

LOGICS FROM $\sqrt{\cdot}$ QUASI-MV ALGEBRAS

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1. INTRODUCTION

In the usual representation of quantum computational processes, a quantum circuit is identified with an appropriate composition of *quantum gates*, mathematically represented by *unitary operators* acting on pure states of a convenient (n -fold tensor product) Hilbert space $\otimes^n \mathbb{C}^2$ [23] and only taking into account reversible processes. But for many reasons this is unduly restrictive. On the one hand, it does not encompass realistic physical states described by mixtures. In fact, a quantum system rarely is in a pure state. This may be caused, for example, by an incomplete efficiency in the preparation procedure, or else by manipulations on the system as measurements over pure states, both of which produce statistical mixtures. Besides, systems cannot be completely isolated from the environment, leading to decoherence of their states. Non-pure states, namely *mixed states*, are described by *density operators*. On the other hand, there are interesting processes that cannot be encoded in unitary evolutions, such as measurements in the middle of the process. Several authors [2, 11, 12, 25] have paid attention to a more general model of quantum computational processes, where pure states and unitary operators are replaced by density operators and quantum operations, respectively. Within this approach, evolution over time is no longer seen as necessarily reversible.

The connection between quantum computational logic with mixed states and fuzzy logic arises when we single out a system of quantum gates that, once interpreted under a probabilistic semantics, yield some kind of operation in the real interval $[0, 1]$ associated to fuzzy logic, such as continuous t-norms. Quasi-MV algebras and $\sqrt{\cdot}$ quasi-MV algebras, the object of the present paper, are a case in point. *Quasi-MV algebras* (for short, qMV algebras) were introduced in [22] with the aim of providing a convenient abstraction of the algebra over the set of all density operators of the Hilbert space \mathbb{C}^2 , endowed with a suitable stock of quantum logical gates. Independent of their original quantum computation motivation, qMV algebras are of (purely algebraic) interest as a generalisation of MV algebras to the quasibsubtractive and weakly τ -regular case (in the sense of [21]). Later, $\sqrt{\cdot}$ *quasi-MV algebras* (for short, $\sqrt{\cdot}$ qMV algebras) were introduced as term expansions of qMV algebras by

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an operation of square root of the inverse [18]. The papers cited above contain the basics of the structure theory for these varieties, including appropriate standard completeness theorems w.r.t. the algebras over the complex numbers that constituted the starting point of the whole investigation. The algebraic properties of qMV algebras and $\sqrt{\cdot}$ qMV algebras have been investigated in greater detail in subsequent papers (e.g. [24], [5], [20]).

Although the study of qMV algebras and $\sqrt{\cdot}$ qMV algebras has its roots in an inquiry into the structure of quantum computational circuits – and thus, to all intents and purposes, in a problem that is logical rather than algebraic – the work referred to so far sidesteps, as it were, the logical aspects connected with these structures. On the other hand, [6] contains a detailed investigation into some abstract logics arising quite naturally out of qMV algebras; the results obtained in this paper will be summarised in Section 3 below. The purpose of the present article is to extend the scope of this investigation to $\sqrt{\cdot}$ qMV algebras. We will therefore introduce, mutually compare and (in some cases) axiomatise several logics arising from the variety of $\sqrt{\cdot}$ qMV algebras and from some of its important subclasses (Section 4); subsequently, we will investigate the same logics by resorting to the methods and techniques of abstract algebraic logic (Section 5). In the interests of space, we will not recapitulate the basic notions of the theory of $\sqrt{\cdot}$ qMV algebras: for a comprehensive introduction, the reader could consult the self-contained papers [18], [24] and [5]. Some acquaintance with the concepts, results, and notation introduced in these papers is presupposed in what follows. As to the rest, except where indicated otherwise, we keep to the terminological and notational conventions typically adopted in universal algebra and abstract algebraic logic. In particular, we will consider sentential logics formulated in languages with a denumerable stock of propositional variables p_0, p_1, \dots and finitely many connectives. As customary, we do not distinguish between logical languages (and their associated connectives) and algebraic similarity types (and their associated operation symbols); so, $\mathbf{Fm}(\mathcal{L})$ will denote e.g. both the set of all formulas of \mathcal{L} (seen as a logical language) and the set of all terms of \mathcal{L} (seen as an algebraic similarity type). The letters x, y, z, \dots or p, q, r, \dots are used variously to denote variables of \mathcal{L} , while the letters $\alpha, \beta, \gamma, \phi, \psi, \dots$ are used as metavariables for arbitrary members of $\mathbf{Fm}(\mathcal{L})$. The expression $\alpha[p_1, \dots, p_n]$ means that the set of propositional variables occurring in α is a (proper or improper) subset of $\{p_1, \dots, p_n\}$, while $\alpha[\beta_1/\alpha_1, \dots, \beta_n/\alpha_n]$ denotes the result of replacing one or more occurrences of β_i in α by α_i . If \mathbf{A} is an algebra of type \mathcal{L} and $a_1, \dots, a_n \in A$, by $\alpha^{\mathbf{A}}(a_1, \dots, a_n)$ we denote the result of the application of the term operation¹ $\alpha^{\mathbf{A}}$ to the elements a_1, \dots, a_n . Superscripts are dropped when not strictly necessary, in the interests of irredundancy. We will feel free to employ self-explanatory vectorial notations whenever the lengths of the indicated strings are clear from the context.

Since most logics we deal with share the same language, we will be rather sloppy as regards the distinction between abstract logics (or deductive systems) and their associated consequence relations. Except when indicated otherwise, these terms will be regarded as

¹If $\alpha(x_1, \dots, x_n) \in \mathbf{Fm}(\mathcal{L})$, and v is a valuation on \mathbf{A} mapping each x_i to $a_i \in A$, then the *term operation* $\alpha^{\mathbf{A}} = \alpha^{\mathbf{A}}(a_1, \dots, a_n)$ induced on \mathbf{A} by α is $v(\alpha(x_1, \dots, x_n))$.

synonymous. Further, $\&$, \hookrightarrow are used as symbols for the conjunction and the implication, respectively, of the (classical) metalanguage by means of which we describe our logical and algebraic structures.

2. BASIC NOTIONS FROM ABSTRACT ALGEBRAIC LOGIC

In this section we briefly recap several notions from abstract algebraic logic to be used in the present paper. Standard references for the material that follows include [4, 15].

A *deductive system* (informally, a *logical system* or a *logic*) is a pair $S = \langle \mathcal{L}, \vdash_S \rangle$, where \mathcal{L} is an algebraic language type, and \vdash_S is a finitary and substitution-invariant consequence relation over $\mathcal{P}(\mathbf{Fm}(\mathcal{L}))$. If \mathbb{K} is a class of algebras of a given type \mathcal{L} , and S a deductive system of the same type. \mathbb{K} is called an *algebraic semantics* for S if there exist finite families of equations $\{\delta_1 \approx \epsilon_1, \dots, \delta_r \approx \epsilon_r\}$ in one variable s.t. for all $\Gamma \cup \{\phi, \psi\} \subseteq \mathbf{Fm}(\mathcal{L})$, and $l = 1, \dots, r$

$$\Gamma \vdash_S \phi \quad \text{iff} \quad \{\delta_t(\xi) \approx \epsilon_t(\xi) : \xi \in \Gamma \text{ and } t = 1, \dots, r\} \models_{\mathbb{K}} \delta_l(\phi) \approx \epsilon_l(\phi).$$

\mathbb{K} is said to be *equivalent* to S if there exists a finite system $\{\Delta_1, \dots, \Delta_m\}$ of formulae in two variables s.t. for all $\Gamma \cup \{\phi, \psi\} \subseteq \mathbf{Fm}(\mathcal{L})$:

$$\{\delta_t(\phi \Delta_i \psi) \approx \epsilon_t(\phi \Delta_i \psi) : i = 1, \dots, m \text{ and } t = 1, \dots, r\} \models_{\mathbb{K}} \phi \approx \psi$$

and

$$\phi \approx \psi \models_{\mathbb{K}} \{\delta_t(\phi \Delta_i \psi) \approx \epsilon_t(\phi \Delta_i \psi) : i = 1, \dots, m \text{ and } t = 1, \dots, r\}.$$

The equations $\delta_l \approx \epsilon_l$, $l = 1, \dots, r$, are called *defining equations* for S and \mathbb{K} , while the family $\{\Delta_1, \dots, \Delta_m\}$ of composite binary connectives is called a system of *equivalence formulae* for S and \mathbb{K} . A deductive system is said to be *algebraisable* iff it has an equivalent variety semantics. By virtue of [4, Corollary 2.11], \mathbb{K} may be identified with the quasivariety generated by \mathbb{K} , that is the class of algebras obtained by closing \mathbb{K} under isomorphic images (**I**), subalgebras (**S**), products (**P**), and ultraproducts (**P_u**), and containing the trivial algebra; in other words, \mathbb{K} is an *equivalent quasivariety semantics*. If \mathbb{K} is a variety – i.e. a class of algebras closed under homomorphic images (**H**), subalgebras (**S**), and products (**P**), or equivalently a class defined by means of a set of identities – then \mathbb{K} is an *equivalent variety semantics*, and S is said to be *strongly algebraisable*.

Let \mathbf{A} be an algebra of type \mathcal{L} , and F a subset of A . An \mathcal{L} -*matrix* (or simply a *matrix*, when \mathcal{L} is understood) is a tuple $\langle \mathbf{A}, F \rangle$. For any matrix $\langle \mathbf{A}, F \rangle$ and $\Gamma \cup \{\phi\} \subseteq \mathbf{Fm}(\mathcal{L})$, let $\models_{\langle \mathbf{A}, F \rangle} \phi$ be the relation defined by $\Gamma \models_{\langle \mathbf{A}, F \rangle} \phi$ if:

$$h(\phi) \in F \text{ for any } \psi \in \Gamma \text{ implies } h(\phi) \in F, \text{ for any } h : \mathbf{Fm}(\mathcal{L}) \rightarrow \mathbf{A}.$$

