{I}_n} \in \mathbf{F}(n,d)$ such that $S_{\mathcal{F}} = S_0^{-1}$. Observe that Proposition 2.6 and Corollary 3.4 imply that $\mathcal{SD}(\mathcal{F}) = U(S_0\,,\,m)$ and $\Lambda(\mathcal{D}(\mathcal{F})\,) = \Lambda(\lambda\,,\,m)$, which in turn show the first identity.

If $S_0 \notin \mathcal{G}l(d)^+$, let us denote by $S_1 = S_0 + I \in \mathcal{G}l(d)^+$. It is easy to see that

$$U(S_1\,,\,m) = \{S + I : S \in U(S_0\,,\,m)\} \implies \Lambda(U(S_1\,,\,0)) = \Lambda(U(S_0\,,\,0)) + \mathbb{1}_d\ .$$

Similarly, it is straightforward to prove that $\Lambda(\lambda + \mathbb{1}_d\,,\,m) = \Lambda(\lambda\,,\,m) + \mathbb{1}_d$. Finally observe that $\lambda(S_1) = \lambda(S_0) + \mathbb{1}_d = \lambda + \mathbb{1}_d$. Hence, by the first part of the proof, we get that

$$\Lambda(U(S_0\,,\,m)) = \Lambda(U(S_1\,,\,m)) - \mathbb{1}_d = \Lambda(\lambda + \mathbb{1}_d\,,\,m) - \mathbb{1}_d = \Lambda(\lambda\,,\,m)\ .$$

The second identity is a straightforward consequence of the first one. □

Let $S_0 \in \mathcal{M}_d(\mathbb{C})^+$ and let $t \ge t_0 = \operatorname{tr}(S_0)$. We shall describe the geometrical structure of minimal elements in $U_t(S_0\,,\,m)$ with respect to submajorization for any $m < d$. We shall see that, under some mild assumptions, there exists a unique $S_t \in U_t(S_0\,,\,m)$ such that $\lambda(S_t) = \nu_{\lambda\,,\,m}(t)$ (the vector of Theorem 4.15 defined in 4.13). In order to do this we fix some notations and establish a series of preliminary results.

Notations. We fix a matrix $S \in \mathcal{M}_d(\mathbb{C})^+$ with $\lambda(S) = \lambda = (\lambda_1\,,\,\dots\,,\,\lambda_d) \in \mathbb{R}_+^{d\,\downarrow}$. We shall also fix an orthonormal basis $\{h_i\}_{i\in\mathbb{I}_d}$ of $\mathbb{C}^d$ such that

$$S\,h_i = \lambda_i\,h_i \quad \text{for every} \quad i \in \mathbb{I}_d\ .$$

Any other such basis will be denoted as a "ONB of eigenvectors for $S\,,\,\lambda$".

Lemma 5.4. *Let $B \in \mathcal{M}_d(\mathbb{C})^+$ and $r \in \mathbb{I}_{d-1}$ such that $\lambda(S+B) = (\lambda_1\,,\,\dots\,,\,\lambda_r\,,\,\alpha)$, for some $\alpha \in \mathbb{R}_{\ge 0}^{d-r\,\downarrow}$ such that $\alpha_1 \le \lambda_r$. Let $\mathcal{M}_r \stackrel{def}{=} \operatorname{span}\{h_i : i \in \mathbb{I}_r\}$ and $P = P_{\mathcal{M}_r}$. Then*

$$PB = BP = PBP = 0\ .$$

Proof. Since $\operatorname{rk} P = r$ and $\operatorname{tr}(PSP) = \sum\limits_{i=1}^{r} \lambda_i$, then the Ky Fan theorem (3) assures that

$$0 \le \operatorname{tr}(PBP) = \operatorname{tr}(P(S+B)P) - \operatorname{tr}(PSP) \le \sum_{i=1}^{r} \lambda_i(S+B) - \sum_{i=1}^{r} \lambda_i = 0\ .$$

Since $B \ge 0$, we have that $\operatorname{tr}(PBP) = 0 \implies PBP = 0 \implies BP = PB = 0$. □

Proposition 5.5. *Let $B \in \mathcal{M}_d(\mathbb{C})^+$ and $r \in \mathbb{I}_{d-1}$ such that $\lambda(S+B) = (\lambda_1\,,\,\dots\,,\,\lambda_r\,,\,c\,\mathbb{1}_{d-r})$, for some $c \in [\lambda_{r+1}\,,\,\lambda_r]$. Then B is unique and given by*

$$B = \sum_{i=1}^{d-r} (c - \lambda_{r+i})\,h_{r+i} \otimes h_{r+i} \quad \textit{so that} \quad S + B = \sum_{i=1}^{r} \lambda_i \cdot h_i \otimes h_i + c \cdot \sum_{i=r+1}^{d} h_i \otimes h_i\,.$$

Proof. Let $\mathcal{M}_r \overset{\text{def}}{=} \operatorname{span}\{h_i : i \in \mathbb{I}_r\}$ and $P = P_{\mathcal{M}_r}$. By Lemma 5.4, $BP = PB = 0$. Hence

$$P(S+B)P = (S+B)P = SP = \sum_{i=1}^{r} \lambda_i \, h_i \otimes h_i \overset{\text{Eq. (4)}}{\Longrightarrow} (S+B)Q = c\,Q \ ,$$

where $Q = I - P$. Hence $B = BQ = c\,Q - S\,Q = \sum\limits_{i=1}^{d-r} (c - \lambda_{r+i})\, h_{r+i} \otimes h_{r+i}$. □

Remark 5.6. In Lemma 5.4, we allow the case where $\lambda_r = \lambda_{r+1} = \alpha_1$. In this case we could change h_r by h_{r+1} (or any other eigenvector for λ_r) as a generator for $\mathcal{M}_r$. The proof of the Lemma assures that we get another projector P' which also satisfies that $BP' = 0$.