If M is a class of matrices, then $\Gamma \models_M \phi$ if $\Gamma \models_{\langle \mathbf{A}, F \rangle} \phi$, for every $\langle \mathbf{A}, F \rangle \in M$. Let S be a deductive system of type \mathcal{L} and \mathbf{A} an algebra of the same type, with $F \subseteq A$. The subset F of A is called an *S-deductive filter*, or just a *deductive filter* when S is understood, if $\Gamma \vdash_S \phi$ implies $\Gamma \models_{\langle \mathbf{A}, F \rangle} \phi$, for all $\Gamma, \{\phi\} \subseteq \text{Fm}(\mathcal{L})$. The elements in F are said to be *designated*. If F is an S -deductive filter, then the \mathcal{L} -matrix $\langle \mathbf{A}, F \rangle$ is called a *matrix model* of S . A congruence θ on \mathbf{A} is said to be *compatible* with F in case $a \in F$ and $a\theta b$ implies $b \in F$. The largest congruence on \mathbf{A} compatible with F is called the *Leibniz congruence* on \mathbf{A} over F , and is denoted by $\Omega_{\mathbf{A}}(F)$. The natural map $\Omega_{\mathbf{A}}$ that assigns to every deductive filter F its corresponding Leibniz congruence $\Omega_{\mathbf{A}}(F)$ is called the *Leibniz operator* on \mathbf{A} . A matrix $\langle \mathbf{A}, F \rangle$ is said to be *reduced* if $\Omega_{\mathbf{A}}(F)$ is the identity relation on A . A matrix model $\langle \mathbf{A}, F \rangle$ is said *reduced* if $\Omega_{\mathbf{A}}(F)$ is reduced. A deductive system is called *regularly algebraizable* if every reduced matrix model has exactly one designated element. A logic is called *protoalgebraic* if the Leibniz operator is monotone on the set of deductive filters, i.e. if $F, G \subseteq A$ and $F \subseteq G$, then $\Omega_{\mathbf{A}}(F) \subseteq \Omega_{\mathbf{A}}(G)$. Finally, a deductive system S is said to be *selfextensional* if it enjoys the *weak Frege principle*:

$$\text{if } \phi \vdash_S \psi \text{ and } \psi \vdash_S \phi, \text{ then for any formula } \delta \text{ with the variable } p, \\ \delta[p/\phi] \vdash_S \delta[p/\psi] \text{ and } \delta[p/\psi] \vdash_S \delta[p/\phi].$$

3. LOGICS FROM QW ALGEBRAS

As observed in the introduction, qMV algebras are generalisations of MV algebras. The variety of MV algebras is not, properly speaking, the equivalent variety semantics of infinite-valued Łukasiewicz logic, just as the variety of Boolean rings is not, properly speaking, the equivalent variety semantics of the classical propositional calculus. The rôle of the equivalent variety semantics of infinite-valued Łukasiewicz logic, in fact, is played by the term equivalent variety of Wajsberg algebras, introduced and investigated in [16]. The similarity type of Wajsberg algebras - containing implication, negation and a truth constant - seems better suited for logical purposes than does the type of MV algebras. The following concept of *quasi-Wajsberg algebra* does the same job in our case: quasi-Wajsberg algebras are to qMV algebras just as Wajsberg algebras are to MV algebras.

Definition 1. A quasi-Wajsberg algebra is an algebra $\mathbf{A} = \langle A, \rightarrow, \neg, 1 \rangle$ of type $\langle 2, 1, 0 \rangle$ satisfying the following identities:

$$\begin{aligned} \text{QW1.} \quad & 1 \rightarrow (x \rightarrow y) \approx x \rightarrow y \\ \text{QW2.} \quad & (x \rightarrow y) \rightarrow ((y \rightarrow z) \rightarrow (x \rightarrow z)) \approx 1 \\ \text{QW3.} \quad & (x \rightarrow y) \rightarrow y \approx (y \rightarrow x) \rightarrow x \\ \text{QW4.} \quad & (\neg x \rightarrow \neg y) \rightarrow (y \rightarrow x) \approx 1 \\ \text{QW5.} \quad & \neg \neg x \approx x \\ \text{QW6.} \quad & 1 \rightarrow \neg(1 \rightarrow x) \approx \neg(1 \rightarrow x) \end{aligned}$$

We denote the class of quasi-Wajsberg algebras by \mathbf{qW} .

Clearly, quasi-Wajsberg algebras form a variety. In contrast to Wajsberg algebras, one may verify that in general a quasi-Wajsberg algebra need not satisfy $1 \rightarrow x \approx x$. A prominent example follows:

Example 2. \mathbf{S} is the quasi-Wajsberg algebra $\langle [0, 1] \times [0, 1], \rightarrow^{\mathbf{S}}, \neg^{\mathbf{S}}, 1^{\mathbf{S}} \rangle$, where:

- $\langle a, b \rangle \rightarrow^{\mathbf{S}} \langle c, d \rangle = \langle \min(1, 1 - a + c), \frac{1}{2} \rangle$;
- $\neg^{\mathbf{S}} \langle a, b \rangle = \langle 1 - a, 1 - b \rangle$;
- $1^{\mathbf{S}} = \langle 1, \frac{1}{2} \rangle$.

True to form, \mathbf{qW} is term equivalent to the variety \mathbf{qMV} of quasi-MV algebras:

Theorem 3.

- (1) If $\mathbf{A} = \langle A, \rightarrow, \neg, 1 \rangle$ is a quasi-Wajsberg algebra, then $f(\mathbf{A}) = \langle A, \oplus, \neg, 0, 1 \rangle$, where $a \oplus b = \neg a \rightarrow b$ and $0 = \neg 1$, is a quasi-MV algebra.
- (2) If $\mathbf{A} = \langle A, \oplus, \neg, 0, 1 \rangle$ is a quasi-MV algebra, then $g(\mathbf{A}) = \langle A, \rightarrow, \neg, 1 \rangle$, where $a \rightarrow b = \neg a \oplus b$, is a quasi-Wajsberg algebra.
- (3) f and g are mutually inverse correspondences.

Having singled out the variety of algebras at the centre of our interest, we now proceed to extract some logics from it. To this purpose, we need a general framework which is somewhat reminiscent of the one whose foundations are laid down in [3]. There, a theory is proposed for varieties of algebras where *partial orders* can be defined by means of equations. Here, on the other hand, we focus on the more general case of *preorders*. We define, in fact:

Definition 4. Let \mathbf{A} be an algebra of type \mathcal{L} and let P be a preorder on \mathbf{A} . P is called an equationally defined preorder for \mathbf{A} iff there are \mathcal{L} -formulae $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n$ of appropriate arities s.t., for any $a, b \in A$,

$$\langle a, b \rangle \in P \text{ iff for any } i \leq n, \text{ and for any } \vec{c} \in A, \alpha_i^{\mathbf{A}}(a, b, \vec{c}) = \beta_i^{\mathbf{A}}(a, b, \vec{c}).$$

In this case we also say that $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n$ define the preorder P on \mathbf{A} . If \mathbb{K} is a class of algebras of type \mathcal{L} and $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n$ define a preorder P on every algebra \mathbf{A} in \mathbb{K} , we say that P is an equationally defined preorder for \mathbb{K} .

An example of equationally defined preorder (which is actually a partial order) is the Wajsberg algebra ordering, where, for any a, b in a Wajsberg algebra \mathbf{A} , $\langle a, b \rangle \in P$ iff $a \rightarrow b = 1$.

The previous definition allows one to introduce, for every equationally defined preorder on a class of algebras, at least two logics: a logic whose valid inferences preserve the property of being evaluated as a designated maximum element (the top element, in case of partial orders); and a logic whose valid inferences are such that their conclusions are always

evaluated above any element which is below the evaluations of all the premisses. If the elements of the algebras at issue are thought of as “degrees of truth”, in the spirit of some many-valued logics, the former logic can be seen as a logic of *preservation of absolute truth*, while the latter can be regarded as a logic of *preservation of degrees of truth* [13]. More specifically:

Definition 5. *Let \mathbb{K} be a class of algebras of type \mathcal{L} , and let \leq be a fixed but arbitrary equationally defined preorder on \mathbb{K} . Further, let $\gamma_1, \dots, \gamma_n, \alpha$ be \mathcal{L} -formulae, and let c be a constant (which is added to \mathcal{L} if it is not already present) denoting a designated maximum element of \leq in any $\mathbf{A} \in \mathbb{K}$. We introduce two finitary logics:*

- the truth-preserving logic $\vdash_{\mathbb{K}}^{\leq, c}$, defined by

$$\gamma_1, \dots, \gamma_n \vdash_{\mathbb{K}}^{\leq, c} \alpha \quad \text{iff} \quad \mathbb{K} \models \gamma_1 \approx c \ \& \ \dots \ \& \ \gamma_n \approx c \Rightarrow \alpha \approx c,$$

- the logic of degrees of truth $\vdash_{\mathbb{K}}^{\leq}$, defined by

$$\gamma_1, \dots, \gamma_n \vdash_{\mathbb{K}}^{\leq} \alpha \quad \text{iff} \quad \mathbb{K} \models p \leq \gamma_1 \ \& \ \dots \ \& \ p \leq \gamma_n \Rightarrow p \leq \alpha.$$

For example, if \mathbb{K} is the variety \mathbb{W} of Wajsberg algebras and \leq is the induced lattice order that is equationally definable for every Wajsberg algebra, $\vdash_{\mathbb{W}}^{\leq, 1}$ is nothing but the infinite-valued Łukasiewicz logic, while $\vdash_{\mathbb{W}}^{\leq}$ is the “Łukasiewicz logic that preserves degrees of truth” of [14].

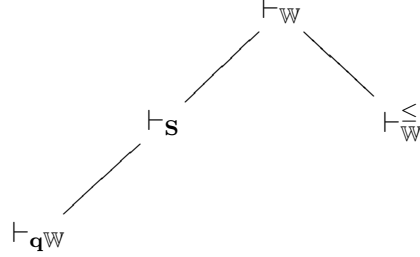
What happens, instead, if the variety of interest is $q\mathbb{W}$? In this case, according to [22], we have an induced *preorder* \leq which is equationally definable for every $q\mathbb{W}$ algebra. This immediately yields at least two logics worthy of investigation, $\vdash_{q\mathbb{W}}^{\leq, 1}$ and $\vdash_{q\mathbb{W}}^{\leq}$, in full similarity with the previous case. However, there is an important dissimilarity with the case of Wajsberg algebras: while the standard Wajsberg algebra $\mathbf{W}_{[0,1]}$ on the closed real unit interval generates \mathbb{W} as a quasivariety – i.e. a quasi-identity holds in \mathbb{W} iff it holds in $\mathbf{W}_{[0,1]}$ – whence no new logic is obtained if we replace the whole of \mathbb{W} by the singleton $\{\mathbf{W}_{[0,1]}\}$, the standard square $q\mathbb{W}$ algebra \mathbf{S} generates \mathbb{W} as a variety, but not as a quasivariety. It is therefore natural to consider the additional logics $\vdash_{\mathbf{S}}^{\leq, 1}$ and $\vdash_{\mathbf{S}}^{\leq}$ (notice that brackets are omitted from $\{\mathbf{S}\}$ for the sake of simplicity).

The above-mentioned logics of degrees of truth are both, as a matter of fact, old hats; they coincide with the Łukasiewicz logic that preserves degrees of truth. We have, namely:

Theorem 6. *Let $\alpha_1, \dots, \alpha_n, \alpha$ be arbitrary formulas. The following statements are equivalent:*

- (1) $\alpha_1, \dots, \alpha_n \vdash_{\mathbb{W}}^{\leq} \alpha$,
- (2) $\alpha_1, \dots, \alpha_n \vdash_{q\mathbb{W}}^{\leq} \alpha$,
- (3) $\alpha_1, \dots, \alpha_n \vdash_{\mathbf{S}}^{\leq} \alpha$.

The situation changes, on the other hand, if we consider truth-preserving logics. The logics $\vdash_{\mathbf{qW}}^{\leq,1}$ and $\vdash_{\mathbf{S}}^{\leq,1}$ (hereafter denoted, respectively, by $\vdash_{\mathbf{qW}}$ and $\vdash_{\mathbf{S}}$) are actually different logics, and both are in turn distinct from $\vdash_{\mathbf{W}}$. The mutual relationships among these distinct logics are depicted in the following diagram:



$\vdash_{\mathbf{qW}}$ and $\vdash_{\mathbf{S}}$ admit of Hilbert-style axiomatisations. $\vdash_{\mathbf{qW}}$ can be axiomatised by means of the postulates listed below:

(A1)	$\alpha \rightarrow (\beta \rightarrow \alpha)$	(A2)	$(\alpha \rightarrow \beta) \rightarrow ((\beta \rightarrow \gamma) \rightarrow (\alpha \rightarrow \gamma))$
(A3)	$((\alpha \rightarrow \beta) \rightarrow \beta) \rightarrow ((\beta \rightarrow \alpha) \rightarrow \alpha)$	(A4)	$(\neg\alpha \rightarrow \neg\beta) \rightarrow (\beta \rightarrow \alpha)$
(A5)	1	(qMP)	$1 \rightarrow \alpha, 1 \rightarrow (\alpha \rightarrow \beta) \vdash 1 \rightarrow \beta$
(Reg)	$\alpha \vdash 1 \rightarrow \alpha$	(AReg1)	$1 \rightarrow (\alpha \rightarrow \beta) \vdash \alpha \rightarrow \beta$
(AReg2)	$1 \rightarrow \neg(\alpha \rightarrow \beta) \vdash \neg(\alpha \rightarrow \beta)$	(AReg3)	$1 \rightarrow \neg 1 \vdash \neg 1$
(Inv1)	$\alpha \vdash \neg\neg\alpha$	(Inv2)	$\neg\neg\alpha \vdash \alpha$
(Flat)	$\alpha, \neg 1 \vdash \neg\alpha$		

Notice that this calculus has the same axioms as $\vdash_{\mathbf{W}}$ and has the rules that we call *quasi Modus Ponens* (qMP), *regularity* (Reg), *antiregularity* (AReg1-3), *involution* (Inv1-2) and *flatness* (Flat). To obtain a Hilbert-style calculus for $\vdash_{\mathbf{S}}$, it suffices to add to these postulates the *ex falso quodlibet* rule

$$\neg 1 \vdash \alpha. \quad (\text{EFQ})$$

Neither $\vdash_{\mathbf{qW}}$ nor $\vdash_{\mathbf{S}}$ is a protoalgebraic logic. This circumstance renders the characterisation of their reduced matrix models a nontrivial problem. Fortunately, this problem can be solved in both cases. To do so, we need the following definitions:

Definition 7.

- (1) A *qW algebra* \mathbf{A} is said to be *narrow* iff, for every $a \in A$, a/χ has at most two elements and for every $a \in \mathcal{R}(\mathbf{A}) - \{0, 1\}$, a/χ has exactly one element.
- (2) A *qW algebra* \mathbf{A} is said to be *almost Wajsberg* if it is narrow and, in addition, it is *trivial if flat*.

The two classes \mathbf{nqW} of narrow qW algebras and \mathbf{aW} of almost Wajsberg algebras are elementary classes, i.e. they are defined by a set of first order formulas with identity, but fail to be closed under direct products. We have that:

Theorem 8. *The following conditions are equivalent:*

- (1) $\langle \mathbf{A}, F \rangle$ is a reduced model of $\vdash_{\mathbf{qW}}$,
- (2) \mathbf{A} is a member of \mathbf{nqW} and $F = \{1\}$.

Theorem 9. *The following conditions are equivalent:*

- (1) $\langle \mathbf{A}, F \rangle$ is a reduced model of $\vdash_{\mathbf{s}}$,
- (2) \mathbf{A} is a member of \mathbf{aW} and $F = \{1\}$.

Using the methods of second order abstract algebraic logic (for which see [15]) it is possible to characterise also the reduced full generalised matrix models of both logics.

4. LOGICS FROM $\sqrt{\cdot}$ qW ALGEBRAS

4.1. $\sqrt{\cdot}$ qW algebras. Quasi-MV algebras are perhaps only of limited interest for quantum computation, except for their being a “baby example” which it may be easier to cope with first, before proceeding to the investigation of richer structures. $\sqrt{\cdot}$ quasi-MV algebras are more interesting from this point of view, in that they contain the genuinely quantum operation of *square root of the inverse*, which provably admits of no fuzzy counterpart (see e.g. [10]). If we want to investigate the logic of $\sqrt{\cdot}$ quasi-MV algebras, it is expedient to follow the route adopted in the previous section: we introduce a term equivalent variant where implication is a primitive operation. We do this in the following

Definition 10. *A $\sqrt{\cdot}$ quasi-Wajsberg algebra (for short, $\sqrt{\cdot}$ qW algebra) is an algebra $\mathbf{A} = \langle A, \rightarrow, \sqrt{\cdot}, 1 \rangle$ of type $\langle 2, 1, 0 \rangle$ such that, upon defining $\neg a = \sqrt{\cdot} \sqrt{\cdot} a$ for all $a \in A$, the following conditions are satisfied:*

- SQ1. $\langle A, \rightarrow, \neg, 1 \rangle$ is a quasi-Wajsberg algebra;
- SQ2. $1 \rightarrow \sqrt{\cdot}(a \rightarrow b) = \sqrt{\cdot}(1 \rightarrow \sqrt{\cdot}(a \rightarrow b))$ for all $a, b \in A$.

A $\sqrt{\cdot}$ qW algebra is called *Cartesian* if the following quasiequation is satisfied:

$$\mathbf{C}. \quad 1 \rightarrow x \approx 1 \rightarrow y \ \& \ 1 \rightarrow \sqrt{\cdot}x \approx 1 \rightarrow \sqrt{\cdot}y \hookrightarrow x \approx y.$$

A $\sqrt{\cdot}$ qW algebra is called *flat* if the equation $1 \approx 0$ is satisfied.

We denote by $\sqrt{\cdot}\mathbf{qW}$ and \mathbf{FW} , respectively, the varieties of $\sqrt{\cdot}$ qW algebras and flat $\sqrt{\cdot}$ qW algebras, and by \mathbf{CW} the quasivariety of Cartesian $\sqrt{\cdot}$ qW algebras.

Hereafter, we will sometimes use the following convenient abbreviations:

$$\begin{aligned} \Box \alpha &:= 1 \rightarrow \alpha \\ \alpha \Rightarrow \beta &:= \sqrt{\cdot} \alpha \rightarrow \sqrt{\cdot} \beta \\ \alpha \leftrightarrow \beta &:= \{\alpha \rightarrow \beta, \beta \rightarrow \alpha\} \end{aligned}$$

Adopting the strategy of Theorem 3, it is possible to prove that:

Lemma 11.

- (1) $\sqrt{\cdot}\mathbf{qW}$ is term equivalent to $\sqrt{\cdot}\mathbf{qMV}$;
- (2) \mathbb{CW} is term equivalent to the quasivariety \mathbb{C} of Cartesian $\sqrt{\cdot}\mathbf{qMV}$ algebras;
- (3) \mathbb{FW} is term equivalent to the variety \mathbb{F} of flat $\sqrt{\cdot}\mathbf{qMV}$ algebras.

Proof. We capitalise on the fact that \mathbf{qW} is provably term equivalent to \mathbf{qMV} (Theorem 3). The only additional translation we need is for the constant k , which is not in the signature of $\sqrt{\cdot}\mathbf{qW}$. However, we can translate k as $\Box\sqrt{\cdot}\Box p_0$. The remainder of the proof is straightforward and is omitted. \square

As a consequence of Lemma 11, most of the structure theory of $\sqrt{\cdot}\mathbf{qMV}$ algebras and Cartesian $\sqrt{\cdot}\mathbf{qMV}$ algebras carries over, respectively, to $\sqrt{\cdot}\mathbf{qW}$ and \mathbb{CW} . In the following, therefore, we will feel free to borrow definitions and results from such a setting without special mention. In particular, we observe that the weak order \leq and the strong order \preceq [19], defined by

$$\begin{aligned} a \leq b & \text{ iff } 1^{\mathbf{A}} = a \rightarrow^{\mathbf{A}} b; \\ a \preceq b & \text{ iff } 1^{\mathbf{A}} = a \rightarrow^{\mathbf{A}} b \text{ and } 1^{\mathbf{A}} = a \Rightarrow^{\mathbf{A}} b \end{aligned}$$

($\mathbf{A} \in \sqrt{\cdot}\mathbf{qW}$ and $a, b \in A$) are equationally defined preorders for the variety $\sqrt{\cdot}\mathbf{qW}$.

4.2. A notable class of $\sqrt{\cdot}\mathbf{qW}$ algebras. In this subsection we will focus on a class - a quasivariety, indeed - of $\sqrt{\cdot}\mathbf{qW}$ algebras which is especially noteworthy from a logical viewpoint. As we will see, the 1-assertional logic of this class not only satisfies modus ponens for the \mathbf{qW} implication, but has interesting connections with infinite-valued Łukasiewicz logic. We give here a first characterisation of this class.

Lemma 12. *For a Cartesian $\sqrt{\cdot}\mathbf{qW}$ algebra \mathbf{A} the following are equivalent:*

- (1) $1^{\mathbf{A}}/\chi = \{1^{\mathbf{A}}\}$;
- (2) $\mathbf{A} \models p \approx 1 \ \& \ p \rightarrow q \approx 1 \hookrightarrow q \approx 1$;
- (3) $\mathbf{A} \models \Box p \approx 1 \hookrightarrow p \approx 1$.

Proof. 1. and 3. are obviously equivalent.

2. implies 3. Suppose that $a \in A$ and $\Box a = 1$. By properties of \mathbf{qW} algebras, we have that

$$\Box a \rightarrow a = \Box(a \rightarrow a) = a \rightarrow a = 1,$$

whence by 2. $a = 1$.

3. implies 2. Suppose $a = a \rightarrow b = 1$. This means $\Box b = 1$, which implies $b = 1$ in virtue of 3. \square

Definition 13. A Cartesian \sqrt{I} qW algebra which satisfies any one of the equivalent conditions of Lemma 12 is called a 1-Cartesian algebra.