Similarly, in Proposition 5.5 we allow the case where $\lambda_r = \lambda_{r+1} = c$. By the previous comments, the projection P in the proof of Proposition 5.5 is not unique. Nevertheless, in this case the positive perturbation B is also unique, because we have that $\operatorname{rk} B < d - m$ (this follows from the fact that $(c - \lambda_{r+1})\, h_{r+1} \otimes h_{r+1} = 0$). △

Lemma 5.7. *Let $m \in \mathbb{I}_{d-1}$ and $B \in \mathcal{M}_d(\mathbb{C})^+$ with $\operatorname{rk} B \le d - m$. Assume that*

$$\lambda(S+B) = (c\mathbb{1}_{d-m}\,,\, \lambda_1\,,\,\dots\,,\,\lambda_m) \ ,$$

for some $c \ge \lambda_1$. Then there exists an ONB $\{v_i\}_{i\in\mathbb{I}_d}$ of eigenvectors for S, λ such that

$$B = \sum_{i=1}^{d-m} (c - \lambda_{m+i})\, v_{m+i} \otimes v_{m+i} \qquad \text{so that} \qquad S + B = \sum_{i=1}^{m} \lambda_i \cdot v_i \otimes v_i + c \cdot \sum_{i=m+1}^{d} v_i \otimes v_i \ . \tag{37}$$

If we assume further that $\lambda_m > \lambda_{m+1}$ then B is unique, and Eq. (37) holds for any ONB of eigenvectors for S, λ.

Proof. Note that, since $\operatorname{rk} B \le d - m$, then

$$\sum_{i=1}^{d-m} \lambda_i(B) = \operatorname{tr} B = \operatorname{tr}(B+S) - \operatorname{tr} S = c\,(d-m) - \sum_{j=m+1}^{d} \lambda_j \ . \tag{38}$$

Take a subspace $\mathcal{M} \subseteq \mathbb{C}^n$ such that $R(B) \subseteq \mathcal{M}$ and $\dim \mathcal{M} = d - m$. Denote by $Q = P_{\mathcal{M}}$. Then $QBQ = B$, and the Ky-Fan inequalities (3) for $S + B$ assure that

$$\begin{aligned} \operatorname{tr}(QSQ) &= \operatorname{tr}(Q(S+B)Q) - \operatorname{tr} B \\ &\le \sum_{i=1}^{d-m} \lambda_i(S+B) - \operatorname{tr} B = c\,(d-m) - \operatorname{tr} B \overset{(38)}{=} \sum_{j=m+1}^{d} \lambda_j \ . \end{aligned}$$

The equality in Ky-Fan inequalities (for $-S$) force that $\mathcal{M} = \operatorname{span}\{v_{m+1}\,,\,\dots\,,\,v_d\}$, for some ONB $\{v_i\}_{i\in\mathbb{I}_d}$ of eigenvectors for S, λ (see the remark following Eq. (4)). Thus, we get that $Q\,S = S\,Q = \sum\limits_{i=1}^{d-m} \lambda_{m+i}\, v_{m+i} \otimes v_{m+i}$. Since $R(B) \subseteq \mathcal{M}$ then $P \overset{\text{def}}{=} I - Q \le P_{\ker B}$, and

$$B\,P = 0 \implies P(S+B)P = S\,P = \sum_{i=1}^{m} \lambda_i\, v_i \otimes v_i \overset{Eq.(4)}{\Longrightarrow} (S+B)\,Q = c\,Q \ .$$

Therefore we can now compute

$$B = B\,Q = (S+B)Q - SQ = \sum_{i=1}^{d-m} (c - \lambda_{m+i})\, v_{m+i} \otimes v_{m+i} \ . \tag{39}$$

Finally, if we further assume that $\lambda_m > \lambda_{m+1}$ then the equality $\mathcal{M} = \operatorname{span}\{v_{m+1}\,,\,\dots\,,\,v_d\}$ is independent of the choice of the ONB of eigenvectors for S, λ. Thus, in this case B is uniquely determined by (39). □

Proposition 5.8. *Let $m \in \mathbb{I}_{d-1}$ and $B \in \mathcal{M}_d(\mathbb{C})^+$ with $\operatorname{rk} B \le d-m$. Let $c \in \mathbb{R}$ such that $\lambda_{r+1} \le c < \lambda_r$, for some $r < m$. Assume that*

$$\lambda(S+B) = \Big(\lambda_1\,,\,\dots\,,\,\lambda_r\,,\,c\,\mathbb{1}_{d-m}\,,\,\lambda_{r+1}\,,\,\dots\,,\,\lambda_m\Big)\ .$$

Then there exists an ONB $\{v_i\}_{i\in\mathbb{I}_d}$ of eigenvectors for S, λ such that

$$B = \sum_{i=1}^{d-m} (c-\lambda_{m+i})\, v_{m+i}\otimes v_{m+i} \qquad \text{so that} \qquad S+B = \sum_{i=1}^{m} \lambda_i\cdot v_i\otimes v_i + c\cdot \sum_{i=m+1}^{d} v_i\otimes v_i\ .$$

If we further assume that $\lambda_m > \lambda_{m+1}$ then B is unique.

Proof. Consider the subspace $\mathcal{M}_r = \operatorname{span}\{h_1\,,\,\dots\,,\,h_r\}$ and $P = P_{\mathcal{M}_r}$. By Lemma 5.4, we know that $P\,B = B\,P = 0$. Let $S_1 = S\big|_{\mathcal{M}_r^\perp}$ and $B_1 = B\big|_{\mathcal{M}_r^\perp}$ $(= B)$ considered as operators in $L(\mathcal{M}_r^\perp)$. Then S_1 and B_1 are in the conditions of Lemma 5.7, so that there exists an ONB $\{w_i\}_{i\in\mathbb{I}_{d-r}}$ of $\mathcal{M}_r^\perp$ of eigenvectors for S_1, $(\lambda_{r+1}\,,\,\dots\,,\,\lambda_d)$ such that

$$B = B_1 = \sum_{i=1}^{d-m} (c-\lambda_{m+i})\, w_{m+i}\otimes w_{m+i}\ .$$

Finally, let $\{v_i\}_{i\in\mathbb{I}_d}$ be given by $v_i = h_i$ for $1\le i\le r$ and $v_{r+i} = w_i$ for $r+1\le i\le d$. Then $\{v_i\}_{i\in\mathbb{I}_d}$ has the desired properties. Notice that if we further assume that $\lambda_m > \lambda_{m+1}$ then Lemma 5.7 implies that B_1 is unique and therefore B is unique, too. $\square$