It is readily checked that 1-Cartesian algebras form a proper quasivariety (to see that they are not closed w.r.t. products, it suffices to consider $\mathbf{D}_r \times \mathbf{D}_r$), which will be referred to by 1CW. Examples of 1-Cartesian algebras include:

- The pivoted rotation² $\mathbf{Rt}(\mathbf{W}_n)$ of any finite Wajsberg chain \mathbf{W}_n ;
- the disc standard algebra \mathbf{D}_r ;
- the strongly Cartesian algebra $\mathbf{N}_{\mathbf{S}_r}$ [24];
- the algebra \mathbf{O}_r , i.e. the subalgebra of the square standard algebra \mathbf{S}_r generated by the points in the open unit square

$$\left\{ \langle a, b \rangle \in [0, 1]^2 : a, b \notin \{0, 1\} \right\},$$

which comprises just the open square plus the boundary elements

$$\left\langle 0, \frac{1}{2} \right\rangle, \left\langle \frac{1}{2}, 1 \right\rangle, \left\langle 1, \frac{1}{2} \right\rangle, \left\langle \frac{1}{2}, 0 \right\rangle.$$

All of the previously listed algebras are examples of 1-Cartesian subalgebras of \mathbf{S}_r .

Lemma 14. 1CW is generated by \mathbf{O}_r .

Proof. We first prove that 1CW is generated by the class of 1-Cartesian subalgebras of \mathbf{S}_r . If \mathbb{K} is a class of \sqrt{I} qW algebras, let

$$\mathbf{S}_1(\mathbb{K}) = \mathbf{S}(\mathbb{K}) \cap 1\text{CW}.$$

We claim that, for any class \mathbb{K} of Wajsberg algebras: (i) $\mathbf{S}_1 \wp \mathbf{S}(\mathbb{K}) \subseteq \mathbf{SS}_1 \wp(\mathbb{K})$; (ii) $\mathbf{S}_1 \wp \mathbf{P}(\mathbb{K}) \subseteq \mathbf{PS}_1 \wp(\mathbb{K})$; (iii) $\mathbf{S}_1 \wp \mathbf{P}_U(\mathbb{K}) \subseteq \mathbf{P}_U \mathbf{S}_1 \wp(\mathbb{K})$. In fact, as regards (i),

$$\begin{aligned} \mathbf{S}_1 \wp \mathbf{S}(\mathbb{K}) &\subseteq \mathbf{S}_1 \mathbf{S} \wp(\mathbb{K}) && \text{proved in [5]} \\ &= \mathbf{S}_1 \wp(\mathbb{K}) && \text{properties of } \mathbf{S}_1 \\ &= \mathbf{SS}_1 \wp(\mathbb{K}) && \text{properties of } \mathbf{S}_1 \end{aligned}$$

Moreover, remark that the isomorphism of Lemma 42 in [5], when restricted to 1-Cartesian subalgebras of pair algebras over products of a given class \mathbb{K} of Wajsberg algebras, clearly

²For the general concept of pivoted rotation see e.g. [20]; the pivoted rotation $\mathbf{Rt}(\mathbf{W}_n)$ is the \sqrt{I} qW algebra obtained from the n -element Wajsberg chain replacing its fixpoint k for negation by a median cloud k/χ in 1-1 correspondence with the chain itself.

yields products of 1-Cartesian subalgebras of pair algebras over members of \mathbb{K} . This takes care of (ii), and a similar argument establishes (iii).

Now, for any Wajsberg algebra \mathbf{A} , we have that $\mathbf{A} \in \text{ISPP}_{\mathcal{U}}(\mathbf{W}_{[0,1]})$, whence

$$\begin{aligned} \mathbf{S}_1 \wp(\mathbf{A}) &\subseteq \mathbf{S}_1 \wp \text{ISPP}_{\mathcal{U}}(\mathbf{W}_{[0,1]}) && \text{monotonicity of } \mathbf{S}_1 \wp \\ &\subseteq \text{ISPP}_{\mathcal{U}} \mathbf{S}_1 \wp(\mathbf{W}_{[0,1]}) && \text{by the above argument} \\ &= \mathbf{Q}(\mathbf{S}_1 \wp(\mathbf{W}_{[0,1]})) && \text{by } \mathbf{Q} = \text{ISPP}_{\mathcal{U}} \\ &= \mathbf{Q}(\mathbf{S}_1(\mathbf{S}_r)) && \text{by } \wp(\mathbf{W}_{[0,1]}) = \mathbf{S}_r \end{aligned}$$

Our main conclusion now follows from the fact that \mathbf{O}_r is the largest algebra in $\mathbf{S}_1(\mathbf{S}_r)$. \square

4.3. Presentation and mutual comparison of our logics. We observed earlier that the $\sqrt{\cdot}$ qW weak order \leq and the $\sqrt{\cdot}$ qW strong order \preceq are equationally defined preorders for the variety $\sqrt{\cdot}\mathbf{qW}$. As a consequence, against the background of our theory of logics from equationally defined preorders (Section 3), we are allowed to consider at least the following logics arising out of $\sqrt{\cdot}$ qW algebras (the constant \top is added, when necessary, to the type of $\sqrt{\cdot}\mathbf{qW}$ and denotes the element $\langle 1, 1 \rangle$ in the standard algebra \mathbf{S}_r):

- the *weak truth-preserving logic* $\vdash_{\sqrt{\cdot}\mathbf{qW}}^{\leq, 1}$;
- the *weak logic of degrees of truth* $\vdash_{\sqrt{\cdot}\mathbf{qW}}^{\leq}$;
- the *standard weak truth-preserving logic* $\vdash_{\mathbf{S}}^{\leq, 1}$;
- the *standard weak logic of degrees of truth* $\vdash_{\mathbf{S}}^{\leq}$;
- the *standard strong truth-preserving logic* $\vdash_{\mathbf{S}}^{\preceq, \top}$;
- the *standard strong logic of degrees of truth* $\vdash_{\mathbf{S}}^{\preceq}$.

In view of the considerations of Subsection 4.2, we may add to this list also:

- the *MP truth-preserving logic* $\vdash_{\mathbf{O}}^{\leq, 1}$.

Remark that the subscripts in \mathbf{S}_r and \mathbf{O}_r have been dropped for the sake of notational simplicity.

Most logics in the above schema look noteworthy in some respect. Of course, $\vdash_{\sqrt{\cdot}\mathbf{qW}}^{\leq, 1}$ is the 1-assertional logic of the variety $\sqrt{\cdot}\mathbf{qW}$; likewise, since the standard algebra \mathbf{S}_r generates the quasivariety \mathbf{CW} of Cartesian algebras [5], $\vdash_{\mathbf{S}}^{\leq, 1}$ is nothing but the 1-assertional logic of the quasivariety \mathbf{CW} . By Lemma 14, $\vdash_{\mathbf{O}}^{\leq, 1}$ is the 1-assertional logic of the quasivariety $1\mathbf{CW}$. $\vdash_{\mathbf{S}}^{\preceq, \top}$ is tightly related in its definition to the \top -assertional logic of the variety \mathbb{GQC} , introduced in [17]. Finally, $\vdash_{\mathbf{S}}^{\leq}$ and $\vdash_{\mathbf{S}}^{\preceq}$ are generalisations to a finite but otherwise arbitrary number of premisses of the “weak” and the “strong” consequence relations of [10], as restricted to the language of $\sqrt{\cdot}\mathbf{qW}$. The logic $\vdash_{\sqrt{\cdot}\mathbf{qW}}^{\leq}$, on the other hand, seems

prima facie to have no special motivation. However, this is not an issue as we can prove that:

Lemma 15. $\vdash_{\sqrt{r}\mathbf{q}\mathbb{W}}^{\leq} = \vdash_{\mathbf{S}}^{\leq}$.

Proof. For the nontrivial inclusion, first observe that any $\sqrt{r}\mathbf{q}\mathbb{W}$ quasiequation which holds in \mathbf{S}_r but fails in $\sqrt{r}\mathbf{q}\mathbb{W}$ fails in a flat $\sqrt{r}\mathbf{q}\mathbb{W}$ algebra. In fact, let ε be a quasiequation which holds in \mathbf{S}_r . It follows that ε holds in \mathbb{CW} , for - as already pointed out - \mathbf{S}_r generates \mathbb{CW} as a quasivariety. Now, let ε fail in $\mathbf{A} \in \sqrt{r}\mathbf{q}\mathbb{W}$. By the direct embedding theorem for $\sqrt{r}\mathbf{q}\mathbb{W}$ algebras, \mathbf{A} embeds into a product $\mathbf{C} \times \mathbf{F}$, for $\mathbf{C} \in \mathbb{CW}$, $\mathbf{F} \in \mathbb{FW}$. Since ε cannot fail in \mathbf{C} , it must fail in \mathbf{F} , which establishes our claim. As a consequence, if the quasiequation $p \leq \gamma_1 \ \& \ \dots \ \& \ p \leq \gamma_n \hookrightarrow p \leq \alpha$ holds in \mathbf{S}_r but fails in $\sqrt{r}\mathbf{q}\mathbb{W}$, then it should fail in some flat algebra, which is impossible since in any flat algebra \mathbf{A} , $\leq = A^2$. \square

Hereafter, we will consistently denote the logic of the preceding lemma by means of the label $\vdash_{\mathbf{S}}^{\leq}$. Since this choice makes a clean sweep of any possible ambiguity, we will also write $\vdash_{\sqrt{r}\mathbf{q}\mathbb{W}}$ in place of $\vdash_{\sqrt{r}\mathbf{q}\mathbb{W}}^{\leq,1}$, to simplify the notation somewhat. Also, in the following $\sqrt{r}^n \alpha$ will denote the formula α preceded by n ($n \geq 0$) occurrences of the sign \sqrt{r} .

The standard strong logic of degrees of truth has a remarkable property:

Lemma 16. *If $\Gamma \vdash_{\mathbf{S}}^{\leq} \sqrt{r}^n p$, then $\sqrt{r}^m p \in \Gamma$ for some $m \equiv n \pmod{4}$.*

Proof. If there is no m s.t. $\sqrt{r}^m p \in \Gamma$, then all $\gamma \in \Gamma$ are either constants, or contain at least an occurrence of \rightarrow , or are of the form $\sqrt{r}^r q$ for some $r \in N$ and $q \neq p$. Now, let $\overrightarrow{\langle a, b \rangle}$ be s.t.

$$\begin{aligned} p^{\mathbf{S}_r}(\overrightarrow{\langle a, b \rangle}) &= \begin{cases} \langle 0, 0 \rangle & \text{if } n \equiv 0 \pmod{4} \\ \langle 0, 1 \rangle & \text{if } n \equiv 1 \pmod{4} \\ \langle 1, 1 \rangle & \text{if } n \equiv 2 \pmod{4} \\ \langle 1, 0 \rangle & \text{if } n \equiv 3 \pmod{4} \end{cases} \\ q^{\mathbf{S}_r}(\overrightarrow{\langle a, b \rangle}) &= \langle \tfrac{1}{2}, \tfrac{1}{2} \rangle, \text{ for } q \neq p \end{aligned}$$

Then, if \bigwedge denotes the meet induced by the strong order, for some $c \in [0, 1]$, $\bigwedge_{\gamma \in \Gamma} \gamma^{\mathbf{S}_r}(\overrightarrow{\langle a, b \rangle}) = \langle c, \tfrac{1}{2} \rangle$, while $\sqrt{r}^n p^{\mathbf{S}_r}(\overrightarrow{\langle a, b \rangle}) = \langle 0, 0 \rangle$, against the hypothesis. On the other side, whenever $\{\sqrt{r}^{r_1} p, \dots, \sqrt{r}^{r_k} p\} \subseteq \Gamma$ but none of the r_1, \dots, r_k is congruent to n modulo 4, then choose $0 < c < 1$ and let $\overrightarrow{\langle a, b \rangle}$ be s.t.

$$\begin{aligned} \sqrt{r}^n p^{\mathbf{S}_r}(\overrightarrow{\langle a, b \rangle}) &= \langle c, 0 \rangle; \\ q^{\mathbf{S}_r}(\overrightarrow{\langle a, b \rangle}) &= \langle \tfrac{1}{2}, \tfrac{1}{2} \rangle, \text{ for } q \neq p. \end{aligned}$$

(Observe that it is always possible to find such a tuple $\overrightarrow{\langle a, b \rangle}$.) Then $\pi_2 \left(\bigwedge_{\gamma \in \Gamma} \gamma^{\mathbf{S}_r} \left(\overrightarrow{\langle a, b \rangle} \right) \right) = \min \left(\frac{1}{2}, c, 1 - c, 1 \right) > 0$, against the hypothesis. \square

Lemma 17.

- (i) $\vdash_{\mathbf{S}}^{\prec} < \vdash_{\mathbf{S}}^{\leq}$;
- (ii) $\vdash_{\mathbf{S}}^{\prec} < \vdash_{\sqrt{\cdot}\mathbf{q}\mathbb{W}}$;
- (iii) $\vdash_{\sqrt{\cdot}\mathbf{q}\mathbb{W}} < \vdash_{\mathbf{S}}^{\leq, 1}$;
- (iv) $\vdash_{\mathbf{S}}^{\leq, 1} < \vdash_{\mathbf{O}}^{\leq, 1}$;
- (v) $\vdash_{\mathbf{S}}^{\prec} < \vdash_{\mathbf{S}}^{\prec, \top}$;
- (vi) $\vdash_{\mathbf{S}}^{\leq} < \vdash_{\mathbf{O}}^{\leq, 1}$;
- (vii) *no other inclusion between the previous logics holds.*

Proof. (i) We prove that $\vdash_{\mathbf{S}}^{\prec} \leq \vdash_{\mathbf{S}}^{\leq}$. Suppose $\Gamma \vdash_{\mathbf{S}}^{\prec} \alpha$ and suppose that for every $\gamma \in \Gamma$, $\gamma^{\mathbf{S}_r} \left(\overrightarrow{\langle a, b \rangle} \right) \geq_{\mathbf{S}_r} \langle c, \frac{1}{2} \rangle$, which means

$$c \leq \min_{\gamma \in \Gamma} \left\{ \pi_1 \left(\gamma^{\mathbf{S}_r} \left(\overrightarrow{\langle a, b \rangle} \right) \right) \right\}.$$

This implies that, for every $\gamma \in \Gamma$,

$$\gamma^{\mathbf{S}_r} \left(\overrightarrow{\langle a, b \rangle} \right) \succeq_{\mathbf{S}_r} \left\langle c, \bigwedge_{\gamma \in \Gamma} \left\{ \pi_2 \left(\gamma^{\mathbf{S}_r} \left(\overrightarrow{\langle a, b \rangle} \right) \right) \right\} \right\rangle$$

whence by hypothesis $\alpha^{\mathbf{S}_r} \left(\overrightarrow{\langle a, b \rangle} \right) \geq_{\mathbf{S}_r} \langle c, \frac{1}{2} \rangle$, as the relation $\geq_{\mathbf{S}_r}$ depends only on the first component. That $\vdash_{\mathbf{S}}^{\prec} < \vdash_{\mathbf{S}}^{\leq}$ can be seen by checking that $p \vdash_{\mathbf{S}}^{\prec} \Box p$, while it is not the case that $p \vdash_{\mathbf{S}}^{\leq} \Box p$.

(ii) Suppose that $\Gamma \vdash_{\mathbf{S}}^{\prec} \alpha$ while it is not the case that $\Gamma \vdash_{\sqrt{\cdot}\mathbf{q}\mathbb{W}} \alpha$. We claim that if the hypothesis is true, then either $\Gamma \vdash_{\mathbf{S}}^{\leq, 1} \alpha$ or else there are a variable p and a natural number n s.t. $\alpha = \sqrt{\cdot}^n p$. In fact, if α is either a constant or contains \rightarrow , then if $\left\{ \gamma^{\mathbf{S}_r} \left(\overrightarrow{\langle a, b \rangle} \right) : \gamma \in \Gamma \right\} \subseteq \left\{ \langle 1, \frac{1}{2} \rangle \right\}$ it must be the case that $\alpha^{\mathbf{S}_r} \left(\overrightarrow{\langle a, b \rangle} \right) \geq_{\mathbf{S}_r} \langle 1, \frac{1}{2} \rangle$ and thus $\alpha^{\mathbf{S}_r} \left(\overrightarrow{\langle a, b \rangle} \right) = \langle 1, \frac{1}{2} \rangle$, for such an element must be regular. This establishes our claim.

Now, if there are a variable p and a natural number n s.t. $\alpha = \sqrt{\cdot}^n p$, then by Lemma 16 $\sqrt{\cdot}^m p$ is in Γ for some $m \equiv n \pmod{4}$. Then, $\Gamma \vdash_{\sqrt{\cdot}\mathbf{q}\mathbb{W}} \alpha$, which is impossible. On the other

hand, if $\Gamma \vdash_{\mathbf{S}}^{\leq, 1} \alpha$, then the sequent is falsified in a flat algebra, i.e. there are $\mathbf{A} \in \mathbb{FW}$ and $\vec{a} \in A$ s.t. for all $\gamma \in \Gamma$, $\gamma^{\mathbf{A}}(\vec{a}) = 0^{\mathbf{A}}$, while $\alpha^{\mathbf{A}}(\vec{a}) \neq 0^{\mathbf{A}}$. But then there are a variable p and a natural number n s.t. $\alpha = \sqrt{r^n} p$, and we fall back into the previous case.

The inclusion is proper since $p \vdash_{\sqrt{r}\mathbf{q}\mathbb{W}} \Box p$, while it is not the case that $p \vdash_{\mathbf{S}}^{\prec} \Box p$.

(iii) The inclusion is trivial. Moreover, $0 \vdash_{\mathbf{S}}^{\leq, 1} p$ while it is not the case that $0 \vdash_{\sqrt{r}\mathbf{q}\mathbb{W}} p$.

(iv) Once again, the inclusion is trivial. Moreover, $\Box p \vdash_{\mathbf{O}}^{\leq, 1} p$ while it is not the case that $\Box p \vdash_{\mathbf{S}}^{\leq, 1} p$.

(v) The inclusion follows from the fact that $\langle 1, 1 \rangle$ is a maximum for $\succeq^{\mathbf{S}_r}$, which is a partial order. Moreover, $p \rightarrow q \vdash_{\mathbf{S}}^{\prec, \top} r$ while it is not the case that $p \rightarrow q \vdash_{\mathbf{S}}^{\prec} r$.

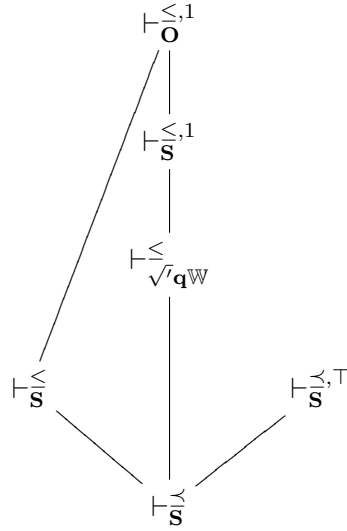
(vi) Suppose that for all $\langle \overrightarrow{a, b} \rangle$, $\bigwedge_{\gamma \in \Gamma} \left\{ \pi_1 \left(\gamma^{\mathbf{S}_r} \left(\langle \overrightarrow{a, b} \rangle \right) \right) \right\} \leq \pi_1 \left(\alpha^{\mathbf{S}_r} \left(\langle \overrightarrow{a, b} \rangle \right) \right)$, and suppose that for all $\gamma \in \Gamma$, $\pi_1 \left(\gamma^{\mathbf{O}_r} \left(\langle \overrightarrow{c, d} \rangle \right) \right) = 1$. Since \mathbf{O}_r is a subalgebra of \mathbf{S}_r , it follows that $\pi_1 \left(\alpha^{\mathbf{O}_r} \left(\langle \overrightarrow{c, d} \rangle \right) \right) = 1$, whence $\alpha^{\mathbf{O}_r} \left(\langle \overrightarrow{c, d} \rangle \right) = \langle 1, \frac{1}{2} \rangle$ as $\langle 1, \frac{1}{2} \rangle / \chi = \{ \langle 1, \frac{1}{2} \rangle \}$. The inclusion is proper as $k \vdash_{\mathbf{O}}^{\leq, 1} p$ yet it is not the case that $k \vdash_{\mathbf{S}}^{\leq} p$.

(vii) Our claim follows from the next table (where Y denotes ‘Yes’ and N denotes ‘No’):

	$\vdash_{\sqrt{r}\mathbf{q}\mathbb{W}}$	$\vdash_{\mathbf{S}}^{\leq, 1}$	$\vdash_{\mathbf{O}}^{\leq, 1}$	$\vdash_{\mathbf{S}}^{\prec, \top}$	$\vdash_{\mathbf{S}}^{\leq}$
$p \vdash \Box p$	Y	Y	Y	N	Y
$\Box p, p \rightarrow q \vdash \Box q$	Y	Y	Y	Y	N
$0 \vdash p$	N	Y	Y	Y	Y
$k \vdash p$	N	Y	Y	Y	N
$\Box p \vdash p$	N	N	Y	Y	Y
$p \rightarrow q \vdash r$	N	N	N	Y	N

□

The next figure summarises the inclusion relationships of Lemma 17.



We proceed by providing a characterization of the entailments in $\vdash_{\sqrt{\cdot}\mathbf{q}\mathbb{W}}$ whose consequents are propositional variables.