Remark 5.9. With the notations of Lemma 5.7 assume that $\lambda_m = \lambda_{m+1}$. In this case B is not uniquely determined. Next we obtain a parametrization of the set of all operators $B \in \mathcal{M}_d(\mathbb{C})^+$ such that $\lambda(S+B) = (c\mathbb{1}_{d-m}\,,\,\lambda_1\,,\,\dots\,,\,\lambda_m)$. Consider $p = (d-m) - \#\{i:\ \lambda_i < \lambda_{m+1}\}$ and notice that in this case we have that $1\le p < \#\{i:\ \lambda_i = \lambda_{m+1}\} = \dim\ker(S-\lambda_{m+1}I)$. Then, for every $B\in\mathcal{M}_d(\mathbb{C})^+$ as above there corresponds a subspace $\mathcal{N} = \operatorname{span}\{h_i:\ m+1\le i\le m+p\} \subset \ker(S-\lambda_m I)$ with $\dim\mathcal{N} = p$ such that

$$B = (c-\lambda_m)\ P_{\mathcal{N}} + \sum_{i=p+1}^{d-m} (c-\lambda_{m+i})\, h_{m+i}\otimes h_{m+i}\ . \tag{40}$$

Conversely, for every subspace $\mathcal{N}\subset\ker(S-\lambda_m I)$ with $\dim\mathcal{N} = p$ then the operator $B\in\mathcal{M}_d(\mathbb{C})^+$ given by (40) satisfies that $\lambda(S+B) = (c\mathbb{1}_{d-m}\,,\,\lambda_1\,,\,\dots\,,\,\lambda_m)$. Since the previous map $B\mapsto P_{\mathcal{N}}$ is bijective, we see that the set of all such operators B is parametrized by the set of projections $P_{\mathcal{N}}$ such that $\mathcal{N}\subset\ker(S-\lambda_m I)$ is a p-dimensional subspace. Moreover, this map is actually an homeomorphism between these sets, with their usual metric structures.

Finally, if we let $k = \#\{i:\ \lambda_i > \lambda_m\}$ then the set of operators $S+B$ such that $B\in\mathcal{M}_d(\mathbb{C})^+$ with $\mathrm{rk}B \le m-d$ and such that $\lambda(S+B) = (c\mathbb{1}_{d-m}\,,\,\lambda_1\,,\,\dots\,,\,\lambda_m)$ is given by

$$S+B = \sum_{i=1}^{k} \lambda_i\cdot h_i\otimes h_i + \lambda_m\cdot P_{\mathcal{N}'} + c\cdot\big(P_{\mathcal{N}} + \sum_{i=p+1}^{d-m} h_i\otimes h_i\big)\ ,$$

where $\mathcal{N}\subset\ker(S-\lambda_m I)$ is a subspace with $\dim\mathcal{N} = p$ and $\mathcal{N}' = \ker(S-\lambda_{m+1}I)\cap\mathcal{N}^\perp$.

As a consequence of the proof of Proposition 5.8, we have a similar description of the operators B of its statement. $\triangle$

Recall from Definition 4.13 the descrption of vector $\nu_{\lambda\,,\,m}(t)$.

Theorem 5.10. *Let $S_0 \in \mathcal{M}_d(\mathbb{C})^+$ with $\lambda = \lambda(S_0)$. For an integer $m < d$ and a number $t \geq \operatorname{tr} S_0$, let us denote by $r' = \max\{r_{\lambda,m}(t), m\}$ and $c = c_{\lambda,m}(t)$. Then*

1. *The vector $\nu_{\lambda,m}(t) \in \Lambda_t(U(S_0, m))$.*
2. *For every matrix $S \in U_t(S_0, m)$ the following conditions are equivalent:*
   (a) *$\lambda(S) = \nu_{\lambda,m}(t)$ (i.e. S is $\prec_w$-minimal in $U_t(S_0, m)$).*
   (b) *There exists $\{v_i\}_{i \in \mathbb{I}_d}$ an ONB of eigenvectors for S_0, λ such that*
   $$B = S - S_0 = \sum_{i=1}^{d-r'} (c - \lambda_{r'+i})\, v_{r'+i} \otimes v_{r'+i} \ . \tag{41}$$
3. *If we further assume any of the following conditions:*
   - *$m \leq 0$,*
   - *$m \geq 1$ and $\lambda_m > \lambda_{m+1}$, or*
   - *$m \geq 1$ and $\lambda_m = \lambda_{m+1}$ but $t \leq s^*(\lambda, m)$ (see Definition 4.8),*

   then B and hence also S are unique. Moreover, in these cases Eq. (41) holds for any ONB of eigenvectors for S_0, λ.

Proof. It follows from Lemmas 4.14, 5.7 and Propositions 5.5, 5.8. (b) $\implies$ (a) in Item 2 follows by Definition 4.13 and the fact that both S_0 and B are diagonal on the same basis. $\square$

6 Minimizing potentials in $\mathcal{D}_t(\mathcal{F})$

Fix a system $\mathcal{F} = \{f_i\}_{i \in \mathbb{I}_n} \in \mathbf{F}(n, d)$. By Corollary 3.4, $\mu \in \Lambda(\mathcal{D}(\mathcal{F})) \implies \mu \geqslant \lambda(S_{\mathcal{F}}^{-1})$. Therefore the canonical dual $\mathcal{F}^\# \in \mathcal{D}(\mathcal{F})$, which by (9) satisfy that $S_{\mathcal{F}^\#} = S_{\mathcal{F}}^{-1}$, has a strong minimality property: Its vector $\lambda \overset{\text{def}}{=} \lambda(S_{\mathcal{F}^\#}) = \lambda(S_{\mathcal{F}}^{-1})$ satisfies that

$$\lambda \leqslant \mu \implies \lambda \prec_w \mu \quad \text{for every} \quad \mu \in \Lambda(\mathcal{D}(\mathcal{F})) \ .$$