Lemma 18. *Let Γ be a finite or empty subset of formulas. Then $\Gamma \vdash_{\sqrt{\cdot}\mathbf{q}\mathbb{W}} p$ iff either:*

(i) *for some $n \in N$, $\sqrt{\cdot}^{4n} p \in \Gamma$;*

or:

(ii) *for some $n \in N$ and for some $1 \leq m \leq 3$, $\sqrt{\cdot}^{4n+m} p \in \Gamma$, and $\Gamma \vdash_{\sqrt{\cdot}\mathbf{q}\mathbb{W}} 0$.*

Proof. From right to left, (i) obviously implies the consequent of the implication. Assume that $\sqrt{\cdot}^{4n+m} p, \Delta \vdash_{\sqrt{\cdot}\mathbf{q}\mathbb{W}} 0$, and fix a $\sqrt{\cdot}\mathbf{q}\mathbb{W}$ algebra \mathbf{A} and $\vec{a} \in A$ s.t. $\sqrt{\cdot}^{4n+m}(\vec{a}) = 1^{\mathbf{A}}, \{\delta^{\mathbf{A}}(\vec{a}) : \delta \in \Delta\} \subseteq \{1^{\mathbf{A}}\}$. Then $0(\vec{a}) = 1^{\mathbf{A}}$, i.e. \mathbf{A} is flat, whence $\sqrt{\cdot}^{4n+m} p(\vec{a}) = 1^{\mathbf{A}}$ implies $p(\vec{a}) = 1^{\mathbf{A}}$.

From left to right, assume $\Gamma \vdash_{\sqrt{\cdot}\mathbf{q}\mathbb{W}} p$ and that for no $n \in N$ we have that $\sqrt{\cdot}^{4n} p \in \Gamma$. For $\gamma \in \Gamma$, let $q_1^0, \dots, q_{j_0}^0$ be all the variables different from p occurring in γ s.t. for some $n \in N$ we have that $\sqrt{\cdot}^{4n} q_i^0 \in \Gamma$. Similarly, let:

- $q_1^1, \dots, q_{j_1}^1$ be the variables different from p occurring in γ s.t. for some $n \in N$ we have that $\sqrt{\cdot}^{4n+1} q_i^1 \in \Gamma$;
- $q_1^2, \dots, q_{j_2}^2$ be the variables different from p occurring in γ s.t. for some $n \in N$ we have that $\sqrt{\cdot}^{4n+2} q_i^2 \in \Gamma$;

- $q_1^3, \dots, q_{j_3}^3$ be the variables different from p occurring in γ s.t. for some $n \in N$ we have that $\sqrt{r^{4n+3}} q_i^3 \in \Gamma$.

Now let

$$\tilde{\Gamma} = \left\{ \left(\left(\left(\gamma \left[\frac{\vec{q}_i^0}{1} \right] \right) \left[\frac{\vec{q}_i^1}{\sqrt{r}1} \right] \right) \left[\frac{\vec{q}_i^2}{0} \right] \right) \left[\frac{\vec{q}_i^3}{\sqrt{r}0} \right] : \gamma \in \Gamma \right\}.$$

This means that, for $a < b$, q_i^a gets replaced before q_i^b , so that e.g. any variable s s.t. $s, \sqrt{r}s, \neg s, \sqrt{r}\neg s \in \Gamma$ is uniformly replaced by 1 throughout. As we can see, $\Gamma \vdash_{\sqrt{r}\mathbf{q}\mathbb{W}} p$ implies that $\tilde{\Gamma} \vdash_{\sqrt{r}\mathbf{q}\mathbb{W}} p$, whence $\tilde{\Gamma} \vdash_{\sqrt{r}\mathbf{q}\mathbb{W}} p$. Remark that p is the sole variable s.t. for some $n, \sqrt{r^n} p \in \tilde{\Gamma}$. We distinguish two cases:

(1) All $\gamma \in \tilde{\Gamma}$ either contain \rightarrow or are constants. Consider the flat algebra \mathbf{F}_{020} and assign p the value $a \neq 1$, and any other variable the value 0. Necessarily, for all $\gamma(p, \vec{p}_i) \in \tilde{\Gamma}$ we have that $\gamma^{\mathbf{F}_{020}}(a, 0, \dots, 0) = 0^{\mathbf{F}_{020}} = 1^{\mathbf{F}_{020}}$, since all $\gamma \in \tilde{\Gamma}$ either contain \rightarrow or are constants. It follows that $p^{\mathbf{F}_{020}}(a, 0, \dots, 0) = 1^{\mathbf{F}_{020}}$; on the other hand, $p^{\mathbf{F}_{020}}(a, 0, \dots, 0) = a \neq 1^{\mathbf{F}_{020}}$, a contradiction.

(2) There are an n and an $1 \leq m \leq 3$ s.t. $\sqrt{r^{4n+m}} p \in \tilde{\Gamma}$, whence also $\sqrt{r^{4n+m}} p \in \Gamma$. We need to check that $\Gamma \vdash_{\sqrt{r}\mathbf{q}\mathbb{W}} 0$. The circumstance that $\sqrt{r^{4n+m}} p \in \Gamma$ implies that either $\Gamma \vdash_{\sqrt{r}\mathbf{q}\mathbb{W}} \sqrt{r}p$ or $\Gamma \vdash_{\sqrt{r}\mathbf{q}\mathbb{W}} \neg p$ or $\Gamma \vdash_{\sqrt{r}\mathbf{q}\mathbb{W}} \sqrt{r}\neg p$. Recall we supposed that $\Gamma \vdash_{\sqrt{r}\mathbf{q}\mathbb{W}} p$. Remembering that $\sqrt{r}\mathbf{q}\mathbb{W}$ satisfies the quasiequation $p \approx \Box p \ \& \ p \approx \sqrt{r}p \hookrightarrow p \approx k$, we have all of the following entailments:

$$\begin{aligned} p, \sqrt{r}p &\vdash \sqrt{r}\mathbf{q}\mathbb{W}0; \\ p, \neg p &\vdash \sqrt{r}\mathbf{q}\mathbb{W}0; \\ p, \sqrt{r}\neg p &\vdash \sqrt{r}\mathbf{q}\mathbb{W}0; \end{aligned}$$

and thus our conclusion readily follows. \square

In a similar way, we prove:

Lemma 19. *Let Γ be a finite or empty subset of formulas. Then $\Gamma \vdash_{\sqrt{r}\mathbf{q}\mathbb{W}} \sqrt{r^n} p$ iff either:*

(i) *for some $m \equiv n \pmod{4}$, $\sqrt{r^m} p \in \Gamma$;*

or:

(ii) *for some m not equivalent to $n \pmod{4}$ $\sqrt{r^m} p \in \Gamma$, and $\Gamma \vdash_{\sqrt{r}\mathbf{q}\mathbb{W}} 0$.*

4.4. Hilbert systems. Our next aim consists in formulating Hilbert-style axiomatisations for the logics $\vdash_{\mathbf{S}}^{\leq,1}$ and $\vdash_{\mathbf{O}}^{\leq,1}$. We will therefore define deductive systems \mathbf{C} and \mathbf{C}_1 whose consequence relations will prove to be sound and complete w.r.t. $\vdash_{\mathbf{S}}^{\leq,1}$ and $\vdash_{\mathbf{O}}^{\leq,1}$, respectively. Incidentally, recall that a *regular formula* [20] is a formula which denotes a regular element in every $\sqrt{I}\mathbf{qW}$ algebra.

Definition 20. *The deductive systems $\vdash_{\mathbf{C}}$ and $\vdash_{\mathbf{C}_1}$ are formulated in the signature $\langle \rightarrow, \sqrt{I}, 1 \rangle$. $\vdash_{\mathbf{C}}$ has the following postulates:*

- Q1. *The axioms and rules of the logic $\vdash_{\mathbf{qW}}$ (Section 3);*
- Q2. $\alpha \Rightarrow \beta$, *for α, β regular formulae*
- Q3 – 4. $\Box \sqrt{I} \Box \alpha \leftrightarrow \sqrt{I} \Box \sqrt{I} \Box \alpha$
- Q5 – 6. $\neg \Box \sqrt{I} \Box \alpha \leftrightarrow \sqrt{I} \Box \sqrt{I} \Box \alpha$
- MP*. $\alpha, \alpha \rightarrow \beta, \alpha \Rightarrow \beta, \beta \Rightarrow \alpha \vdash \beta$
- GR. $\alpha, \beta \vdash \alpha \Rightarrow \beta$

$\vdash_{\mathbf{C}_1}$ extends $\vdash_{\mathbf{C}}$ by the rule

$$\text{MP. } \alpha, \alpha \rightarrow \beta \vdash \beta$$

Observe that $\vdash_{\mathbf{C}_1}$ can be viewed as the expansion of Łukasiewicz logic by the connective \sqrt{I} and, simultaneously, as its extension by the axioms Q2-Q6. The rule MP*, as well as all the inference rules of $\vdash_{\mathbf{S}}$, become redundant in $\vdash_{\mathbf{C}_1}$.

Lemma 21. *The following hold:*

- L1. $\alpha \vdash_{\mathbf{C}} 1 \Rightarrow \alpha$
- L2. $\alpha \vdash_{\mathbf{C}} \alpha \Rightarrow 1$
- L3. $\alpha \rightarrow 1, 1 \rightarrow \alpha, \alpha \Rightarrow 1, 1 \Rightarrow \alpha \vdash_{\mathbf{C}} \alpha$

Proof. L1 and L2 follow from GR and Q1 (more precisely, the provability of 1).

For L3, let $\alpha = 1$ in MP*. Since $\vdash_{\mathbf{C}} 1$, we get that $1 \rightarrow \alpha, 1 \Rightarrow \alpha, \alpha \Rightarrow 1 \vdash_{\mathbf{C}} \alpha$, whence our conclusion. \square

Corollary 22. *Let $\Delta(p, q)$ be $\{p \rightarrow q, q \rightarrow p, p \Rightarrow q, q \Rightarrow p\}$. Moreover, let $\delta(p) = p$ and $\epsilon(p) = 1$. Then, for every formula α :*

$$\alpha \dashv\vdash_{\mathbf{C}} \Delta(\delta(\alpha), \epsilon(\alpha))$$

Proof. From L1, L2 and L3, together with Q1. \square

Theorem 23.

- (i) $\Gamma \vdash_{\mathbf{C}} \alpha$ *iff* $\Gamma \vdash_{\mathbf{S}}^{\leq,1} \alpha$;

$$(ii) \Gamma \vdash_{\mathbf{C}_1} \alpha \text{ iff } \Gamma \vdash_{\mathbf{O}}^{\leq, 1} \alpha.$$

Proof. (i) The left-to-right direction of our theorem follows from the fact that all the postulates of $\vdash_{\mathbf{C}}$ are sound in the class of Cartesian $\sqrt{\cdot}$ qW algebras; consequently they hold in $\vdash_{\mathbf{S}}^{\leq, 1}$.

For the right-to-left direction, we proceed as follows. Let $\Delta(p, q)$ be

$$\{p \rightarrow q, q \rightarrow p, p \Rightarrow q, q \Rightarrow p\},$$

and let $\delta(p) = p$ and $\epsilon(p) = 1$. $\vdash_{\mathbf{S}}^{\leq, 1}$ is the 1-assertional logic of the relatively 1-regular quasivariety $\mathbb{C}\mathbb{W}$, whereby it is regularly algebraisable with $\Delta(p, q)$ as a system of equivalence formulae and $\delta(p) \approx \epsilon(p)$ as defining equation [9]. It is therefore axiomatisable by means of the following postulates:

- (1) $\Delta(\zeta_1, \eta_1), \dots, \Delta(\zeta_n, \eta_n) \vdash_{\mathbf{S}}^{\leq, 1} \Delta(\zeta, \eta)$, for every quasiequation $\zeta_1 \approx \eta_1 \ \& \ \dots \ \& \ \zeta_n \approx \eta_n \hookrightarrow \zeta \approx \eta$ in the axiomatisation of $\mathbb{C}\mathbb{W}$
- (2) $\alpha, \Delta(\alpha, \beta) \vdash_{\mathbf{S}}^{\leq, 1} \beta$
- (3) $\alpha, \beta \vdash_{\mathbf{S}}^{\leq, 1} \Delta(\alpha, \beta)$
- (4) $\vdash_{\mathbf{S}}^{\leq, 1} \Delta(\alpha, \alpha)$
- (5) $\Delta(\alpha, \beta), \Delta(\beta, \gamma) \vdash_{\mathbf{S}}^{\leq, 1} \Delta(\alpha, \gamma)$
- (6) $\Delta(\alpha, \beta), \Delta(\gamma, \delta) \vdash_{\mathbf{S}}^{\leq, 1} \Delta(\alpha \rightarrow \gamma, \beta \rightarrow \delta)$
- (7) $\Delta(\alpha, \beta) \vdash_{\mathbf{S}}^{\leq, 1} \Delta(\sqrt{\cdot}\alpha, \sqrt{\cdot}\beta)$

It is sufficient to show that all of 1.-7. above are derivable in $\vdash_{\mathbf{C}}$.

For the rules in 1., we distinguish three cases. If $\zeta \approx \eta$ is an equation in the axiomatisation of $\mathbf{q}\mathbb{W}$, we prove $\vdash_{\mathbf{C}} \Delta(\zeta, \eta)$ using Q1 and Q2. For example, given the equation $\Box(\alpha \rightarrow \beta) \approx \alpha \rightarrow \beta$ (axiom qW1), we get both $\vdash_{\mathbf{C}} \Box(\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \beta)$ and $\vdash_{\mathbf{C}} (\alpha \rightarrow \beta) \rightarrow \Box(\alpha \rightarrow \beta)$ by the completeness of the axiom system for $\vdash_{\mathbf{q}\mathbb{W}}$, while $\vdash_{\mathbf{C}} \Box(\alpha \rightarrow \beta) \Rightarrow (\alpha \rightarrow \beta)$ and $\vdash_{\mathbf{C}} (\alpha \rightarrow \beta) \Rightarrow \Box(\alpha \rightarrow \beta)$ are obtained as consequences of Q2. If $\zeta \approx \eta$ is the equation

$$\Box\sqrt{\cdot}\Box\alpha \approx \sqrt{\cdot}\Box\sqrt{\cdot}\Box\alpha$$

we resort to Q3-Q6. Finally, the Cartesian quasiequation

$$\Box\alpha \approx \Box\beta \ \& \ \Box\sqrt{\cdot}\alpha \approx \Box\sqrt{\cdot}\beta \hookrightarrow \alpha \approx \beta$$

is taken care of with the help of Q1 and the completeness of the axiom system for $\vdash_{\mathbf{q}\mathbb{W}}$.

2. follows from MP*.

3. follows from the completeness of the axiom system for $\vdash_{\mathbf{q}\mathbb{W}}$ and GR.

4., 5. and 7. follow from the completeness of the axiom system for $\vdash_{\mathbf{qW}}$.

Finally, as regards 6., we must show that

$$\begin{array}{ccc} \alpha \rightarrow \beta & \beta \rightarrow \alpha & (\alpha \rightarrow \gamma) \rightarrow (\beta \rightarrow \delta) \\ \gamma \rightarrow \delta & \delta \rightarrow \gamma & (\beta \rightarrow \delta) \rightarrow (\alpha \rightarrow \gamma) \\ \alpha \Rightarrow \beta & \beta \Rightarrow \alpha & (\alpha \rightarrow \gamma) \Rightarrow (\beta \rightarrow \delta) \\ \gamma \Rightarrow \delta & \delta \Rightarrow \gamma & (\beta \rightarrow \delta) \Rightarrow (\alpha \rightarrow \gamma) \end{array} \vdash_{\mathbf{C}}$$

the completeness of the axiom system for $\vdash_{\mathbf{qW}}$ suffices for the former two conclusions; the latter two follow from Q2.

(ii) We follow the same procedure as in (i). We only have to take care of the additional rule

$$\begin{array}{l} \Box\alpha \rightarrow 1 \quad 1 \rightarrow \Box\alpha \quad \vdash_{\mathbf{C}_1} \quad \{\alpha \rightarrow 1, 1 \rightarrow \alpha, \alpha \Rightarrow 1, 1 \Rightarrow \alpha\} \\ \Box\alpha \Rightarrow 1 \quad 1 \Rightarrow \Box\alpha \end{array}$$

However, the first formula in the conclusion set is a thesis of Łukasiewicz logic. The second formula can be obtained by the rule $\Box\alpha \vdash \alpha$ from $1 \rightarrow \Box\alpha$. The last two formulas are obtained as follows: α is derived from $1 \rightarrow \Box\alpha$ by applying twice the inference rule $\Box\alpha \vdash \alpha$; then, since 1 is a thesis of Łukasiewicz logic, our conclusions follow from GR. \square

5. LOGICS FROM $\sqrt{\cdot}$ QW ALGEBRAS AND ABSTRACT ALGEBRAIC LOGIC

We are now going to place the above logics within the Leibniz hierarchy. This is easily done for $\vdash_{\mathbf{S}}^{\leq, 1}$ and $\vdash_{\mathbf{O}}^{\leq, 1}$: being the 1-assertional logics of relatively point regular quasivarieties, they are regularly algebraisable and their equivalent quasivariety semantics are, respectively, \mathbf{CW} and $1\mathbf{CW}$ [9].

Since \mathbf{CW} and $1\mathbf{CW}$ are relatively 1-regular quasivarieties, characterising the matrix models of their 1-assertional logics boils down to providing a description of the deductive filters of such logics on members of, respectively, \mathbf{CW} and $1\mathbf{CW}$ (cf. [8], [1]). This is, indeed, the next goal we set ourselves.

Definition 24. Let \mathbf{A} be a Cartesian $\sqrt{\cdot}$ qW algebra. A subset $F \subseteq A$ is called an implicative filter of \mathbf{A} iff it obeys the following conditions:

- F1 $1 \in F$;
- F2 $a, b \in F$ implies $a \otimes b \in F$;
- F3 $a \in F$ implies $\Box a \in F$;
- F4 $0 \in F$ implies $F = A$;
- F5 $a, a \rightarrow b, a \Rightarrow b, b \Rightarrow a \in F$ implies $b \in F$;
- F6 $a, b \in F$ implies $a \Rightarrow b \in F$.

An implicative filter of \mathbf{A} is called strong iff it obeys the additional condition

$$F7 \quad a, a \rightarrow b \in F \text{ implies } b \in F.$$

Lemma 25. (i) Let \mathbf{A} be a Cartesian $\sqrt{\cdot}$ qW algebra, and let $F \subseteq A$ be an implicative filter of such. Suppose moreover that a, b are regular elements of A s.t. $a, a \rightarrow b \in F$. Then $b \in F$. (ii) In the definition of strong implicative filter, the conditions F2, F3, F4, F5 are redundant.

Proof. (i) If a, b are regular, then $a \Rightarrow b, b \Rightarrow a = 1 \in F$, whence we are done by F5. (ii) Remark that, for any $a, b \in A$, we have that

$$a \rightarrow (b \rightarrow (a \otimes b)) = 0 \rightarrow a = a \rightarrow \Box a = 1 \in F,$$

whence F2, F3 and F4 are taken care of. The redundancy of F5 is obvious. \square

It is immediate that implicative filters are closed w.r.t. the weak preorder \leq . On the other hand, it is immediate that strong implicative filters are closed w.r.t. the partial order \leq defined by

$$a \leq b \text{ iff } a \leq b \text{ and } \Box \sqrt{\cdot} a = \Box \sqrt{\cdot} b.$$

Theorem 26. Let \mathbf{A} be a Cartesian $\sqrt{\cdot}$ qW algebra, and let $F \subseteq A$.

- (i) F is an implicative filter of \mathbf{A} iff it is a $\vdash_{\mathbf{S}}^{\leq, 1}$ -filter on \mathbf{A} ;
- (ii) F is a strong implicative filter of \mathbf{A} iff it is a $\vdash_{\mathbf{O}}^{\leq, 1}$ -filter on \mathbf{A} .

Proof. (i) Left to right. In virtue of Theorem 23, it suffices to prove that every implicative filter is closed w.r.t. the postulates of the logic \mathbf{C} - in other words, if $\alpha_1, \dots, \alpha_n \vdash_{\mathbf{C}} \alpha$ is a (possibly 0-premiss) rule of such logic, then if $\vec{a} \in A$ and $\alpha_i(\vec{a}) \in A$ for every $i \leq n$, it follows that $\alpha(\vec{a}) \in A$. This can be done in the usual inductive way. The axioms of $\vdash_{\mathbf{qW}}$ cause no problem, since they always evaluate at 1 and thus they accounted for by F1. The rule (Reg) preserves membership in the filter by F3. The rules (AReg1-3) are accounted for by F1 and Lemma 25.(i). (Inv1-2) are trivial, while (EFQ) is in order by F4. This much suffices for the postulates of $\vdash_{\mathbf{qW}}$.

Q2: If α, β are regular formulae, then $\alpha(\vec{a}), \beta(\vec{a})$ are regular elements for any $\vec{a} \in A$, whence $\alpha(\vec{a}) \Rightarrow \beta(\vec{a}) = 1 \in F$.

Q3-6: All of these axioms are handled similarly; let us consider Q3 by way of example. Since $k \leq \sqrt{\cdot} k$ in any $\sqrt{\cdot}$ qW algebra, we have that for any $a \in A$

$$\Box \sqrt{\cdot} \Box a \rightarrow \sqrt{\cdot} \Box \sqrt{\cdot} \Box a = k \rightarrow \sqrt{\cdot} k = 1 \in F.$$

MP*: by F5.

GR: by F6.