By this fact, $\mathcal{F}^\#$ is the global minimum in terms of a family of frame potentials on $\Lambda(\mathcal{D}(\mathcal{F}))$ (see Definition 6.1 below). But the canonical dual might not be the optimal choice from an applied point of view (e.g. the frame operator of $\mathcal{F}^\#$ can be ill-conditioned). The problem we focus in is to find dual frames $\mathcal{G}_t \in \mathcal{D}(\mathcal{F})$ that are minimal with respect to submayorization whiting the set of dual frames $\mathcal{G} \in \mathcal{D}(\mathcal{F})$ such that $\operatorname{tr}(S_\mathcal{G}) > \operatorname{tr}(S_{\mathcal{F}}^{-1})$. Notice that this problem is equivalent to the problem of finding minimal pseudo inverses of $T_\mathcal{F}$ with Frobenius norm great than $t^{1/2}$. As we shall see, these optimal duals $\mathcal{G}_t$ have minimal condition number and, in some cases are are tight frames. In order to study this problem, we fix general notations:

Let $\mathcal{F} \in \mathbf{F}(n, d)$. Fix $t \geq \operatorname{tr} S_{\mathcal{F}}^{-1}$ and consider the sets

- $\mathcal{D}_t(\mathcal{F}) = \{\mathcal{G} \in \mathcal{D}(\mathcal{F}) : \operatorname{tr} S_\mathcal{G} \geq t\}$.
- $\mathcal{SD}(\mathcal{F}) = \{S_\mathcal{G} : \mathcal{G} \in \mathcal{D}(\mathcal{F})\}$ and $\mathcal{SD}_t(\mathcal{F}) = \{S_\mathcal{G} : \mathcal{G} \in \mathcal{D}_t(\mathcal{F})\}$.
- $\Lambda_t(\mathcal{D}(\mathcal{F})) = \{\lambda(S_\mathcal{G}) : \mathcal{G} \in \mathcal{D}_t(\mathcal{F})\}$.

Recall from Corollary 4.2 that if $\mathcal{F} \in \mathbf{F}(n, d)$ with $\lambda = \lambda(S_{\mathcal{F}}^{-1})$, $m = 2d - n$ and $t \geq \operatorname{tr} \lambda$, then $\Lambda_t(\mathcal{D}(\mathcal{F})) = \Lambda_t(\lambda, m)$, and it has a unique majorization minimzer $\nu_{\lambda,m}(t)$ (see Definition 4.13).

Definition 6.1. *Let $f : [0,\infty) \to [0,\infty)$ be an increasing convex function. Following [22] we consider the (generalized) frame potential associated to f, denoted P_f, given by*

$$P_f(\mathcal{F}) = \operatorname{tr}(f(S_\mathcal{F})) \quad \textit{for} \quad \mathcal{F} = \{f_i\}_{i\in\mathbb{I}_n} \in \mathcal{H}^n \ . \qquad \triangle$$

Of course, one the most important such potentials is the Benedetto-Fickus (BF) frame potential obtained by setting $f(x) = x^2$ for $x \geq 0$. As shown in [22, Sec. 4] these potentials (which are related with the so-called entropic measures of frames) share many properties with the BF-potential. Indeed, under certain restrictions both the spectral and geometric structures of minimizers of these potentials coincide (see [22]).

Theorem 6.2 (Spectral structure of Global minima)**.** *Let $\mathcal{F} = \{f_i\}_{i\in\mathbb{I}_n} \in \mathbf{F}(n,d)$ with $\lambda = \lambda(S_\mathcal{F}^{-1})$, $m = 2\,d - n$ and $t \geq \operatorname{tr}\lambda$. Then $\nu_{\lambda\,,\,m}(t) \in \Lambda_t(\mathcal{D}(\mathcal{F}))$ and we have that:*

1. *If $\mathcal{G}_t \in \mathcal{D}_t(\mathcal{F})$ is such that $\lambda(S_{\mathcal{G}_t}) = \nu_{\lambda\,,\,m}(t)$ we have that*

$$P_f(\mathcal{G}_t) \leq P_f(\mathcal{G}) \quad \textit{for every } \mathcal{G} \in \mathcal{D}_t(\mathcal{F}) ,$$

   and every increasing convex function $f : [0,\infty) \to [0,\infty)$.

2. *If we assume further that f is strictly convex then, for every global minimizer $\mathcal{G}_t'$ of $P_f(\cdot)$ on $\mathcal{D}_t(\mathcal{F})$ we get that $\lambda(\mathcal{G}_t') = \nu_{\lambda\,,\,m}(t)$.*

Proof. As an immediate consequence of Theorem 4.15 we see that $\nu = \nu_{\lambda\,,\,m}(t) \in \Lambda_t(\mathcal{D}(\mathcal{F}))$ is such that $\nu \prec_w \mu$ for every $\mu \in \Lambda_t(\mathcal{D}(\mathcal{F}))$. By the remarks in Section 2.2 we conclude that, if $\mathcal{G}_t$ is as above and $\mathcal{G} \in \mathcal{D}_t(\mathcal{F})$ then

$$P_f(\mathcal{G}_t) = \operatorname{tr}(f(S_{\mathcal{G}_t})) = \operatorname{tr} f(\nu) \leq \operatorname{tr} f(\lambda(S_\mathcal{G})) = P_f(\mathcal{G}) \ ,$$

since $\lambda(S_\mathcal{G}) \in \Lambda_t(\mathcal{D}(\mathcal{F}))$. Assume further that f is strictly convex and let $\mathcal{G}_t'$ be a global minimizer of $P_f(\cdot)$ on $\mathcal{D}_t(\mathcal{F})$. Then, we have that

$$\nu_{\lambda\,,\,m}(t) \prec_w \lambda(S_{\mathcal{G}_t'}) \quad \text{but} \quad \operatorname{tr} f(\lambda(S_{\mathcal{G}_t'})) = P_f(S_{\mathcal{G}_t'}) \leq P_f(S_{\mathcal{G}_t}) = \operatorname{tr}\ f(\nu_{\lambda\,,\,m}(t)) \ .$$

These last facts imply (see [3]) that $\lambda(S_{\mathcal{G}_t'}) = \nu_{\lambda\,,\,m}(t)$ as desired. □

Next we describe the geometric structure of the global minimizers of the (generalized) frame potential $P_f(\cdot)$ in $\mathcal{D}_t(\mathcal{F})$, in terms of their frame operators.