The right-to-left direction is easily disposed of by checking that the closure conditions in Definition 24 correspond to sound $\vdash_{\mathbf{S}}^{\leq,1}$ -rules. Only F5 requires a modicum of proof: suppose then $a = a \rightarrow b = a \Rightarrow b = b \Rightarrow a = 1$. This implies $\Box b = \Box 1$ and $\Box \sqrt{b} = \Box \sqrt{1}$, whence $b = 1$ by the Cartesian quasiequation.

(ii) Trivial, in the light of Theorem 23 and Lemma 25.(ii). \square

Bar $\vdash_{\mathbf{S}}^{\leq,1}$ and $\vdash_{\mathbf{O}}^{\leq,1}$, none of the other logics we have considered so far is even protoalgebraic. We prove this result first for $\vdash_{\sqrt{\cdot}\mathbf{qW}}$.

Lemma 27. *Let \mathbf{A} be a flat $\sqrt{\cdot}\mathbf{qW}$ algebra.*

- (i) *If $F \subseteq A$ is a nonempty $\vdash_{\sqrt{\cdot}\mathbf{qW}}$ -filter on \mathbf{A} , then $0 \in F$;*
- (ii) *If $F \subseteq A$ contains 0 and is closed w.r.t. $\sqrt{\cdot}$, then F is a $\vdash_{\sqrt{\cdot}\mathbf{qW}}$ -filter on \mathbf{A} .*

Proof. (i) It follows from the fact that $\alpha \vdash_{\sqrt{\cdot}\mathbf{qW}} 1$ and that $1^{\mathbf{A}} = 0^{\mathbf{A}}$.

(ii) Let F contain 0. Since flat algebras satisfy the equation $p \rightarrow q \approx 0$, F is closed w.r.t. all entailments whose consequents either contain \rightarrow or are constants. What we need to prove, then, is that if $\{\gamma^{\mathbf{A}}(\vec{a}) : \gamma \in \Gamma\} \subseteq F$ and $\Gamma \vdash_{\sqrt{\cdot}\mathbf{qW}} \sqrt{r^n} p$, then $\sqrt{r^n} p^{\mathbf{A}}(\vec{a}) \in F$. Anyway, Lemma 18 implies that for some m , $\sqrt{r^m} p \in \Gamma$. Since F is closed w.r.t. $\sqrt{\cdot}$, this implies our conclusion. \square

Theorem 28. $\vdash_{\sqrt{\cdot}\mathbf{qW}}$ *is not protoalgebraic.*

Proof. It suffices to show that there is an algebra of type $\langle 2, 1, 0 \rangle$ s.t. the Leibniz operator is not monotone on its $\vdash_{\sqrt{\cdot}\mathbf{qW}}$ -filters. So, choose the flat algebra \mathbf{F}_{120} :

$$0 \text{ --- } a \text{ --- } b \text{ --- } \sqrt{b}$$

By Lemma 27, $\{0\}$ and $\{0, a\}$ are $\vdash_{\sqrt{\cdot}\mathbf{qW}}$ -filters of such. Anyway, it is easily checked that $\Omega(\{0\})$ is the congruence whose cosets are $\{0\}$ and $\{a, b, \sqrt{b}\}$, while $\Omega(\{0, a\})$ is the congruence whose cosets are $\{0, a\}$ and $\{b, \sqrt{b}\}$. Thus, $\Omega(\{0\})$ is not included in $\Omega(\{0, a\})$. \square

As a consequence, $\vdash_{\mathbf{S}}^{\leq}$ is not protoalgebraic, being a sublogic of $\vdash_{\sqrt{\cdot}\mathbf{qW}}$. We also observe that

Lemma 29. $\vdash_{\mathbf{S}}^{\leq, \top}$ *is not protoalgebraic.*

Proof. There can be no set of formulae Δ s.t. $\vdash_{\mathbf{S}}^{\leq, \top} \Delta(p, p)$: in fact, such formulae should contain at least an occurrence of \rightarrow , and this would constrain their evaluation on the standard algebra to the set $\{\langle a, b \rangle : a = \frac{1}{2} \text{ or } b = \frac{1}{2}\}$. \square

Finally, we complete our chart by showing that $\vdash_{\mathbf{S}}^{\leq}$ is not protoalgebraic.

Recall from [7] that q-lattices are a non-idempotent generalisation of lattices. It was observed in [18] that any $\sqrt[n]{q}W$ algebra has a term reduct $\langle A, \mathbb{m}, \mathbb{u} \rangle$ which is a q-lattice. Filters of q-lattices are defined in the same way as in lattices: they are upward closed subsets which are closed w.r.t. the pseudo-meet operation \mathbb{m} . It is immediate to verify that the following two conditions are jointly equivalent to the upward closure requirement:

- (i) If $a \in F$, then $a \mathbb{u} b \in F$;
- (ii) If $a \mathbb{u} a \in F$, then $a \in F$.

Lemma 30. $\vdash_{\mathbf{S}}^{\leq}$ -filters on a $\sqrt[n]{q}W$ algebra \mathbf{A} are the filters of the q-lattice term reduct of \mathbf{A} .

Proof. From right to left, suppose that F is a q-lattice filter of \mathbf{A} and let $\alpha_1(\vec{c}), \dots, \alpha_n(\vec{c}) \in F, \alpha_1, \dots, \alpha_n \vdash_{\mathbf{S}}^{\leq} \beta$. Then $\alpha_1(\vec{c}) \mathbb{m} \dots \mathbb{m} \alpha_n(\vec{c}) \leq \beta(\vec{c})$, whence $\beta(\vec{c}) \in F$, as F is a q-lattice filter. From right to left, closure w.r.t. \mathbb{m} follows from the fact that $\alpha, \beta \vdash_{\mathbf{S}}^{\leq} \alpha \mathbb{m} \beta$, while upward closure follows from the fact that $\alpha \vdash_{\mathbf{S}}^{\leq} \alpha \mathbb{u} \beta$ and $\alpha \mathbb{u} \alpha \vdash_{\mathbf{S}}^{\leq} \alpha$. \square

Theorem 31. $\vdash_{\mathbf{S}}^{\leq}$ is not protoalgebraic.

Proof. Fix $n \geq 1$ and let $\mathbf{A} = \mathbf{Rt}(\mathbf{W}_{2n+1})^\omega$. Also, let

$$a = \left\langle \frac{1}{m} : \exists n \in \omega (m = 2n + 1) \right\rangle$$

and let $F = [a']$ be the $\vdash_{\mathbf{S}}^{\leq, 1}$ -filter generated by a . This is both a $\vdash_{\mathbf{S}}^{\leq, 1}$ -filter on \mathbf{A} and a q-lattice filter on the same algebra - since $b, c \in F$ implies $b \otimes (b' \oplus c) = b \mathbb{m} c \in F$. Recall that the equivalence formulae for $\vdash_{\mathbf{S}}^{\leq, 1}$ are

$$\{p \rightarrow q, q \rightarrow p, p \Rightarrow q, q \Rightarrow p\}.$$

We now prove that $\langle a, 0 \rangle \in \Omega(F)$. In fact, $a \rightarrow 0 = \neg a \in F$ by definition, while

$$0 \rightarrow a = a \Rightarrow 0 = 0 \Rightarrow a = 1 \in F.$$

For $X \subseteq A$, let $F_{\leq}(X)$ denote the smallest q-lattice filter of A containing X . Then we have also that $\langle a, 0 \rangle \in \Omega(F_{\leq}(F))$. However, $a \notin F_{\leq}(F)$, for F and $F_{\leq}(F)$ contain the same regular elements. If $\vdash_{\mathbf{S}}^{\leq}$ were protoalgebraic, Ω should be monotonic on deductive filters of \mathbf{A} , whence $\langle a, 0 \rangle \in \Omega(F_{\leq}(F, a))$. Since $a \in F_{\leq}(F, a)$, by compatibility also $0 \in F_{\leq}(F, a)$. Since $F_{\leq}(F)$ is a q-lattice filter, by standard q-lattice filter generation $0 = b \mathbb{m} a$ for

some $b \in [a']$. From this last fact we obtain that there exists n s.t. $n(\neg(a)) \leq b$, i.e. $n(\neg(a)) \mathbin{\&}\! a = 0$, by q-lattice properties. But this is false, given the way a was defined. \square

Recall the definition of selfextensional logic from Section 2 (see also [15]). Apparently, we have that $\vdash_{\mathbf{S}}^{\prec}$ is selfextensional, while $\vdash_{\mathbf{S}}^{\leq}$, $\vdash_{\mathbf{S}}^{\leq,1}$ and $\vdash_{\mathbf{O}}^{\leq,1}$ are not. For,

- p and $\Box p$ are $\vdash_{\mathbf{S}}^{\leq}$ -interderivable, while $\sqrt{\cdot} p$ and $\sqrt{\cdot} \Box p$ are not;
- $\sqrt{\cdot} 0$ and k are $\vdash_{\mathbf{S}}^{\leq,1}$ -interderivable, while $\sqrt{\cdot} \sqrt{\cdot} 0$ and $\sqrt{\cdot} k$ are not;
- the latter counterexample also shows the failure of selfextensionality for $\vdash_{\mathbf{O}}^{\leq,1}$.

We finally prove that $\vdash_{\mathbf{S}}^{\prec, \top}$ is selfextensional.

Theorem 32. $\vdash_{\mathbf{S}}^{\prec, \top}$ is selfextensional.

Proof. Remark that, if $\alpha \vdash_{\mathbf{S}}^{\prec, \top} \beta$, then either i) α contains \rightarrow ; or (ii) α contains a constant; or (iii) α, β are of the form $\sqrt{\cdot}^n p$ for some n and some p . In fact,

$$\sqrt{\cdot}^m q \vdash_{\mathbf{S}}^{\prec, \top} \gamma \rightarrow \delta$$

is always falsifiable by choosing a s.t. $\sqrt{\cdot}^m q(a) = \langle 1, 1 \rangle$. As a corollary, if α, β are $\vdash_{\mathbf{S}}^{\prec, \top}$ -interderivable, then either $\alpha(\vec{a})$ and $\beta(\vec{a})$ are regular elements for any $\vec{a} \in A \in \sqrt{\cdot} \mathbf{qW}$, or else we have that $\alpha = \sqrt{\cdot}^m p, \beta = \sqrt{\cdot}^n q$ for n, m s.t. $|m - n| \equiv 0 \pmod{4}$. In the former case, supposing that α, β and γ, δ are pairwise $\vdash_{\mathbf{S}}^{\prec, \top}$ -interderivable, $\sqrt{\cdot} \alpha, \sqrt{\cdot} \beta$ are vacuously such, and likewise for $\alpha \rightarrow \gamma, \beta \rightarrow \delta$. In the latter case, $\sqrt{\cdot} \alpha, \sqrt{\cdot} \beta$ are $\vdash_{\mathbf{S}}^{\prec, \top}$ -interderivable as the two formulae denote necessarily the same element, and analogously for $\alpha \rightarrow \gamma, \beta \rightarrow \delta$. \square

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