Theorem 6.3 (Geometric Structure of global minima)**.** *Let $\mathcal{F} \in \mathbf{F}(n,d)$, $m = 2d-n$, let $t \geq \operatorname{tr}\ S_\mathcal{F}^{-1}$ and denote by $\lambda = \lambda(S_\mathcal{F}^{-1})$. Let $f : [0,\infty) \to [0,\infty)$ an increasing and strictly convex function.*

1. *If $\mathcal{G} \in \mathcal{D}_t(\mathcal{F})$ is a global minimum of P_f in $\mathcal{D}_t(\mathcal{F})$ then there exists $\{h_i\}_{i\in\mathbb{I}_d}$, an ONB of eigenvectors for $S_\mathcal{F}^{-1}$, λ such that*

$$S_\mathcal{G} = S_\mathcal{F}^{-1} + \sum_{i=1}^{d-r'} \Big(c_{\lambda\,,\,m}(t) - \lambda_{r'+i} \Big)\, h_{r'+i} \otimes h_{r'+i} \ ,$$

   where $r' = \max\{r_{\lambda\,,\,m}(t)\,,\, m\}$.

2. *If we further assume any of the conditions of item 3 of Theorem 5.10, there exists a unique $S_t \in \mathcal{SD}_t(\mathcal{F})$ such that if $\mathcal{G}$ is a global minimum of P_f in $\mathcal{D}_t(\mathcal{F})$ then $S_\mathcal{G} = S_t$.*

Proof. It is a consequence of Theorems 5.10 and 6.2. □

Remark 6.4. Let $\lambda \in \mathbb{R}_+^{d\,\downarrow}$ and $m \in \mathbb{I}_{d-1}$. Then there exist $t \in \mathbb{R}_{>0}$ and a constant vector

$$c\,\mathbb{1}_d \in \Lambda_t(\lambda\,,\,m) \iff \lambda_1 = \lambda_m\ .$$

In this case $c = \lambda_1$ and $t = d\,\lambda_1$. Indeed, if $\nu = c\,\mathbb{1}_d \in \Lambda_t(\lambda\,,\,m)$ then, by Eq. (19),

$$c = \nu_d \le \lambda_m \le \lambda_1 \le \nu_1 = c \implies \lambda_1 = \lambda_m = c\ .$$

Conversely, if $\lambda_1 = \lambda_m$ and $t = d\,\lambda_1$, then it is easy to see that $\lambda_1\,\mathbb{1}_d \in \Lambda_t(\lambda\,,\,m)$. Observe that $\lambda_1\,\mathbb{1}_d$ is the vector $\nu_{\lambda\,,\,m}(t)$ of Theorem 4.15 for such a λ and $t = d\,\lambda_1 = s^*(\lambda\,,\,m)$.

Therefore a frame $\mathcal{F} = \{f_i\}_{i\in\mathbb{I}_n} \in \mathbf{F}(n,d)$ has a dual frame which is tight if and only if

- $m = 2\,d - n \le 0$. Recall that in this case $\nu_\lambda(t) = \frac{t}{d}\cdot\mathbb{1}_d$ for every $t \ge d\,\lambda_1$.
- $m \in \mathbb{I}_{d-1}$ and $\lambda_{d-m+1}(S_\mathcal{F}) = \lambda_d(S_\mathcal{F})$ i.e., the last m eigenvalues of its frame operator are equal among them. Indeed, just recall that $\Lambda(\mathcal{D}(\mathcal{F})\,) = \Lambda(\lambda\,,\,m)$ for $\lambda = \lambda(S_\mathcal{F}^{-1})$.

In particular, if $m \in \mathbb{I}_{d-1}$ then there is a Parseval dual frame for $\mathcal{F}$ if and only if

$$\lambda_{d-m+1}(S_\mathcal{F}) = \lambda_d(S_\mathcal{F}) = 1 \iff S_\mathcal{F} \ge I_d \quad \text{and} \quad \operatorname{rk}(I_d - S_\mathcal{F}) \le d - m = \dim\ker T_\mathcal{F}^*\ .$$

Observe that the equivalence preserves if $2d \le n$. In this case there is a Parseval dual frame for $\mathcal{F} \iff S_\mathcal{F} \ge I_d$, because the restriction $\dim\ker T_\mathcal{F}^* = n - d \ge d \ge \operatorname{rk}(I_d - S_\mathcal{F})$ is irrelevant. This characterization was already proved by Han in [16], even for the infinite dimensional case. △

7 Optimal completions with prescribed norms

In this section we show how our previous results and techniques allow us to partially solve a frame completion problem posed in [15]. In order to describe this problem let us fix some notations and terminology:

7.1. In what follows, we fix the following data: A space $\mathcal{H} \cong \mathbb{C}^d$.

1. A sequence of vectors $\mathcal{F}_0 = \{f_i\}_{i\in\mathbb{I}_{n_0}} \in \mathcal{H}^{n_0}$.
2. An integer $n > n_0$. We denote by $k = n - n_0$. We assume that $\operatorname{rk}S_{\mathcal{F}_0} \ge d - k$.
3. A sequence $\mathbf{a} = \{\alpha_i\}_{i\in\mathbb{I}_n} \in \mathbb{R}_{>0}^n$ such that $\|f_i\|^2 = \alpha_i$ for every $i \in \mathbb{I}_{n_0}$.
4. We shall denote by $t = \sum\limits_{i\in\mathbb{I}_n} \alpha_i$ and by $\mathbf{b} = \{\alpha_i\}_{i=n_0+1}^n \in \mathbb{R}_{>0}^k$.
5. The vector $\lambda = \lambda(S_{\mathcal{F}_0}) \in \mathbb{R}_+^{d\,\downarrow}$.
6. The integer $m = d - k = (d + n_0) - n$. Observe that $d - m = k = n - n_0$. △

The problem is to find a sequence $\mathcal{F}_1 = \{f_i\}_{i=n_0+1}^n \in \mathcal{H}^k$ with $\|f_i\|^2 = \alpha_i$ for $n_0 + 1 \le i \le n$, such that the the mean square error of the resulting completed frame $\mathcal{F} = (\mathcal{F}_0\,,\,\mathcal{F}_1) = \{f_i\}_{i\in\mathbb{I}_n} \in \mathbf{F}(n,d)$, namely $\operatorname{tr}(S_\mathcal{F}^{-1})$, is minimal among all possible such completions.

There are other possible ways to measure robustness of the completed frame $\mathcal{F}$ as above. For example, we can consider optimal (minimizing) completions, with prescribed norms, for the Benedetto-Fickus' potential. This last fact raises the question of whether the minimizers corresponding to the mean square error or to the Benedetto-Fickus' potential coincide.

We shall show that this is the case. Indeed, notice that the mean square error of the completed frame $\mathcal{F}$ corresponds to the generalized potential $P_f(\mathcal{F})$, where $f(x) = x^{-1},\ x > 0$. Although f is not an increasing function we shall show that our majorization techniques apply to this strictly convex potential. In order to consider the problems described above, we define the following sets.

Definition 7.2. Given the data $\mathcal{F}_0 = \{f_i\}_{i\in\mathbb{I}_{n_0}}$ and $\mathbf{a} = \{\alpha_i\}_{i\in\mathbb{I}_n}$ as in 7.1, consider the sets

$$\mathcal{C}_{\mathbf{a}}(\mathcal{F}_0) = \big\{ \{f_i\}_{i\in\mathbb{I}_n} \in \mathbf{F}(n,d) : \{f_i\}_{i\in\mathbb{I}_{n_0}} = \mathcal{F}_0 \quad \text{and} \quad \|f_i\|^2 = \alpha_i \ \text{ for } \ i \geq n_0+1 \big\} ,$$

$$\mathcal{SC}_{\mathbf{a}}(\mathcal{F}_0) = \{S_{\mathcal{F}} : \ \mathcal{F} \in \mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)\} \quad \text{and} \quad \Lambda_{\mathbf{a}}(\mathcal{F}_0) = \{\lambda(S) : \ S \in \mathcal{SC}_{\mathbf{a}}(\mathcal{F}_0)\} . \qquad \triangle$$

We shall show that under certain hypothesis on the initial sequence $\mathcal{F}_0$,the difference $k = n - n_0$ and the sequence $\mathbf{a}$ (which includes the case where the final sequence $\{\alpha_i\}_{i=n_0+1}^n$ is uniform) we can explicitly compute the completing sequence $\mathcal{F}_1 = \{f_i\}_{i=n_0+1}^n \in \mathcal{H}^k$ that is optimal for a family of entropic measures of $\mathcal{F} = \{f_i\}_{i\in\mathbb{I}_n}$ including the mean square error.

Proposition 7.3 ([1, 21])**.** Let $B \in \mathcal{M}_d(\mathbb{C})^+$ with $\lambda(B) \in \mathbb{R}_+^{d\,\downarrow}$ and let $\mathbf{b} = (\beta_i)_{i\in\mathbb{I}_k} \in \mathbb{R}_{>0}^k$. Then there exists a sequence $\mathcal{G} = \{g_i\}_{i\in\mathbb{I}_k} \in \mathcal{H}^k$ with frame operator $S_{\mathcal{G}} = B$ and such that $\|g_i\|^2 = \beta_i$ for every $i \in \mathbb{I}_k$ if and only if $\mathbf{b} \prec \lambda(B)$ (completing with zeros if $k \neq d$). $\square$

Proposition 7.4. *Fix the data $\mathcal{F}_0 = \{f_i\}_{i\in\mathbb{I}_{n_0}}$, $\mathbf{a} = \{\alpha_i\}_{i\in\mathbb{I}_n}$ and $t = \operatorname{tr}(\mathbf{a})$ as in 7.1. Then*

$$\mathcal{SC}_{\mathbf{a}}(\mathcal{F}_0) = \big\{S \in \mathcal{G}l(d)^+ \ : \ S \geq S_0 \quad \text{and} \quad \mathbf{b} = (\alpha_i)_{i=n_0+1}^n \prec \lambda(S - S_0)\big\} \subseteq U_t(S_{\mathcal{F}_0} \, , \, d-k) ,$$

where $U_t(S_{\mathcal{F}_0} \, , \, d-k)$ is defined in 5.1.

Proof. Observe that if $\mathcal{F} = (\mathcal{F}_0 \, , \, \mathcal{F}_1) \in \mathbf{F}(n\, , \, d)$, then $S_{\mathcal{F}} = S_{\mathcal{F}_0} + S_{\mathcal{F}_1}$. Denote by $S_0 = S_{\mathcal{F}_0}$ and $B = S - S_0$, for any $S \in \mathcal{G}l(d)^+$. Applying Proposition 7.3 to the matrix B (which must be nonnegative if $S \in \mathcal{SC}_{\mathbf{a}}(\mathcal{F}_0)$), we get the first equality.

The inclusion $\mathcal{SC}_{\mathbf{a}}(\mathcal{F}_0) \subseteq U_t(S_{\mathcal{F}_0} \, , \, d-k)$ follows using that, if $\mathcal{F} = (\mathcal{F}_0 \, , \, \mathcal{F}_1) \in \mathbf{F}(n\, , \, d)$, then $\operatorname{rk} B = \operatorname{rk} S_{\mathcal{F}_1} \leq k = d - (d-k)$. On the other hand, recall that $S \in \mathcal{SC}_{\mathbf{a}}(\mathcal{F}_0) \implies \operatorname{tr} S = t$. $\square$

In order to apply the results of sections 4 and 5 to the problems of this section, we need to recall and restate some objects and notations:

Definition 7.5. Fix the data $\mathcal{F}_0 = \{f_i\}_{i\in\mathbb{I}_{n_0}}$ and $\mathbf{a} = \{\alpha_i\}_{i\in\mathbb{I}_n}$ as in 7.1. Recall that $t = \operatorname{tr}\mathbf{a}$, $\lambda = \lambda(S_{\mathcal{F}_0})$ and $m = d-k$. We rename some notions of previous sections:

1. The vector $\nu(\mathcal{F}_0 \, , \, \mathbf{a}) = \nu_{\lambda\, , \, m}(t) \in \mathbb{R}_{\geq 0}^d$ (see Definition 4.13).
2. The integer $r = r(\mathcal{F}_0 \, , \, \mathbf{a}) \stackrel{\text{def}}{=} \max\{r_{\lambda\, , \, m}(t)\, , \, m\}$ (see Definition 4.11). Note that $d - r \leq k$.
3. The number $c = c(\mathcal{F}_0 \, , \, \mathbf{a}) \stackrel{\text{def}}{=} c_{\lambda\, , \, m}(t)$ (see Definition 4.11).
4. Now we consider the vector $\mu = \mu(\mathcal{F}_0 \, , \, \mathbf{a}) \stackrel{\text{def}}{=} \Big(c(\mathcal{F}_0 \, , \, \mathbf{a}) - \lambda_{r+j}\Big)_{j\in\mathbb{I}_{d-r}} \in (\mathbb{R}_{\geq 0}^{d-r})^{\uparrow}$. Observe that $\operatorname{tr} \mu = \operatorname{tr} \nu_{\lambda\, , \, m}(t) - \operatorname{tr} \lambda = t - \sum\limits_{i\in\mathbb{I}_{n_0}} \|f_i\|^2 = \operatorname{tr} \mathbf{a} - \sum\limits_{i\in\mathbb{I}_{n_0}} \alpha_i = \operatorname{tr} \mathbf{b}$. $\triangle$

Throughout the rest of this section we shall denote by $S_0 = S_{\mathcal{F}_0}$ the frame operator of $\mathcal{F}_0$. Recall that we call $\mathbf{b} = \{\alpha_i\}_{i=n_0+1}^n \in \mathbb{R}_{>0}^k$, $t = \operatorname{tr} \mathbf{a} = \operatorname{tr} S_0 + \operatorname{tr} \mathbf{b}$ and $m = d-k$.

Theorem 7.6. *Fix the data of 7.1 and 7.5. If we assume that $\mathbf{b} \prec \mu(\mathcal{F}_0 \, , \, \mathbf{a})$ then*

1. *The vector $\nu = \nu(\mathcal{F}_0 \, , \, \mathbf{a}) \in \Lambda_{\mathbf{a}}(\mathcal{F}_0)$.*
2. *We have that $\nu \prec \beta$ for every other $\beta \in \Lambda_{\mathbf{a}}(\mathcal{F}_0)$.*

3. *Let* $\{h_i\}_{i\in\mathbb{I}_d}$ *be any ONB of eigenvectors for* $S_0\,,\,\lambda$. *Let*

$$r = r(\mathcal{F}_0\,,\,\mathbf{a}) \quad \text{and} \quad B = \sum_{i=1}^{d-r} \mu_i\, h_{r+i}\otimes h_{r+i}\ . \tag{42}$$

Given any sequence $\mathcal{F}_1 = \{f_i\}_{i=n_0+1}^{n} \in \mathcal{H}^k$ *such that* $\mathcal{F} = (\mathcal{F}_0\,,\,\mathcal{F}_1) \in \mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)$, *then*

$$S_{\mathcal{F}_1} = B \implies \lambda(S_{\mathcal{F}}) = \nu(\mathcal{F}_0\,,\,\mathbf{a})\ . \tag{43}$$

By the hypothesis $\mathbf{b} \prec \mu = \mu(\mathcal{F}_0\,,\,\mathbf{a})$ *such a* $\mathcal{F}_1$ *must exist.*

4. *Moreover, if any of the conditions of item 3 of Theorem 5.10 hold, then*

   (a) *Any ONB of eigenvectors for* $S_0\,,\,\lambda$ *produces the same operator* B *via* (42).

   (b) *Any* $\mathcal{F} = (\mathcal{F}_0\,,\,\mathcal{F}_1) \in \mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)$ *satisfies that* $\lambda(S_{\mathcal{F}}) = \nu(\mathcal{F}_0\,,\,\mathbf{a}) \iff S_{\mathcal{F}_1} = B$.

Proof. Since the elements of $\mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)$ must be frames, we have first to show that $\nu(\mathcal{F}_0\,,\,\mathbf{a}) > 0$. Following the definition of $\nu(\mathcal{F}_0\,,\,\mathbf{a})$, there are two possibilities: In one case $\nu_d = \lambda_m > 0$, because we know from the data given in 7.1 that $\operatorname{rk} S_0 \geq m$. In all other cases $\nu_d = c(\mathcal{F}_0\,,\,\mathbf{a}) \geq c_\lambda(t) > 0$, since $\mathbf{b} > 0 \implies t > \operatorname{tr} S_0$ (see 4.12 and Definitions 4.4 and 4.11).

By Proposition 7.4, we know that the hypothesis $\mathbf{b} \prec \mu = \mu(\mathcal{F}_0\,,\,\mathbf{a})$ assures that there exists a sequence $\mathcal{F}_1 = \{f_i\}_{i=n_0+1}^{n} \in \mathcal{H}^k$ such that $\mathcal{F} = (\mathcal{F}_0\,,\,\mathcal{F}_1) \in \mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)$ and $S_{\mathcal{F}_1} = B$. Then

$$\lambda(S_{\mathcal{F}}) = \lambda(S_{\mathcal{F}_0} + S_{\mathcal{F}_1}) = \lambda(S_{\mathcal{F}_0} + B) = \nu(\mathcal{F}_0\,,\,\mathbf{a})\ ,$$

by Theorem 5.10. Observe that $\Lambda_{\mathbf{a}}(\mathcal{F}_0) \subseteq \Lambda(U_t(S_{\mathcal{F}_0}\,,\,m)\,) = \Lambda_t(\lambda\,,\,m)$, by Propositions 7.4 and 5.3. Hence the majorization of item 2 follows from Theorem 4.15. Finally, the uniqueness results of item 4 follow from Theorem 5.10. $\square$

Corollary 7.7. *Fix the data of 7.1 and 7.5. If we assume that* $\mathbf{b} \prec \mu(\mathcal{F}_0\,,\,\mathbf{a})$ *then*

1. *Any* $\mathcal{F} \in \mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)$ *such that* $\lambda(S_{\mathcal{F}}) = \nu(\mathcal{F}_0\,,\,\mathbf{a})$ *satisfies that*

$$P_f(\mathcal{F}) \leq P_f(\mathcal{G}) \quad \text{for every } \mathcal{G} \in \mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)\,,$$

   and every convex function $f : (0,\infty) \to (0,\infty)$.

2. *If* f *is strictly convex then, for every global minimizer* $\mathcal{F}'$ *of* $P_f(\cdot)$ *on* $\mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)$ *we get that* $\lambda(S_{\mathcal{F}'}) = \nu(\mathcal{F}_0\,,\,\mathbf{a})$.

Proof. It follows from Theorem 7.6 and the majorization facts of 2.1. $\square$

Remark 7.8. The data $\nu(\mathcal{F}_0\,,\,\mathbf{a})$, $r(\mathcal{F}_0\,,\,\mathbf{a})$, $c(\mathcal{F}_0\,,\,\mathbf{a})$ and $\mu(\mathcal{F}_0\,,\,\mathbf{a})$ are essential for Theorem 7.6, both for checking the hypothesis $\mathbf{b} \prec \mu(\mathcal{F}_0\,,\,\mathbf{a})$ and for the construction of the matrix B of (42), which is the frame operator of the optimal extensions of $\mathcal{F}_0$. (Notice that the vector $\mu(\mathcal{F}_0\,,\,\mathbf{a})$ measures how restrictive is the hypothesis $\mathbf{b} \prec \mu(\mathcal{F}_0\,,\,\mathbf{a})$). Fortunately, these objects can be easily computed according to the following algorithm:

1. The numbers $t = \operatorname{tr}\,\mathbf{a}$ and $m = d - k$ are included in the data 7.1.

2. The main point is to compute the irregularity $r = r(\mathcal{F}_0\,,\,\mathbf{a}) = \max\{r_{\lambda\,,\,m}(t)\,,\,m\}$. If $m \leq 0$ then (20) allows us to compute $r_\lambda(t)$; if $m \geq 1$, the number $s^*(\lambda\,,\,m)$ allows us to compute $r_{\lambda\,,\,m}(t)$ as in Definition 4.11.

3. Once r is obtained, we can see that the wideness of the allowed weights $\mathbf{b}$ depends directly of the dispersion on the eigenvalues $(\lambda_{r+1}\,,\,\dots\,,\,\lambda_d)$ of $S_{\mathcal{F}_0}$.

4. Indeed, the number $t_1 \stackrel{\text{def}}{=} \operatorname{tr} \mathbf{b} = t - \operatorname{tr} S_{\mathcal{F}_0}$ is known data. Also $\operatorname{tr} \mu(\mathcal{F}_0 \, , \mathbf{a}) = t_1$. Hence $c(\mathcal{F}_0 \, , \mathbf{a})$ and $\mu(\mathcal{F}_0 \, , \mathbf{a})$ can be directly computed: Let $s = \sum_{r+1}^d \lambda_i$. Then

$$t_1 = \operatorname{tr} \mu = (d-r)\, c(\mathcal{F}_0 \, , \mathbf{a}) - s \implies c(\mathcal{F}_0 \, , \mathbf{a}) = \frac{t_1 + s}{d-r} = \frac{\operatorname{tr} \mathbf{b} + \sum_{r+1}^d \lambda_i}{d-r} \quad .$$

And we have the vector $\mu = \mu(\mathcal{F}_0 \, , \mathbf{a}) = \Big(c(\mathcal{F}_0 \, , \mathbf{a}) - \lambda_{r+j} \Big)_{j \in \mathbb{I}_{d-r}} \in (\mathbb{R}_{\geq 0}^{d-r})^\uparrow$. Then

$$\mathbf{b} \prec \mu \iff \sum_{i=1}^{p} \mathbf{b}_i^\downarrow + \lambda_{d-i+1} \leq \frac{p}{d-r} \Big(\operatorname{tr} \mathbf{b} + \sum_{r+1}^{d} \lambda_i \Big) \quad \text{for} \quad 1 \leq p < d-r \ ,$$

since the last inequalities $s + \sum_{i=1}^p \mathbf{b}_i^\downarrow \leq s + \operatorname{tr} \mathbf{b}$ (for $d - r \leq p \leq k$) clearly hold. △

It is interesting to note that the closer $\mathcal{F}_0$ is to be tight (at least in the last r entries of λ), the more restrictive Theorem 7.6 becomes; but in this case $\mathcal{F}_0$ and $\mathcal{F}_0^\#$ are already "good".

On the other hand, if $\mathcal{F}_0$ is far from being tight then the sequence $(\lambda_{r+1} \, , \, \dots \, , \lambda_d)$ has more dispersion and the hypothesis $\mathbf{b} \prec \mu(\mathcal{F}_0 \, , \mathbf{a})$ becomes less restrictive. It is worth mentioning that in the uniform case $\mathbf{b} = b\, \mathbb{1}_k$, Theorem 7.6 can always be applied.

Observe that as the number k of vectors increases (or as the weights α_i increase) the trace t grows and the numbers r and m become smaller, taking into account more entries λ_i of $\lambda(\mathcal{F}_0)$. This fact offers a criterion for choosing a convenient data k and $\mathbf{b}$ for the completing process.

Remark 7.9 (Construction of optimal completions for the mean square error)**.** Consider the data in 7.1. Apply the algorithm described in Remark 7.8 and assume that $\mathbf{b} \prec \mu(\mathcal{F}_0 \, , \mathbf{a})$. Then construct B as in Eq. (42). In order to obtain an optimal completion of $\mathcal{F}_0$ with prescribed norms we have to construct a sequence $\mathcal{F}_1 \in \mathcal{H}^k$ with frame operator B and norms given by the sequence $\mathbf{b}$ (which is minimal for the mean square error by Corollary 7.7). But once we know B and the weights $\mathbf{b}$ we can apply the results in [7] in order to concretely construct the sequence $\mathcal{F}_1$. △

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